Particle Accelerators: Transverse Beam Dynamics

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Guiding charged particle

- (Varying) electric field changes their energy
  - determines the energy and arrival time
- Dipole magnets change their direction
  - defines a reference trajectory
- Consider motion with respect to reference
  - six-dimensional: $x, x', y, y', \tau, dp/p$
  - small deviation $\rightarrow$ linear oscillatory motion *if stable*
- Ensemble of particles $\rightarrow$ Distribution, Moments
- We will here focus on one transverse direction

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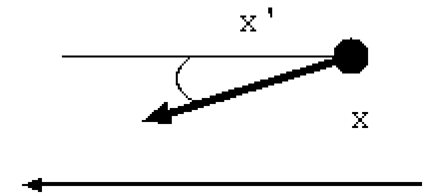
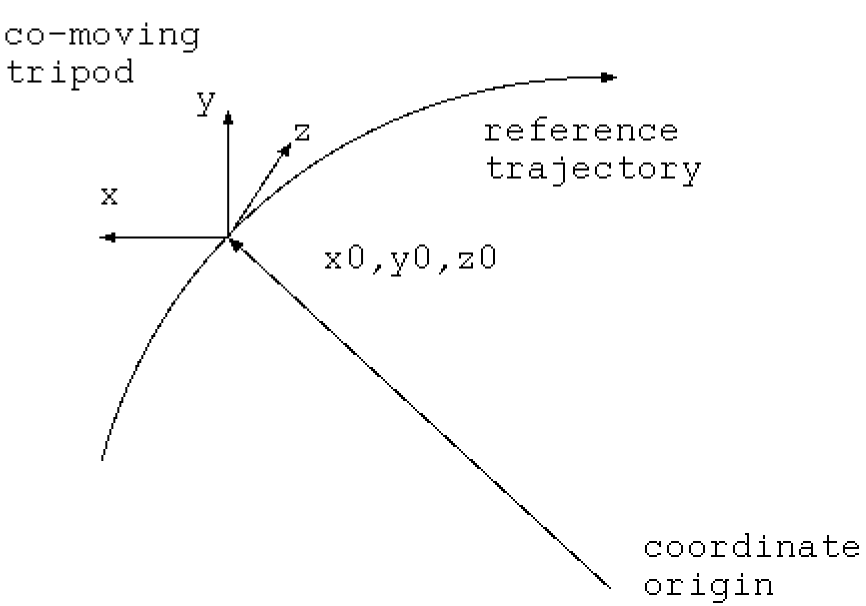
Single Particles



- **Reference trajectory** is determined by design and labelled by x_0, y_0, z_0
- and reference time t_0



- Describe position of a particle by how much it differs from the reference particle
- 3D space → 6D phase space of positions and momenta
- New coordinates:
  - horizontal: x, $x'=p_x/p_0$
  - vertical: y, $y'=p_y/p_0$
  - longitudinal: τ, $\delta=\Delta p/p_0$
- Subtle differences in MAD that vanish in ultra-relativistic limit.

Tranverse Beam Optics

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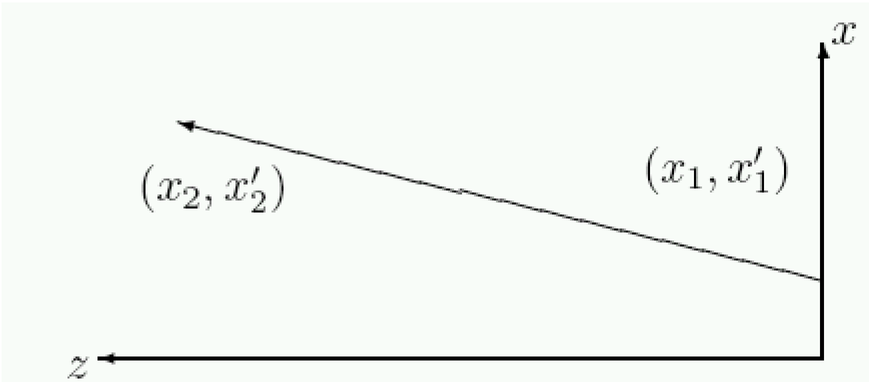
- So far we talked about the particles and how to represent them.
- Now we need to talk about the elements that make them go from one place to the other.
- Or affect the beam in any other way, for example accelerate, or do
- Nothing, a.k.a. beam pipe
- Magnets: quadrupoles, dipoles
- We will consider only one degree of freedom, i.e. horizontal x,x'
- Need stable situation $\rightarrow$ restoring force

Drift Space



- Particles go on a straight line in the absence of external forces
- From right to left (will become clear in a minute)
- Map particle with initial coordinate x_1 and x'_1 to final coordinates x_2 and x'_2

$$\begin{aligned} x_2 &= x_1 + L x_1' \\ x_2' &= x_1' \end{aligned}$$

- Linear set of equations
- Equivalent for vertical y, y'
- Write this in matrix form

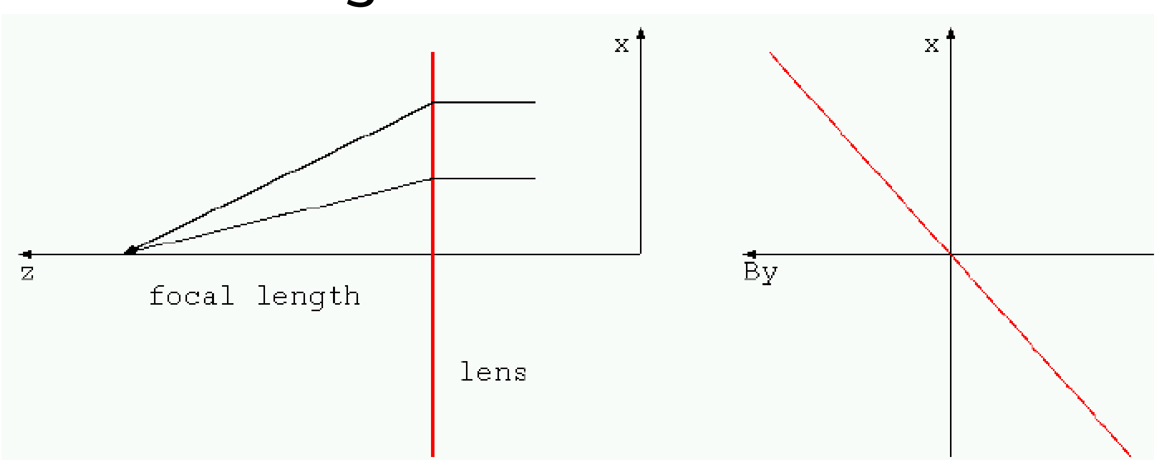
$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

- Concatenation of two drift spaces works, it's a matrix multiplication, try it!

Thin Quadrupole

- Similar to an optical lens
- Twice the distance gets twice the deflection such that both rays make it through the focus
- Linear restoring force
- Requires linearly rising magnetic field (turn head to left) → **linear** again

- Transfer matrix maps input x_1 and x'_1 to output coordinates x_2 and x_2'
- Parallel rays $(x_1,0)$ going through thin quad and drift of length *f* go through zero, independent of x_1



$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

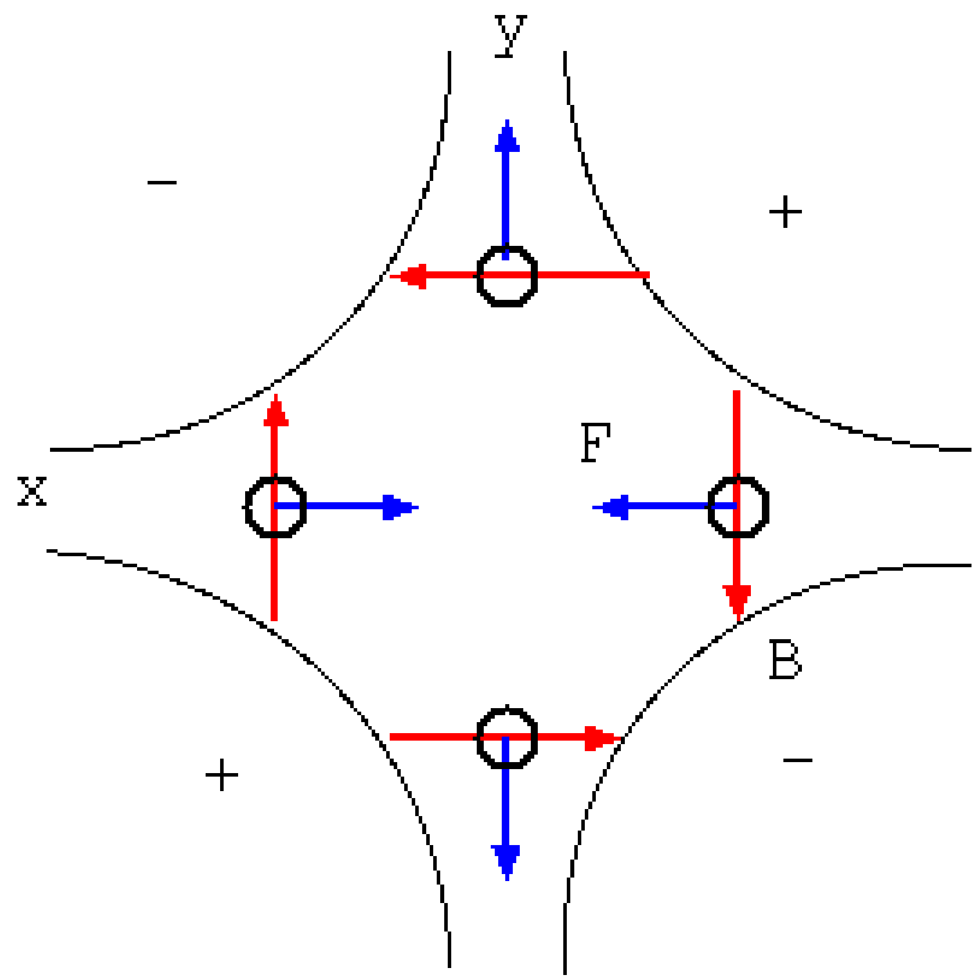
Quadrupoles, both planes

- Quads focus in one plane, but defocus in the other
- A consequence of Maxwell's equations $dB_y/dx = dB_x/dy$

$$\nabla \times \vec{B} = 0$$

- The 4 x 4 matrix therefore looks like

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 1 \end{pmatrix}$$

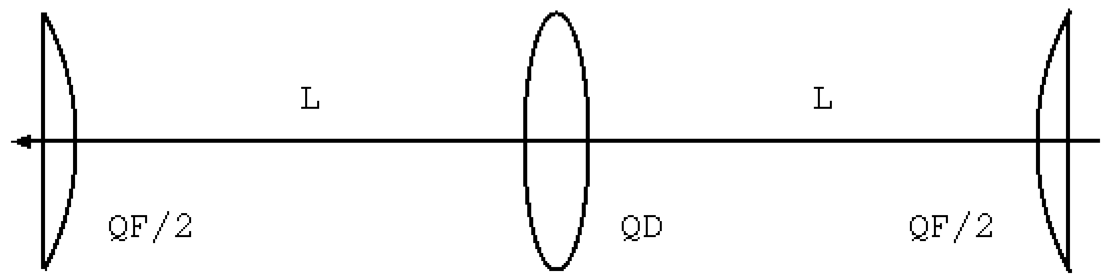


Beamlines and Rings

- Now we have the most important elements of an accelerator: dipoles, drifts and quadrupoles.
- There is a fundamental difference between beamlines and rings
  - **Beamlines:** initial value problem
    - transient problem
  - **Rings:** periodic boundary condition
    - equilibrium (which might not exist)
    - The beam bites its tail
  - We will stick to periodic systems (rings)

FODO lattice

- Consider straight beamline with alternating drifts and quads:



- Represent by a transfer matrix

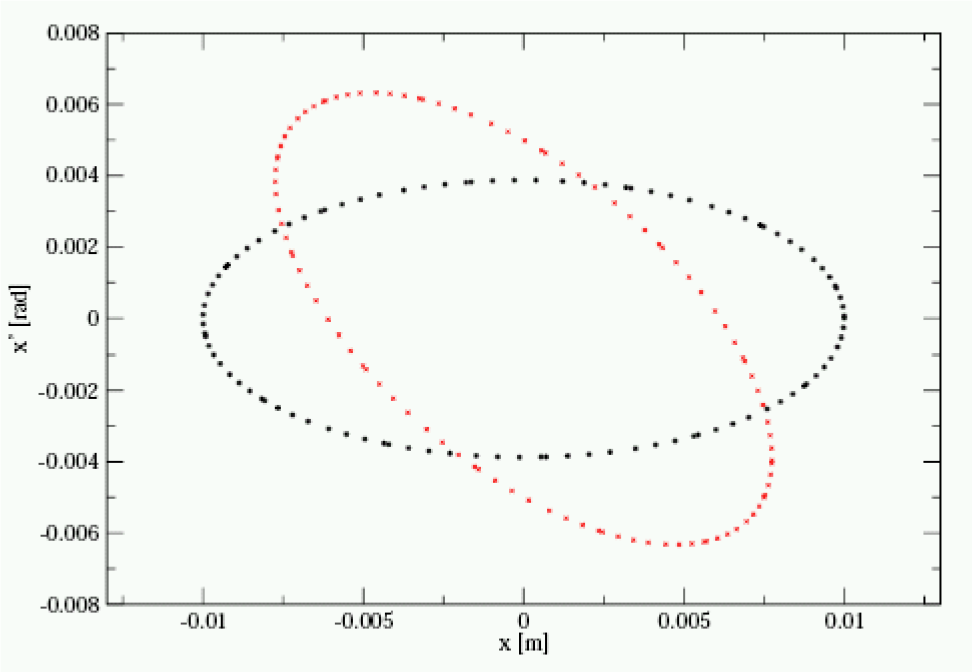
$$M = \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}$$

- Multiplying up all matrices yields

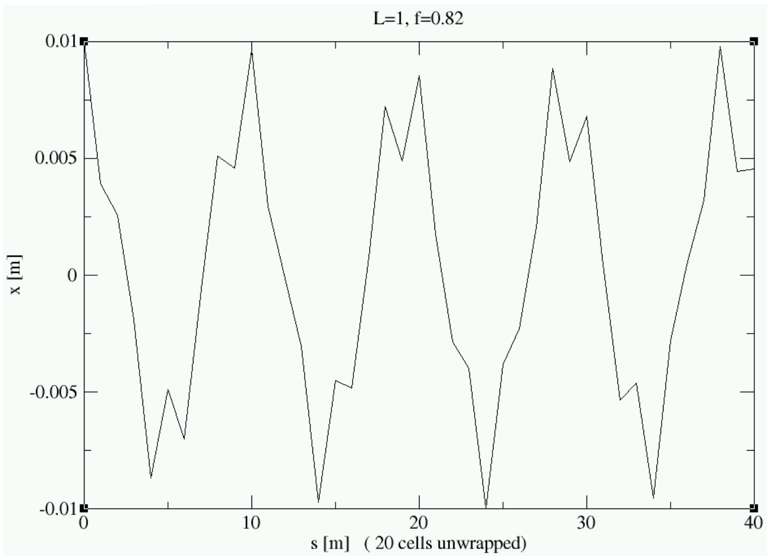
$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ -\frac{1}{4f^2} + \frac{L^2}{16f^3} & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

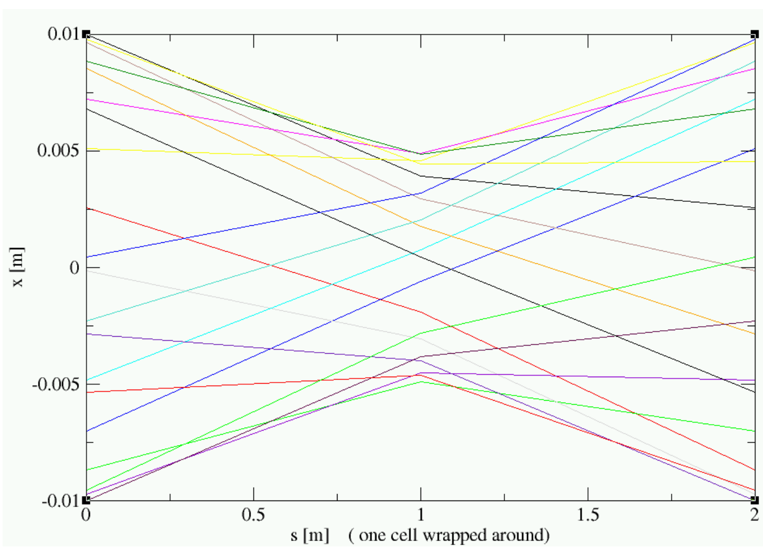
Particle motion in FODO

- Assume that we have an infinite sequence of FODO cells → periodic boundary conditions
- Use L = 1 m, f = ± 1 m.
- Start at x=0.01 m, x'=0
- Iterate 100 times
- Observe x and x' in QF
- Center of first straight
- Poincare map



Betatron Oscillations





- Unwind the transverse motion of a single particle along s → oscillations
- called betatron oscillations
- If plotted modulo cell length, we observe that the trajectory of a single particle has an envelope
- remember β as the maximum excursion at a given location s
- Envelope is given by β(s)
- Action variable J scales amplitude, but not shape

Parametrizing the Transfer matrix

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- Observation:
  - Motion is oscillatory → rotation
  - Matrices have unit determinant → 3 parameters
  - Two parameters for 'shape'

$$M = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1+\alpha^2}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}.$$

Periodic !

- Rotation matrix describes a circle and α, β describe how it is deformed into an ellipse
- By construction the same α, β as used before

Calculations for FODO

- Try out the parametrization for periodic FODO

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1+\alpha^2}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ -\frac{1}{4f^2} + \frac{L^2}{16f^3} & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

- Sum of diagonal elements $\cos\mu = 1 - \frac{L^2}{8f^2}$
- Use 1-cosμ = 2 $\sin^2(\mu/2)$ $\sin(\mu/2) = \frac{L}{4f}$
- From the 12 element $\beta = L\frac{1 + L/4f}{\sin\mu} = L\frac{1 + \sin(\mu/2)}{\sin\mu}$
- Difference of diagonal elements: α=0
- Can calculate μ, α, β from transfer matrix!

Tune

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- Imagine having calculated the transfer matrix for and arbitrary ring
- and do the trick from the slide before
- μ is the phase advance for one turn
- Q=μ/2π is the number of oscillations per turn
- and is called **tune** of the ring
- Q_x: horizontal, Q_y: vertical
- The tune Q is independent of where we start calculating the transfer matrix

Why are beta-functions useful?

- They give an indication where the particle amplitude is large → **envelope**
- They tell where the beam is **sensitive** to perturbations
  - use the parametrization in (#)
  - calculate R_{12} which tells you the response of the particle position x at (2) to a kick θ at (1)
  - $x(\text{at } 2) = R_{12}\, \theta(\text{at } 1)$
  - and $$R_{12} = \sqrt{\beta_1 \beta_2} \sin \mu$$

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Stability

- Observe that the tune is calculated from the sum of the diagonal elements of the transfer matrix
- $2\cos(2\pi Q) = M_{11} + M_{22} = Tr(M)$
- This can actually fail if Tr(M) > 2 or < -2
- an constitutes the limit of stability
- Lesson: You can place your quadrupoles in a stupid way such that no stable operation is possible.

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The Beam or Sigma-matrix

- So far, we mostly discussed the magnets setup, the lattice and single particle motion
- Now we have a look at the beam as a ensemble om many particles
- Determine how the moments of a distribution propagate through a beamline or ring
- Consider transfer matrix R_{ij} and state vector of individual particle x_i where i,j is 2,4,6, depending on what dynamics we discuss

Propagating Moments 1

- Single particle is mapped by $\bar{x}_i = \sum_{j=1}^{n} R_{ij} x_j$
- The first moment is mapped according to

$$\begin{aligned} \bar{X}_i &= \langle \bar{x}_i \rangle = \langle \sum_{j=1}^{n} R_{ij} x_j \rangle = \sum_{j=1}^{n} R_{ij} \langle x_j \rangle \\ &= \sum_{j=1}^{n} R_{ij} X_j \end{aligned}$$

- First Moment or center-of-gravity maps just like a single particle.

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Propagating Moments 2

- Central second moment (COM removed) is

$$\sigma_{ij} = \langle (x_i - X_i)(x_j - X_j) \rangle$$

- For simplicity use X_i=0, then σ_{ij} propagates like

$$\begin{aligned}\bar{\sigma}_{ij} &= \langle \bar{x}_i \bar{x}_j \rangle = \langle \sum_{k=1}^{n} R_{ik} x_k \sum_{l=1}^{n} R_{jl} x_l \rangle = \sum_{k=1}^{n} R_{ik} \sum_{l=1}^{n} R_{jl} \langle x_k x_l \rangle \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} R_{ik} R_{jl} \sigma_{kl}\end{aligned}$$

- or in matrix form $$\bar{\sigma} = R \sigma R^T$$
- Theory behind TRANSPORT and other codes

Emittance and Beam beta-function

- Take the determinant of $\bar{\sigma} = R\sigma R^T$

$$\det\bar{\sigma} = \det(R\sigma R^T) = \det(R)\det(\sigma)\det(R^T) = \det\sigma$$

- The determinant of the sigma-matrix is constant
- Let's call it emittance squared $\varepsilon^2 = \det\sigma$
- Sigma matrix is 2x2 and symmmetric → three independent parameters, choose ε, α, β, such that

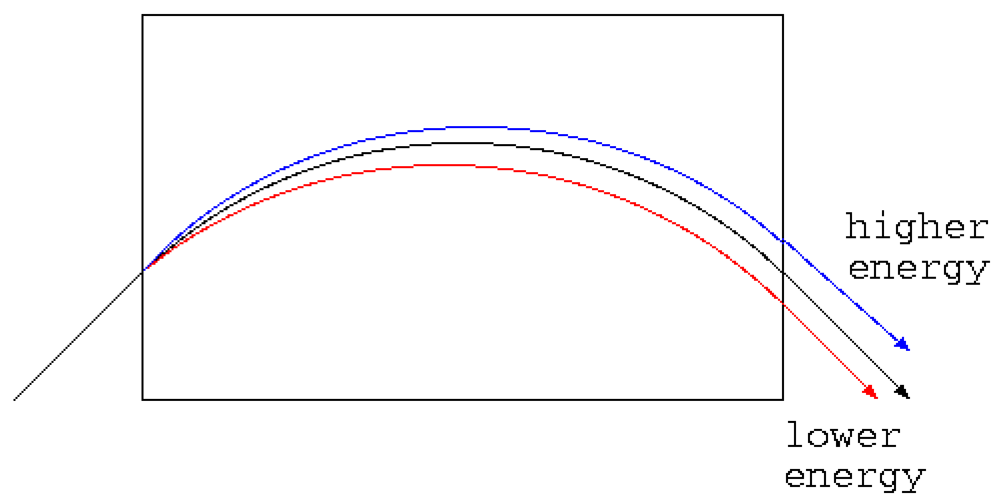
$$\sigma = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- Why is this useful? What is the relation of α, β to the previously defined quantities?

Dispersion

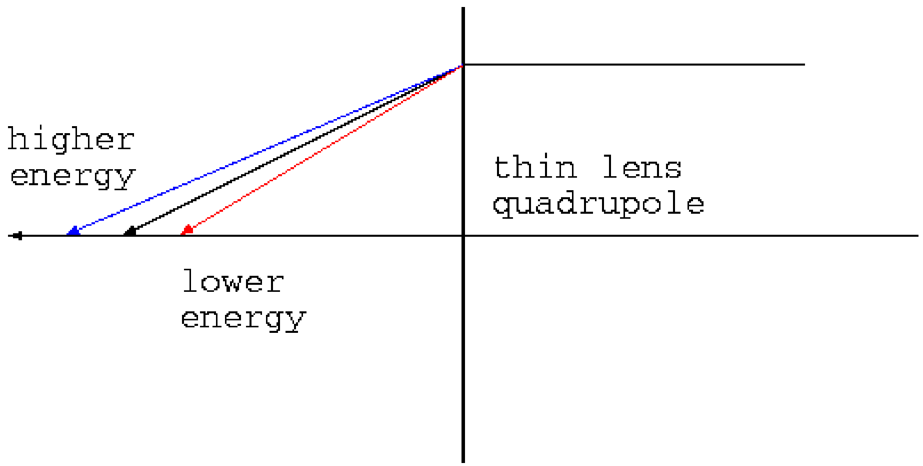
The Pink Floyd effect

- Dipoles deflect particles with different energies by a dfifferent angle
- Sorting according to dp/p
- Spectrometer-function



Chromaticity

- Quadrupole focusing is energy dependent
- higher energy larger focal length f

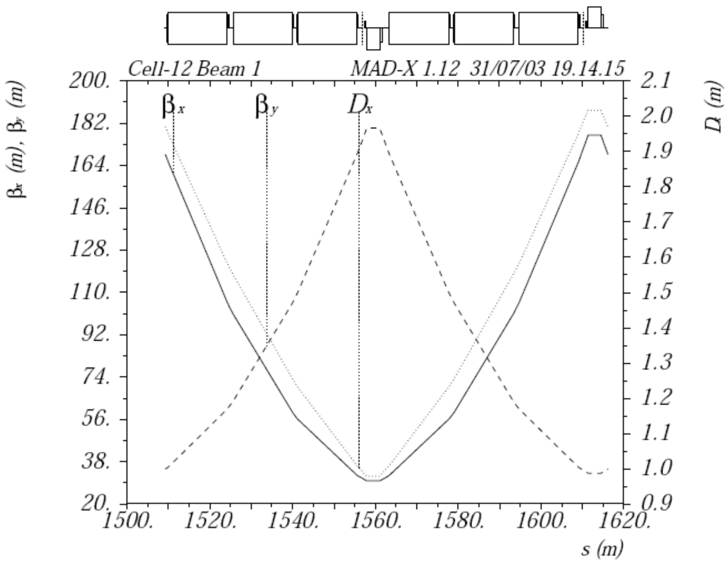


$$f(\delta) = f(1+\delta) \quad \text{or} \quad k_1 \to \frac{k_1}{1+\delta} \approx k_1 - k_1\delta$$

- quadrupole error kδ

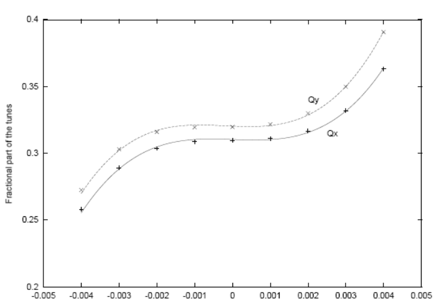
- Beam parameters become energy dependent
  - Tune
  - Beam size

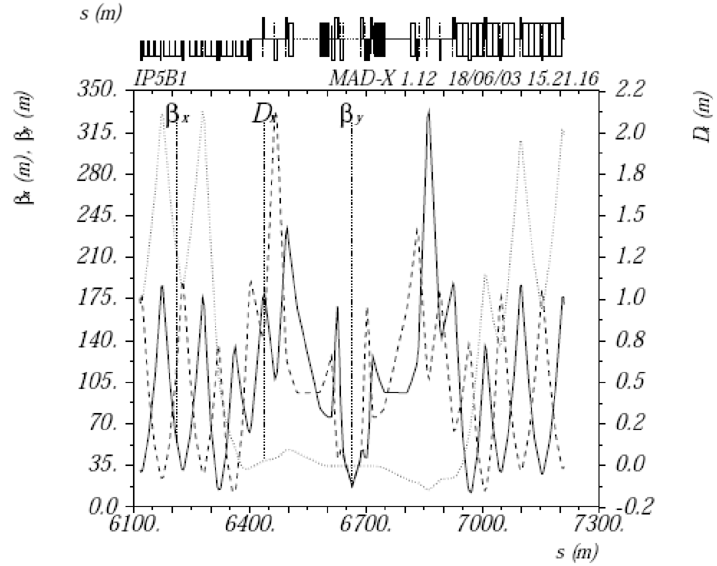
Example: LHC

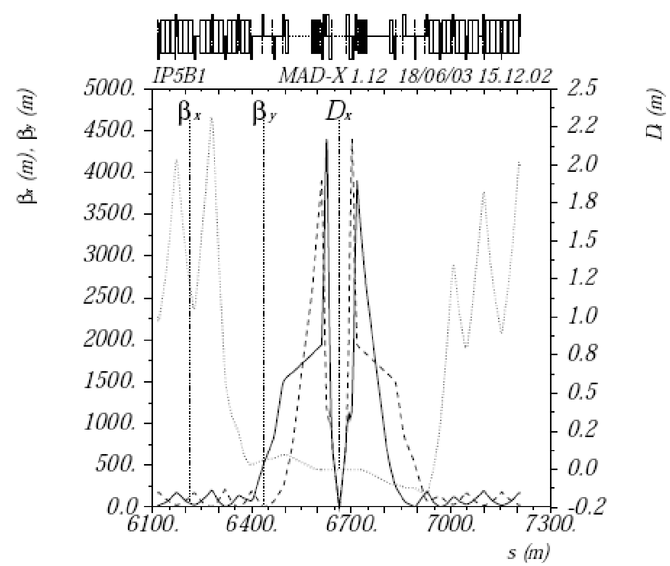


- Arc cell, FODO
- IR1,5: ATLAS, CMS
  - injection, collision

Tune vs. dp/p







Summary

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- Beta functions all over the place
  - unique in a circular or periodic systems
  - appears in parametrization of the transfer matrix
  - appears in parametrization of the sigma matrix
  - describes beam size $\sigma_{11} = \sigma_x^2 = \varepsilon\beta$
  - describes the envelope of the single particle motion and the maximum amplitude.
  - describes sensitivity to perturbations
- Courant-Snyder action variable J: single particle
- Emittance: rms property of the beam