Putting together all equations, in the early universe with radiation-dominated background in terms of the ~ fields:

$$\begin{cases} \partial_i \tilde{E}^i = \tilde{q} \qquad \partial_i \tilde{B}^i = 0 \\ \partial_\eta \tilde{E}_i - \delta_{ijk}\partial_j \tilde{B}^k = -\tilde{J}_i \\ \partial_\eta \tilde{B}_i + \delta_{ijk}\partial_j \tilde{E}^k = 0 \end{cases}$$

$$\begin{cases} \partial_\eta \tilde{q} + \partial_i \tilde{J}^i = 0 \\ \tilde{J}^\delta = \tilde{q}\tilde{v}^\delta + \gamma\tilde{\sigma}\left[\tilde{E}^\delta + \delta^\delta{}_{\beta\gamma}\tilde{B}^\gamma\tilde{v}^\beta - \tilde{E}\tilde{v}^\beta\tilde{v}^\delta\right] \end{cases}$$

which become, if the fluid is uncharged:

$$\begin{cases} \tilde{q} = \tilde{v}_i \tilde{J}^i = \gamma\tilde{\sigma}\tilde{v}_i\tilde{E}^i \\ \tilde{J}^i = \gamma\tilde{\sigma}\left[\tilde{E}^i + \delta^i{}_{jk}\tilde{v}^j\tilde{B}^k\right] \end{cases}$$

The combined equations for the co-evolution of the EM field and the cosmic plasma in radiation era assuming non-relativistic bulk motion:

$$\begin{cases} \partial_\eta\left(\tilde{\rho}\left(1+\frac{4}{3}v^2\right)\right) + \frac{4}{3}\partial_i\left(\tilde{\rho}\gamma^2\tilde{v}^i\right) = \tilde{E}^i\tilde{J}_i \\ \frac{4}{3}\left[\partial_\eta\left(\tilde{\rho}\tilde{v}^i(1+v^2)\right) + \partial_j\left(\tilde{\rho}\tilde{v}^i\tilde{v}^j\right)\right] + \partial_i\tilde{p} = \\ = \tilde{q}\tilde{E}^i + (\tilde{J}\times\tilde{B})^i + \tilde{\nu}\left(\partial_j\partial^j\tilde{v}^i + \frac{1}{3}\partial^i\partial_j\tilde{v}^j\right) \end{cases}$$

THE INDUCTION EQUATION

non-relativistic bulk motion and neutral fluid. Suppress also ~.

$$\partial_\eta E_i - \delta_{ijk}\partial_j B^k = -\sigma\left[E_i + \delta_{ijk}v^j\tilde{B}^k\right]$$

magnetic diffusivity $\eta_M = \frac{1}{\sigma}$

$$\left(1+\eta_M\frac{\partial}{\partial\eta}\right)\underline{E} = \eta_M\nabla\times\underline{B} - \underline{v}\times\underline{B}$$

The displacement current can be neglected for: $1 \gg \frac{\eta_M}{\tau}$ where τ is a typical evolution time (conformal time interval). Let us assume this is the case for the moment:

$$\frac{\partial\underline{B}}{\partial\eta} = -\nabla\times\left[\eta_M\nabla\times B - \underline{v}\times\underline{B}\right]$$

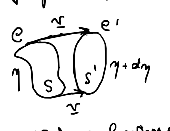
$$= \nabla\times\underline{v}\times\underline{B} + \frac{\Delta\underline{B}}{\sigma} \qquad (\nabla\times\nabla\times a = \nabla(\nabla\cdot a) - \Delta a)$$

what does this mean?

Magnetic Reynolds number: $\boxed{R_M = \frac{vL}{\eta_M}}$ (L: coordinate (comoving) length scale)

determines which term dominates, where L is the typical scale of the problem with fluid velocity v. We will evaluate it later at Recomb.

•) Suppose the advection term dominates $R_M \gg 1$:
→ we are in the situation of "flux-freezing"



$$\phi_B = \int_S \underline{B}\cdot d\underline{S}$$

$$\frac{d\phi_B}{d\eta} = \underbrace{\int_S \frac{\partial\underline{B}}{\partial\eta}\cdot d\underline{S}}_{B \text{ changes}} + \underbrace{\oint_C (\underline{B}\times\underline{v})\cdot d\underline{\ell}}_{\text{translation of the circuit changes location of boundary}} + \cancel{\int_S (\nabla\cdot\underline{B})\underline{v}\cdot d\underline{S}} =$$

Reynolds transport theorem (Jackson)

$$\overset{\text{Stokes}}{=} \int_S \frac{\partial\underline{B}}{\partial\eta}\cdot d\underline{S} + \int_S \nabla\times(\underline{B}\times\underline{v})\,d\underline{S} =$$

$$= \int_S \frac{\Delta\underline{B}}{\sigma}\cdot d\underline{S} \xrightarrow[\sigma\to\infty]{} 0$$

the flux is constant if the fluid is highly conductive

B is amplified

•) Suppose the diffusion term dominates:

$$\frac{\partial\underline{B}}{\partial\eta} = \frac{\Delta\underline{B}}{\sigma}$$

then $\underline{B}$ decays on the diffusion timescale

$$\boxed{\tau_d \sim L^2\sigma}$$

how long is this time? Is this effect relevant?

CONDUCTIVITY IN THE EARLY UNIVERSE

Thermal history of the early universe

relativistic energy density: $\rho_{rad}(T) = \frac{\pi^2}{30}g(T)T^4$

$$g(T) = \sum_b g_b\left(\frac{T_b}{T}\right)^4 + \frac{7}{8}\sum_f g_f\left(\frac{T_f}{T}\right)^4$$

entropy density conserved $s(T) = g_s(T)\,a(T)^3 T^3$
(expansion is adiabatic $d(sa^3) = \text{const} \to a \propto 1/T$)

$$g_s(T) = \sum_b g_b\left(\frac{T_b}{T}\right)^3 + \frac{7}{8}\sum_f g_f\left(\frac{T_f}{T}\right)^3$$

Friedmann eq: $\left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho_{rad} = \frac{8\pi^3 G}{90}g(T)T^4$

age of the universe: $\frac{1}{a}\frac{da}{dt} = H(t) \propto T^2 \propto a^{-2}$

$$t = \int_0^t dt' = \int_0^a \frac{da'}{a'}\frac{1}{H(a')} = \int_0^a \frac{da'}{a'}\left(\frac{a'}{a}\right)^2\frac{1}{H(a)} = \frac{H(a)^{-1}}{a^2}\frac{1}{2}a^2 = \frac{H(a)^{-1}}{2}$$

$$t = \frac{1}{2}\sqrt{\frac{90}{8\pi^3 G}}\,g^{-1/2}T^{-2} \overset{\uparrow}{=} \frac{1}{2}\sqrt{\frac{90}{8\pi^3}}\frac{1}{\sqrt g}\left(\frac{m_{pl}}{GeV}\right)\left(\frac{1\,MeV}{T}\right)^2\frac{GeV}{(1\,MeV)^2}\frac{sec}{1.52\cdot10^{24}}$$

$$G = \frac{1}{m_{pl}^2} = \frac{1}{(1.2211\cdot10^{19})^2 GeV^2} \qquad = 2.4\frac{1}{\sqrt g}\left(\frac{1\,MeV}{T}\right)^2 sec$$

$$GeV^{-1} = \frac{sec}{1.52\cdot10^{24}} \qquad \begin{matrix}(\text{horizon: } d_H = H^{-1}) \\ (\text{age: } t_H = \frac{H^{-1}}{2})\end{matrix}$$

- Kolb and Turner
- Husdal 1609.04979

Ignoring the contribution of BSM particles:

$T > m_t = 175\,GeV$ all particles are relativistic and have the same T: $g = g_s$

$$g_b = \underset{\substack{(s=1)\\(g=2)}}{\gamma} + \begin{cases} Z_0\ (s=1) \\ W^\pm\ (g=3) \\ H\ (s=0) \\ \ \ (g=1)\end{cases} + \underset{\substack{(s=1)\\(g=2)}}{8\text{ gluons}}$$

$$= 1\cdot2 + \begin{cases}3\cdot3 \\ 1\cdot1\end{cases} + 8\cdot2 = 28$$

$$g_f = \begin{cases}\tau^\pm\ (s=\frac12) \\ \mu^\pm\ (g=4) \\ e^\pm\ \ 2\cdot2\end{cases} + \begin{cases}\nu_\tau\ (s=\frac12) \\ \nu_\mu\ (g=2) \\ \nu_e\end{cases} + \overset{\sim 175\,GeV}{\begin{cases} t \\ b\ (s=\frac12) \\ c\ (c=3) \\ s\ (g=2\cdot \\ u\ \ 2\cdot3=12) \\ d \end{cases}}$$

$$= 3\cdot4 + 3\cdot2 + 12\cdot6 = 90$$

$$\boxed{g = 106.75}$$

g gradually changes as T falls below particle masses. When also the pions get non-rel one has: $e^\pm, \gamma, \nu$; $g_b = 2$; $g_f = 10$; $\boxed{g = 10.75}$

ν stays relativ. but their T becomes different than equilibrium temperature: $g \neq g_s$; $g_b = 2$; $g_f = 10$; $g_s = 2 + \frac{7}{8}\cdot4 + \frac{7}{8}\cdot6\cdot\left(\frac{T_\nu}{T}\right)^3$

g doesn't change; baryons mesons

10^{19} GeV	10^{16} GeV	100 GeV	150 MeV	1 MeV
Planck	GUT?	EWPT	QCDPT	ν decoupling
	Inflation →	20 ps	30 μs	1 sec

$\underbrace{\qquad\qquad}_{\text{Baryogenesis?}}$

also $e^\pm$ become non relativistic
$g_b = 2$
$g_f = 6$
$g_s = 2 + \frac{7}{8}6\left(\frac{T_\nu}{T}\right)^3 = 3.9$
$T_\nu = \left(\frac{4}{11}\right)^{1/3}T$ (from entropy conservation)
$g = 2 + \frac{7}{8}6\left(\frac{T_\nu}{T}\right)^4 = 3.36$ ($N_{eff} = 3.04$ instead of 3)

$g = g_s = 2$?

0.5 MeV	0.08 MeV	0.8 eV	0.32 eV	0.26 eV	? meV	T_0 =
e^+e^- annihilation	BBN	matter radiation equality	Recombination	+ γ decoupling	ν non rel.	2.73 K – 2.35 10^{-4} eV
6 sec	200 sec	$6\cdot10^4$ yr	$z_{rec} = 1300$	$z_{dec} = 1100$		$1.34\cdot10^{10}$ yr
		$t_{eq} = 3.4\cdot10^5$	378 000 yr			

Non-relativistic regime: $\boxed{T < m_e}$

$$J_{TOT} = -em_e\underline{v}_e + em_p\underline{v}_p \overset{\uparrow}{=} e^2 X_e m_b\left(\frac{\tau_e}{m_e} + \frac{\tau_p}{m_p}\right)\underline{E}$$

•) $\underline{v} = \pm e\frac{\tau}{m}\underline{E}$ τ: mean free time between collisions

•) $m_e = m_p = X_e m_b$ $X_e = \frac{m_e}{m_e + m_H}$ ionization fraction

$$\sigma = e^2 X_e m_b\left(\frac{\tau_e}{m_e} + \frac{\tau_p}{m_p}\right)$$

dominated by the particle species with longest mean-free time (but shortest for each particle!)

$$\tau \sim \frac{l_{mfp}}{\langle v\rangle} \sim \frac{1}{n\sigma_{cs}}\underbrace{\sqrt{\frac{m}{T}}}_{\text{thermal velocity}}$$

Coulomb and Thomson scatterings alternate.

Recombination decreases X_e drastically decreasing conductivity, but Reionization re-increases it

Arnold et al hep-ph/0010177

Relativistic regime: take into account more particles, and sum over those with $m \leq T$ and that do EM interaction OR scatter with particles doing EM interaction

$m_e < T < T_{QCD}$: relativistic leptons and anti leptons (longest τ)
scatterings like $e^+e^- \to \gamma\gamma$, $e^-l^- \to e^-l^-$, $e^+e^- \to q\bar q$ (...)

$T_{QCD} < T < m_W$: include also quarks scattering leptons: conductivity is reduced

$T > m_W$: add also $W^\pm, Z$ bosons, restore EW symmetry → $U(1)_Y$ field conductivity further reduced

As an estimate, for relativistic particles:

$$\tau \sim \underset{\substack{\uparrow\\ \text{(physical scale)}}}{l_{mfp}} \sim \frac{1}{n\sigma_{cs}} \overset{\uparrow}{\sim} \frac{1}{n}\frac{T^2}{\pi e^4\ln\Lambda}$$

$$\sigma_{cs} \sim \pi b^2\underbrace{\ln\Lambda}_{\text{accounts for the range of impact parameters}} \sim \pi\frac{e^4}{T^2}\ln\Lambda$$

impact parameter $b \gtrsim \frac{e^2}{T}$

and therefore $\sigma \sim e^2 n\frac{\tau}{T} \sim \frac{e^2 n}{T}\frac{1}{n}\frac{T^2}{\pi e^4\ln\Lambda}$

$$\sigma \sim \frac{T}{e^2\ln 1/e}$$

(e becomes $g' = 0.36$ hypercharge coupling for $T > M_W$)

DO MAGNETIC FIELDS DIFFUSE?

$\tau_d \sim L^2\sigma$ what is the scale that gets dissipated in one Hubble time w.r.t the Hubble scale?

$$\frac{L}{H^{-1}} \sim \sqrt{\frac{t_H}{H^{-2}\sigma}} = \sqrt{\frac{H^{-1}/2}{H^{-2}\sigma}} = \sqrt{\frac{H}{2\sigma}}$$

physical now: $\frac{\tau_d^p}{a} \sim \frac{L^{p2}}{a^2}a\sigma$

Non-relativistic regime: (ATT! physical σ)

$$\sigma > \sigma_{Rec} \simeq \frac{3T_{Rec}^{3/2}}{4\sqrt{2\pi}e^2\ln\Lambda\sqrt{m_e}} \simeq 3\cdot10^{-4}\,eV$$

complicated dependence on T but decreasing

At Recombination, dominant scattering is Coulomb of e^-: Spitzer

$$\frac{L}{H^{-1}}(T > T_{rec}) < \sqrt{\frac{H_0\sqrt{\Omega_m^0(1+z)^3 + \Omega_{rad}^0(1+z)^4 + \Omega_\Lambda}}{2\cdot3\cdot10^{-4}\,eV}}$$

$$\frac{H_0}{h} = 2.13\cdot10^{-33}\,eV$$

$= 2.6\cdot10^{-13}$ so the scale that gets dissipated is tiny w.r.t. the horizon

In the relativistic case the situation is similar:

$$\frac{L}{H^{-1}}(T > m_e) \sim \sqrt{\frac{1}{2}\sqrt{\frac{8\pi^3 G}{90}}\sqrt{g(T)}\frac{T^2}{T}e^2\ln(1/e)}$$

$$\sim \sqrt{\frac{1}{2}\sqrt{\frac{8\pi^3}{90}}\underset{\substack{\downarrow\\10.75}}{\sqrt{g(T)}}e^2\ln(1/e)\frac{T}{m_{pl}}}$$

$$\sim 0.54\sqrt{\frac{T}{m_{pl}}} \ll 1 \qquad \text{(the pre-factor can change)}$$

let's go back to the condition for neglecting the displacement current:

$$\frac{\tau}{\eta_M} \gg 1 \Rightarrow t_H\sigma \gg 1 \quad \frac{H^{-1}}{2}\sigma \gg 1 \quad \frac{2H}{\sigma} \ll 1$$

already evaluated at Recombination:

$$2\frac{H_0\sqrt{\Omega_m^0(1+z)^3 + \Omega_{rad}^0(1+z)^4 + \Omega_\Lambda}}{3\cdot10^{-4}\,eV} \simeq 3\cdot10^{-25} \quad \text{ok}$$

One can also evaluate the magnetic Re n°

$$R_M = \frac{vL}{\eta_M} \sim \sigma V_A^2 t_H \sim \frac{3}{4}\sigma\frac{B^2}{\rho_{rad}}\frac{H^{-1}}{2} = \frac{3}{4}\frac{B^2}{\rho_{rad}}\underbrace{\frac{\sigma}{2H}}_{\text{already evaluated at Recomb: } 0.3\cdot10^{25}}$$

Alfvén speed: $V_A = \frac{B}{\sqrt{4/3\,\rho_{rad}}}$

$L \sim V_A t_H$

at Recomb.

$$= \frac{3}{4}0.3\cdot10^{25}\left(\frac{B}{10^{-9}G}\right)^2\frac{(10^{-9}G)^2}{\rho_{rad}(T_{rec})} \simeq 10^6\left(\frac{B}{10^{-9}G}\right)^2$$

$$\frac{(10^{-9}G)^2}{\rho_{rad}(T)} = \frac{10^{-18}\cdot8\pi\cdot1.9\cdot10^{-40}\,GeV^4}{\frac{\pi^2}{30}g(T)T^4} =$$

$$= \frac{1.45\cdot10^{-56}}{g(T)}\left(\frac{GeV}{T}\right)^4 = \frac{1.45\cdot10^{-20}}{g(T)}\left(\frac{eV}{T}\right)^4$$

So we have established that MF do not diffuse. Once they are generated in the early universe they remain in a situation of flux freezing.