STOCHASTIC MF IN THE EARLY UNIVERSE

Uniform MF pervading the universe breaks the hom. & isotropy of FLRW universe and furthermore cannot be generated by local processes operating within the causal horizon at a given time T (i.e. the distance a γ has travelled since the "beginning":

$$d_H = a(\eta)\eta = a(\eta)\int_0^{\eta(t)} \frac{dt}{a(t)} = a(\eta)\int_0^{\eta(t)} \frac{dt}{t^{\frac{2}{3(1+w)}}} \begin{cases} \xrightarrow{w=1/3} 2t \\ \xrightarrow{w=0} 3t \end{cases})$$

the correct way to describe a primordial MF is assuming it is a stochastic variable, statistically homo. and isot. such that $\langle \underline{B}(\underline{x})\rangle = 0$ and with two-point correlation function given by: $(P_{ij} = \delta_{ij} - \hat{r}_i\hat{r}_j)$

$$\boxed{\langle B_i(\underline{x}) B_j(\underline{x}+\underline{r})\rangle = M_N P_{ij} + M_L \hat{r}_i\hat{r}_j + \varepsilon_{ijk}\hat{r}_k M_H}$$

where $\langle\ldots\rangle$ is averaging over realisations, meaning that under ergodic hypothesis $\langle\ldots\rangle = \frac{1}{V}\int_V d^3x$ becomes spatial average

and $M_N(r)$, $M_L(r)$, $M_H(r)$ are $\begin{cases}\text{normal}\\ \text{longitudinal}\\ \text{helical}\end{cases}$ components.

The divergenceless of the MF implies:

$$\nabla\cdot\underline{B} = 0 \rightarrow \frac{\partial}{\partial(\underline{x}+\underline{r})_j}\langle B_i(\underline{x})B_j(\underline{x}+\underline{r})\rangle = 0 \qquad \underline{y} = \underline{x}+\underline{r}$$

$$\frac{\partial}{\partial y_j}\left[M_N P_{ij} + M_L\hat{r}_i\hat{r}_j + \varepsilon_{ijk}\hat{r}_k M_H\right] =$$

$$= \frac{\partial r}{\partial y_j}\frac{dM_N}{dr}P_{ij} - M_N\frac{\partial \hat{r}_i\hat{r}_j}{\partial y_j} + \frac{\partial r}{\partial y_j}\frac{dM_L}{dr}\hat{r}_i\hat{r}_j +$$

$$+ M_L\frac{\partial(\hat{r}_i\hat{r}_j)}{\partial y_j} + \varepsilon_{ijk}\frac{\partial\hat{r}_k}{\partial y_j}M_H + \varepsilon_{ijk}\hat{r}_k\frac{dM_H}{dr}\frac{\partial r}{\partial y_j} =$$

$$= -M_N\frac{2}{r}\hat{r}_i + \frac{dM_L}{dr}\hat{r}_i + M_L\frac{2}{r}\hat{r}_i = 0 \qquad M_N = \frac{r}{2}\frac{dM_L}{dr} + M_L$$

$$\uparrow$$

$$\frac{\partial\sqrt{(y-x)^i(y-x)_i}}{\partial y_j} = \hat{r}_j \qquad \frac{\partial(\hat{r}_i\hat{r}_j)}{\partial y_j} = \hat{r}_j\frac{\partial\hat{r}_i}{\partial y_j} + \hat{r}_i\frac{\partial\hat{r}_j}{\partial y_j} =$$

$$= \hat{r}_j\frac{1}{r}(\delta_{ij} - \hat{r}_i\hat{r}_j) + \hat{r}_i\frac{2}{r} = \frac{2}{r}\hat{r}_i$$

$$\boxed{M_N = \frac{1}{2r}\frac{d}{dr}(r^2 M_L)} \quad \text{divergence-less field}$$

Power spectrum:

$$\langle B_i(\underline{k})B_j^*(\underline{q})\rangle = \int d^3x\, e^{i\underline{k}\cdot\underline{x}}\int d^3y\, e^{-i\underline{q}\cdot\underline{y}}\langle B_i(\underline{x})B_j(\underline{y})\rangle$$

$$= \int d^3x\int d^3r\, e^{i\underline{k}\cdot\underline{x} - i\underline{q}\cdot\underline{x} - i\underline{q}\cdot\underline{r}}\,\xi_{ij}(|\underline{r}|) =$$

$$= (2\pi)^3\delta(\underline{k}-\underline{q})\int d^3r\, e^{-i\underline{q}\cdot\underline{r}}\left[M_N P_{ij} + M_L\hat{r}_i\hat{r}_j + \varepsilon_{ijk}\hat{r}_k M_H\right]$$

$$\boxed{\equiv (2\pi)^3\delta(\underline{k}-\underline{q})\left[(\delta_{ij} - \hat{k}_i\hat{k}_j)P_B(k) + i\varepsilon_{ije}\hat{k}_e P_H(k)\right]}$$

if $i=j$: $\displaystyle P_B(k) = \pi\int r^2 dr\int_{-1}^{1} d\mu\, e^{-iqr\mu}(2M_N + M_L)$

$$= 2\pi\int r^2 dr\,(2M_N + M_L)\frac{\sin qr}{qr}$$

multiply by $-i\varepsilon_{ijm}\hat{k}_m$:

$$2P_H(k) = 2\pi\int r^2 dr\int_{-1}^{1} d\mu\, e^{-iqr\mu}(-2i\mu M_H)$$

$$P_H(k) = \pi\int r^2 dr\, M_H\, 4\,\frac{qr\cos(qr) - \sin(qr)}{(qr)^2}$$

$P_B(k)$ and $P_H(k)$ are the symmetric and antisymmetric (helical) parts of the MF power spectrum (spectral density).

$P_B(k)$ is related to the MF energy density:

$$\rho_B = \frac{\langle B_i(\underline{x})B_i(\underline{x})\rangle}{2} =$$

$$\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}e^{-i(\underline{k}-\underline{q})\cdot\underline{x}}(2\pi)^3\delta(\underline{k}-\underline{q})\,2P_B(k) =$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P_B(k) \equiv \begin{cases}\int_0^\infty dk\, E_B(k) & \text{energy density P.S.}\\ \int_0^\infty \frac{dk}{k}\rho_B(k) & \text{energy density P.S. per log interval}\end{cases}$$

$P_H(k)$ is related to the MF helicity:

$H = \int_V \underline{A}\cdot\underline{B}\, d^3x$ where $\underline{B} = \nabla\times\underline{A}$ vector potential (and $\underline{E} = -\nabla\phi + \frac{\partial\underline{A}}{\partial\eta}$)

$\downarrow$

volume in which the MF flux lines are fully contained: $\underline{B}\cdot\underline{n} = 0$, so that H is gauge invariant:

if $\underline{A}\rightarrow\underline{A}' = \underline{A} + \nabla\Lambda$ $\qquad \underline{B} = \nabla\times\underline{A}' = \nabla\times\underline{A}$

$\phi\rightarrow\phi' = \phi - \frac{\partial\Lambda}{\partial\eta}$ $\qquad \underline{E} = -\nabla\phi' + \frac{\partial\underline{A}'}{\partial\eta} =$

$$= -\nabla\phi - \frac{\partial\nabla\Lambda}{\partial\eta} + \frac{\partial\underline{A}}{\partial\eta} + \frac{\partial\nabla\Lambda}{\partial\eta}$$

$$= -\nabla\phi + \frac{\partial\underline{A}}{\partial\eta}$$

Since H depends on $\underline{A}$ it could be gauge dependent, but:

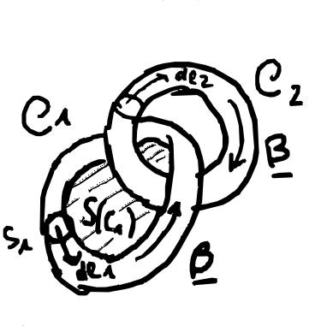
$$H' = \int_V \underline{A}'\cdot\underline{B}\,d^3x = \int_V \underline{A}\cdot\underline{B}\,d^3x + \underbrace{\int_V \nabla\Lambda\cdot\underline{B}\,d^3x}$$

$= H$

$$\int_V \nabla\cdot(\Lambda\underline{B})d^3x = \int_{\partial V}\Lambda\underline{B}\cdot\underline{n}\,d^2x = 0$$

(because $\nabla\cdot\underline{B} = 0$)

helicity is related to the topological properties of the MF configuration: it represents the links and twists of the flux tubes:



$$\phi_1 = \int_{S_1}\underline{B}\cdot d\underline{S} \rightarrow \underline{B}\,d^3x = \underline{B}\cdot d\underline{S}\cdot d\underline{\ell}_1$$

$$H = \int_V \underline{A}\cdot\underline{B}\,d^3x = \phi_1\oint_{C_1}\underline{A}\cdot d\underline{\ell}_1 + \phi_2\oint_{C_2}\underline{A}\cdot d\underline{\ell}_2 =$$

$$= \phi_1\int_{S(C_1)}\nabla\times\underline{A}\cdot d\underline{S} + \phi_2\int_{S(C_2)}\nabla\times\underline{A}\cdot d\underline{S} = 2\phi_1\phi_2$$

Helicity DENSITY connected to $P_H(k)$:

$$h = \frac{1}{V}H = \frac{1}{V}\int_V d^3x\,\underline{A}\cdot\underline{B} = \langle\underline{A}(\underline{x})\cdot\underline{B}(\underline{x})\rangle =$$

$$= \int\frac{d^3k}{(2\pi)^3}e^{-i\underline{k}\cdot\underline{x}}\int\frac{d^3q}{(2\pi)^3}e^{i\underline{q}\cdot\underline{x}}\langle A_i(\underline{k})B_i^*(\underline{q})\rangle$$

$$\underline{B} = \nabla\times\underline{A} \rightarrow \nabla\times\underline{B} = \nabla\times(\nabla\times\underline{A}) = \nabla(\nabla\cdot\underline{A}) - \Delta\underline{A}$$

$$\int\frac{d^3k}{(2\pi)^3}\varepsilon_{ije}\partial_j\left(e^{-i\underline{k}\cdot\underline{x}}\right)B_e(\underline{k}) = \int\frac{d^3k}{(2\pi)^3}(-i\varepsilon_{ije}k_j)B_e(\underline{k})$$

$$-i\varepsilon_{ije}k_j B_e(\underline{k}) = \underbrace{-k_i k_e A_e(\underline{k})}_{\text{choose Coulomb gauge } \nabla\cdot\underline{A} = 0} + k^2 A_i(\underline{k})$$

$$A_i(\underline{k}) = -i\varepsilon_{ije}\frac{\hat{k}_j}{k}B_e(\underline{k})$$

$$h = \int\frac{d^3k}{(2\pi)^3}e^{-i\underline{k}\cdot\underline{x}}\int\frac{d^3q}{(2\pi)^3}e^{i\underline{q}\cdot\underline{x}}\left(-i\varepsilon_{ije}\frac{\hat{k}_j}{k}\langle B_e(\underline{k})B_i^*(\underline{q})\rangle\right)$$

$$= \int\frac{d^3k}{(2\pi)^3}\left\{-i\underset{\downarrow}{\varepsilon_{ije}}\frac{\hat{k}_j}{k}\left[\cancel{(\delta_{ei} - \hat{k}_e\hat{k}_i)P_B(k)} + i\varepsilon_{eim}\hat{k}_m P_H(k)\right]\right\}$$

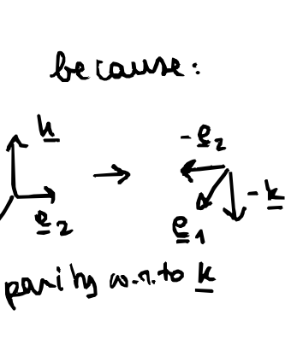
$$\varepsilon_{eij}\varepsilon_{eim} = 2\delta_j^m \qquad \hat{k}_j\hat{k}^j = 1$$

$$= \frac{2\cdot 4\pi}{(2\pi)^3}\int\frac{k^2 dk}{k}P_H(k) = \int\frac{k\,dk}{\pi^2}P_H(k)$$

One can also use a basis $(\underline{e}^+, \underline{e}^-, \hat{k})$ with respect to a right-handed orthonormal basis $(\underline{e}_1, \underline{e}_2, \hat{k})$ with $\underline{e}_2 = \hat{k}\times\underline{e}_1$ it is:

$$\underline{e}^\pm = \frac{1}{\sqrt{2}}(\underline{e}_1 \pm i\underline{e}_2)$$

Properties: $\begin{cases}\underline{e}^\pm\cdot\underline{e}^\mp = 1\\ \underline{e}^\pm\cdot\underline{e}^\pm = 0\\ \underline{e}^\pm(\underline{k}) = \underline{e}^\mp(-\underline{k})\\ \hat{k}\times\underline{e}^\pm = i(\mp\underline{e}^\pm)\\ (\underline{e}^\pm)^* = \underline{e}^\mp\end{cases}$ because:



Transverse vector such as $\underline{B}$ $(\nabla\cdot\underline{B} = 0 \rightarrow \underline{k}\cdot\underline{B}(\underline{k}) = 0)$

$$\underline{B}(\underline{k}) = B^+\underline{e}^+ + B^-\underline{e}^-$$

note that: $\underline{B}(\underline{x})\in\mathbb{R}\rightarrow\underline{B}(\underline{k}) = \int d^3x\, e^{i\underline{k}\cdot\underline{x}}\underline{B}(\underline{x}) = \underline{B}^*(-\underline{k})$

and therefore $B^* = B^{+*}\underline{e}^-(\underline{k}) + B^{-*}\underline{e}^+(\underline{k}) = B^{+*}\underline{e}^+(-\underline{k}) + B^{-*}\underline{e}^-(-\underline{k})$
$= B^+(-\underline{k})\,\underline{e}^+(-\underline{k}) + B^-(-\underline{k})\,\underline{e}^-(-\underline{k})$

$$(B^\pm(\underline{k}))^* = B^\pm(-\underline{k})$$

under parity: $\underline{B}(\underline{x}) = \underline{B}(-\underline{x})$ (pseudovector)

$$\int\frac{d^3k}{(2\pi)^3}e^{-i\underline{k}\cdot\underline{x}}B(\underline{k}) = \int\frac{d^3k}{(2\pi)^3}e^{i\underline{k}\cdot\underline{x}}B(\underline{k}) \Rightarrow B(\underline{k})\rightarrow B(-\underline{k})$$

parity is reflection $\hat{k}\rightarrow-\hat{k}$

$$\underline{B}(-\underline{k}) = B^+\underline{e}^-(\underline{k}) + B^-\underline{e}^+(\underline{k}) \quad \begin{cases}B^+\rightarrow B^-\\ B^-\rightarrow B^+\end{cases}$$

This basis provides a "new interpretation" of the presence of helicity:

$$\langle\underline{B}(\underline{k})\cdot\underline{B}^*(\underline{k})\rangle = 2(2\pi)^3 P_B(k)\,\delta(\underline{k}-\underline{q}) \qquad \overset{\underline{e}^-(\underline{k})}{\uparrow} \quad \overset{\underline{e}^+(\underline{k})}{\uparrow}$$

$$= \langle(B^+(\underline{k})\underline{e}^+(\underline{k}) + B^-(\underline{k})\underline{e}^-(\underline{k}))(B^+(-\underline{k})\underline{e}^+(-\underline{k}) + B^-\underline{e}^-(-\underline{k}))\rangle$$

$$= \langle B^+(\underline{k})B^+(-\underline{k})\rangle + \langle B^-(\underline{k})B^-(-\underline{k})\rangle =$$

$$= \langle B^+(\underline{k})B^{+*}(\underline{k})\rangle + \langle B^-(\underline{k})B^{-*}(\underline{k})\rangle$$

$P_B(k)$ is related to the sum of the average value of the two ± components

$$-i\varepsilon_{ijm}\hat{k}_m\langle B_i(\underline{k})B_j^*(\underline{k})\rangle = 2(2\pi)^3 P_H(k)\quad\delta(\underline{k}-\underline{q})$$

$$= i\langle\underline{B}(\underline{k})\cdot(\hat{k}\times\underline{B}^*(\underline{k}))\rangle =$$

$$= i\langle(B^+\underline{e}^+ + B^-\underline{e}^-)\cdot(\hat{k}\times(B^+(-\underline{k})\underline{e}^-(\underline{k}) + B^-(-\underline{k})\underline{e}^+(\underline{k})))\rangle$$

$$= i\langle(B^+\underline{e}^+ + B^-\underline{e}^-)(B^+(-\underline{k})\,i\,\underline{e}^-(\underline{k}) - iB^-(-\underline{k})\underline{e}^+(\underline{k}))\rangle =$$

$$= -\langle B^+(\underline{k})B^{+*}(\underline{k})\rangle + \langle B^-(\underline{k})B^{-*}(\underline{k})\rangle$$

$P_H(k)$ is related to the difference of the average values of the two ± components, and therefore changes sign under parity, while $P_B(k)$ remains the same under parity.

Also $\boxed{|P_H(k)| \leq P_B(k)}$ and if one helicity mode is zero one has equality: FULLY HELICAL FIELD.

Another important aspect of helicity is that it is almost independent on time:

$$\frac{\partial}{\partial\eta}(\underline{A}\cdot\underline{B}) = (-\underline{E} + \nabla V)\cdot\underline{B} + \underline{A}\cdot(-\nabla\times\underline{E}) =$$

$$= -\underline{E}\cdot\underline{B} + \nabla\cdot(V\underline{B}) - V\cancel{\nabla\cdot\underline{B}} - \nabla\cdot(\underline{E}\times\underline{A}) - \underline{E}\cdot\nabla\times\underline{A} =$$

$\uparrow$

Mx eq: $\nabla\times\frac{\partial\underline{A}}{\partial\eta} + \nabla\times\underline{E} = 0$

$$\frac{\partial\underline{A}}{\partial\eta} + \underline{E} = \nabla V$$

$$= -2\underline{E}\cdot\underline{B} - \nabla\cdot(\underline{E}\times\underline{A} + V\underline{B})$$

$$\underline{E} = -\underline{v}\times\underline{B} + \frac{\underline{J}}{\sigma}$$

$\downarrow$

$$\frac{dH}{d\eta} = \int_V d^3x\left(-\frac{2}{\sigma}\underline{J}\cdot\underline{B}\right) - \int_{\partial V}(\underline{E}\times\underline{A} + \cancel{V\underline{B}})\cdot\hat{n}\,dS =$$

$$\frac{1}{\sigma}\left[\int_V d^3x(-2\underline{J}\cdot\underline{B}) - \int_{\partial V}\cancel{(\underline{J}\times\underline{A})\cdot\hat{n}}\,dS\right] +$$

$$+ \int_{\partial V}\underbrace{((\underline{v}\times\underline{B})\times\underline{A})}_{\cancel{(\underline{v}\cdot\underline{A})\underline{B}\cdot\hat{n}} - \cancel{(\underline{v}\cdot\underline{B})\underline{A}}}\cdot\hat{n}\,dS$$

$\underline{J}\parallel\underline{E}$, $\underline{v}\times\underline{B}\perp\underline{B}$
$\underline{A} = \nabla\times\underline{B}\perp\underline{B}$
$\underline{J}\times\underline{A}\parallel\underline{B}$

$$\frac{dH}{d\eta} = -\frac{1}{\sigma}\left[\int_V d^3x\,2(\nabla\times\underline{B})\cdot\underline{B}\right]$$

So in ideal MHD $\sigma\rightarrow\infty$ helicity is conserved, independently on the bulk velocity $\underline{v}$.

This is quite logical: H quantifies the topology of the MF, and when the MF is perfectly advected the topology cannot change.

SHAPE OF THE M.F. SPECTRUM:

In general one has power laws:

$$P_B(k) = A\begin{cases}(k/k_*)^{n_B} & k \ll k_*\\ (k/k_*)^{-m_B} & k \gg k_*\end{cases} \quad \text{and same structure for } P_H(k)$$

k_* is a characteristic wavenumber determined by the MF generation process and by the MF evolution, together with the slopes n_B and m_B.

There are several ways to interpolate the two power laws such as for example:

$$P_B(k) = A\frac{(k/k_*)^{n_B}}{\left[m_B + n_B(k/k_*)^{(n_B+m_B)\alpha}\right]^{1/\alpha}}$$

this spectral shape is designed to have a stationary point at k_*: $\left.\frac{dP_B(k)}{dk}\right|_{k_*} = 0$

The amplitude A can be expressed in terms of $\langle B^2\rangle$

$$\rho_B = \frac{\langle B^2\rangle}{2} = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P_B(k)$$

In order for energy density to be finite, one needs to require: $\boxed{\begin{matrix}n_B > -3\\ m_B > 3\end{matrix}}$

Correspondingly for the helical part to be well defined one has: $\boxed{\begin{matrix}n_H > -2\\ m_H > 2\end{matrix}}$

Another useful quantity is the "integral scale" that can be defined as:

$$\lambda_I = 2\pi\frac{1}{\rho_B}\int_0^\infty dk\, k\, P_B(k)$$

This only exists if $\boxed{n_B > -2}$ and $\boxed{n_H > -1}$

It can also be interesting to quantify the MF strength on a given scale λ. We then volume average the MF on a region of size λ

$$B_{i\lambda}(\underline{x}) = \frac{1}{(\sqrt{2\pi}\lambda)^3}\int d^3y\, B_i(\underline{y})\, e^{-\frac{(\underline{x}-\underline{y})^2}{2\lambda^2}}$$

$$B_\lambda^2 \equiv \langle B_{i\lambda}B_{i\lambda}\rangle = \frac{1}{(\lambda\sqrt{2\pi})^6}\int d^3y\int d^3z\langle B_i(\underline{y})B_i(\underline{z})\rangle e^{-\frac{(\underline{x}-\underline{y})^2 + (\underline{x}-\underline{z})^2}{2\lambda^2}}$$

$\uparrow$

MF energy density on scale λ

$$= \frac{1}{(\lambda\sqrt{2\pi})^6}\int d^3y\int d^3z\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}e^{-\frac{(\underline{x}-\underline{y})^2}{2\lambda^2} - i\underline{k}\cdot\underline{y}}$$

$$e^{-\frac{(\underline{x}-\underline{z})^2}{2\lambda^2} + i\underline{k}\cdot\underline{z}}\langle B_i(\underline{k})B_i^*(\underline{q})\rangle = \frac{1}{(\lambda\sqrt{2\pi})^6}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}$$

$$(\sqrt{2\pi}\lambda)^6 e^{-\frac{k^2\lambda^2}{2}}e^{-i\underline{k}\cdot\underline{x}}e^{-\frac{q^2\lambda^2}{2}}e^{i\underline{q}\cdot\underline{x}}\,2(2\pi)^3\delta(\underline{k}-\underline{q})P_B(k)$$

$$= \int\frac{d^3k}{(2\pi)^3}2P_B(k)\,e^{-k^2\lambda^2} = \frac{1}{\pi^2}\int dk\, k^2 P_B(k)\, e^{-k^2\lambda^2}$$

In order to get B_λ^2, the MF spectrum is convolved with a Gaussian filter function which limits the integration of the energy spectrum to wavelengths $k < \lambda$.

What are the typical values of n_B and m_B?

n_B is constrained by CAUSALITY if the MF is generated by a process operating during radiation domination or any other phase in which the causal horizon does not diverge (like during inflation). In this case, the MF cannot be correlated on scales $L > H_*^{-1}$ where $*$ denotes the time of generation

$$\langle B_i(\underline{x})B_j(\underline{x}+\underline{r})\rangle = 0 \quad \text{for } |\underline{r}| \geq L \geq H^{-1} \qquad (*)$$

$$P_B(k\rightarrow 0) = 2\pi\int_0^\infty r^2 dr\,(2M_N + M_L)\left(1 - \frac{(kr)^2}{6} + O(kr)^4\right)$$

$$= 2\pi\int r^2 dr\,(2M_N + M_L) - \frac{\pi}{3}k^2\int r^4 dr\,(2M_N + M_L)$$

The divergence-free condition:

$$M_N = \frac{1}{2r}\frac{d}{dr}(r^2 M_L) \qquad 2r^2 M_N = 2r^2 M_L + r^3\frac{dM_L}{dr}$$

$$\int_0^\infty dr\left(3r^2 M_L + r^3\frac{dM_L}{dr}\right) = r^3 M_L\Big|_0^\infty = 0 \quad \text{because of } (*)$$

$$\text{and } P_B(k\rightarrow 0) = -\frac{\pi}{3}k^2\int_0^\infty dr\, r^2\frac{d}{dr}(r^3 M_L) = \frac{2}{3}\pi k^2\int_0^\infty dr\, r^4 M_L$$

So one has $\boxed{n_B = 2 \text{ at least.}}$

Another way to see this:

$\langle B_i(\underline{x})B_j(\underline{x}+\underline{r})\rangle$ compact support so F.T. analytic: $\left[\underbrace{(\delta_{ij} - \hat{k}_i\hat{k}_j)P_B(k)} + \underbrace{i\varepsilon_{ije}\hat{k}_e P_H(k)}\right]$

analytic in $k\rightarrow 0$ means $\boxed{n_B \geq 2}$ and even

analytic in $k\rightarrow 0$ means $\boxed{n_H \geq 1}$ and odd

Combined with $|P_H(k)| \leq P_B(k)$ means

$$\boxed{n_H \geq n_B} \text{ so } \boxed{n_H \geq 3}$$

The part of the spectrum at high k is in general determined by the evolution, and for example could be Kolmogorov: $m_B = 11/3$

The important point is that it is independent on the initial conditions under which the MF has been generated. Those determine instead n_B, n_H and also k_*.

So one example of MF spectrum can be:

$$P_B(k) = A\frac{(k/k_*)^2}{\left[\frac{11}{3} + 2(k/k_*)^2\right]^{17/6}} \quad \begin{cases}n_B = 2\\ m_B = \frac{11}{3}\\ \alpha = \frac{6}{17}\end{cases} \text{ Von Karman spectrum}$$

$$\text{with } A = \frac{32\sqrt{2}\,\pi^{3/2}\,\Gamma\left(\frac{17}{6}\right)}{3\left(\frac{3}{11}\right)^{1/3}\Gamma\left[\frac{1}{3}\right]}\frac{\langle B^2\rangle}{k_*^3}, \quad k_* \text{ is the peak,}$$

$$\lambda_I = \frac{4\pi}{\langle B^2\rangle}\int_0^\infty dk\, k\, P_B(k) = \sqrt{\frac{8}{33}\pi}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{2\pi}{k_*} \text{ is close}$$

to k_*^{-1} and it is the scale at which most of the MF energy is contained.

As we will see, in general MF evolve with the fluid and one also needs the

VELOCITY POWER SPECTRUM

This is very similar to the MF power spectrum except that the velocity field doesn't need to be divergence free. Physically this means that there can be a compressional component in the velocity field, as for example sound waves.

$$\langle v_i(\underline{x})v_j(\underline{x}+\underline{r})\rangle = V_N P_{ij} + V_L\hat{r}_i\hat{r}_j + \varepsilon_{ijk}\hat{r}_k V_H$$

Power spectrum:

$$\langle v_i(\underline{k})v_j^*(\underline{q})\rangle =$$

$$= (2\pi)^3\delta(\underline{k}-\underline{q})\int d^3r\, e^{-i\underline{q}\cdot\underline{r}}\left[V_N P_{ij} + V_L\hat{r}_i\hat{r}_j + \varepsilon_{ijk}\hat{r}_k H\right]$$

- $\displaystyle 2\pi\int r^2 dr\int_{-1}^{1} d\mu\, e^{-iqr\mu}B_N\delta_{ij} = 2\pi\,\delta_{ij}\int r^2 dr\, 2B_N j_0$

$\underline{q}\cdot\underline{r} = qr\cos\theta$
$\mu = \cos\theta \quad \sin\theta\, d\theta = d\mu$
$\displaystyle\int_{-1}^{1} d\mu\, e^{-iqr\mu} = \frac{2\sin(qr)}{qr} = 2j_0(qr)$

- $\displaystyle 2\pi\int r^2 dr\int_{-1}^{1} d\mu\, e^{-iqr\mu}(B_L - B_N)\hat{r}_i\hat{r}_j =$

$$\hat{r}_i\hat{r}_j = \frac{1-\mu^2}{2}(\delta_{ij} - \hat{q}_i\hat{q}_j) + \mu^2\hat{q}_i\hat{q}_j$$

$$\int_{-1}^{1} d\mu\, e^{-iqr\mu}\left(\frac{1-\mu^2}{2}\right) = 2\frac{\sin qr}{(qr)^3} - 2\frac{\cos qr}{(qr)^2} = 2\frac{j_1}{qr}$$

$$\int_{-1}^{1} d\mu\, e^{-iqr\mu}\mu^2 = -4\frac{j_1}{qr} + 2j_0$$

$$= 2\pi\int r^2 dr\,(B_L - B_N)\left[(\delta_{ij} - \hat{q}_i\hat{q}_j)2\frac{j_1}{qr} + \hat{q}_i\hat{q}_j\left(2j_0 - 4\frac{j_1}{qr}\right)\right]$$

$$\int d^3r\, e^{-i\underline{q}\cdot\underline{r}}\left[B_N P_{ij} + B_L\hat{r}_i\hat{r}_j\right] = 4\pi\int r^2 dr\left[B_N j_0\delta_{ij}\right.$$

$$\left. + (B_L - B_N)\left[(\delta_{ij} - \hat{q}_i\hat{q}_j)\frac{j_1}{qr} + \hat{q}_i\hat{q}_j\left(j_0 - 2\frac{j_1}{qr}\right)\right]\right]$$

$$= \delta_{ij}\,4\pi\int dr\, r^2\left[B_N j_0 + (B_L - B_N)\frac{j_1(qr)}{qr}\right] +$$

$$+ \hat{q}_i\hat{q}_j\,4\pi\int dr\, r^2(B_L - B_N)\left[j_0 - 3\frac{j_1(qr)}{qr}\right]$$

so there is a new longitudinal term in the power spectrum:

$$\boxed{\begin{aligned}\langle v_i(\underline{k})v_j^*(\underline{q})\rangle = (2\pi)^3\delta(\underline{k}-\underline{q})\Big[&\delta_{ij}P_{KS}(q) +\\ &+ \hat{q}_i\hat{q}_j P_{KV}(q) + i\varepsilon_{ijm}\hat{k}_m P_{KH}(k)\Big]\end{aligned}}$$

$$\frac{\rho_K}{\rho_{tot}} = \frac{\langle v^2\rangle}{2} = \frac{1}{(2\pi)^2}\int_0^\infty dk\, k^2\left[3P_{KS}(k) + P_{KV}(k)\right]$$

$$\equiv \begin{cases}\int_0^\infty dk\, E_K(k) & \text{energy density P.S.}\\ \int_0^\infty \frac{dk}{k}\rho_K(k) & \text{energy density P.S. per log interval}\end{cases}$$

And one can define λ_K and the power spectra as for the MF.

During inflation there is no cosmic plasma: therefore, bulk fluid motion can develop only after Reheating and therefore by processes that operate sub-horizon. The correlation function has therefore compact support and the argument above gives:

Analiticity of $P_{KS}(k)$: $P_{KS}(k < k_*) \propto k^{n_{KS}}$ with integer $\boxed{n_{KS} \geq 0}$

Analiticity of $P_{KV}(k)$: $\boxed{n_{KV} \geq 2}$

This means, for the total kinetic energy spectrum:

$$E_K(k \ll k_*) = k^2(3P_{KS}(k) + P_{KV}(k)) \propto k^{n_K}$$

with $\boxed{n_K \geq 0}$