EVOLUTION

We have seen that MF diffusion is negligible in the early universe and therefore the MF is advected by bulk fluid motion. However, the MF itself can also create fluid motion.

If we can treat the early universe plasma as a single fluid (i.e. the mean free paths of all the particles are smaller than the relevant scales in the problem – we will see when this assumption breaks down) then the evolution of the MF is coupled to the one of the fluid and the system is described by the following set of eqs – further assuming that bulk velocity is non-relativistic and the fluid is neutral:

$$\frac{\partial \underline{B}}{\partial \eta} = \nabla\times \underline{v}\times\underline{B} + \cancel{\frac{\Delta \underline{B}}{\sigma}} \longrightarrow$$

In the MHD limit of a perfectly conducting fluid

$$\underline{J} = \sigma\left[\underline{E}+\underline{v}\times\underline{B}\right] \to \underline{E} = -\underline{v}\times\underline{B}$$

$$\cancel{\frac{\partial \underline{E}}{\partial\eta}} - \nabla\times\underline{B} = -\underline{J}$$

$$\partial_\eta\left(\rho\left(1+\frac{4}{3}v^2\right)\right) + \frac{4}{3}\nabla\cdot(\rho\,\underline{v}) = \overbrace{-(\underline{v}\times\underline{B})\cdot\underline{J}}^{\text{higher order}}$$

$$\frac{4}{3}\left[\partial_\eta(\rho\,\underline{v}) + \frac{1}{2}(\underline{v}\cdot\nabla\rho)\underline{v} + \rho(\underline{v}\cdot\nabla)\underline{v} + \rho(\nabla\cdot\underline{v})\underline{v}\right] + \frac{1}{3}\nabla\rho =$$

$$+(\nabla\times\underline{B})\times\underline{B} + \nu\left[\Delta\underline{v} + \frac{1}{3}\Delta(\nabla\cdot\underline{v})\right]$$

From these equations one appreciates first of all that B produces velocity in the fluid via the $\nabla\times\underline{B}\times\underline{B}$ term, and the velocity produces $\underline{B}$ via the induction eq. the system is coupled and can become highly non-linear. This depends on the importance of the non-linear vs viscous terms, that is represented by the kinetic Reynolds number

meaning the relative importance of the advection and viscous terms:

$$\underset{\substack{\nearrow\\ \text{comoving}\\ \text{(coordinate)}\\ \text{scale}}}{\frac{\rho v^2}{L}} \leftrightarrow \frac{3}{4}\nu\frac{v}{L^2} \qquad \boxed{Re = \frac{\rho v^2}{L}\frac{4}{3}\frac{L^2}{\nu v} = \frac{vL}{\nu/\frac{4}{3}\rho}}$$

We have already demonstrated that $Re_M \gg 1$ in the early universe. One can also define the Prandtl number as the ratio:

$$\boxed{P_M = \frac{Re_M}{Re} = \sigma\frac{\nu}{\rho}}$$

meaning whether ohmic dissipation is more or less important than the kinetic one.

We therefore need to study

VISCOSITY IN THE EARLY UNIVERSE

The Reynolds number contains kinematic viscosity $\boxed{\nu_k = \frac{\nu}{\frac{4}{3}\rho}}$ where ν is shear visc.

Remember that these are all rescaled quantities: using $\tilde{\nu} = a^3\nu$, $\tilde{\rho} = a^4\rho$ one finds $\tilde{\nu}_k = \frac{\tilde{\nu}}{\frac{4}{3}\tilde{\rho}} = \frac{1}{a}\nu_k$ so the Re

$Re = \frac{vL}{\tilde{\nu}_k} = \frac{vL_p}{a\frac{\nu_k}{a}} = Re$ is the same with rescaled or physical quantities

that in the early universe can be expressed as $\boxed{\nu = \frac{4}{15}\frac{\pi^2}{30}g_d T^4 \ell^d_{mfp}}$ NOT RESCALED!

and therefore $\boxed{\nu_k = \frac{1}{5}\frac{g_d}{g_{tot}}\ell^d_{mfp}}$ d: diffusing particle

So it appears that the larger viscosity comes always from the particle with the longest mean free path → weakest interaction.

- $\boxed{T > m_W}$ EW symmetry restored, and like for the hypercharge conductivity, transport is dominated by right-handed leptons interaction. The handwaving argument gives:

$$\underset{\text{physical}}{\ell_{mfp}} \sim \frac{1}{n\sigma_{cs}} \underset{\uparrow}{\sim} \frac{\pi}{\zeta(3)\left(g_b+\frac{3}{4}g_f\right)}\frac{1}{g'^4\ln(1/g')}\frac{1}{T}$$

$$n = \frac{\zeta(3)}{\pi^2}T^3\left(g_b+\frac{3}{4}g_f\right);\quad \sigma_{cs}\sim\pi b^2\ln\Lambda \sim \pi\frac{g'^4}{T^2}\ln 1/g'$$

impact parameter $b \gtrsim \frac{g'^2}{T}$

so the kinematic viscosity goes as T^{-1}.
A detailed evaluation gives:

$$\nu_k = \frac{90}{4\pi^2}\left(\frac{5}{2}\right)^3\zeta(5)^2\left(\frac{12}{\pi}\right)^5\left(\frac{3/2}{9\pi^2+224(5+\nu_2)}\right)\frac{1}{g_{tot}}\frac{1}{g'^4\ln(1/g')}\frac{1}{T}$$

$$\sim \frac{20}{GeV}\left(\frac{GeV}{T}\right) \qquad (@\,T = m_W = 80\text{ GeV}: 0.25\text{ GeV}^{-1})$$

- $\boxed{1\text{ MeV} < T < m_W}$ ν decoupling. after the EWPT the particle with the largest mean free path are neutrinos, until they decouple

$$\ell_{mfp} \simeq \frac{8\pi^3}{27\zeta(3)}\frac{1}{g_f G_F^2 T^5} \qquad G_F = \frac{\sqrt{2}}{8}\frac{g'^2}{m_W^2} \simeq 1.166\,10^{-5}\frac{1}{GeV^2}$$

with $\sigma_{cs} \sim \frac{9}{2\pi}G_F^2T^2$ Fermi cross section $\sim \frac{G_F^2 E}{2\pi}$ and $n \sim \frac{\zeta(3)}{\pi^2}\frac{3}{4}g_f T^3$ $E \sim 3k_BT$

$$\nu_k \simeq \frac{8\pi^3}{135}\frac{g_\nu}{g_t^2}\frac{1}{G_F^2T^5} \quad \begin{cases} g_t = 84 & T > T_{QCDPT}\\ g_t = 10 & T < T_{QCDPT}\end{cases}$$

$$\simeq \begin{cases} 10^7\left(\frac{GeV}{T}\right)^5\frac{1}{GeV} & T > T_{QCDPT} \quad \left(\begin{smallmatrix}@T=m_W:\\ 3\cdot10^{-3}\,GeV^{-1}\end{smallmatrix}\right)\\ 8\cdot10^8\left(\frac{GeV}{T}\right)^5\frac{1}{GeV} & T < T_{QCDPT} \quad \left(\begin{smallmatrix}@T=0.5\,MeV\\ 2\cdot10^{25}\,GeV^{-1}\end{smallmatrix}\right)\end{cases}$$

- $\boxed{T < 0.5\text{ MeV}}$ After ν decoupling and e+e- annihilation, all particles are non-relativistic and photons are those with the longest mean-free path, corresponding to Thomson scattering off e^-:

$$\ell^\gamma_{mfp} = \frac{1}{n_e\sigma_T} \simeq \frac{3\pi}{8\alpha^2}\frac{(0.511\cdot10^{-3})^2}{\eta_B\,\beta\,\zeta(3)}\left(\frac{GeV}{T}\right)^3\frac{1}{GeV}$$

$$n_e = \eta_B n_\gamma = 6\cdot10^{-10}\frac{8\pi}{\pi^2}\zeta(3)T^3$$

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2}$$

$$\nu_k \simeq \frac{1}{5}\frac{2}{3.36}\ell^\gamma_{mfp} \simeq 5\cdot10^5\left(\frac{GeV}{T}\right)^3\frac{1}{GeV} \quad \left(\begin{smallmatrix}@T=0.5\,MeV\\ 4\cdot10^{15}\,GeV^{-1}\end{smallmatrix}\right)$$

the kinematic viscosity has a drastic reduction by 10 orders of magnitude after ν decoupling. It follows the above formula until Recombination, after which γ decouple and one cannot apply the above formulas any longer.

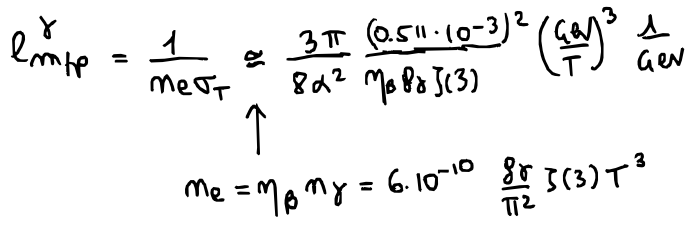
Now let's evaluate the Reynolds number assuming "plausible" values:

$$L = 0.01\,H^{-1} \qquad \frac{\rho_k}{\rho_{rad}} \sim \frac{v^2}{2} \sim 0.01$$

$$Re = \frac{10^{-4}H^{-1}}{\nu_k} = 10^{-4}\sqrt{\frac{90}{8\pi^3}}\frac{m_{pe}}{GeV}\begin{cases}\frac{(106.75)^{-\frac{1}{2}}}{20}\left(\frac{T}{GeV}\right) & T > T_W\\ \frac{(106.75)^{-\frac{1}{2}}}{10^{-3}}\left(\frac{T}{GeV}\right)^3 & T > T_{QCD}\\ \frac{(10.75)^{-\frac{1}{2}}}{8\cdot10^8}\left(\frac{T}{GeV}\right)^3 & T < T_{QCD}\\ \frac{(3.36)^{-\frac{1}{2}}}{5\cdot10^5}\left(\frac{T}{GeV}\right) & T < m_e\end{cases}$$

$$\frac{3H^2}{8\pi G} = \frac{\pi^2}{30}g\,T^4$$

$$H = \left(\frac{8\pi^3}{90}g\right)^{\frac{1}{2}}\frac{GeV}{m_{pe}}\left(\frac{T}{GeV}\right)^2 GeV$$



there are two periods when Re < 1:

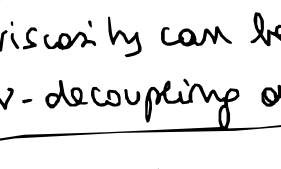
- 0.5 MeV < T < 15 MeV when viscosity is due to ν
- T < 13 eV around Recombination, when γ start to increase their m.f.p.

These numbers obviously depend on our choice of L and v but the main point is:

> The MF evolution in the early universe is mainly characterised by turbulent evolution, a part in two stages where the viscosity can become important: around ν-decoupling and around γ-decoupling

The situation goes more or less as follows:

1) Some process acts to generate the MF during the radiation dominated epoch before ν-decoupling (or "MF" modes are amplified during inflation and re-enter the horizon in the radiation dom. era). the initial condition is therefore a stochastic MF with spectrum:



with $n_B > -3$ and L is the characteristic scale of the MF ($k_* = \frac{2\pi}{L}$) (where the energy is concentrated in the case of a causal generation process). A velocity spectrum can already be associated as well or not, it depends on the generation mechanism. For this simple spectrum things can be calculated easily:

$$\frac{\langle B^2\rangle}{2} = \frac{1}{2\pi^2}\int_0^{\frac{2\pi}{L}}dk\,k^2 P_B(k) = \frac{1}{2\pi^2}\left(\frac{L}{2\pi}\right)^{n_B}\frac{A}{n_B+3}\left(\frac{2\pi}{L}\right)^{n_B+3}$$

$$A\left(\frac{kL}{2\pi}\right)^{n_B} \quad \hookrightarrow A = \frac{n_B+3}{8\pi}\langle B^2\rangle L^3$$

$$P_B(k) = \frac{n_B+3}{8\pi}\langle B^2\rangle L^3\left(\frac{kL}{2\pi}\right)^{n_B}$$

$$B^2_\lambda = \frac{1}{\pi^2}\int_0^\infty dk\,k^2 P_B(k)\,e^{-k^2\lambda^2} =$$

$$= \frac{n_B+3}{2(2\pi)^{n_B+3}}\Gamma\left(\frac{n_B+3}{2}\right)\langle B^2\rangle\left(\frac{L}{\lambda}\right)^{n_B+3}$$

Energy on a scale λ scales as λ^{-n_B+3}.

The MF then generates bulk velocity through the term $\nabla\times\underline{B}\times\underline{B}$. The bulk velocity affects in turns the MF because of the term $\nabla\times\underline{v}\times\underline{B}$. Re is in general very high: this statement depends on v, L but is true on a wide range of values, so MHD turbulence in general arises. This leads to equipartition of the MF and kinetic energy densities:

$$\frac{\partial v}{\partial\eta} \sim \frac{1}{\frac{4}{3}\rho}\nabla\times\nabla\times\underline{B} \to v \sim \underset{\substack{\nearrow\\ \text{conformal time}\\ \text{interval}}}{\tau}\frac{B^2k}{\frac{4}{3}\rho} \sim \tau k V_A^2$$

Equipartition is reached after a time:

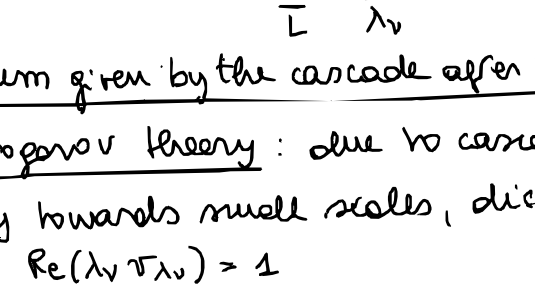
$V_A \sim v \to \boxed{\tau_k \sim \frac{1}{kV_A}}$ "Eddy turnover time" on scale k

And indeed from the induction equation:

$$\frac{\partial B}{\partial\eta} \sim kvB \to kv\tau = 1 \to v \sim V_A$$

Small scales (large k) go non-linear first.

After a time $\boxed{\tau_L \sim \frac{L}{V_A}}$ all small scales have gone non linear and an inertial range has appeared in the spectra:



Spectrum given by the cascade after τ_L in Kolmogorov theory: due to cascading of energy towards small scales, dissipated where $Re(\lambda_\nu v_{\lambda_\nu}) \simeq 1$

ε energy transfer per unit time given by:

$$\varepsilon \sim \frac{v_\ell^2}{\tau_\ell} \sim \frac{v_\ell^3}{\ell} \underset{\uparrow}{\sim} \frac{\left(P_v(1/\ell)\,1/\ell^3\right)^{3/2}}{\ell}$$

$$v_\ell^2 = \frac{1}{\pi^2}\int_0^\infty dk\,k^2 P_v(k)\,e^{-k^2\ell^2}$$

$$P(k) \sim \varepsilon^{2/3}k^{-2/3}k^{-3} \sim \varepsilon^{2/3}k^{-11/3} \qquad \boxed{n_K = n_B = -\frac{11}{3}}$$

How do the MF and velocity spectra evolve?

If the MF generation process is over, there is no more energy injection and energy is dissipated at the very small scales. This is called "Freely decaying turbulence" and depends on many factors: initial conditions, helicity, spectral indexes, conserved quantities... here the very simple vanilla picture (not yet clear how well validated by simulations):

As for the MF, the kinetic energy scales as:

$$v_\lambda^2\,\lambda^{n_v+3} \sim \langle v^2\rangle L^{n_v+3}$$

$$\frac{d\langle v^2\rangle}{d\eta} \sim \frac{\langle v^2\rangle}{\tau_L} \sim \frac{\langle v^2\rangle}{L/\sqrt{\langle v^2\rangle}} \sim \frac{\langle v^2\rangle^{3/2}}{\langle v^2\rangle^{-\frac{1}{n_K+3}}} \sim \langle v^2\rangle^{\frac{3n_v+11}{2(n_v+3)}}$$

$$\langle v^2\rangle \sim \eta^{\left(-\frac{3n_v+11}{2(n_v+3)}+1\right)^{-1}} \sim \eta^{-\frac{2(n_v+3)}{n_v+5}}$$

The same reasoning can apply for $\frac{\langle B^2\rangle}{\rho_{rad}}$ but with n_B. However, $\langle v^2\rangle \sim \frac{\langle B^2\rangle}{\rho_{rad}}$ because of equipartition, and therefore one chooses the spectral index that is dominant, i.e. n_v whenever $n_v = 0$ and $n_B = 2$.

The scale evolves as: $L \sim \langle v^2\rangle^{-\frac{1}{n_v+3}} \sim \eta^{\frac{2}{n_v+5}}$

So one has:

$$\langle v^2\rangle \sim \langle B^2\rangle \propto \eta^{-2p} \quad\text{with } p = \frac{(n_v+3)}{n_v+5},\ q = \frac{2}{n_v+5}$$

$$L \propto \eta^q \qquad\text{and therefore } p+q = 1$$

What happens at large scales?

$$P_B\left(k < \frac{2\pi}{L}\right) = \frac{n_B+3}{8\pi}\langle B^2\rangle L^3\left(\frac{kL}{2\pi}\right)^{n_B} \quad\text{and the same for } P_v(k)$$

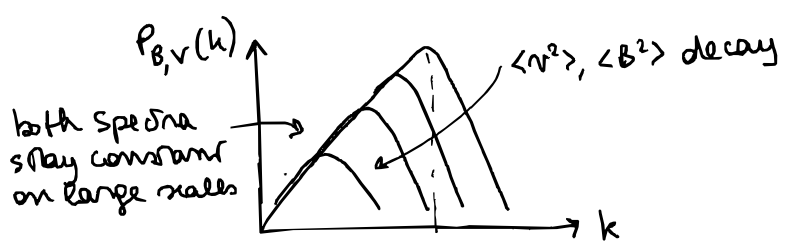
$$\langle B^2\rangle L^3 L^{n_B} \propto \eta^{-\frac{2(n_v+3)}{n_v+5}+\frac{3\cdot2}{n_v+5}+\frac{2n_B}{n_v+5}} \propto \eta^{\frac{2(n_B-n_v)}{n_v+5}}$$

So $P_v(k)$ is always constant at large scales, but $P_B(k)$ is in general not, if $n_B \neq n_v$!

- For incompressible turbulence $n_B = n_v = 2$

$$\langle v^2\rangle \sim \langle B^2\rangle \propto \eta^{-10/7} \quad\text{decays}$$

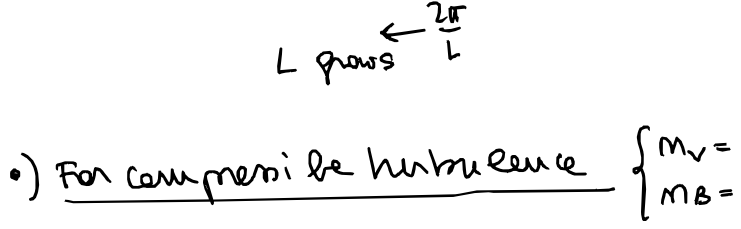
$$L \propto \eta^{2/7} \quad\text{grows}$$

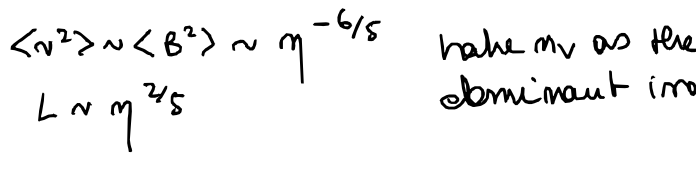


- For compressible turbulence $\begin{cases} n_v = 0\\ n_B = 2\end{cases}$

$$\langle v^2\rangle \sim \langle B^2\rangle \sim \eta^{-6/5} \quad\text{take } n_v \text{ as the dominant index}$$

$$L \sim \eta^{2/5}$$





"inverse cascade" observed numerically

This inverse cascade is always observed also for helical evolution and in that case it is connected to the conservation of helicity:

Evolution if the initial MF has helicity

$$h = \langle\underline{A}(x)\cdot\underline{B}(x)\rangle \underset{\uparrow}{\sim} \frac{B^2}{k} = \text{const in time}$$

$$\underline{B} = \nabla\times\underline{A}$$

$$\boxed{\langle B^2\rangle \sim \langle v^2\rangle \sim L^{-1}}$$

Inserting this into the laws found above:

$$\langle v^2\rangle \sim \langle B^2\rangle \propto \eta^{-2p} \quad\text{with } p = \frac{(n_v+3)}{n_v+5},\ q = \frac{2}{n_v+5}$$

$$L \propto \eta^q$$

it means $-2p+q = 0$ and therefore equivalent to $n_v^{equiv} = -2$. So the decay that conserves helicity is

$$\langle v^2\rangle \sim \langle B^2\rangle \propto \eta^{-2/3} \qquad L \propto \eta^{2/3}$$

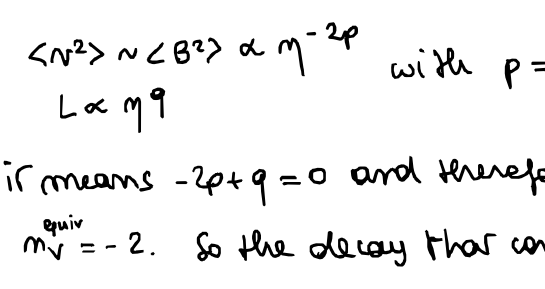
At large scales one has:

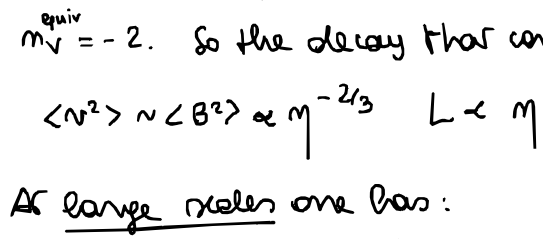
$$\langle B^2\rangle L^3 L^{n_B} \propto \eta^{\frac{2(n_B-n_v^{equiv})}{n_v^{equiv}+5}} \propto \eta^{\frac{2(n_B+2)}{3}} \qquad \boxed{\text{it grows for } n_B > -2}$$

and now for the velocity as well the situation changes because $n_v = 0$ but evolution $n_v^{equiv} = -2$

$$\langle v^2\rangle L^3 L^{n_v} \propto \eta^{\frac{2(n_v-n_v^{equiv})}{n_v^{equiv}+5}} \propto \eta^{\frac{2(n_v+2)}{3}} \qquad \boxed{\text{also grows for } n_v^{equiv} > -2}$$

So if the MF is helical, there is inverse cascade in both the magnetic and velocity spectra also when turbulence is incompressible





but note that despite the appearance, both total energies are still decreasing with time:

$$\langle v^2\rangle \sim \langle B^2\rangle \propto \eta^{-2/3}$$

Note that:

- For a MF that is partially helical, it evolves as if incompressible, so energy is dissipated but helicity is conserved. Eventually the field becomes maximally helical and obeys the decay above.
- All this is valid only for T ≲ MeV as before there is the chiral magnetic effect to be added to the equations (changes n° of in left-right chiral particles influence MF helicity)
- and even there, not sure it is indeed valid because conservation of local helicity fluctuation can play a role and we have ignored them

let's go now to point 2)

2) During the free MHD decay, MF and kinetic energy densities are dissipated at very small scales, where Re ~ 1. Viscosity grows, so the scale at which Re ~ 1 grow as well:

$$Re(\ell) = \frac{v\ell^{\to\text{comoving}}}{\tilde{\nu}_{k\to\text{rescaled}}} \underset{\uparrow}{\sim} \frac{V_A\,\ell}{\ell^d_{mfp}\,a^{-1}} \propto \frac{1}{\eta} \qquad \boxed{\text{it decays!}}$$

equipartition $V_A \sim v$

the evolution of the numerator is

$$\ell \propto \eta^{\frac{2}{5+n_v}},\ V_A \propto \eta^{\frac{-3+n_v}{5+n_v}} \to \ell V_A \propto \eta^{-(n+1)} \underset{\substack{\uparrow\\ \text{worst case}\\ \text{scenario: } n_v = 0}}{\propto} 1/\eta$$

with $a \propto \eta \propto \frac{1}{T}$ the denominator goes:

$$\frac{\ell^d_{mfp}}{a} = \ell^d_{cmfp} \propto T\cdot T^{-(1,3,5)} \propto \eta^{(0,2,4)} \underset{\substack{\uparrow\\ \text{worst case}\\ \text{scenario}}}{\propto} \eta^0$$

So little by little the inertial range is reduced as Re ≃ 1 at progressively larger scales, and when it reaches the integral scale L one enters the viscous regime.

In this regime, the Lorentz force term induces a small velocity balancing the friction term:

$$(\nabla\times\underline{B})\times\underline{B} \sim \nu_k\,\rho\,\Delta v$$

$$\nu_k k^2 v \sim k V_A^2 \to v \sim \frac{V_A^2}{\nu_k k} \sim Re(k)V_A \ll V_A$$

This velocity is too small to advect the MF efficiently, and the suppression of plasma motion removes the coupling between the MF and the fluid, so the MF is frozen:

$$\frac{\partial B}{\partial\eta} \sim kvB \ll 1 \qquad B \sim \text{const}$$

no more equipartition, $\langle v^2\rangle$ is still dissipated

3) At some point the ℓ^d_{mfp} grows to reach the characteristic scale L, so the particle species d is too weakly coupled to provide a true viscosity and one enters the free-streaming regime. This regime should be treated via Boltzmann eqs, but a more simplified treatment consists in modelling the occasional collisions with the fluid particles with a viscous force given by:

$$\underline{F} = -\alpha\,\underline{v} \quad\text{with } \boxed{\alpha = \frac{4}{3}\frac{\rho_d}{\ell_{mfp}}} \text{ to be}$$

inserted in the mom. conservation equation. As we have seen for the viscosity, there are two phases in which the evolution goes through the free-streaming regime:

- ν-decoupling, $d = \nu$

$$\alpha_K = \frac{\alpha}{\frac{4}{3}\rho_{tot-\nu}} = \frac{\rho_\nu}{\rho_{tot-\nu}}\frac{1}{\ell^\nu_{mfp}} = \frac{\rho_\nu}{\rho_{rad}-\rho_\nu}\frac{1}{\ell^{\nu c}_{mfp}}$$

(here conf. inv. applies so ℓ^c_{mfp} is comoving)

- γ-decoupling, $d = \gamma$, which occurs in the matter era so the conservation eqs. are different because conf. inv. doesn't hold any longer. It is then useful to write them in terms of the coordinate system $dt = a\,d\eta$ so with physical time and distance $d\underline{r} = a\,d\underline{x}$. One then defines the LAB MF: $B_{Li} = \frac{B_i}{a} = \frac{\tilde{B}_i}{a^2}$ and the induction and mom. cons. equations become:

$$\frac{\partial(a^2\underline{B}_L)}{\partial t} = \frac{1}{a}\nabla\times\left[\underline{v}\times(a^2\underline{B}_L)\right] \to B_L \propto \frac{1}{a^2}$$

if no advection, flux freezing in expanding universe

$$\frac{\partial\underline{v}}{\partial t} + \underset{\substack{\downarrow\\ \text{physical Hubble parameter}}}{H(t)}\underline{v} + \underline{v}\cdot\nabla\underline{v} =$$

$$= -\frac{1}{a\rho_b}\nabla\rho_b + \frac{1}{\rho_b}(\nabla\times\underline{B}_L)\times\underline{B}_L - \frac{1}{a}\nabla\phi - \frac{4}{3}\frac{\rho_\gamma}{\rho_b}\frac{1}{\ell^\gamma_{mfp}}\underline{v}$$

physical mfp

the particles coupled to the MF and the photons are the baryons

without the sustain from the photons, baryons become compressible

perturbations in the baryon density (negligible before) can become important and react to gravity

The drag term is now due to Thomson scattering $-\frac{4}{3}\frac{\rho_\gamma}{\rho_b}n_b\sigma_T\underline{v}$

One enters this regime from the viscous one: $\underline{v}$ is already small, and then all non-lin. terms can be dropped and v is given by: (ρ_x is now either $\rho_{tot-\nu}$ or ρ_b depending on the regime we are in)

$$k\frac{B^2}{\rho_x} \sim \frac{4}{3}\frac{\rho_d}{\rho_x}\frac{1}{\ell^d_{mfp}}v \quad \begin{cases}\tilde{B},\tilde{\rho},\text{ and } k,\ell^c_{mfp} \text{ comoving}\\ B_L,\rho,\text{ and } k,\ell^p_{mfp} \text{ physical}\end{cases}$$

$$v \sim k\underbrace{V_{Ad}^2}_{B^2/\rho_d}\ell^d_{mfp} \lesssim V_{Ad}^2 \ll 1$$

$\ell_{mfp} \sim L$ and the only remaining scales are $k^{-1} > L$

So not surprisingly v is small, however, it is increasing with ℓ_{mfp}, because the damping due to free-streaming is decreasing as the particle is more and more decoupled: $\alpha_K \propto 1/\ell_{mfp}$

So the free-streaming regime must end at some point, when the particle decouples

As v increases, the MF is advected again and the evolution proceeds through damped MHD waves (Alfvén or compressible), with the MF surviving up to the scale:

$$\underset{\substack{\text{scale that}\\ \text{did not oscillate}\\ \text{appreciably}\\ \text{on timescale } \tau}}{k_D v\tau \sim 1} \longrightarrow k_D^2 V_{Ad}^2\ell^d_{mfp}\tau \sim 1$$

$$\uparrow\ v \sim kV_{Ad}^2\ell^d_{mfp}$$

So the MF survives damping at a scale

$L_D \sim V_{Ad}\sqrt{\ell_{mfp}\tau} \longrightarrow$ for γ decoupling this is $L_D \sim V_{Ad}L_S$

At this point, the Reynolds number is:

$$Re\left(\frac{1}{k}\right) \simeq \frac{\rho_x k v^2}{\alpha v} = \frac{kv}{\alpha/\rho_x} \sim \frac{\rho_x}{\rho_d}kv\,\ell^d_{mfp} \sim \frac{\rho_x}{\rho_d}k^2V_{Ad}^2\ell^{d\,2}_{mfp}$$

$$\alpha = \frac{4}{3}\frac{\rho_d}{\ell_{mfp}} \qquad v \sim kV_{Ad}^2\ell^d_{mfp}$$

$$\lesssim \frac{\rho_x}{\rho_d}k_D^2V_{Ad}^2\ell^{d\,2}_{mfp} \sim \frac{\rho_x}{\rho_d}\frac{\ell^d_{mfp}}{t_H} < 1$$

particle not decoupled yet!

$$\frac{g_{tot}-g_\nu}{g_\nu} = \frac{4.75}{6}$$

$$\frac{\rho_b}{\rho_\gamma} = \frac{4}{3}R_b^{rec} \sim 0.8 \text{ or smaller}$$

But the velocity is still increasing, and when it reaches $\boxed{v \sim V_A}$ the situation changes again:

- the scale up to which the MF survives is: $k_DV_At_H \sim 1$
- the Reynolds number is:

$$Re\left(\frac{1}{k}\right) \sim \frac{\rho_x}{\rho_d}kV_A\ell^d_{mfp} \sim \frac{\rho_d}{\rho_x} \gtrsim 1$$

$$V_A \sim v \sim kV_{Ad}^2\ell^d_{mfp} \to kV_A\left(\frac{\rho_x}{\rho_d}\right)^2\ell^d_{mfp} \sim 1$$

and one goes back to the turbulent phase, provided that the fluid picture still applies, meaning that there are still particles strongly coupled among themselves on the relevant MF scales.

- After the ν-decoupling phase: the MF has basically frozen during the evolution, on all scales larger than $L^c_D \sim V_A/\mathcal{H}$, so the turbulent evolution starts again from the same value of MF as before the onset of the viscous phase, with characteristic scale given by L^c_D
- After the γ decoupling phase: the MF does not evolve appreciably again a part from structure formation, so one remains with:

$$L^p_D \sim V_At_H \sim \frac{B_L^{rec}}{\sqrt{\rho_b^{rec}}}\frac{2}{H_{rec}} \quad \begin{cases}B_L^{rec} = B_L^0\left(\frac{a^0}{a_{rec}}\right)^2\\ \rho_b^{rec} = \rho_b^0(1+z_{rec})^3\end{cases}$$

$$L^c_D \sim (1+z_{rec})B^0_L(1+z_{rec})^2\frac{1}{\sqrt{h^2\Omega_b}}\frac{h}{\sqrt{\rho^0_c}}\frac{1}{(1+z_{rec})^{3/2}}$$

$$\frac{2h}{H_0\sqrt{h^2\Omega_{m}(1+z_{rec})^3+h^2\Omega_{rad}(1+z_{rec})^4}} \sim$$

$$\sim B^0_L(1+z_{rec})^{3/2}\frac{1}{\sqrt{h^2\Omega_b}}\frac{2}{\sqrt{h^2\Omega_m(1+z_{rec})^3+h^2\Omega_{rad}(1+z_{rec})^4}}$$

$$\frac{h^2}{H_0^2}\sqrt{\frac{8\pi}{3}}\frac{1}{m_{pe}} \sim B^0_L\frac{1.69\cdot10^{66}}{GeV^3}$$

$$\frac{L^c_D}{Mpc} \sim \frac{B^0_L}{10^{-8}\,Gauss}\frac{10^{-8}\sqrt{8\pi}\sqrt{1.9\cdot10^{-40}}\,GeV^3}{3\cdot10^{24}\cdot5\cdot10^{13}}\frac{1.69\cdot10^{66}}{GeV^3}$$

$$\sim 7\frac{B^0_L}{10^{-8}\,Gauss} \qquad \boxed{B^0_L \sim \left(\frac{L^c_D}{Mpc}\right)10^{-9}\,Gauss}$$

$$L^p \sim V_At_H$$

$$aL^c \sim V_At_H \qquad L^c \sim \boxed{V_A}\frac{1}{aH}$$

$$V_A = \frac{B}{\sqrt{\rho}}$$

$$\swarrow \qquad \searrow$$

$$(\text{const})\frac{\tilde{B}}{\sqrt{\tilde{\rho}}} \qquad \frac{B_L}{\sqrt{\rho}}$$

$$\downarrow \text{red}$$

$$\frac{\tilde{B}}{a^2\sqrt{\rho}} = \frac{\tilde{B}}{\sqrt{\tilde{\rho}}}$$

EVOLUTION IN THE LINEAR REGIME

•) DURING THE RADIATION ERA (all ~ quantities)

$$\frac{B^2}{\rho_{rad}} = \left(\frac{B}{10^{-9}\,\text{Gauss}}\right)^2 \frac{(10^{-9}\,\text{Gauss})^2}{\rho_{rad}} = \qquad \tilde{\rho}_{rad} = a_0^4\,\rho^0_{rad}$$

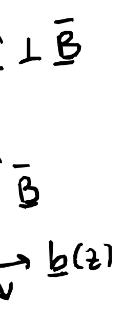
$$\left(\frac{B}{10^{-9}\,\text{Gauss}}\right)^2 \frac{1}{h^2\Omega_{rad}} \frac{h^2}{\rho_c^0}\, 10^{-18}\, 8\pi\, 1.9\cdot 10^{-40}\,\text{GeV}^4$$

$$= \frac{6\cdot 10^{-11}}{h^2\Omega_{rad}} \left(\frac{B}{10^{-9}\,\text{Gauss}}\right)^2 \simeq 2.4\cdot 10^{-6}\left(\frac{B}{10^{-9}\,\text{Gauss}}\right)^2$$

Dissipation of magnetic energy does not alter the total ρ_{rad} even for MF of order nanogauss.

In principle one cannot assume $\underline{\bar{B}} + \underline{b}$ with $\underline{B_0}$ constant and $\underline{b} \ll \underline{\bar{B}}$ as usually done in the linearised MHD equations (Jackson chapter 5). Suppose however that $\bar{B} \sim \sqrt{B\lambda^2}$ with λ very large, and suppose that $b \sim \bar{B}$.

Furthermore assume $\underline{b} \perp \underline{B_0} \parallel \hat{z}$, $\underline{V} \perp \underline{\bar{B}}$ and assume everything depends only on z: $\nabla\cdot\underline{V} = \partial_z V_z = 0$



higher order

$$\partial_\eta\tilde{\rho} + \frac{4}{3}\tilde{\rho}\,\cancel{\partial_i\tilde{v}^i} + \cancel{\frac{2}{3}\tilde{v}^i\partial_i\tilde{\rho}} = \cancel{\tilde{E}^i\tilde{J}_i} \rightarrow \partial_\eta\tilde{\rho} = 0$$

meaning that the "bck" energy density isn't perturbed by the MF $\rightarrow \rho \propto a^{-4}$

$$\partial_\eta\tilde{v}^i = -\overset{v^z\partial_z v}{\cancel{\tilde{v}^j\partial_j\tilde{v}^i}} + \cancel{\frac{1}{3}\tilde{v}^i\partial_j\tilde{v}^j} + \cancel{\frac{1}{6}\frac{\tilde{v}^i}{\tilde{\rho}}(\tilde{v}^j\partial_j\tilde{\rho})}$$

$$\cancel{-\frac{1}{4}\frac{1}{\tilde{\rho}}\partial_i\tilde{\rho}} = \frac{3}{4}\frac{1}{\tilde{\rho}}(\underline{\tilde{J}}\times\underline{\tilde{B}})^i + \tilde{\nu}_k\left(\partial_j\partial^j\tilde{v}^i + \cancel{\frac{1}{3}\partial^i\partial_j\tilde{v}^j}\right)$$

$$\hookrightarrow \boxed{\partial_\eta\underline{V} = \frac{3}{4}\frac{1}{\rho}\left(-\bar{B}\,\partial_z\underline{b}\right) + \nu_k\Delta\underline{V}}$$

$$\uparrow$$

$$[\nabla\times(\underline{\bar{B}} + \underline{b})]\times(\underline{\bar{B}} + \underline{b}) = \nabla\times\underline{b}\times\underline{\bar{B}} + \nabla\times\underline{b}\times\underline{b} =$$

$$= (-\partial_z b_y\bar{B}, -\partial_z b_x\bar{B}, 0) + (0, 0, -b_y\partial_z b_y \cancel{+ b_x\partial_z b_x})$$

$$\partial_\eta\underline{\tilde{B}} = \nabla\times(\underline{\tilde{v}}\times\underline{\tilde{B}}) \rightarrow \boxed{\partial_\eta\underline{b} = -\bar{B}\,\partial_z\underline{V}}$$

$$\uparrow$$

$$\underline{v}\times\underline{B} = \underline{v}\times\underline{\bar{B}} + \underline{v}\times\underline{b} = (v_y\bar{B}, v_x\bar{B}, 0) + (0, 0, \cancel{v_x b_y + v_y b_x})$$

$$\nabla\times(\underline{v}\times\underline{B}) = (-\bar{B}\,\partial_z v_x, -\bar{B}\,\partial_z v_y, 0)$$

$$\partial_\eta^2\underline{b} = -\bar{B}\,\partial_z\left(-\frac{3}{4}\frac{\bar{B}}{\rho}\partial_z\underline{b} + \nu_k\partial_z^2\underline{v}\right) =$$

$$= -\frac{\bar{B}}{\rho}\partial_z\left[-\frac{3}{4}\partial_z\underline{b} + \rho\nu_k\left(-\frac{1}{\bar{B}}\partial_\eta\partial_z\underline{b}\right)\right]$$

$$\boxed{\partial_\eta^2 b - \nu_k\partial_z^2\partial_\eta b - V_A^2\partial_z^2 b = 0} \quad \left(\begin{array}{l}V_A \text{ defined}\\ \text{here with}\\ 4/3\text{ factor}\end{array}\right)$$

Damped wave eq. for the amplitude of the MF component b.

Suppose now $b(\eta, z) = f(\eta)\,e^{ikz}$

$$\frac{d^2f}{d\eta^2} + \nu_k k^2\frac{df}{d\eta} + k^2V_A^2 f(\eta) = 0$$

damped Alfvén waves with $k \parallel B_0 \parallel V_A$ moving at the Alfvén speed V_A

Setting $k^2V_A^2 = \omega_0^2$ driving term
$\nu_k k^2 = D(\eta)$ damping term

the solutions are of the same type as those of a damped harmonic oscillator

- $\omega_0 \gg D(\eta)$ oscillatory damped wave
- $D(\eta) \gg \omega_0$ overdamped mode: → **MF isn't damped and survives!!**

$$\frac{D(\eta)}{\omega_0} = \frac{\nu_k k}{V_A} \underset{\uparrow}{=} \frac{1}{5}\frac{\rho_d}{\rho_{tot}}\ell^{\gamma c}_{mfp}\,k\,\frac{745}{B_{-9}} \simeq 149\,\frac{2}{3.36}\,\ell^{\gamma c}_{mfp}\,\frac{k}{B_{-9}}$$

$$\begin{cases}\nu_k = \frac{1}{5}\frac{\rho_d}{\rho_{tot}}\ell^{d,c}_{mfp}\\[4pt] \frac{B^2}{\rho_{rad}} \simeq \frac{B^2}{h^2\Omega_{rad}}\left(\frac{a}{a_0}\right)^4\frac{h^2}{\rho_c^0} = 2.4\cdot 10^{-6}\left(\frac{B_0}{10^{-9}\,\text{Gauss}}\right)^2 \equiv 2.4\cdot 10^{-6}\,B_{-9}^2\\[4pt] V_A = \frac{B}{\sqrt{\frac{4}{3}\rho}} = \sqrt{\frac{3}{4}2.4\cdot 10^{-6}}\,B_{-9} \simeq 0.0013\,B_{-9}\end{cases}$$

$\ell^{\gamma c}_{mfp}k < 1$ for the fluid approx to hold, but there still are modes which can satisfy $D(\eta) \gg \omega_0$, especially when $B < B_{-9}$

*) Solution when $\omega_0 \gg D(\eta)$

$$f(\eta) \equiv e^{-\frac{1}{2}\int D\,d\eta'}w(\eta) \rightarrow e^{-\frac{1}{2}\int D\,d\eta'}\Big[\omega_0^2 w(\eta) - \frac{D^2}{2}w(\eta)$$

$$\dot{f}(\eta) = -\frac{1}{2}Df + e^{-\frac{1}{2}\int D\,d\eta'}\dot{w} \qquad + \cancel{D\dot{w}}(\eta) - \frac{1}{2}\dot{D}\,w(\eta)$$

$$+ \frac{1}{4}D^2 w(\eta) - \cancel{\frac{1}{2}D\dot{w}}$$

$$- \cancel{\frac{1}{2}D\dot{w}} + \ddot{w}\Big] = 0$$

new equation:

$$\ddot{w} + \left(\omega_0^2 - \cancel{\frac{D^2}{4}} - \cancel{\frac{\dot{D}}{2}}\right)w = 0$$

Full solution for the b component:

$$\boxed{b(z, \eta) = e^{-\frac{1}{2}\int D\,d\eta'}\,e^{\pm i\omega_0\eta + ikz}}$$

damped oscillations with frequency $\omega_0 = kV_A$

and damping term: $e^{-\frac{1}{2}k^2/k_D^2(\eta)}$ with

$$k_D(\eta) = \left(\int_{\eta_{in}}^{\eta}\frac{1}{5}\frac{\rho_\gamma}{\rho_{tot}}\ell^{\gamma c}_{mfp}\,d\eta'\right)^{-\frac{1}{2}}$$

$\ell^{\gamma c}_{mfp} = \frac{1}{a n_e\sigma_T} \propto T^{-2}$ growing with time so damping cutting out larger and larger scales with time

*) Solution when $D(\eta) \gg \omega_0$: $\ddot{f} + D\dot{f} + \omega_0^2 f = 0$

f adjust itself so that $\ddot{f} \to 0$ and $\boxed{D\dot{f} \simeq -\omega_0^2 f}$

$$\boxed{b(z, \eta) = e^{ikz}f(\eta_{in})\,e^{-\int_{\eta_{in}}^{\eta}\frac{\omega_0^2}{D}d\eta'}}$$

no more time oscillations! The oscillator reaches a terminal velocity regime

$$-\int_{\eta_{in}}^{\eta}\frac{\cancel{k^2}V_A^2}{\nu_k\cancel{k^2}}d\eta' \simeq -V_A^2\int_{\eta_{in}}^{\eta}\ell^{\gamma,c}_{mfp}\,d\eta'$$

η_{in}: time at which first $\ddot{f} \simeq 0$

this damping term can be very small for modes with $\dot{f}$ at η_{in} already small: they then remain frozen and resist damping.

(oscillator in strong friction released from rest acquire small velocity before friction balances the driving force)

the final outcome is that these Alfvén waves do not all get erased:

- some are frozen in the overdamped regime, if k is small enough
- some do not oscillate appreciably, if $kV_A\mathcal{H}^{-1} \lesssim 1$

There can also be a compressible (magnetosonic) mode which, however, can only be analysed with a fully linearised setting, assuming $b \ll \bar{B}$. Since this is not valid in the cosmological setting, we don't study it. The linearized analysis indicates that it is not relevant because it is damped much earlier: it oscillates much faster, with $c_s = \frac{1}{\sqrt{3}}$ sound speed

A phenomenon similar to Silk damping, that is damping the acoustic waves of the baryon-photon fluid (visible in CMB at small scales)

•) DURING THE FREE-STREAMING REGIME

As soon as γ start to free-stream, there is a big drop in the pressure of the baryons (that were formerly sustained by the coupling with the γ) and the MF pressure can suddenly become important.

The incompressible Alfvén waves studied above can become compressible, and the compressible modes can become gravitationally unstable.

When does this happen?

$$p_b = n_b T = \eta\, n_\gamma T = 6\cdot 10^{-10}\,\zeta(3)\,\frac{2}{\pi^2}T^4$$

$$p_B = \frac{B^2}{6} = \frac{1}{6}\,2.4\cdot 10^{-6}\,B_{-9}^2\,\rho_{rad} =$$

$$= \frac{2.4\cdot 10^{-6}}{6}h^2\Omega_{rad}\frac{\rho_c^0}{h^2}\left(\frac{T}{T_0}\right)^4 B_{-9}^2$$

$$\boxed{p_B \gg p_b} \text{ for } B_{-9} \gg \sqrt{\frac{12\cdot 10^{-10}\zeta(3)}{\pi^2}\frac{6}{2.4\cdot 10^{-6}}\frac{1}{h^2\Omega_{rad}}\frac{h^2}{\rho_c^0}T_0^4}$$

$$= 0.02$$

So the evolution depends on the amplitude of the MF:

•) for $B_0 \gg B_{crit} \simeq 2\cdot 10^{-11}$ Gauss one enters the regime in which the modes become compressible: as before, need linear analysis we don't study this

•) for $B_0 \ll B_{crit}$ the Alfvén mode entering the free-streaming regime REMAINS INCOMPRESSIBLE. The situation is therefore similar to what studied before, but applied to the eq:

$$\frac{\partial v}{\partial t} + H(t)\underline{v} + \cancel{\underline{v}\cdot\nabla\underline{v}} =$$

$$= -\frac{1}{a\rho_b}\cancel{\nabla}p_b + \frac{1}{\rho_b}(\nabla\times\underline{B_L})\times\underline{B_L} - \cancel{\frac{1}{a}\nabla\phi} - \frac{4}{3}\frac{\rho_\gamma}{\rho_b}\frac{1}{\ell^{\gamma p}_{mfp}}\underline{v}$$

and with the same setting as above (but now for the lab field) one gets:

$$\ddot{f}(\eta) + \underbrace{\left[aH + \frac{4}{3}\frac{\rho_\gamma}{\rho_b}\frac{1}{\ell^{\gamma c}_{mfp}}\right]}_{D(\eta)}\dot{f}(\eta) + \underbrace{k^2V_A^2\frac{4}{3}\frac{\rho_\gamma}{\rho_b}}_{\omega_0^2}f(\eta) = 0$$

In the damping term, one can neglect the Hubble term: $\frac{D(\eta)}{aH} = \frac{4}{3}\frac{\rho_\gamma}{\rho_b}\frac{\mathcal{H}^{-1}}{\ell^{\gamma c}_{mfp}} \gg 1$ since γ are not yet decoupled

the damping to driving terms ratio is:

$$\frac{D(\eta)}{\omega_0} = \sqrt{\frac{4}{3}\frac{\rho_\gamma}{\rho_b}}\frac{1}{\ell^{\gamma c}_{mfp}kV_A} \underset{\uparrow}{\simeq} 10^3\sqrt{\frac{\rho_\gamma}{\rho_b}}\,\frac{1}{\underbrace{\ell^{\gamma c}_{mfp}k}_{\simeq 1}}\,\frac{1}{B_{-9}}$$

$V_A \simeq 0.0013\,B_{-9}$

$\simeq 1$ for k in the free-streaming regime

$\frac{D(\eta)}{\omega_0} \gg 1$ for $B \ll B_{crit}$:

all modes are in the overdamped regime and the solution is:

$$b(z, \eta) = e^{ikz}f(\eta_{in})\,e^{-\int_{\eta_{in}}^{\eta}\frac{\omega_0^2}{D}d\eta'}$$

where now $\frac{\omega_0^2}{D} = k^2V_A^2\ell^{\gamma c}_{mfp}$ and therefore

$$b(z, \eta) = e^{ikz}f(\eta_{in})\,e^{-k^2/k_D^2}$$

$$\text{with } \boxed{k_D = \left(-\int_{\eta_{in}}^{\eta}V_A^2\ell^{\gamma c}_{mfp}\,d\eta'\right)^{-1/2} = \sqrt{\frac{2}{15}}\frac{1}{V_A}\frac{1}{L_S(\eta)}}$$

So the smallest magnetic scale getting damped at Recombination is much smaller than the mode damped by Silk damping, by a factor $V_A^2 \ll 1$: the MF remains at much smaller scales and can influence CMB spectrum at small scales. This k_D is the upper cutoff in the MF spectrum that is used in CMB papers.

Indeed, after recombination,

$$\frac{D(\eta)}{aH} = \frac{4}{3}\frac{\rho_\gamma}{\rho_b}\frac{\mathcal{H}^{-1}}{\ell^{\gamma c}_{mfp}} \ll 1$$

and the Hubble damping dominates: the (LAB!) MF simply redshifts with the expansion of the universe, until one goes back to the turbulent phase.