Nordita Winter School 2026

Cosmological Magnetic Fields:
Generation, Observation, and Modeling

Cosmological Magnetic Fields Theory Exercise Session 1: Relativistic MHD

Based on A. Roper Pol & A. S. Midiri [2501.05732]

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Swiss National Science Foundation

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General Relativistic MHD

The energy-momentum tensor for a relativistic perfect fluid (in a generic metric) is

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Metric tensor	$g^{\mu\nu}$
Fluid 4-velocity	u^μ
Fluid density	ρ
Fluid pressure	p

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Consider a constant equation of state $p = c_s^2 \rho$

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Choose time coordinate as conformal time

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<u>Rescale physical quantities to comoving ones</u>

$$\tilde{\rho} = a^4 \rho \qquad \tilde{u}^\mu = a\, u^\mu = \gamma(1, \boldsymbol{u}) \qquad \tilde{T}_{pf}^{\mu\nu} = a^6 T_{pf}^{\mu\nu}$$

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$$\tilde{T}^{\mu\nu}_{pf} = (1 + c_s^2)\, \tilde{\rho}\, \tilde{u}^\mu \tilde{u}^\nu + c_s^2\, \tilde{\rho}\, \eta^{\mu\nu}$$

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Consider radiation domination $c_s^2 = 1/3$

$\rightarrow \tilde{f}^\nu_H = 0$

Consider a constant equation of state $p = c_s^2 \rho$

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Relativistic MHD in FLRW

$$\tilde{T}^{\mu\nu}_{pf} = \frac{4}{3}\,\tilde{\rho}\,\tilde{u}^{\mu}\tilde{u}^{\nu} + \frac{1}{3}\,\tilde{\rho}\,\eta^{\mu\nu}$$

$$\partial_\mu \tilde{T}^{\mu\nu}_{pf} = \tilde{f}^{\nu}_{visc} + \tilde{f}^{\nu}_{Lor}$$

$\equiv -\partial_\mu \tilde{\pi}^{\mu\nu}$ $\equiv -\partial_\mu \tilde{T}^{\mu\nu}_{EM}$

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$$\tilde{f}^{0}_{\rm visc} = 2\,\tilde{\nu}\,(1+c_{\rm s}^2)\,\tilde{\rho}\left[\tilde{S}^{ij}\tilde{S}_{ij} - \tfrac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\right],$$

$$\tilde{\boldsymbol{f}}_{\rm visc} = \tilde{\nu}\,(1+c_{\rm s}^2)\,\tilde{\rho}\left(\boldsymbol{\nabla}^2\boldsymbol{u} + \tfrac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + 2\,(\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}\right),$$

$$\tilde{S}^{ij} \;=\; \tfrac{1}{2}(\partial^i u^j + \partial^j u^i)$$

$$\tilde{\sigma}^{ij} = \tilde{S}^{ij} - \frac{1}{3}(\nabla\cdot u)\,\delta^{ij}$$

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$$\tilde{f}^{0}_{\mathrm{Lor}} = \tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{J}}, \qquad \tilde{f}^{i}_{\mathrm{Lor}} = \tilde{J}^{0}\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}}.$$

$$\tilde{\boldsymbol{J}}^{0} = \gamma(\tilde{\rho}_{e} + \tilde{\sigma}\boldsymbol{u}\cdot\tilde{\boldsymbol{E}}), \qquad \tilde{\boldsymbol{J}}^{i} = \gamma(\tilde{\rho}_{e}\boldsymbol{u} + \tilde{\sigma}[\tilde{\boldsymbol{E}} + \boldsymbol{u}\times\tilde{\boldsymbol{B}}])$$

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$$\tilde{T}^{\mu\nu}_{pf} = \frac{4}{3}\,\tilde{\rho}\,\tilde{u}^{\mu}\tilde{u}^{\nu} + \frac{1}{3}\,\tilde{\rho}\,\eta^{\mu\nu}$$

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$$\tilde{f}^{0}_{\rm Lor} = \tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{J}}, \qquad \tilde{f}^{i}_{\rm Lor} = \tilde{J}^0\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}}.$$

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$$\partial_\tau\tilde{\boldsymbol{E}} = \boldsymbol{\nabla}\times\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{J}}, \qquad \partial_\tau\tilde{\boldsymbol{B}} = -\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}, \qquad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}} = \tilde{J}^0, \qquad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0,$$ **Maxwell**

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$$= \tilde{\mathcal{F}}^{\nu}(\tilde{\rho}, \boldsymbol{u}, \tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}})$$

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4 variables ($\tilde{\rho}$ & $\boldsymbol{u}$) and 4 equations for the fluid → closed system!

Relativistic MHD in FLRW

$$\tilde{T}^{\mu\nu}_{pf} = \frac{4}{3}\,\tilde{\rho}\,\tilde{u}^\mu \tilde{u}^\nu + \frac{1}{3}\,\tilde{\rho}\,\eta^{\mu\nu}$$

$$\tilde{u}^\mu = \gamma(1, \boldsymbol{u})$$

$$\partial_\mu \tilde{T}^{\mu\nu}_{pf} = \tilde{f}^\nu_{visc} + \tilde{f}^\nu_{Lor} = \tilde{\mathcal{F}}^\nu(\tilde{\rho}, \boldsymbol{u}, \tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}})$$

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4 variables ($\tilde{\rho}$ & $\boldsymbol{u}$) and 4 equations for the fluid → closed system!

How can we solve them numerically?

Relativistic MHD in FLRW: conservation form

$$T^{\mu\nu}_{pf} = \frac{4}{3}\,\rho\,u^{\mu}u^{\nu} + \frac{1}{3}\,\rho\,\eta^{\mu\nu}$$

$$u^{\mu} = \gamma(1, \boldsymbol{u})$$

$$\partial_{\mu} T^{\mu\nu}_{pf} = \mathcal{F}^{\nu}(\rho, \boldsymbol{u}, \boldsymbol{E}, \boldsymbol{B})$$

**all quantities are comoving*

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$$\begin{cases} \partial_0 T_{pf}^{00} = -\,\partial_j T_{pf}^{j0} + \mathcal{F}^{0} \\ \partial_0 T_{pf}^{0i} = -\,\partial_j T_{pf}^{ji} + \mathcal{F}^{i} \end{cases}$$

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We need to choose the 4 variables to solve for

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T^{00}, T^{0i} seems to be a natural choice

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T^{00}, T^{0i} seems to be a natural choice

However, we first need to express ρ, u and T^{ij} in terms of the 4 variables T^{00}, T^{0i}

**all quantities are comoving*

Exercise no. 1: CONSERVATION FORM

$$T^{\mu\nu} = \frac{4}{3}\,\rho\, u^{\mu} u^{\nu} + \frac{1}{3}\,\rho\,\eta^{\mu\nu} \qquad u^{\mu} = \gamma(1, \boldsymbol{u}) \qquad \gamma^2 = 1/(1-u^2)$$

$$\partial_\mu T^{\mu\nu} = \mathcal{F}^{\nu}(\rho, \boldsymbol{u}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow \begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} + \mathcal{F}^0 \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} + \mathcal{F}^i \end{cases}$$

Exercise no. 1: CONSERVATION FORM

$$T^{\mu\nu} = \frac{4}{3}\,\rho\, u^{\mu}u^{\nu} + \frac{1}{3}\,\rho\,\eta^{\mu\nu} \qquad u^{\mu} = \gamma(1, \boldsymbol{u}) \qquad \gamma^2 = 1/(1-u^2)$$

$$\partial_\mu T^{\mu\nu} = \mathcal{F}^{\nu}(\rho, \boldsymbol{u}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow \begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} + \mathcal{F}^0 \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} + \mathcal{F}^i \end{cases}$$

A) Express $\rho, \boldsymbol{u}$ and T^{ji} in terms of T^{00}, T^{0i} and γ

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$$T^{00} = \frac{4}{3}\rho\,\gamma^2 - \frac{1}{3}\rho$$

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$$\rho = \frac{3T^{00}}{4\,\gamma^2 - 1}$$

$$u^i = \frac{T^{0i}}{T^{00}}\frac{4\gamma^2 - 1}{4\,\gamma^2}$$

$$T^{ji} = \frac{4\gamma^2 - 1}{4\gamma^2}\,\frac{T^{0j}T^{0i}}{T^{00}} + \frac{T^{00}}{4\gamma^2 - 1}\,\delta^{ji}$$

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B) Express γ^2 in terms of T^{00}, T^{0i}

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$$r^2 = \frac{T^{0i}T^{0i}}{[T^{00}]^2} = \frac{(4/3)^2\gamma^4 u^2}{[(4/3)\gamma^2 - 1/3]^2}$$

$$u^2 = 1 - 1/\gamma^2$$

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$$\gamma^2 = \frac{1}{2(1-r^2)}\left[1 - \frac{r^2}{2} + \sqrt{1 - \frac{3}{4}r^2}\,\right]$$

Fluid dynamics in the **conservation form***

$$r^2 = T^{0i}T^{0i}/(T^{00})^2$$

$$\gamma^2 = \frac{1}{2(1-r^2)}\left[1 - \frac{2r^2c_s^2}{1+c_s^2} + \sqrt{1 - \frac{4r^2c_s^2}{(1+c_s^2)^2}}\right]$$

$$T^{ji} = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\,\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\,T^{00}\,\frac{c_s^2}{\gamma^2(1+c_s^2) - c_s^2} = T^{ji}[T^{0\mu}]$$

*general c_s^2

Fluid dynamics in the **conservation form***

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$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

CONSERVATION FORM

*general c_s^2

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

How do we solve them in the lattice?

Fluid dynamics in the **conservation form**

$$
\begin{cases}
\partial_0 T^{00} = -\,\partial_j T^{j0} \\
\partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}]
\end{cases}
$$

How do we solve them in the lattice?

After discretizing the derivatives we get equations of the form

$$\partial_0 X^{\mu} = \mathcal{K}^{\mu}[X^{\nu}] \longleftarrow$$

The RHS is a function of the fields themselves

Fluid dynamics in the **conservation form**

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The RHS is a function of the fields themselves

Natural algorithm for timestepping → **explicit Runge-Kutta**

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}] \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}] \end{cases}$$

space discretization

$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_\mathrm{j} \mathrm{T}^{\mathrm{j}0}$$

$$\mathcal{K}^i[T^{0\mu}] \equiv \nabla_\mathrm{j} \mathrm{T}^{\mathrm{ji}}$$

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$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}} \mathrm{T}^{\mathrm{j}0}$$

For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

$$\mathcal{K}^i[T^{0\mu}] \equiv \nabla_{\mathrm{j}} \mathrm{T}^{\mathrm{ji}}$$

Fluid dynamics in the **conservation form**

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FORWARD DERIVATIVE

$$\nabla_i^+ f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\, \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x})}{\delta x} \to \partial_i f(\boldsymbol{x}) \Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x)$$

$\hat{\boldsymbol{\imath}}$ unit vectors in the three spatial directions

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}] \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}] \end{cases}$$

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For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

BACKWARD DERIVATIVE

$$\nabla_i^- f(\boldsymbol{x}) = \frac{f(\boldsymbol{x}) - f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}})}{\delta x} \rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x)$$

$\hat{\boldsymbol{\imath}}$ unit vectors in the three spatial directions

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}] \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}] \end{cases}$$

space discretization

$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}} \mathrm{T}^{\mathrm{j}0}$$

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For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

NEUTRAL DERIVATIVE

$$\nabla_i^{(0)} f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\ \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\ \hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

$$\partial_0 T^{00} = -\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}]$$

$$\partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}]$$

space discretization

$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_\mathrm{j} \mathrm{T}^{\mathrm{j}0}$$

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For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

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$$\nabla_i^{(0)} f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\, \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Simpler at higher orders if fields «live» at lattice sites

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x} + \delta x\, \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}})}{2\delta x} \quad \to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Fluid dynamics often requires higher order spatial derivatives (shocks, nonlinearities…)

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x} + \delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\,\hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(4)} = \frac{-f(\boldsymbol{x} + 2\delta x\,\hat{\boldsymbol{\imath}}) + 8f(\boldsymbol{x} + \delta x\,\hat{\boldsymbol{\imath}}) - 8f(\boldsymbol{x} - \delta x\,\hat{\boldsymbol{\imath}}) + f(\boldsymbol{x} - 2\delta x\,\hat{\boldsymbol{\imath}})}{12\delta x}$$

$$\to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^4)$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(4)} = \frac{-f(\boldsymbol{x}+2\delta x\,\hat{\boldsymbol{\imath}}) + 8f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - 8f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}}) + f(\boldsymbol{x}-2\delta x\,\hat{\boldsymbol{\imath}})}{12\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(6)} = \frac{f(\boldsymbol{x}+3\delta x\,\hat{\boldsymbol{\imath}}) - 9f(\boldsymbol{x}+2\delta x\,\hat{\boldsymbol{\imath}}) + 45f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - 45f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}}) + 9f(\boldsymbol{x}-2\delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x}-3\delta x\,\hat{\boldsymbol{\imath}})}{60\delta x}$$

$$\to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^6)$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_{\mathrm{j}}^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

Runge-Kutta order $(\Delta t)^N$

Neutral derivative order $(\Delta t)^M$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

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A special property of the **CONSERVATION FORM**

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{ji} \end{cases}$$

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A special property of the **CONSERVATION FORM**

When using periodic boundary conditions we have that (*Gauss theorem*)

$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} T^{ji} \end{cases}$$

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$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0 \longrightarrow \partial_0 \left[\sum_{all\ lattice\ points\ n} T^{0\mu}(n) \right] = 0$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{j0}} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

Runge-Kutta order $(\Delta t)^N$

Neutral derivative order $(\Delta t)^M$

A special property of the **CONSERVATION FORM**

When using periodic boundary conditions we have that (*Gauss theorem*)

$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0 \longrightarrow \partial_0 \langle T^{0\mu} \rangle = 0$$

Average $T^{0\mu}$ conserved at machine precision!

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

An alternative form is obtained by substituting $T^{\mu\nu}$ with its expression in terms of the fluid primitive variables ρ and $\boldsymbol{u}$

$$T^{\mu\nu} = (1 + c_s^2)\rho\, u^\mu u^\nu + c_s^2 \rho\, \eta^{\mu\nu}$$

$$T^{00} = \rho(1 + c_s^2)\gamma^2 - c_s^2\, \rho$$

$$T^{0i} = \rho(1 + c_s^2)\gamma^2 u^i$$

$$T^{ji} = \rho(1 + c_s^2)\gamma^2 u^j u^i + c_s^2 \rho\, \delta^{ji}$$

$$u^\mu = \gamma(1, \boldsymbol{u})$$

Exercise no. 2: NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

An alternative form is obtained by substituting $T^{\mu\nu}$ with its expression in terms of the fluid primitive variables ρ and $\boldsymbol{u}$

$$T^{\mu\nu} = (1 + c_s^2)\rho\, u^\mu u^\nu + c_s^2 \rho\, \eta^{\mu\nu}$$

$$T^{00} = \rho(1 + c_s^2)\gamma^2 - c_s^2\,\rho$$

$$T^{0i} = \rho(1 + c_s^2)\gamma^2 u^i$$

$$T^{ji} = \rho(1 + c_s^2)\gamma^2 u^j u^i + c_s^2 \rho\, \delta^{ji}$$

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$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

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$$T^{\mu\nu} = (1 + c_s^2)\rho\, u^\mu u^\nu + c_s^2 \rho\, \eta^{\mu\nu}$$

$$T^{00} = \rho(1 + c_s^2)\gamma^2 - c_s^2\, \rho$$

$$T^{0i} = \rho(1 + c_s^2)\gamma^2 u^i$$

$$T^{ji} = \rho(1 + c_s^2)\gamma^2 u^j u^i + c_s^2 \rho\, \delta^{ji}$$

$$u^\mu = \gamma(1, \boldsymbol{u})$$

Write the fluid equations ($\partial_\mu T^{\mu\nu} = 0$) in the form

$$\begin{cases} \partial_0 \rho = \mathrm{F}^0 \\ \partial_0 \boldsymbol{u} = \mathrm{F}^i \end{cases}$$

with $\mathrm{F}^0, \mathrm{F}^i$ functions of $\rho, \boldsymbol{u}$ and their gradients

Exercise no. 2: NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j(\rho\,\gamma^2\,u^j)$$

$$T^{\mu\nu} = (1+c_s^2)\rho\,u^\mu u^\nu + c_s^2\rho\,\eta^{\mu\nu}$$

$$T^{00} = \rho(1+c_s^2)\gamma^2 - c_s^2\,\rho$$

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$$u^\mu = \gamma(1, \boldsymbol{u})$$

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$$u^\mu = \gamma(1, \boldsymbol{u})$$

$$\partial_0[\rho(1 + c_s^2)\gamma^2 - c_s^2 \rho] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^j\right) = \mathcal{K}^0$$

$$\partial_0\left[\rho(1 + c_s^2)\gamma^2 u^i\right] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^i u^j\right) - c_s^2 \partial_i \rho$$

$$= \mathcal{K}^i$$

Exercise no. 2: NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

A) Express $\partial_0 \gamma^2$ in terms of $\partial_0 \rho$

$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^j\right) = \mathcal{K}^0$$

$$T^{\mu\nu} = (1+c_s^2)\rho\, u^\mu u^\nu + c_s^2\rho\,\eta^{\mu\nu}$$

$$\partial_0\left[\rho(1+c_s^2)\gamma^2 u^i\right] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^i u^j\right) - c_s^2\partial_i\rho$$

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$$T^{00} = \rho(1+c_s^2)\gamma^2 - c_s^2\,\rho$$

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$$T^{00} = \rho(1+c_s^2)\gamma^2 - c_s^2\,\rho$$

$$T^{0i} = \rho(1+c_s^2)\gamma^2 u^i$$

$$\partial_0\gamma^2 = \gamma^4 2\,u_i\partial_0 u^i = \frac{2\gamma^2}{\rho(1+c_s^2)}\left[u_i\mathcal{K}^i - u^2(c_s^2\partial_0\rho + \mathcal{K}^0)\right]$$

$$T^{ji} = \rho(1+c_s^2)\gamma^2 u^j u^i + c_s^2\rho\,\delta^{ji}$$

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Exercise no. 2: NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

B) Use the expression for $\partial_0 \gamma^2$ in the energy equation

$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^j\right) = \mathcal{K}^0$$

$$T^{\mu\nu} = (1+c_s^2)\rho\,u^\mu u^\nu + c_s^2\rho\,\eta^{\mu\nu}$$

$$\partial_0\left[\rho(1+c_s^2)\gamma^2 u^i\right] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^i u^j\right) - c_s^2\partial_i\rho$$

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$$T^{00} = \rho(1+c_s^2)\gamma^2 - c_s^2\,\rho$$

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$$\partial_0[\rho(1 + c_s^2)\gamma^2 - c_s^2 \rho] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^j\right) = \mathcal{K}^0$$

$$\partial_0\left[\rho(1 + c_s^2)\gamma^2 u^i\right] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^i u^j\right) - c_s^2 \partial_i \rho$$

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$$\partial_0 \gamma^2 = \gamma^4 2\, u_i \partial_0 u^i = \frac{2\gamma^2}{\rho(1 + c_s^2)}[u_i \mathcal{K}^i - u^2(c_s^2 \partial_0 \rho + \mathcal{K}^0)]$$

$$\partial_0 \rho = \frac{1}{1 - u^2 c_s^2}[(1 + u^2)\mathcal{K}^0 - 2\, u_i\, \mathcal{K}^i]$$

Exercise no. 2: NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

C) Use $\partial_0\gamma^2$ and $\partial_0\rho$ in the momentum equation

$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^j\right) = \mathcal{K}^0$$

$$T^{\mu\nu} = (1+c_s^2)\rho\,u^\mu u^\nu + c_s^2\rho\,\eta^{\mu\nu}$$

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$$\partial_0\rho = \frac{1}{1-u^2c_s^2}\left[(1+u^2)\mathcal{K}^0 - 2\,u_i\,\mathcal{K}^i\right]$$

$$u^\mu = \gamma(1,\boldsymbol{u})$$

Fluid dynamics in the **non-conservation form**

After a few manipulations we arrive at the **NON-CONSERVATION FORM** of fluid dynamics

$$\partial_0 \ln \rho = -\frac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

$$\partial_0 u_i = -\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\, u_i - \frac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i \frac{c_s^2}{(1-c_s^2 u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

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After discretizing the RHS we are left with a system of equations of the form

$$\partial_0 \ln\rho = \mathcal{G}^0[\ln\rho, u]$$

$$\partial_0 u_i = \mathcal{G}^i[\ln\rho, u]$$

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After discretizing the RHS we are left with a system of equations of the form

$$\partial_0 \ln\rho = \mathcal{G}^0[\ln\rho, u] \longrightarrow$$ The RHS depends on the fluid variables themselves

$$\partial_0 u_i = \mathcal{G}^i[\ln\rho, u]$$ Natural algorithm for timestepping → **explicit Runge-Kutta**

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

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$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \to \left[\nabla_j^{(0)} \mathrm{u}_j\right]^{(N)}$$

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\nabla\cdot\boldsymbol{u} \rightarrow \left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_{\mathrm{j}}\right]^{(N)} \qquad (\boldsymbol{u}\cdot\nabla)\ln\rho \rightarrow u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\ln\rho\right]^{(N)}$$

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$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \to \left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_{\mathrm{j}}\right]^{(N)} \qquad (\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho \to u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\ln\rho\right]^{(N)}$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\, u_i \to u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_i\right]^{(N)}$$

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \to \left[\nabla_{\rm j}^{(0)}\mathrm{u}_{\rm j}\right]^{(N)} \qquad (\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho \to u_{\rm j}\left[\nabla_{\rm j}^{(0)}\ln\rho\right]^{(N)}$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\, u_i \to u_{\rm j}\left[\nabla_{\rm j}^{(0)}\mathrm{u}_i\right]^{(N)} \qquad \nabla_i \ln\rho \to \left[\nabla_i^{(0)}\ln\rho\right]^{(N)}$$

CONSERVATION vs NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

$$r^2 = \frac{T^{0i}T^{0i}}{(T^{00})^2} \qquad \gamma^2 = \frac{1}{2(1-r^2)}\left[1 - \frac{2r^2 c_s^2}{1+c_s^2} + \sqrt{1 - \frac{4r^2 c_s^2}{(1+c_s^2)^2}}\right]$$

$$T^{ji}[T^{0\mu}] = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\, T^{00}\frac{c_s^2}{\gamma^2(1+c_s^2) - c_s^2}$$

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2 u^2}\left[\nabla\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\nabla)\ln\rho\right] \\ \partial_0 u_i = -(\boldsymbol{u}\cdot\nabla)\, u_i - \dfrac{c_s^2}{1+c_s^2}\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\nabla\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\nabla)\ln\rho\right] \end{cases}$$

CONSERVATION vs NON-CONSERVATION FORM

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$$T^{ji}[T^{0\mu}] = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\,\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\,T^{00}\,\frac{c_s^2}{\gamma^2(1+c_s^2)-c_s^2}$$

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \dfrac{c_s^2}{1+c_s^2}\,\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Both forms can be solved with a Runge-Kutta timestepping scheme and neutral derivatives

CONSERVATION vs NON-CONSERVATION FORM

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$$T^{ji}[T^{0\mu}] = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\,T^{00}\frac{c_s^2}{\gamma^2(1+c_s^2)-c_s^2}$$

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{c_s^2}{1+c_s^2}\dfrac{\nabla_i\ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Which one is better?

CONSERVATION vs NON-CONSERVATION FORM

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$$r^2 = \frac{T^{0i}T^{0i}}{(T^{00})^2} \qquad \gamma^2 = \frac{1}{2(1-r^2)}\left[1 - \frac{2r^2 c_s^2}{1+c_s^2} + \sqrt{1 - \frac{4r^2 c_s^2}{(1+c_s^2)^2}}\right]$$

$$T^{ji}[T^{0\mu}] = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\,\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\,T^{00}\,\frac{c_s^2}{\gamma^2(1+c_s^2) - c_s^2}$$

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \dfrac{c_s^2}{1+c_s^2}\,\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i\,\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Which one is better? It depends on the physical problem!

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right) u_i - \dfrac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i \dfrac{c_s^2}{(1-c_s^2 u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

$$
\begin{cases}
\partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\
\partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{\mathrm{c}_s^2}{1+\mathrm{c}_s^2}\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]
\end{cases}
$$

$$\Big\downarrow$$

$$
\begin{cases}
\partial_0 \ln\rho = -(1+c_s^2)\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\
\partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{\mathrm{c}_s^2}{1+\mathrm{c}_s^2}\nabla_i \ln\rho + u_i\;c_s^2\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]
\end{cases}
$$

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \dfrac{c_s^2}{1+c_s^2}\dfrac{\nabla_i \ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \partial_0 \ln\rho = -(1+c_s^2)\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \dfrac{c_s^2}{1+c_s^2}\nabla_i \ln\rho + u_i\;c_s^2\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

$$\begin{cases} \partial_0 \ln\rho = -(1+c_s^2)\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\nabla_i\ln\rho + u_i\;c_s^2\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

lrelativistic_eos=T

Missing factors in Pencil Code (in the default implementation they are 1) from erroneously taking $\partial_0 \gamma^2 = 0$

$$
\begin{cases}
\partial_0 \ln \rho = -(1+c_s^2)\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\
\partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\nabla_i \ln\rho + u_i\;c_s^2\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]
\end{cases}
$$

Sub-relativistic limit $u^2 \ll 1$

NON-CONSERVATION FORM

lrelativistic_eos=T

Missing factors in Pencil Code (in the default implementation they are 1) from erroneously taking $\partial_0 \gamma^2 = 0$

Now corrected → select lrelativistic_eos=T and lrelativistic_eos_corr=T in density run parameters

$$\begin{cases} \partial_0 \ln\rho = -(1+c_s^2)\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ u_i - \dfrac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\nabla_i \ln\rho + u_i\ c_s^2\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$