Nordita Winter School 2026

Cosmological Magnetic Fields:
Generation, Observation, and Modeling

Cosmological Magnetic Fields Theory Exercise Session 3: Statistical description of turbulent velocity and magnetic fields

Based on Monin & Yaglom «Statistical Fluid Mechanics: Mechanics of Turbulence»

UNIVERSITÉ DE GENÈVE

Swiss National Science Foundation

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Elements of hydrodynamic turbulence

What is hydrodynamic turbulence?

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Let us go back to the fluid equations in the non-conservation form

$$\partial_0 \ln \rho = -\frac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

$$\partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \frac{\mathrm{c}_s^2}{1+\mathrm{c}_s^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i\frac{c_s^2}{(1-c_s^2 u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

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Let us focus (for simplicity) on the subrelativistic limit ($c_s^2 \ll 1,\ u^2 \ll 1$)

The momentum equation, using a simple model for the viscosity, is

$$\partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_i + \nu\,\nabla^2 u_i - \frac{\boldsymbol{\nabla}p}{\rho}$$

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The evolution of the velocity is strongly dependent on the interplay between two terms

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$\propto v_{rms}^2 / L$ $\propto \nu \, v_{rms} / L^2$

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To gain insight on the interplay between them we can consider a fluid motion with a characteristic lenght scale L and a characteristic velocity v_{rms}

We then define the Reynolds number as the ratio between nonlinearities and viscosity

$$Re = \frac{\text{nonlinearities}}{\text{viscosity}} = \frac{v_{rms} L}{\nu}$$

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$$\partial_0 u_i = -(\boldsymbol{u} \cdot \boldsymbol{\nabla}) u_i + \nu \nabla^2 u_i - \frac{\boldsymbol{\nabla} p}{\rho} \qquad Re = \frac{\text{nonlinearities}}{\text{viscosity}} = \frac{v_{rms} L}{\nu}$$

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Experiments show that this ratio can be used to distinguish two different regimes

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Small Reynolds number

A small change in the initial conditions causes
a small change in the fluid profiles
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Laminar regime

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Small Reynolds number

A small change in the initial conditions causes a small change in the fluid profiles (ordered flow)

Laminar regime

Large Reynolds number

A small change in the initial conditions can cause instabilities and big changes in the fluid profiles (chaotic flow)

Turbulent regime

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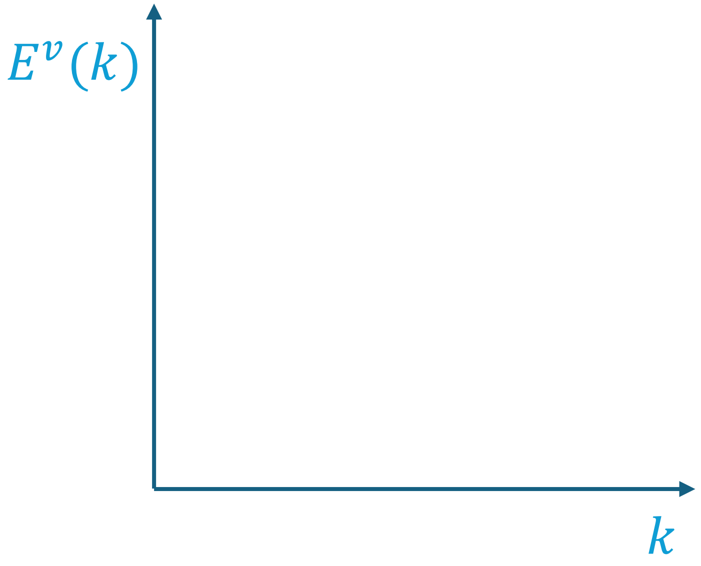

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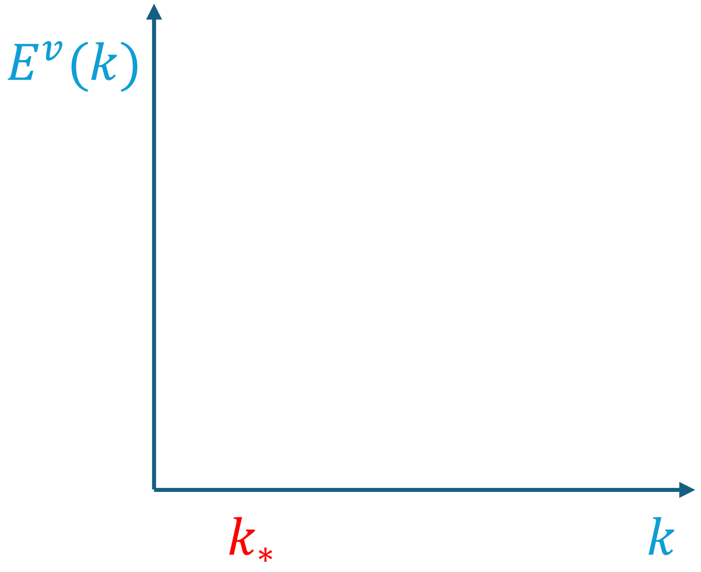

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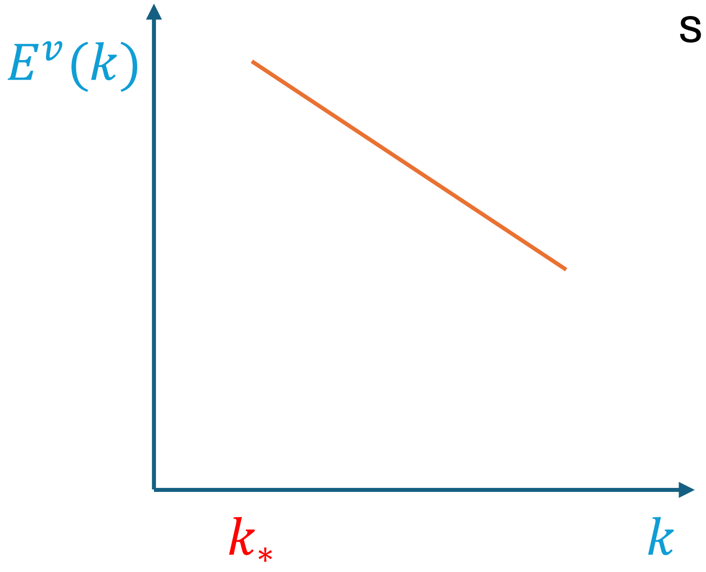

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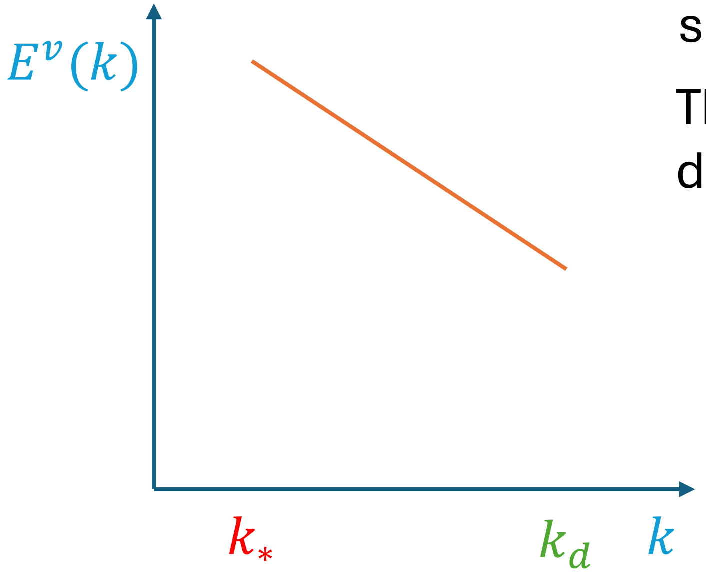

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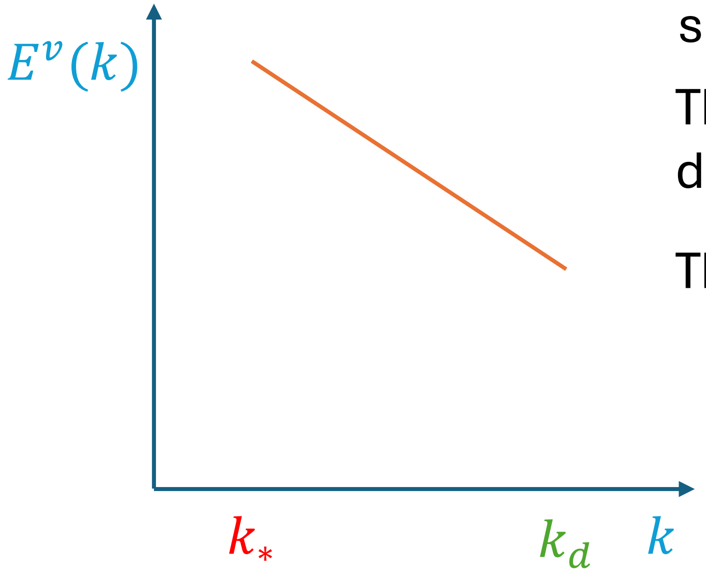

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The energy transfer rate is approximately scale independent

$$const = \epsilon = \frac{E}{\delta\tau_{eddy}} \propto \frac{v_{rms}^2}{\frac{1}{v_{rms}\,k}} \rightarrow v_{rms} \propto \epsilon^{\frac{1}{3}}\,k^{-\frac{1}{3}}$$

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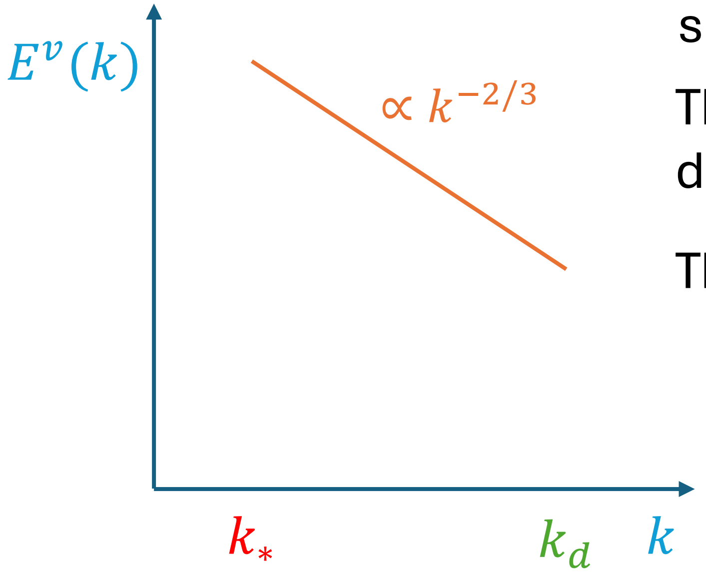

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Which implies the Kolmogorov spectrum for $k_* < k < k_d$ $\quad E^v(k)\,\mathrm{d}\ln k \propto v_{rms}^2 \rightarrow E^v(k) \propto v_{rms}^2 \propto \epsilon^{\frac{2}{3}}\, k^{-\frac{2}{3}}$

Statistical description of turbulence

→ Turbulent velocity and magnetic fields can be described by studying their statistical properties

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$$\langle \, \boldsymbol{u}(\boldsymbol{x}) \, \rangle = \boldsymbol{U}$$

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Since the field is also statistically isotropic, this constant must be equal to zero (invariant under rotations)

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Since the field is also statistically isotropic, this constant must be equal to zero (invariant under rotations)

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The first property of a statistically homogeneous and isotropic random field is that it has zero average

Statistically homogeneous and isotropic random fields

What can we say about the two point function?

$$B_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \langle\, u_i(\boldsymbol{x})\, u_j(\boldsymbol{y})\, \rangle$$

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The scalar products $\boldsymbol{a} \cdot \boldsymbol{b},\ \ \boldsymbol{a} \cdot \boldsymbol{r},\ \ \boldsymbol{b} \cdot \boldsymbol{r}$

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The scalar products $\boldsymbol{a}\cdot\boldsymbol{b},\ \boldsymbol{a}\cdot\boldsymbol{r},\ \boldsymbol{b}\cdot\boldsymbol{r}$, the modulus of $\boldsymbol{r}$

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and the oriented volume of the parallepiped formed by the three vectors $\epsilon_{ijk} a_i b_j r_k = \boldsymbol{r}\cdot(\boldsymbol{a}\times\boldsymbol{b})$

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Which quantities built out of $\boldsymbol{a},\boldsymbol{b},\boldsymbol{r}$ are invariant under rotations? <u>all terms must be linear in $\boldsymbol{a}$ and $\boldsymbol{b}$</u>

The scalar products $\boldsymbol{a}\cdot\boldsymbol{b},\ \boldsymbol{a}\cdot\boldsymbol{r},\ \boldsymbol{b}\cdot\boldsymbol{r}$, the modulus of $\boldsymbol{r}$
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$$\equiv \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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$$= P_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, M_L(r) + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$P_{ij} = \delta_{ij} - \hat{r}_i\, \hat{r}_j$$

Statistically homogeneous and isotropic random fields

$$a_i\, b_j\, B_{ij}(\boldsymbol{r}) = \boldsymbol{a}\cdot\boldsymbol{b}\ f_1(r) + (\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{b}\cdot\boldsymbol{r})\, f_2(r) + \boldsymbol{r}\cdot(\boldsymbol{a}\times\boldsymbol{b})\, f_3(r)$$

$$= a_i\, b_j\, \delta_{ij}\, f_1(r) + a_i\, b_j\, \hat{r}_i\, \hat{r}_j\, r^2 f_2(r) + a_i\, b_j\, \epsilon_{ijk}\, \hat{r}_k\ r\, f_3(r)$$

$$= a_i\, b_j \left[\delta_{ij}\, f_1(r) + \hat{r}_i\, \hat{r}_j\, r^2 f_2(r) + \epsilon_{ijk}\, \hat{r}_k\ r\, f_3(r)\right]$$

$$\longrightarrow \quad B_{ij}(\boldsymbol{r}) = \delta_{ij}\, f_1(r) + \hat{r}_i\, \hat{r}_j\, r^2 f_2(r) + \epsilon_{ijk}\, \hat{r}_k\ r\, f_3(r)$$

$$\equiv \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$= P_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, M_L(r) + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$P_{ij} = \delta_{ij} - \hat{r}_i\, \hat{r}_j$$

$M_H(r) \neq 0$ if there is parity violation [$\boldsymbol{r}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ is not invariant under parity]

Statistically homogeneous and isotropic random fields

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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Compute the two-point correlator in Fourier space

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}') \rangle$$

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$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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Compute the two-point correlator in Fourier space

A) Reduce it to a single integral

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}')\rangle = \iint d^3\boldsymbol{x}\, d^3\boldsymbol{y}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot\boldsymbol{y}}\, \langle\, u_i(\boldsymbol{x}) u_j(\boldsymbol{y})\, \rangle$$

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$$= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{y}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot\boldsymbol{y}}\, B_{ij}(\boldsymbol{r}) \longleftarrow \quad \boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$$

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Compute the two-point correlator in Fourier space

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$$\begin{aligned}
\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}')\rangle &= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{y}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot\boldsymbol{y}}\, \langle\, u_i(\boldsymbol{x}) u_j(\boldsymbol{y})\,\rangle \\
&= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{y}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot\boldsymbol{y}}\, B_{ij}(\boldsymbol{r}) \quad \longleftarrow \quad \boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x} \\
&= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot(\boldsymbol{r}+\boldsymbol{x})}\, B_{ij}(\boldsymbol{r})
\end{aligned}$$

Exercise no. 1

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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Compute the two-point correlator in Fourier space

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$$= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{y}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot\boldsymbol{y}}\, B_{ij}(\boldsymbol{r})$$

← $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$

$$= \iint d^3\boldsymbol{x}\, d^3\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{i\,\boldsymbol{k}'\cdot(\boldsymbol{r}+\boldsymbol{x})}\, B_{ij}(\boldsymbol{r})$$

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Exercise no. 1

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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B) Compute $\mathbb{X}$

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Compute the two-point correlator in Fourier space

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Exercise no. 1

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

Let us go to Fourier space and define $u(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, u(\boldsymbol{x})$

Compute the two-point correlator in Fourier space

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The result must be a rank-2 tensor built from $\boldsymbol{k}$…

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The result must be a rank-2 tensor built from $\boldsymbol{k}$... We will then have the following general structure

$$\mathbb{Y} = A(k)\delta_{ij} + B(k)\, \hat{k}_i \hat{k}_j$$

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If we take the contraction of both with δ_{ij}

$$3A(k) + B(k) = \int d^3\boldsymbol{r}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, [M_L(r) - M_N(r)]$$

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C) Compute $\mathbb{Y}$

$$\mathbb{Y} = \int d^3\boldsymbol{r}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}}\, \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)]$$

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$$= 4\,\pi \int dr\, r^2 j_0(kr)\,\, [M_L(r) - M_N(r)]$$

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Exercise no. 1

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

Let us go to Fourier space and define $u(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, u(\boldsymbol{x})$

Compute the two-point correlator in Fourier space

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}') \rangle \quad = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \int d^3\boldsymbol{r}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} B_{ij}(\boldsymbol{r})$$

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Two-point correlator in Fourier space

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\, \mathcal{F}_{ij}(\boldsymbol{k}) \qquad \mathbb{X}$$

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$$+ \hat{k}_i\, \hat{k}_j\, 4\pi \int dr\, r^2 [M_L(r) - M_N(r)] \left[\frac{\sin kr}{kr} - 3\frac{\sin kr}{(kr)^3} + 3\frac{\cos kr}{(kr)^2}\right]$$

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& + i\, \epsilon_{ijl}\, \hat{k}_l\, 4\pi \int dr\, r^2 M_H(r) \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]
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Two-point correlator in Fourier space

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$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}') \rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\, \mathcal{F}_{ij}(\boldsymbol{k})$$

$$\mathcal{F}_{ij}(\boldsymbol{k}) = \delta_{ij}\, 4\pi \int_0^\infty dr\, r^2 \left\{ M_N(r) \left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)] \left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$+ \hat{k}_i\, \hat{k}_j\, 4\pi \int dr\, r^2 [M_L(r) - M_N(r)] \left[\frac{\sin kr}{kr} - 3\frac{\sin kr}{(kr)^3} + 3\frac{\cos kr}{(kr)^2}\right]$$

$$+ i\, \epsilon_{ijl}\, \hat{k}_l\, 4\pi \int dr\, r^2 M_H(r) \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]$$

$$\equiv \delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl} \hat{k}_l\, F_H(k)$$

Two-point correlator in Fourier space

$$B_{ij}(\boldsymbol{r}) = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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$$+\, i\, \epsilon_{ijl}\, \hat{k}_l\, 4\pi \int dr\, r^2 M_H(r) \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]$$

$$\equiv \delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl} \hat{k}_l\, F_H(k)$$

$$F_L(k) = 4\pi \int_0^\infty dr\, r^2 \left\{ M_L(r) \left[\frac{\sin kr}{kr}\right] - 2[M_L(r) - M_N(r)] \left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right] \right\}$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\Big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl} \hat{k}_l\, F_H(k)\Big]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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CASE 1: purely vortical field $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

CASE 1: purely vortical field $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$

$$0 = \langle\, k_i\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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CASE 1: purely vortical field $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$

$$0 = \langle\, k_i\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle = k_i\,\langle\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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CASE 1: purely vortical field $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$

$$0 = \langle\, k_i\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle \;= k_i\,\langle\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle$$

$$= k_i(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\big]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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CASE 1: purely vortical field $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$

$$0 = \langle\, k_i\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle = k_i\,\langle\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\,\rangle$$

$$= k_i(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

$$= k_j(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')F_L(k)$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

CASE 1: purely vortical field $\nabla \cdot \boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$ $\rightarrow F_L(k) = 0$

$$0 = \langle\, k_i\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\, \rangle = k_i\, \langle\, u_i(\boldsymbol{k})\, u_j^*(\boldsymbol{k}')\, \rangle$$

$$= k_i(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

$$= k_j(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')F_L(k)$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

CASE 1: purely vortical field $\nabla\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0 \quad \rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla \times \boldsymbol{u} = 0 \rightarrow \epsilon_{ijl}k_j u_l(\boldsymbol{k}) = 0$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{y}) \rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}') \rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl} \hat{k}_l\, F_H(k) \big]$$

CASE 1: purely vortical field $\nabla \cdot \boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$ $\rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla \times \boldsymbol{u} = 0 \rightarrow \epsilon_{ijl} k_j u_l(\boldsymbol{k}) = 0$

$$0 = \langle \epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}') \rangle$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle \ = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

CASE 1: purely vortical field $\nabla\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0 \quad \rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla \times \boldsymbol{u} = 0 \rightarrow \epsilon_{ijl}k_j u_l(\boldsymbol{k}) = 0$

$$0 = \langle\epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\,\rangle \ = \epsilon_{ijl}\, k_j\, \langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\,\rangle$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k}) u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl} \hat{k}_l\, F_H(k)\big]$$

CASE 1: purely vortical field $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \to k_i u_i(\boldsymbol{k}) = 0 \quad \to F_L(k) = 0$

CASE 2: purely compressional field $\boldsymbol{\nabla} \times \boldsymbol{u} = 0 \to \epsilon_{ijl} k_j u_l(\boldsymbol{k}) = 0$

$$0 = \langle \epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}') \rangle = \epsilon_{ijl}\, k_j\, \langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}') \,\rangle$$

$$= \epsilon_{ijl}\, k_j (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\big[\delta_{lm}\, F_N(k) + \hat{k}_l\, \hat{k}_m\, [F_L(k) - F_N(k)] + i\, \epsilon_{lmn} \hat{k}_n\, F_H(k)\big]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\big]$$

CASE 1: purely vortical field $\nabla \cdot \boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0 \quad \rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla \times \boldsymbol{u} = 0 \rightarrow \epsilon_{ijl} k_j u_l(\boldsymbol{k}) = 0$

$$0 = \langle \epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle \;= \epsilon_{ijl}\, k_j\, \langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle$$

$$= \epsilon_{ijl}\, k_j (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{lm}\, F_N(k) + \hat{k}_l\, \hat{k}_m\, [F_L(k) - F_N(k)] + i\, \epsilon_{lmn}\hat{k}_n\, F_H(k)\big]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\big(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\big)\; \hat{k}_j\, \hat{k}_n\, k\, F_H(k)]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\big]$$

CASE 1: purely vortical field $\nabla\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0 \quad \rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla\times\boldsymbol{u} = 0 \rightarrow \epsilon_{ijl}k_j u_l(\boldsymbol{k}) = 0$

$$0 = \langle\epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\,\rangle = \epsilon_{ijl}\, k_j\,\langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\,\rangle$$

$$= \epsilon_{ijl}\, k_j(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{lm}\, F_N(k) + \hat{k}_l\, \hat{k}_m\, [F_L(k) - F_N(k)] + i\, \epsilon_{lmn}\hat{k}_n\, F_H(k)\big]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\big(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\big)\ \hat{k}_j\, \hat{k}_n\, k\, F_H(k)]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\big(\delta_{im} - \hat{k}_i\, \hat{k}_m\big)\, k\, F_H(k)]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

CASE 1: purely vortical field $\nabla \cdot \boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0 \quad \rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\nabla \times \boldsymbol{u} = 0 \rightarrow \epsilon_{ijl} k_j u_l(\boldsymbol{k}) = 0 \quad \rightarrow F_N(k) = 0$

$$0 = \langle \epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle = \epsilon_{ijl}\, k_j\, \langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle$$

$$= \epsilon_{ijl}\, k_j (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{lm}\, F_N(k) + \hat{k}_l\, \hat{k}_m\, [F_L(k) - F_N(k)] + i\, \epsilon_{lmn}\hat{k}_n\, F_H(k)\right]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\, \hat{k}_j\, \hat{k}_n\, k\, F_H(k)]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\left(\delta_{im} - \hat{k}_i\, \hat{k}_m\right) k\, F_H(k)]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\big]$$

CASE 1: purely vortical field $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \rightarrow k_i u_i(\boldsymbol{k}) = 0$ $\rightarrow F_L(k) = 0$

CASE 2: purely compressional field $\boldsymbol{\nabla}\times\boldsymbol{u} = 0 \rightarrow \epsilon_{ijl}k_j u_l(\boldsymbol{k}) = 0$ $\rightarrow F_N(k) = 0$

$\rightarrow F_H(k) = 0$

$$0 = \langle \epsilon_{ijl}\, k_j\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle = \epsilon_{ijl}\, k_j\, \langle\, u_l(\boldsymbol{k})\, u_m^*(\boldsymbol{k}')\, \rangle$$

$$= \epsilon_{ijl}\, k_j (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\big[\delta_{lm}\, F_N(k) + \hat{k}_l\, \hat{k}_m\, [F_L(k) - F_N(k)] + i\, \epsilon_{lmn}\hat{k}_n\, F_H(k)\big]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\big(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\big)\, \hat{k}_j\, \hat{k}_n\, k\, F_H(k)]$$

$$= (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')[\epsilon_{ijm}\, k_j\, F_N(k) + i\big(\delta_{im} - \hat{k}_i\, \hat{k}_m\big)\, k\, F_H(k)]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

Two-point correlators of statistically homogeneous and isotropic random fields

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

We have shown that

$$F_N(k) = 4\pi \int_0^\infty dr\, r^2 \left\{M_N(r)\left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$F_L(k) = 4\pi \int_0^\infty dr\, r^2 \left\{M_L(r)\left[\frac{\sin kr}{kr}\right] - 2[M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$F_H(k) = 4\pi \int_0^\infty dr\, r^2 M_H(r)\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]$$

Exercise no. 2

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\,\epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

We have shown that

$$F_N(k) = 4\pi \int_0^\infty dr\, r^2 \left\{ M_N(r)\left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$F_L(k) = 4\pi \int_0^\infty dr\, r^2 \left\{ M_L(r)\left[\frac{\sin kr}{kr}\right] - 2[M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

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Exercise no. 2

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle \; = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

We have shown that

$$F_N(k) = \; 4\pi \int_0^\infty dr\, r^2 \left\{M_N(r)\left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

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Assume that all correlation functions $M_N(r), M_L(r), M_H(r)$ go to zero above a finite scale $r \geq r_0$ (CAUSALITY)

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We want to study the large scales $(k \to 0)$ properties of the different spectra (as consequences of causality)

Exercise no. 2 A) Find the first two nonzero terms in the Taylor expansion of $F_N(k)$ for $k \to 0$

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For $n = 0$ these two terms cancel

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

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For $n = 0$ these two terms cancel
We define $n = m + 1$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Assume that all correlation functions $M_N(r), M_L(r), M_H(r)$ go to zero above a finite scale $r \geq r_0$ (CAUSALITY)

We want to study the large scales ($k \to 0$) properties of the different spectra (as consequences of causality)

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$$= \sum_{n=0}^\infty k^{2n} (-1)^n \left[\frac{1}{(2n+1)!\,(2n+3)}\right] 4\pi \int_0^{r_0} dr\, r^{2n+2} [M_L(r) + 2(n+1)\, M_N(r)]$$

$$\sin x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} x^{2n}$$

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For $k \to 0$

The expansion is valid iff all the integrals converge, which happens only if $M_L(r), M_N(r) \to 0$ for $r \to \infty$ faster than any power law (causality condition)

$$\sin x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} x^{2n}$$

Assume that all correlation functions $M_N(r), M_L(r), M_H(r)$ go to zero above a finite scale $r \geq r_0$ (CAUSALITY)

We want to study the large scales ($k \to 0$) properties of the different spectra (as consequences of causality)

Exercise no. 2 A) Find the first two nonzero terms in the Taylor expansion of $F_N(k)$ for $k \to 0$

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For $k \to 0$ $\quad \simeq \frac{4\pi}{3} \int_0^{r_0} dr\, r^2 [M_L(r) + 2\, M_N(r)] - k^2 \frac{2\pi}{15} \int_0^{r_0} dr\, r^4 [M_L(r) + 4\, M_N(r)] + \mathcal{O}(k^4)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

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Exercise no. 2 B) Find the first two nonzero terms in the Taylor expansion of $F_L(k)$ for $k \to 0$

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

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$$F_N(k) = 4\pi \int_0^\infty dr\, r^2 \left\{M_N(r)\left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

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$$F_H(k) = 4\pi \int_0^\infty dr\, r^2 M_H(r)\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]$$

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For $k \to 0$ $\quad \simeq \frac{4\pi}{3} \int_0^{r_0} dr\, r^2 [M_L(r) + 2\, M_N(r)] - k^2 \frac{2\pi}{15} \int_0^{r_0} dr\, r^4 [3\, M_L(r) + 2\, M_N(r)] + \mathcal{O}(k^4)$

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For $k \to 0$

$$F_N(k) \simeq \frac{4\pi}{3} \int_0^{r_0} dr\, r^2 [M_L(r) + 2\, M_N(r)] - k^2 \frac{2\pi}{15} \int_0^{r_0} dr\, r^4 [M_L(r) + 4\, M_N(r)] + \mathcal{O}(k^4)$$

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For the general case $F_N \neq 0, F_L \neq 0$ we find for $k \to 0$ a flat spectrum

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C) Let us now consider the fully vortical case $F_L(k) = 0, F_N(k) \neq 0$

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$$F_N(k) \simeq \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] - k^2\,\frac{2\pi}{15}\int_0^{r_0} dr\, r^4[M_L(r) + 4\, M_N(r)] + \mathcal{O}(k^4)$$

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How does the leading term in F_N for $k \to 0$ scale with k in this case?

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Since F_L is identically zero, all the terms in its infinite series in powers of k have to vanish, hence we also have

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For $k \to 0$

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Since F_L is identically zero, all the terms in its infinite series in powers of k have to vanish, hence we also have

$$F_L = 0 \to \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] = 0$$

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Since F_L is identically zero, all the terms in its infinite series in powers of k have to vanish, hence we also have

$$F_L = 0 \to \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] = 0 \qquad \to F_N(k) \sim k^2 \text{ for } k \to 0$$

Exercise no. 2

For $k \to 0$

$$F_N(k) \simeq \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] - k^2\frac{2\pi}{15}\int_0^{r_0} dr\, r^4[M_L(r) + 4\, M_N(r)] + \mathcal{O}(k^4)$$

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For the general case $F_N \neq 0, F_L \neq 0$ we find for $k \to 0$ a flat spectrum

Notice that the leading term is the same for both

D) Consider now the fully compressional case $F_N(k) = 0, F_L(k) \neq 0$

How does the leading term in F_L for $k \to 0$ scale with k in this case?

Exercise no. 2

For $k \to 0$

$$F_N(k) \simeq \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] - k^2\, \frac{2\pi}{15}\int_0^{r_0} dr\, r^4[M_L(r) + 4\, M_N(r)] + \mathcal{O}(k^4)$$

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For the general case $F_N \neq 0, F_L \neq 0$ we find for $k \to 0$ a flat spectrum

Notice that the leading term is the same for both

D) Consider now the fully compressional case $F_N(k) = 0, F_L(k) \neq 0$

How does the leading term in F_L for $k \to 0$ scale with k in this case?

For the same reason as in the previous question

$F_N = 0 \to \frac{4\pi}{3}\int_0^{r_0} dr\, r^2[M_L(r) + 2\, M_N(r)] = 0$ $\quad \to F_L(k) \sim k^2$ for $k \to 0$

Exercise no. 2 E) Find the first two nonzero terms in the Taylor expansion of $F_H(k)$ for $k \to 0$

$$\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle = \delta_{ij}\, M_N(r) + \hat{r}_i\, \hat{r}_j\, [M_L(r) - M_N(r)] + \epsilon_{ijk}\, \hat{r}_k\, M_H(r)$$

$$\langle u_i(\boldsymbol{k})u_j^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\delta_{ij}\, F_N(k) + \hat{k}_i\, \hat{k}_j\, [F_L(k) - F_N(k)] + i\, \epsilon_{ijl}\hat{k}_l\, F_H(k)\right]$$

We have shown that

$$F_N(k) = 4\pi \int_0^\infty dr\, r^2 \left\{M_N(r)\left[\frac{\sin kr}{kr}\right] + [M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$F_L(k) = 4\pi \int_0^\infty dr\, r^2 \left\{M_L(r)\left[\frac{\sin kr}{kr}\right] - 2[M_L(r) - M_N(r)]\left[\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right]\right\}$$

$$F_H(k) = 4\pi \int_0^\infty dr\, r^2 M_H(r)\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]$$

Assume that all correlation functions $M_N(r), M_L(r), M_H(r)$ go to zero above a finite scale $r \geq r_0$ (CAUSALITY)

We want to study the large scales ($k \to 0$) properties of the different spectra (as consequences of causality)

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For $n = 0$ these two terms cancel

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For $n = 0$ these two terms cancel
We define $n = m + 1$

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$$= \sum_{n=0}^\infty k^{2n+1} (-1)^n\, 4\pi \int_0^{r_0} dr \frac{r^{2n+3} M_H(r)}{(2n+1)!\,(2n+3)}$$

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For $k \to 0$ $\quad F_H(k) \simeq k\, \frac{4\pi}{3} \int_0^{r_0} dr\, r^3 M_H(r) - k^3 \frac{2\pi}{15} \int_0^{r_0} dr\, r^5 M_H(r) + \mathcal{O}(k^5)$

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