Probability of Presence Versus $\psi^*(x,t)\psi(x,t)$

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Abstract

Postulating the identification of $\psi^*(x,t)\psi(x,t)$ with a physical probability density is unsatisfactory conceptually and overly limited practically. For electrons, there is a simple, calculable relativistic correction proportional to $\nabla\psi^* \cdot \nabla\psi$. In particular, zeroes of the wave function do not indicate vanishing probability density of presence. Effects of this kind arise generically in Lagrangian-based theories implementing the particle concept.

Textbooks on quantum mechanics, when they come to making contact with the empirical world, tend to postulate that there is such a thing as a particle that has amplitudes to be at different times and places – i.e., a wave-function $\psi(x,t)$ – and that $\psi^*\psi$ represents the (unnormalized) probability for it to "be there then". (See for example these three excellent modern texts: [1–3].) Here we will examine that probability postulate critically. We will argue that the choice $\psi^*\psi$ can be modified in ways that are fully consistent with the basic principles of quantum theory. They entail associated changes in the form of the Schrödinger equation and of conservation laws, and in the energy spectrum.

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1 Correction from Dirac Theory

Indeed, the relativistic (Dirac) theory for electrons suggests a specific correction to that probability density. In the Dirac theory there is an underlying 4-component spinor Ψ, and a density

$$\rho = \Psi^\dagger \Psi. \tag{1}$$

Since this density is positive-definite, and associated with a conserved 4-current $j^\mu = \bar{\Psi}\gamma^\mu \Psi$, it is natural to associate ρ with the probability density for finding an electron, especially since there is no reasonable alternative candidate with those properties. In the non-relativistic limit Ψ takes the form

$$\Psi \approx \left(\begin{array}{c} \psi \\ \frac{\sigma \cdot p}{2m} \psi \end{array} \right) \tag{2}$$

where ψ is a two-component spinor. In terms of ψ, then, the probability density is

$$\rho \approx \psi^* \psi + \frac{1}{4m^2} \nabla \psi^* \cdot \nabla \psi. \tag{3}$$

Eqn. (3) includes a correction to the conventionally assumed expression. The correction is of order $\frac{p^2}{m^2} \sim \frac{v^2}{c^2}$, and we can expect it to be small within most practical applications of non-relativistic quantum mechanics. But it has the qualitatively and conceptually significant effect of removing zeroes in the probability distribution at the points where ψ vanishes. More generally, we can anticipate its quantitative significance will emerge at places where the wave function is small while its gradient is large, for example at the edge of a high, steep barrier. As we shall see in examples, it also has striking effects on the highly excited states in bound state problems, which bring in large gradients due to orthogonality constraints.

2 Conceptual Critique

2.1 Structure of Points

At a formal level, the issue arises in the following way. If we assume that there is a dynamical variable x corresponding to the position of a particle, then there will be states $|x\rangle$ that diagonalize it, and for a general state the expansion $|\psi\rangle = \int dx\, \psi(x)|x\rangle$. This general framework does not yet supply enough structure, however, for us to calculate the probability density for

finding the particle at x_0, *viz.* $\langle\psi|\delta(x-x_0)|\psi\rangle$. To do that, according to

$$\langle\psi|\delta(x-x_0)|\psi\rangle = \int\int dx_2 dx_1 \psi^*(x_2)\delta(x_1-x_0)\psi(x_1)\langle x_2|x_1\rangle, \quad (4)$$

we need to have $\langle x_2|x_1\rangle$. The standard prescription

$$\langle x_2|x_1\rangle_s = \delta(x_2-x_1) \quad (5)$$

leads to the standard consequence $\langle\psi|\delta(x-x_0)|\psi\rangle_s = \psi^*(x_0)\psi(x_0)$. In this language, Eqn. (3) corresponds to the choice

$$\langle x_2|x_1\rangle = (1 - \frac{\nabla^2_{x_2}}{4m^2})\delta(x_2-x_1). \quad (6)$$

Comparing the form Eqn. (6) with Eqn. (5), we may say that points, as perceived by an electron, have acquired interior structure.

2.2 Measurement

In other parts of the textbooks, it is often emphasized that we should be careful about assigning physical reality to things that we don't measure. In that spirit: What sort of measurement corresponds to determining the probability that an electron will be found at a given space-time point?

It is difficult for an answer to be better defined than the question it responds to, but here there is a useful answer that leads us back to the preceding choices. That is, we recognize that many practical ways of "locating an electron" involve sensing its interaction with electromagnetic fields. From this perspective, the position of an electron is a theoretical construct for describing the electromagnetic response of systems that can be usefully modeled based on a theory built up from electron particle variables. In this context, it is natural to identify electron density with the density of electric charge (more accurately, the part of the electric charge operator that we ascribe, in the model, to electrons). The electric charge density operator is uniquely determined, so it does provide a definite answer to our question. Other answers might be appropriate to other ways of implementing the question experimentally, but since it is difficult for the answer to be better defined than the question, if we're looking for a specific answer then charge density appears to be the most salient possibility. We will refer to this as "probability of presence", as opposed to simply "probability density", to emphasize how it is to be interpreted physically.

3 Generalization to Quasi-Particles

Above we have spoken of electrons, and invoked fundamental (empty space) electrodynamics. Specifically, Eqn. (6) was reverse-engineered to reproduce the result of approximating the Dirac theory, regarded as fundamental.

Similar conceptual issues, however, arise for other kinds of quantum particles and quasi-particles. To guide our choices in more general cases, it is natural to appeal to the principles of Lagrangian dynamics and locality, since these implement the general principles of quantum theory using path integrals. In cases where the particles are associated with a conserved $U(1)$ quantum number that, in the model that uses them, simply counts them, it is natural to identify the particle probability density with the expectation value of the $U(1)$ charge density operator.

Intuitively, we might associate the "spread" of probability of presence associated with electrons in vacuum with their irreducible uncertainty in position, associated with their small-scale motion, or *zitterbewegung*, that has been integrated out in the non-relativistic description. In that spirit, we might expect correction terms of this kind could be incorporated usefully into the quantum description of particles with extended structure, such as nucleons (for which the Compton and geometric sizes are comparable) or, with significantly larger coefficients, atoms and molecules.

4 Lagrangian Realization

Effective theories based on Lagrangians allow us to realize the general principles of quantum theory and embody appropriate symmetries, including the number symmetries that help to specify the census of ingredients in the models. Furthermore, they lend themselves to quantization using path integrals. Now let us realize our conceptual considerations, and the motivating example of relativistic electron theory, within that framework.

To realize our correction term in this framework is significant for other reasons. For one thing, it allows us to draw out all its implications, including the form of the associated 3-current, the form of the associated energy-momentum tensor density, and the form of the associated contribution to the equations of motion. For another, it brings us into the spirit of Landau-Ginzburg theories, where we identify relevant parameters for the description of material systems based on their appearance in effective Lagrangians [4].

4.1 Mathematical Structure

The conventional probability expression is identical with the charge density expression that arises for

$$L^{(0)} \; = \; \frac{i}{2}\psi^* \overleftrightarrow{\partial_t} \psi. \tag{7}$$

One might anticipate that a contribution to the Lagrangian of the form

$$L^{(1)} \; \propto \; i\nabla\psi^* \overleftrightarrow{\partial_t} \nabla\psi \tag{8}$$

leads to a contribution to the location (i.e. charge) density of the form we're looking for. That is correct, as we will now demonstrate directly in a way that applies more broadly.

As is traditional in problems of this kind, we regard the Lagrangian formally as a function of the fields acted upon by derivatives. Thus the $U(1)$ phase invariance of L is expressed as

$$0 = \frac{\delta L}{\delta\psi}\psi + \frac{\delta L}{\delta\partial_t\psi}\partial_t\psi + \frac{\delta L}{\delta\nabla\psi}\nabla\psi + \frac{\delta L}{\delta\partial_t\nabla\psi}\partial_t\nabla\psi - (\psi \to \psi^*) \tag{9}$$

(including only the terms we will be using) and the equations of motion are

$$0 = \frac{\delta L}{\delta\psi} - \partial_t\frac{\delta L}{\delta\partial_t\psi} - \nabla\frac{\delta L}{\delta\nabla\psi} + \partial_t\nabla\frac{\delta L}{\delta\partial_t\nabla\psi} \tag{10}$$

together with a similar equation with $\psi \to \psi^*$.

Inserting Eqn. (10) into the first half of the right hand side of Eqn. (9) yields six terms, as follows

$$0 = (\nabla\frac{\delta L}{\delta\nabla\psi} + \partial_t\frac{\delta L}{\delta\partial_t\psi} - \partial_t\nabla\frac{\delta L}{\delta\partial_t\nabla})\psi + \frac{\delta L}{\delta\nabla\psi}\nabla\psi + \frac{\delta L}{\delta\partial_t\psi}\partial_t\psi + \frac{\delta L}{\delta\partial_t\nabla\psi}\partial_t\nabla\psi. \tag{11}$$

The first and fourth terms combine as $\nabla(\frac{\delta L}{\delta\nabla\psi}\psi)$, while the second and fifth combine as $\partial_t(\frac{\delta L}{\delta\partial_t\psi}\psi)$. Finally, for the third and sixth we have

$$-\partial_t\nabla\frac{\delta L}{\delta\partial_t\nabla}\psi + \frac{\delta L}{\delta\partial_t\nabla\psi}\partial_t\nabla\psi = \partial_t(\frac{\delta L}{\delta\partial_t\nabla\psi}\nabla\psi) - \nabla(\partial_t\frac{\delta L}{\delta\partial_t\nabla\psi}\psi). \tag{12}$$

Thus we express the first half of the right hand side of Eqn. (9) as

$$\partial_t(\frac{\delta L}{\delta\partial_t\psi}\psi + \frac{\delta L}{\delta\partial_t\nabla\psi}\nabla\psi) + \nabla(\frac{\delta L}{\delta\nabla\psi}\psi - \partial_t\frac{\delta L}{\delta\partial_t\nabla\psi}\psi) \tag{13}$$

and Eqn. (9) itself as

$$0 = \partial_t(\frac{\delta L}{\delta \partial_t \psi}\psi + \frac{\delta L}{\delta \partial_t \nabla \psi}\nabla \psi) + \nabla(\frac{\delta L}{\delta \nabla \psi}\psi - \partial_t \frac{\delta L}{\delta \partial_t \nabla \psi}\psi) - (\psi \rightarrow \psi^*). \quad (14)$$

$L = L^{(0)}$ gives a presence density of the conventional form, while $L = L^{(1)}$ gives a presence density of the form $\nabla \psi^* \cdot \nabla \psi$. In constructing quantum-mechanical models, of course, we can consider including both terms, together with other additions.

4.2 Equation of Motion (Modified Schrödinger Equation)

Now let us work out general consequences of expanding the Schrödinger Lagrangian (i.e., the Lagrangian that has the Schrödinger equation as its equation of motion) to include the additional term discussed above, so that

$$L = \frac{i}{2}\psi^* \overset{\leftrightarrow}{\partial_t} \psi + \frac{a}{2}i\nabla \psi^* \overset{\leftrightarrow}{\partial_t} \nabla \psi - V\psi^*\psi - \frac{1}{2m}\nabla \psi^* \cdot \nabla \psi. \quad (15)$$

Here, to keep things appropriately simple, we have included a (real) scalar potential V but not a vector potential. Also, we will allow V to depend on space, but not on time.

The modified Schrödinger equation – i.e., the equation of motion derived from L – reads

$$\begin{aligned}
0 &= \frac{\delta L}{\delta \psi^*} - \partial_t \frac{\delta L}{\delta \partial_t \psi^*} - \nabla \frac{\delta L}{\delta \nabla \psi^*} + \partial_t \nabla \frac{\delta L}{\delta \partial_t \nabla \psi^*} \\
&= \left(-V\psi + \frac{i}{2}\partial_t \psi\right) + \partial_t\left(\frac{i}{2}\psi\right) \\
&\quad - \nabla\left(-\frac{1}{2m}\nabla \psi + \frac{a}{2}\cdot i\nabla \partial_t \psi\right) - \partial_t \nabla\left(\frac{a}{2}\cdot i\nabla \psi\right) \\
&= -V\psi + i\partial_t \psi + \frac{1}{2m}\nabla^2 \psi - a\cdot i\partial_t \nabla^2 \psi. \quad (16)
\end{aligned}$$

4.3 Conservation Laws

1. Following the logic of the preceding section, we find the equation expressing local charge conservation

$$\begin{aligned}
0 &= \partial_t(\psi^*\psi + a(\nabla \psi^* \cdot \nabla \psi)) \\
&\quad + \nabla\left(\frac{i}{2m}(\nabla \psi^*\psi - \psi^*\nabla \psi) - \frac{a}{2}(\partial_t(\psi\nabla \psi^*) - \nabla(\psi^*\partial_t \psi))\right). \quad (17)
\end{aligned}$$

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Thus, the quantity

$$\rho = \psi^*\psi + a(\nabla\psi^* \cdot \nabla\psi) \tag{18}$$

has suitable properties to represent, after normalization, a probability distribution – namely, it is positive definite and, given appropriate spatial boundary conditions, its integral over space is conserved. On the other hand, the spatial integral

$$\int dx \psi^*(x,t)\psi(x,t)$$

is not independent of time, so $\psi^*\psi$ cannot be interpreted as a probability density. This supports use of ρ, rather than $\psi^*\psi$, as the preferred measure of probability of presence.

2. By adding the product of the equation of motion for ψ with $\partial_t\psi^*$ to its complex conjugate, and re-organizing the terms, we obtain an equation that expresses local energy conservation:

$$
\begin{aligned}
0 &= \partial_t(\psi^*V\psi + \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi) \\
&+ \nabla\big(ia(\partial_t\psi^*\partial_t\nabla\psi - \partial_t\nabla\psi^*\partial_t\psi) - \frac{1}{2m}(\partial_t\psi^*\nabla\psi + \nabla\psi^*\partial_t\psi)\big).
\end{aligned}
\tag{19}
$$

The formal expression of the energy density

$$\varepsilon = \psi^*V\psi + \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi \tag{20}$$

is independent of a, but the formal expression for its flux

$$j_\varepsilon = ia(\partial_t\psi^*\partial_t\nabla\psi - \partial_t\nabla\psi^*\partial_t\psi) - \frac{1}{2m}(\partial_t\psi^*\nabla\psi + \nabla\psi^*\partial_t\psi) \tag{21}$$

does depend on a. Of course, in evaluating the energy density within a given state, we must take account of the normalization of the wave function, which can bring in a dependence.

3. By adding the product of the equation of motion for ψ with $\partial_k\psi^*$ to its complex conjugate, and re-organizing the terms, we obtain an equation that expresses the local change of momentum reacting to the

force field $-\partial_k V$:

$$- \psi\psi^* \partial_k V =$$
$$+ i\partial_t \left(\psi\partial_k\psi^* - \psi^*\partial_k\psi - a(\nabla\psi^*\partial_k\nabla\psi - \nabla\psi\partial_k\nabla\psi^*) \right)$$
$$- \frac{1}{2m}\nabla(\psi^*\partial_k\nabla\psi - \nabla\psi\partial_k\psi^* + \psi\partial_k\nabla\psi^* - \nabla\psi^*\partial_k\psi)$$
$$- ai\nabla(\partial_k\psi^*\partial_t\nabla\psi - \partial_k\psi\partial_t\nabla\psi^* + \psi\partial_k\partial_t\nabla\psi^* - \psi^*\partial_k\partial_t\nabla\psi). \quad (22)$$

From this, we identify the momentum density

$$\pi_k = i\left(\psi\partial_k\psi^* - \psi^*\partial_k\psi - a(\nabla\psi^*\partial_k\nabla\psi - \nabla\psi\partial_k\nabla\psi^*) \right) \quad (23)$$

and its flux

$$j_l^{(\pi_k)} = \frac{1}{2m}(\psi^*\partial_k\partial_l\psi - \partial_l\psi\partial_k\psi^* + \psi\partial_k\partial_l\psi^* - \partial_l\psi^*\partial_k\psi)$$
$$+ ia(\partial_k\psi^*\partial_t\partial_l\psi - \partial_k\psi\partial_t\partial_l\psi^* + \psi\partial_k\partial_t\partial_l\psi^* - \psi^*\partial_k\partial_t\partial_l\psi). \quad (24)$$

The momentum flux density can be interpreted, following standard arguments, as a stress tensor

$$T_{kl} = j_l^{(\pi_k)}. \quad (25)$$

5   Examples

For stationary states, with $\psi(x,t) \propto e^{-iEt}\psi(x)$, the modified Schrödinger equation Eqn. (16) takes the form of the usual Schrödinger equation with a modified coefficient of the Laplacian term, i.e. an effective mass with

$$\begin{aligned}\frac{1}{2m_{\text{eff.}}} &= \frac{1}{2m} - aE \\ m_{\text{eff.}} &= \frac{m}{1 - 2amE}.\end{aligned} \quad (26)$$

In cases where the energy levels of the ordinary Schrödinger equation have a simple analytical dependence on mass, this observation allows us to calculate the energy eigenvalues for the modified Schrödinger equation.

Thus, for the harmonic oscillator with spring constant k and mass m we have the energy eigenvalues

$$\epsilon_n = (n + \frac{1}{2})\sqrt{\frac{k}{m}} \quad (27)$$

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with $n \geq 0$, and we infer for the modified Schrödinger equation the energy eigenvalues

$$E_n = (n + \frac{1}{2})\sqrt{\frac{k(1 - 2amE_n)}{m}} \tag{28}$$

leading to

$$E_n = \epsilon_n(\sqrt{1 + (\epsilon_n am)^2} - \epsilon_n am). \tag{29}$$

The corrected probability densities for the ground state and fifth excited state of the harmonic oscillator are displayed in Figure (1).

Three qualitative features of this result, assuming $a > 0$, deserve comment:

- The energy levels are no longer equally spaced.

- The modifications of the energy levels become more significant as n increases.

- For $n \to \infty$, $\epsilon_n \to \infty$ and we have the leading behavior

$$E_n \to \frac{1}{2am}. \tag{30}$$

The eigenvalues increase monotonically with n, but the discrete spectrum is bounded from above.

When the limiting energy for the discrete spectrum is exceeded, we have a continuous spectrum characterized by negative effective mass. In this regime wave functions can respond to increases in potential by incorporating rapid oscillation. Here we will present a heuristic discussion of that phenomenon. For $m_{\text{eff.}} > 0$ in the classically forbidden region $V > E$ the solution of

$$\left(-\frac{1}{2m_{\text{eff}}}\frac{d^2}{dx^2} + V - E\right)\psi(x) = 0 \tag{31}$$

involves real exponentials $\psi(x) \sim e^{\pm\sqrt{2m_{\text{eff.}}(V-E)}\,x}$. Here the growing exponential behavior cannot easily be sustained in a normalizable wave function, and thus we generally expect damping of amplitude in classically forbidden regions. But if $m_{\text{eff.}} < 0$ the solutions are $\psi(x) \sim e^{\pm i\sqrt{2|m_{\text{eff.}}|(V-E)}\,x}$. Thus they oscillate in the forbidden regions, and they oscillate more rapidly, the more forbidden those regions are. This corresponds to the classical behavior of a negative mass particle accelerating into a rising potential. Related behavior is seen for positive mass in the inverted harmonic oscillator [5].

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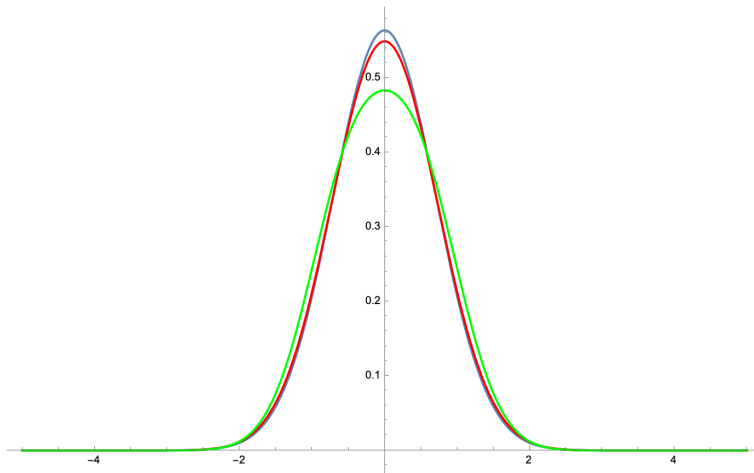
(a) Ground state
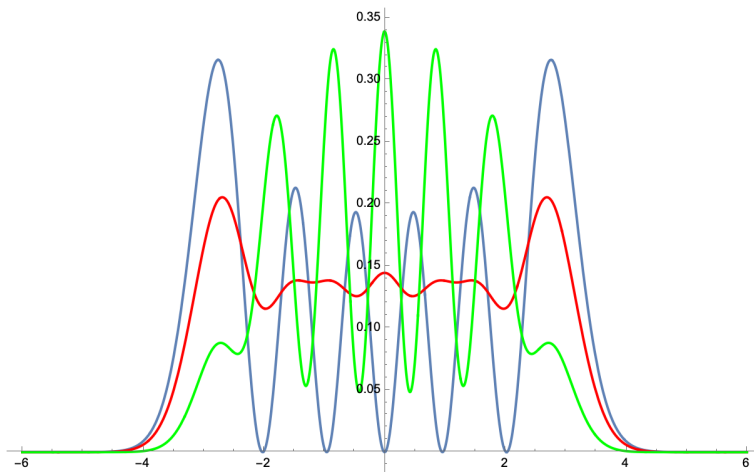
(b) Fifth excited state

Figure 1: Corrected probability density of the harmonic oscillator from Eqn. (18), where the states are modified using the effective mass from Eqn. (26). Here, $m = \omega = 1$, for $a = 0$ (blue), $a = 0.1$ (red), $a = 0.5$ (green).

Note that in the generalized Schrödinger equation it is $m_{\text{eff.}}^{-1}$ that occurs naturally, and in this sense the difference between an infinitely positive and an infinitely negative effective mass is small.

For a particle confined within an infinitely deep well with length L, the energy eigenvalues of the conventional Schrödinger equation are

$$\epsilon_n = \frac{n^2 \pi^2}{2mL^2} \tag{32}$$

with $n \geq 1$, and we infer for the modified Schrödinger equation the energy eigenvalues

$$E_n = \frac{n^2 \pi^2}{2mL^2}(1 - 2amE_n) \tag{33}$$

leading to

$$E_n = \frac{\epsilon_n}{1 + 2am\epsilon_n}. \tag{34}$$

The corrected probability densities for the ground state and fifth excited state of the infinite well are displayed in Figure (2). Here too we find the limiting energy $\frac{1}{2am}$.

The modified free-particle dispersion relation

$$\omega(1 + ak^2) = \frac{k^2}{2m} \tag{35}$$

leads to the phase velocity

$$\frac{\omega}{k} = \frac{k}{2m}\frac{1}{1 + ak^2} \tag{36}$$

and group velocity

$$\frac{d\omega}{dk} = \frac{k}{m}\frac{1}{(1 + ak^2)^2}. \tag{37}$$

These indicate, for $a > 0$, a slowing of propagation. Note here that the momentum associated to a plane wave $e^{i(kx-\omega t)}$ is $k(1 + ak^2)$, while its energy is ω as usual.

6 Hamiltonian and Local Energy

To set up the Hamiltonian formalism in the most straightforward way, we take ψ as the dynamical variable in Eqn. (15). The terms linear in ∂_t, including the term proportional to a, do not contribute numerically to the Hamiltonian. Note that our expression Eqn. (20) for energy density reflects

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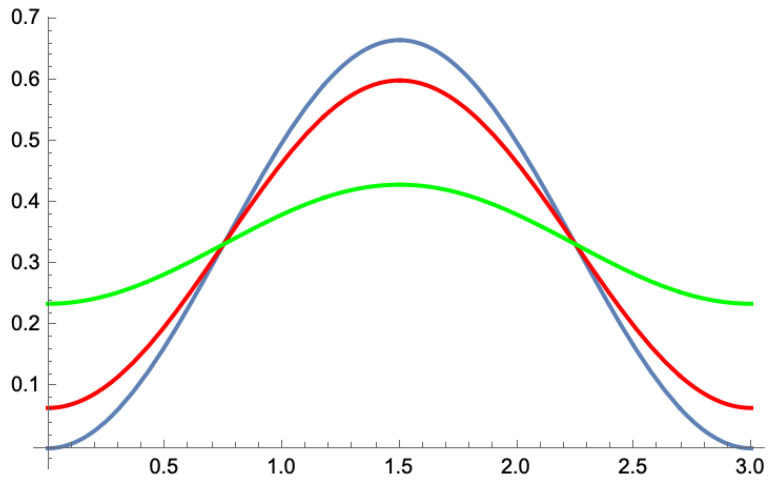
(a) Ground state
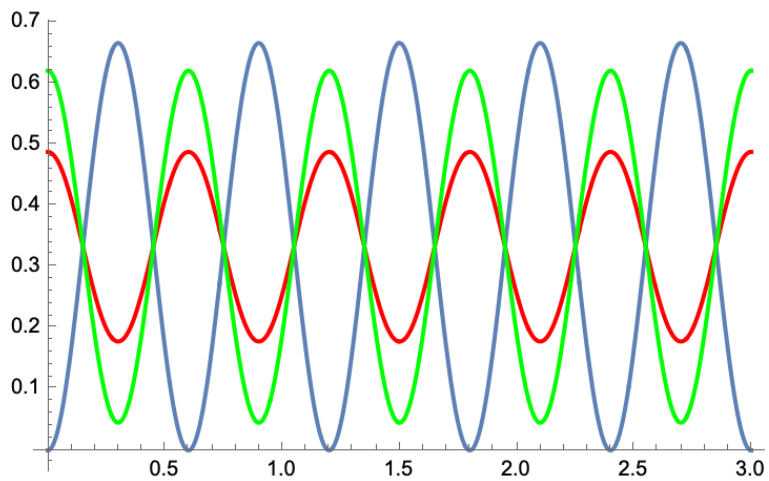
(b) Fifth excited state

Figure 2: Corrected probability density of the infinite well, from Eqn. (18) with $L = 3$, for $a = 0$ (blue), $a = 0.1$ (red), $a = 0.5$ (green).

this fact. In particular, it is bounded below by the minimum of V. Those terms do, however, control the identification of canonical momentum. Indeed, we have

$$\pi_\psi \equiv \frac{\partial L}{\partial \partial_t \psi} = i(1 - a\nabla^2)\psi^*. \tag{38}$$

To express the Hamiltonian in terms of π_ψ we must invert this equation, in the form

$$\psi^* = -i(1 - a\nabla^2)^{-1}\pi_\psi. \tag{39}$$

This brings in non-locality, at a formal level. Note that for $a \geq 0$ the operator $1 - a\nabla^2$ is invertible, and can be expressed simply as a convolution in real space.

In the preceding discussion of examples we identified the E that occurs in the factor e^{-iEt} that accompanies stationary states with energy. That identification is associated with the canonical pairing of energy and time, as realized in the abstract Schrödinger equation

$$i\partial_t \psi = H\psi. \tag{40}$$

Comparing Eqn. (16) with Eqn. (40), we see that our modified Schrödinger equation fits into the abstract paradigm with the Hamiltonian operator

$$H = \frac{1}{1 - a\nabla^2}(V - \frac{1}{2m}\nabla^2). \tag{41}$$

The issue arises, how to reconcile this expression for the Hamiltonian with the local energy density appearing in Eqn. (20). The point is that in evaluating the energy density we must recognize that in view of the modified presence density Eqn. (18) the dual vector ("bra") connected to the Hilbert space vector ("ket") ψ is no longer ψ^*, but rather

$$\langle\psi| = (1 - a\nabla^2)\psi^*. \tag{42}$$

Thus, in the expectation value for energy density

$$\langle\psi|H|\psi\rangle = \psi^*(V - \frac{1}{2m}\nabla^2)\psi, \tag{43}$$

we recover Eqn. (20).

7 Discussion

1. The a-term in Eqn. (15) is polynomial and of low mass dimension, and is therefore, in the spirit of Landau-Ginzburg theory, a very natural term to incorporate into the description of quasi-particles or emergent fields. Since it is quadratic in the field (or, alternatively, wave function) it affects free propagation directly, and brings in qualitatively new features. Another quadratic term of a similar sort, *viz.*

$$\Delta L_0 \;=\; b\psi^* \overset{\leftrightarrow}{\partial_t} \sigma \cdot \overset{\leftrightarrow}{\nabla} \psi \qquad (44)$$

can arise for two-component spinor fields, and is of even lower dimension. It is rotationally invariant, but violates parity. The candidate conserved "probability of presence" current associated to Eqn. (44) as it stands does not define a positive-definite inner product, so it is not suitable to provide a Hilbert space metric. There are related terms, *e. g.*,

$$\Delta L_1 \;=\; \psi^* i \overset{\leftrightarrow}{\partial_t} (1 + ib\sigma \cdot \overset{\leftrightarrow}{\nabla})^2 \psi \,, \qquad (45)$$

which are free of that difficulty.

2. There is no difficulty in extending the preceding discussions to many-body wave functions and to models with more complex conventional interactions. Thus we can infer modified equations of state for quantum ideal gases, modified densities of states, and so forth. Second quantization brings in some formal novelties, but it is straightforward in principle.

3. If we consider the path integral expression for transition amplitudes defined by $\int \mathcal{D}\psi^* \mathcal{D}\psi e^{iL + J(x,t)\mathcal{O}(x,t)}$, for various prescribed probes J coupled to sources $\mathcal{O}$, we see that the expressions we obtained for densities, currents, and stresses have physical interpretations in line with their names. This is essentially the logic of the Schwinger action principle [6]. It gives local forms of the Hellman-Feynman theorem [7] that can readily incorporate the possibility of the unconventional terms considered above, as we have shown directly.

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