Checking Unit Magnitude

April 13, 2024

Abstract

Direct demonstration that the eigenvalues of the unitary evolution associated with a periodic version of Griffiths' original model are of unit magnitude.

We want to show that under the assumption $|\beta| \leq 1$ the roots of the equation

$$\omega^{1-N} \; = \; \frac{\omega + \beta}{1 + \beta^* \omega} \tag{1}$$

have absolute value unity.

Taking the squared magnitude of both sides of Eqn. (1) we have

$$|\omega|^{2-2N} \; = \; \frac{|1 + \frac{\beta}{\omega}|^2}{|\frac{1}{\omega} + \beta^*|^2} \; = \; \frac{1 + \frac{\beta}{\omega} + \frac{\beta^*}{\omega^*} + \frac{|\beta|^2}{|\omega|^2}}{\frac{1}{|\omega|^2} + \frac{\beta}{\omega} + \frac{\beta^*}{\omega^*} + |\beta|^2} \tag{2}$$

If $|\omega| > 1$ then Eqn. (2) implies that the right-hand side has magnitude < 1, or

$$\frac{1}{|\omega|^2} + |\beta|^2 > 1 + \frac{|\beta|^2}{|\omega|^2} \tag{3}$$

leading to

$$(1 - |\beta|^2)(\frac{1}{|\omega|^2} - 1) > 0 \tag{4}$$

which is a contradiction.

A parallel argument excludes $|\omega| < 1$, leaving $|\omega| = 1$ as the only possibility.