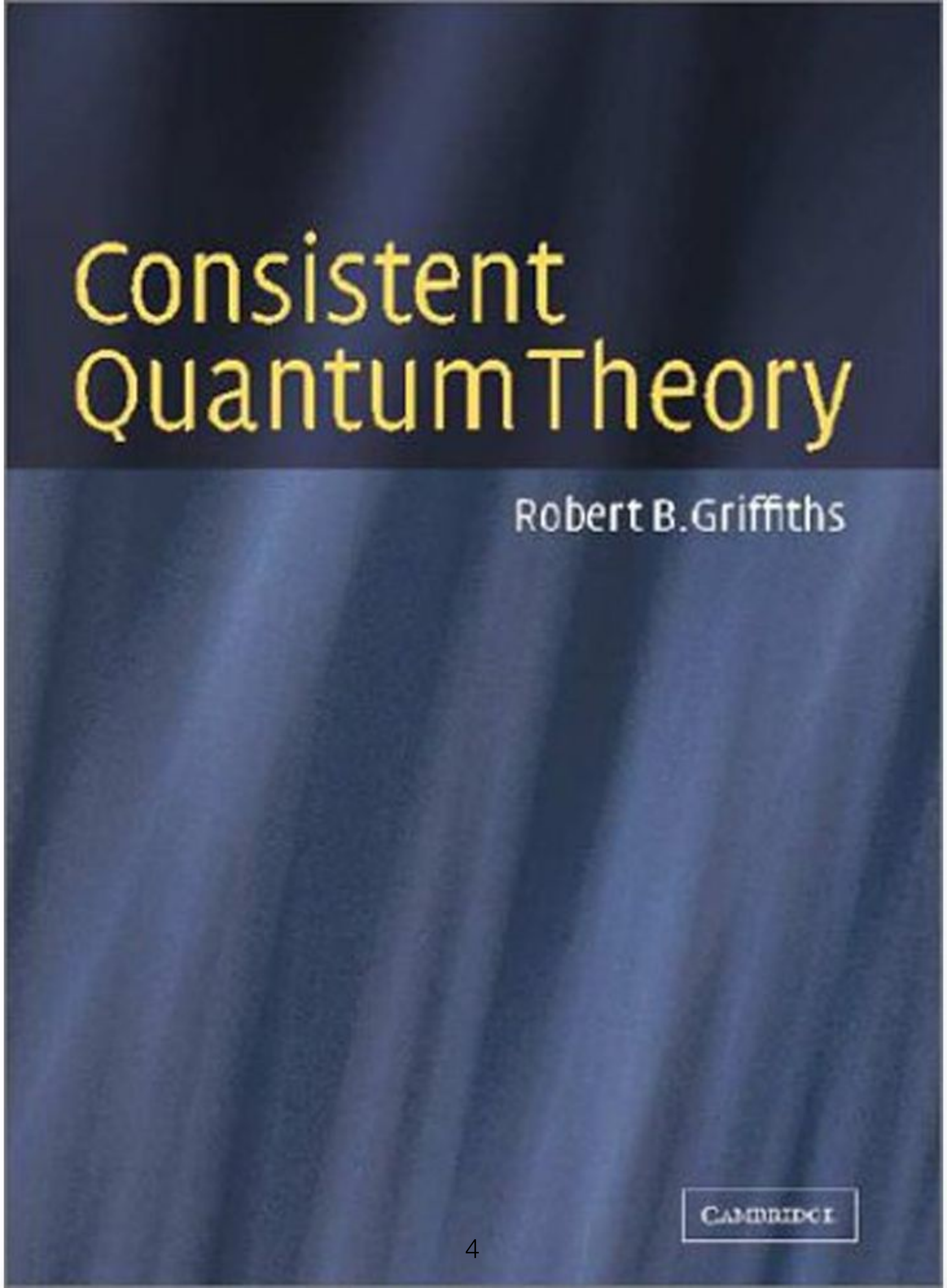
Discrete Space-Time Quantum Mechanics

A Friendly Playground

Z. Yu

(1) Framework

Local QM Made Simple and Flexible


Consistent
Quantum Theory
Robert B. Griffiths
CAMBRIDGE

- Space-time lattice theory

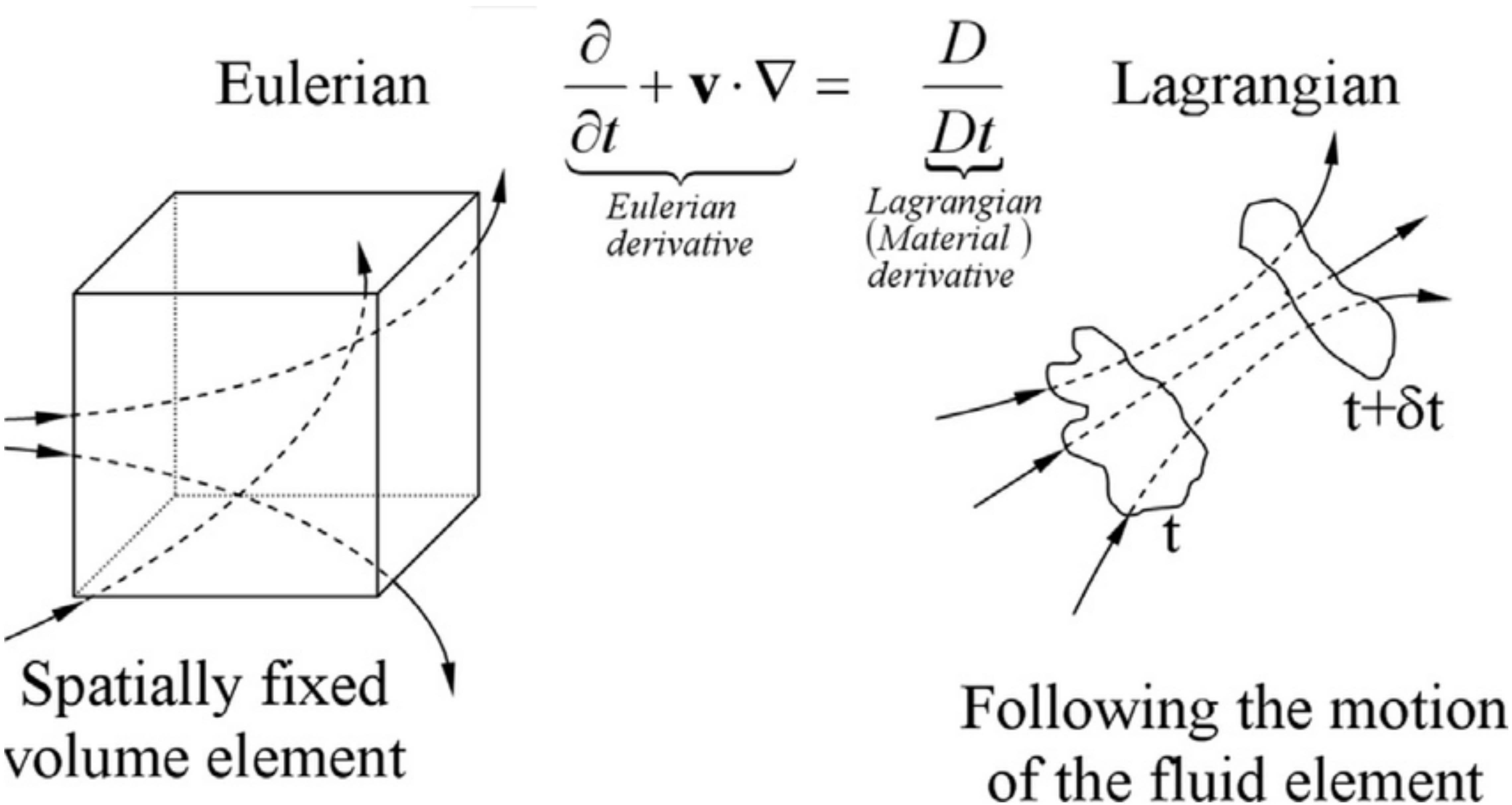
- The framework and general principles of quantum mechanics are fully respected.

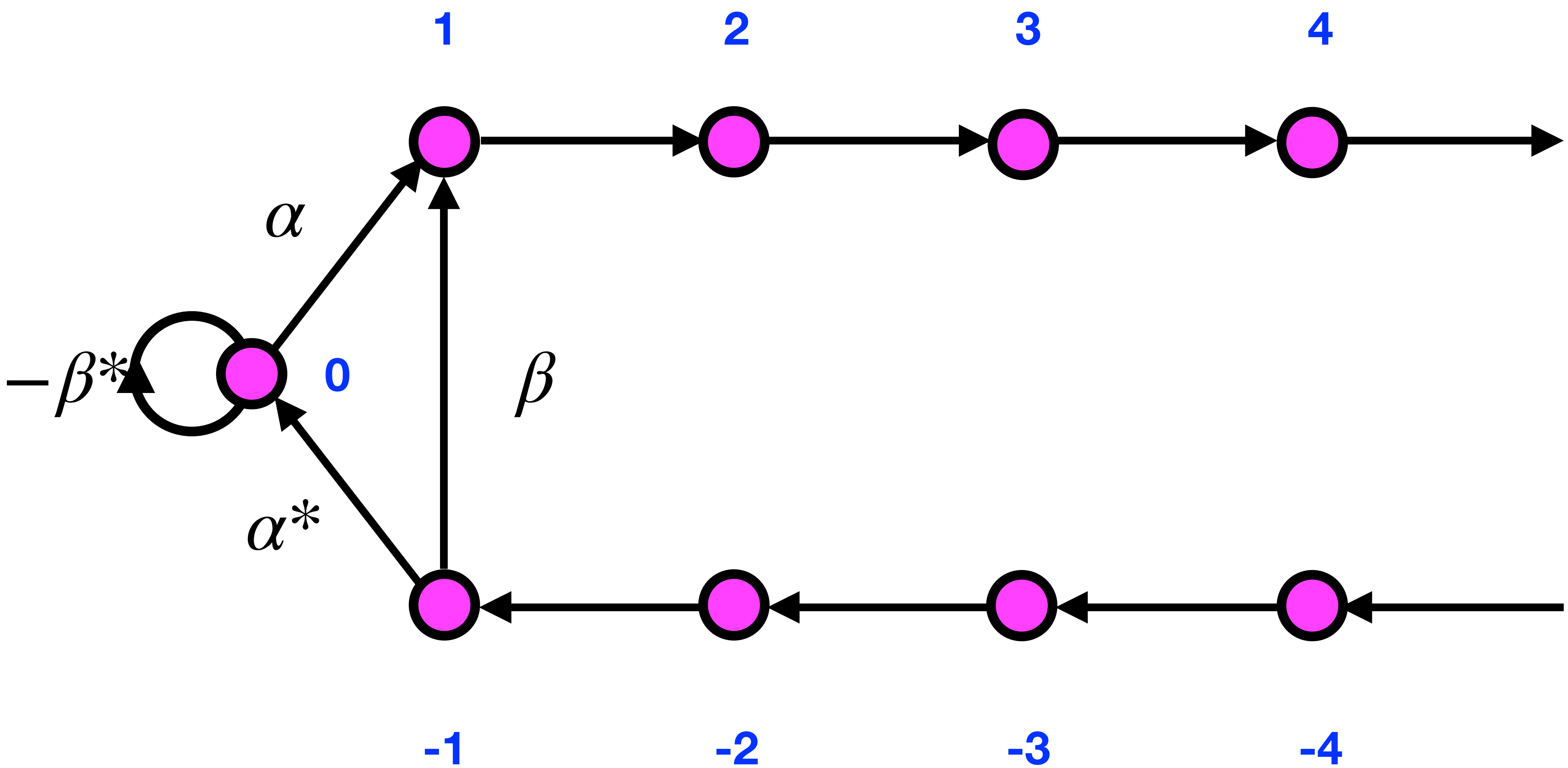
- Simple formulation in the L-picture*

- Toy models of hard-to-treat problems

- $\sim$ *Unitary cellular automata:* “Inner locality”

* E Versus L Pictures





1
2
3
4
α
$-\beta^*$
0
β
α^*
-1
-2
-3
-4

An orthonormal** basis of states: ** !

$$\ldots, |-3\rangle, |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots$$

Interpretation: the particle is at the indicated place.

The dynamics is specified by a unitary operator Ω that implements a unit time step

$$\Omega|n\rangle = |n+1\rangle \text{ for } n \neq -1,0$$

$$\Omega|0\rangle = -\beta^*|0\rangle + \alpha|1\rangle$$

$$\Omega|-1\rangle = \alpha^*|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Since this takes one orthonormal basis to another, it is indeed unitary.

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\beta^* & 0 & 0 \\ 0 & 0 & \beta & \alpha^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(Understood to be extended …)

Interpretation: the particle marches steadily along a line toward points with larger coordinates, except that there is a glitch at 0.

Alternatively, we can interpret 0 as the origin of radial coordinates, with -n indicating distance n and incoming motion, +n indicating distance n and outgoing motion.

Griffiths used this model to display the basic quantum mechanics of decay.

Indeed, if we start at time 0 with $|0\rangle$, then at time n we have

$$(-\beta^*)^n|0\rangle + \alpha\Big((-\beta^*)^{n-1}|1\rangle + (-\beta^*)^{n-2}|2\rangle + \ldots + |n\rangle\Big)$$

This is exponential decay, with an explicit radiation field.

More expansively, we can use it discuss the physics of traps:

When $|\beta| = 1$ (so $\alpha = 0$), $|0\rangle$ is an isolated state. It can neither be entered nor exited.

When $|\beta|$ is close to 1, $|0\rangle$ is a trap. It is difficult to enter, but once you're in, it is difficult to exit.

Traps are lurking everywhere:

We will be taking a deep dive into Griffiths' little model. We'll use it to illustrate the quasi-normal mode concept in its purest and simplest form.

Exploiting the ideas further, we'll display tractable QM models that illustrate and provide sanity checks for ideas about quantum radiation fields and black holes, dynamical geometry and topology change, advanced interferometry …

… and of barrier penetration and tunneling, to address issues about transit time.

(2) Green Function and Quasi-Normal Modes

An Enlightening Exercise

For amplitudes (wave-functions, E picture) we have

$$\begin{aligned} a(n,t+1) &= a(n-1,t) \quad n \le -1 \\ a(0,t+1) &= \alpha a(-1,t) - \beta^* a(0,t) \\ a(1,t+1) &= \beta a(-1,t) + \alpha^* a(0,t) \\ a(n,t+1) &= a(n-1,t) \quad n \ge 2 \end{aligned}$$

For eigenmodes $a(n, t+1) = \omega^{-1} a(n, t)$, giving

$$\begin{aligned} \phi_n^{(\omega)} &= \omega^n \quad n \le -1 \\ \phi_0^{(\omega)} &= \frac{\alpha}{1+\beta^*\omega} \\ \phi_n^{(\omega)} &= \frac{\omega^{n-1}(\omega+\beta)}{1+\beta^*\omega} \quad n \ge 1 \end{aligned}$$

Applying periodic boundary conditions $\phi_{-2} = \phi_{N-2}$ we have

$$\omega^{1-N} = \frac{\omega + \beta}{1 + \beta^* \omega}$$

The solutions are complex numbers of magnitude unity. (This is not quite trivial to prove directly — see the notes.)

But if we take $N \to \infty$, I.e. an open line, then there are other kinds of solution. We can take $\omega = (-\beta^*)^{-1}$! The corresponding mode:

$$\begin{aligned} \phi_n^{(QN)} &= \delta(n) + H(n-1)\alpha^*(-\beta^*)^{-n} \\ a_n^{(QN)}(t) &= \phi_n^{(QN)}(-\beta^*)^t \end{aligned}$$

is not normalizable - not even in the sense of plane waves - but (as we'll see) it has a clear physical interpretation and conceptual utility. It is a quasi-normal mode.

The quasi-normal mode grows in space, but shrinks in time. It is the “completed” form of the radiation field that arises from decay of the trap state.

In many contexts, e.g. in acoustics and microwave engineering, quasi-normal modes arise as one includes dissipative “loss” corrections to idealized normal modes.

Here are getting imaginary energies* by allowing non-normalizable configurations - going outside of Hilbert space!

The Green function, defined by

$a(n,t) \;=\; \sum_{n'} G(n,t;n',t')\,a(n',t')$, is a useful encoding of the dynamics.

It is straightforward in principle to calculate it, piecewise, by following particle trajectories. One finds (with $\tau \equiv t - t' \geq 0$):

$$
\begin{aligned}
G(n,n',\tau) &= \delta(n-n'-\tau)(H(-n-1)H(-n'-1)+H(n-1)H(n'-1)) \\
&+ (-\beta^*)^\tau\delta(n)\delta(n')H(\tau-n) \\
&+ \alpha^*(-\beta^*)^{\tau-n}H(n-1)H(\tau-n)\delta(n') \\
&+ H(-n'-1)H(\tau+n'+1)\big(\beta\delta(\tau-n+n'+1)H(n-1) \\
&+ \alpha(-\beta^*)^{\tau+n'}\delta(n)H(\tau-n+n') \\
&+ |\alpha|^2(-\beta^*)^{\tau-n+n'}H(n-1)H(\tau-n+n')\big)
\end{aligned}
$$

At first sight this is a hideous mess, but using the quasi-normal mode it becomes beautifully transparent.

$$
\begin{aligned}
G(n,n',\tau) &= \delta(n-n'-\tau)(H(-n-1)H(-n'-1)+H(n-1)H(n'-1)) \\
&+ \delta(n')a_n^{(QN)}(\tau)H(\tau-n) \\
&+ H(-n'-1)H(\tau+n'-1)\Big(\beta\delta(\tau-n+n'+1)H(n-1) \\
&+ \alpha a_n^{(QN)}(\tau+n')H(\tau-n+n')\Big)
\end{aligned}
$$

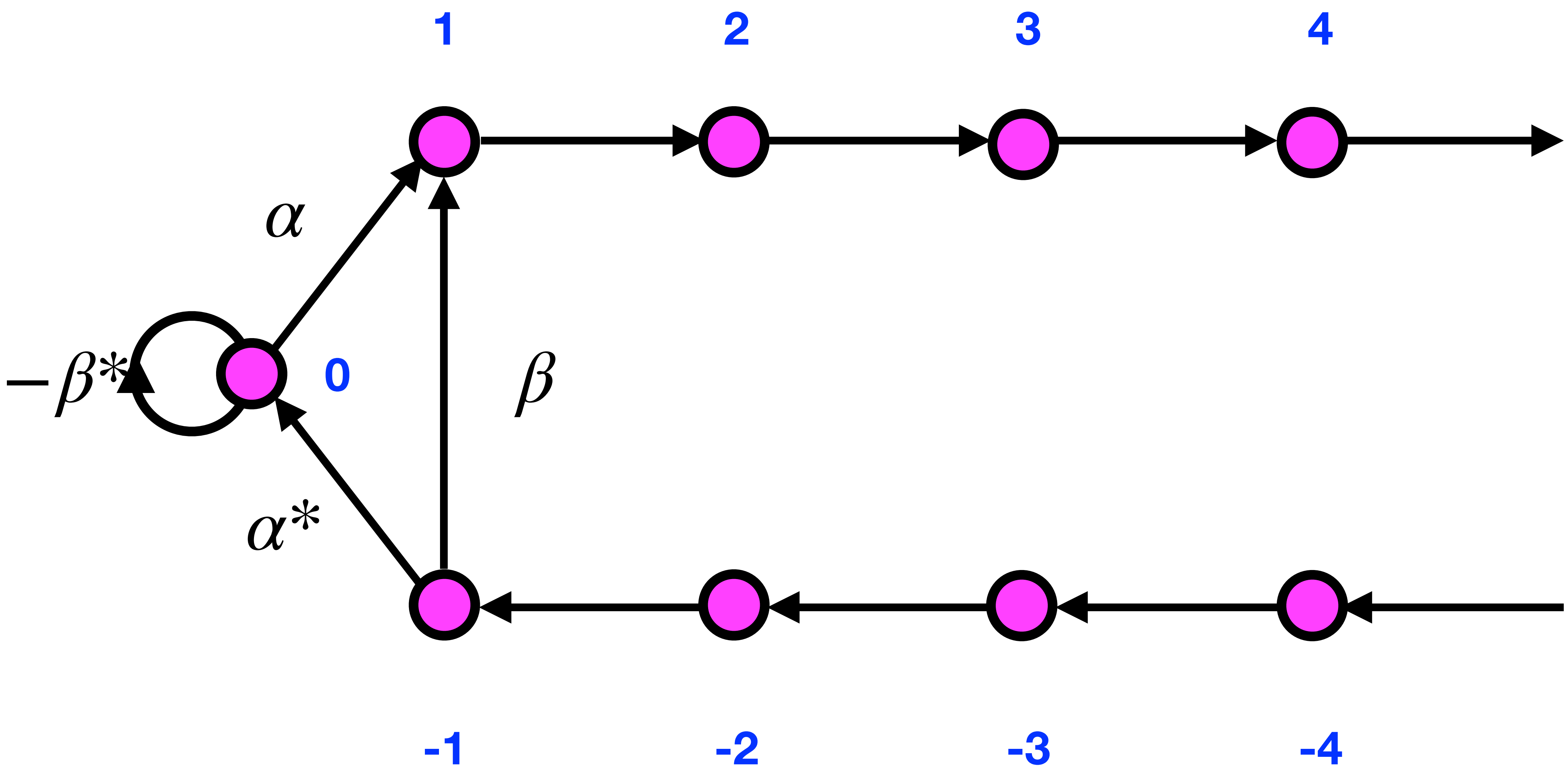
(blue) : free propagation, no encounter with trap

(green) : decay/radiation from trap location

(pink) : (with prefactor) free propagation, hopping over trap

(orange) : (with prefactor) falling into trap, then decay/radiation

Note that the quasi-normal mode appears here with a dynamic cutoff. The cutoff implements a causal light front.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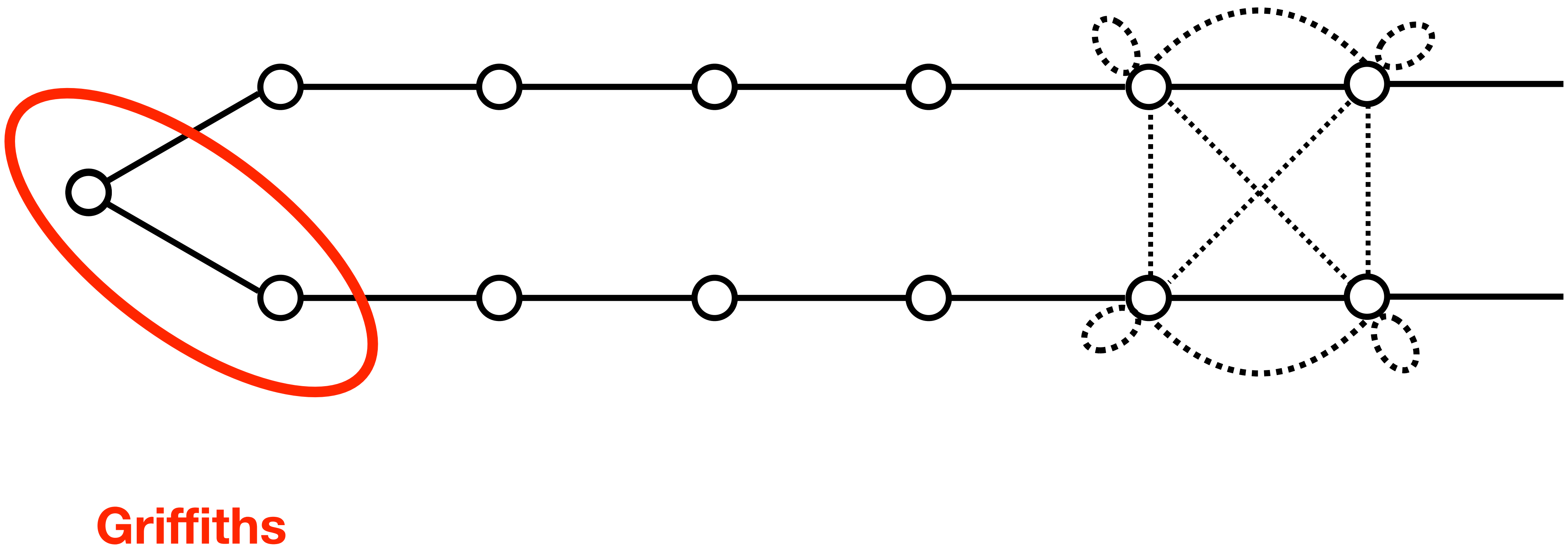
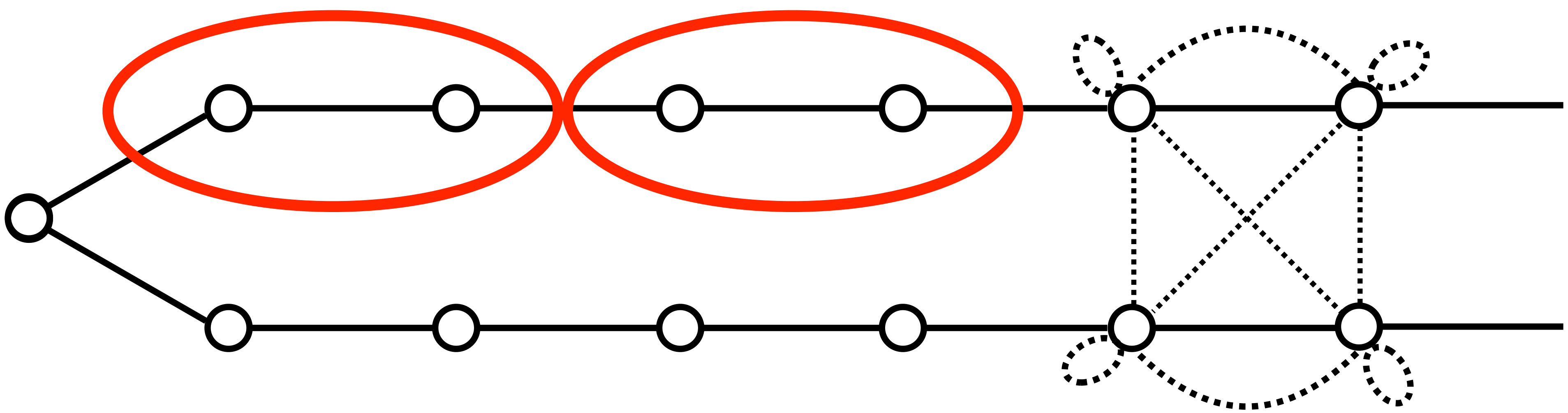
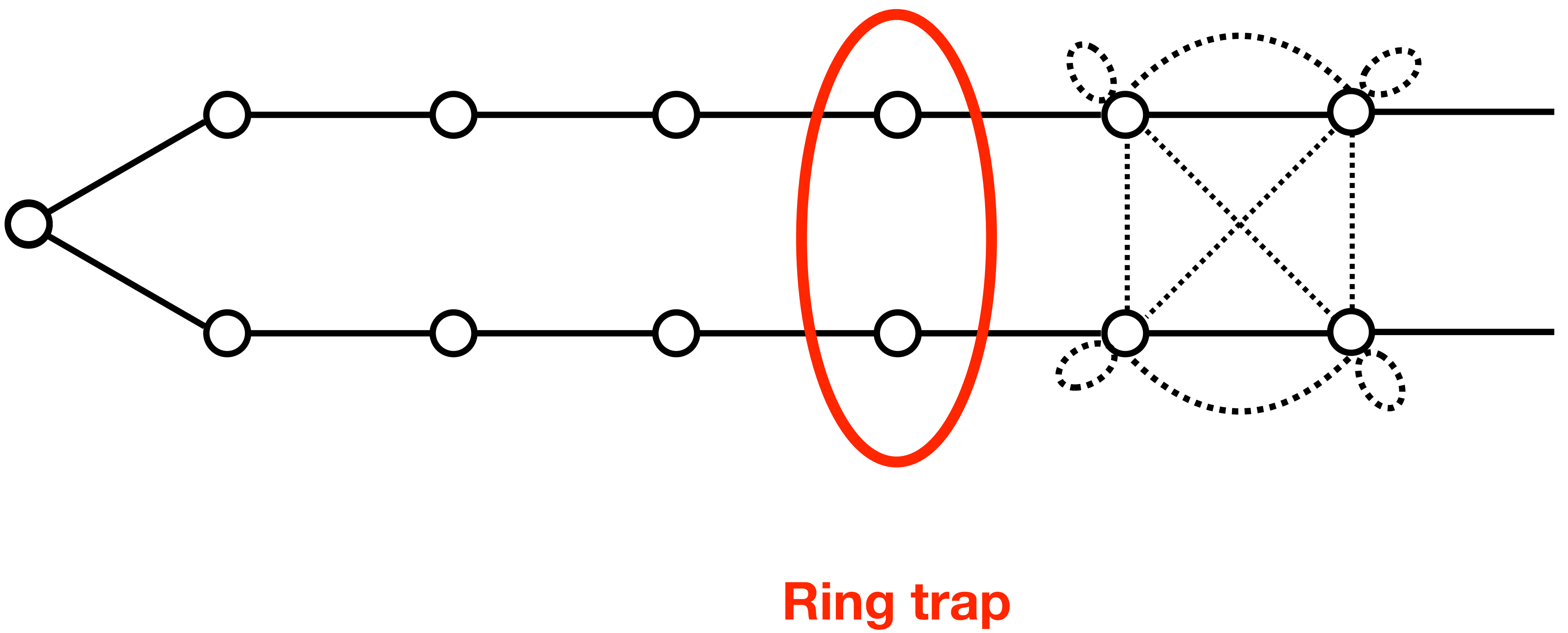
More Complex Traps

- Double trap
- Ring trap
- Dynamic geometry (see below)


Griffiths



Double trap


Ring trap

(3) Many-Particle Traps

Serious Toy Models

By taking tensor products, we can build many-particle versions of the model.

We can incorporate boson or fermion statistics by taking symmetric or antisymmetric tensor products.

If the trap is very trappy ($|\alpha| <<< 1$), then we can have an interesting model of pseudo-information loss.

Namely, we start at time $\tau = 0$ with a large number of particles having moderately small negative coordinates and let it evolve.

Most of the particles will hop over the trap, but (if we started with enough particles) many will accumulate there.

The resulting “object” at $|0\rangle$ will then slowly decay, as particles slowly escape.

The emitted radiation field will display only very subtle signs of how the object was formed: slight offsets in the starting times for the quasi-normal modes!

Nevertheless, the overall process is entirely unitary.

In principle, we can follow the entropy flow in detail …

… and we can in practice, too.

(4) Radiation Field Properties

Building Up To Statistics and Entropy

To probe the radiation field, introduce a “detector” or “internal state” at $|1\rangle$

$$W = (\mathbb{1} - |1\rangle\langle 1|) \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The complete model is

$$\begin{aligned} \Omega_d &\equiv \Omega \otimes \mathrm{I} \\ U &= W\Omega_d \end{aligned}$$

$$\psi_0(0) \;=\; |\,0\rangle \otimes |\,e\rangle$$

$$U^n\,\psi_0(0) \;=\; (-\beta^*)^n\,|\,0\rangle \otimes |\,e\rangle + \alpha((-\beta^*)^{n-1}\,|\,1\rangle \;+\; (-\beta^*)^{n-2}\,|\,2\rangle \;+\; \ldots\,|\,n\rangle) \otimes |\,g\rangle$$

The density matrix for the detector is

$$\rho_0(n) \;=\; \begin{pmatrix} |\beta|^{2n} & 0 \\ 0 & 1 - |\beta|^{2n} \end{pmatrix}$$

For the entropy of the detector, which is also the entropy of the radiation field, we have

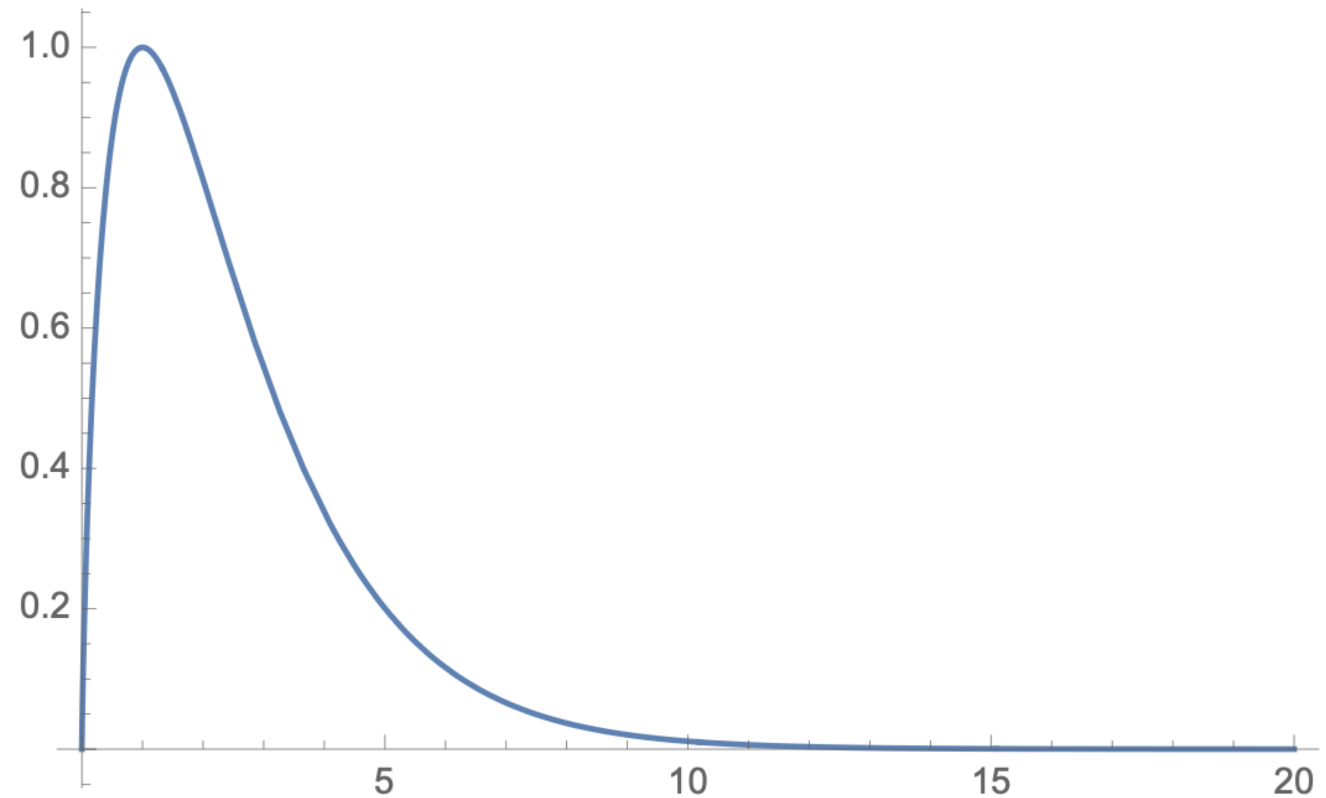
$$\mathrm{Ent}_0(n) = - \left(|\beta|^{2n} \log_2 |\beta|^{2n} + (1 - |\beta|^{2n} (\log_2 (1 - |\beta|^{2n}) \right)$$

or in continuous form

$$\mathrm{CEnt}(t) \;=\; t\, 2^{-t} - (1 - 2^{-t}) \log_2 (1 - 2^{-t})$$

$$t \;\equiv\; n \log_2 |\beta|^{-2}$$

We get this universal function:



as we evolve from pure excitation to pure radiation, with entanglement in between.

By starting with two particles at $|-1\rangle, \ |0\rangle$, and introducing two detectors (or one counter) down the line, we can see the effect of quantum statistics:

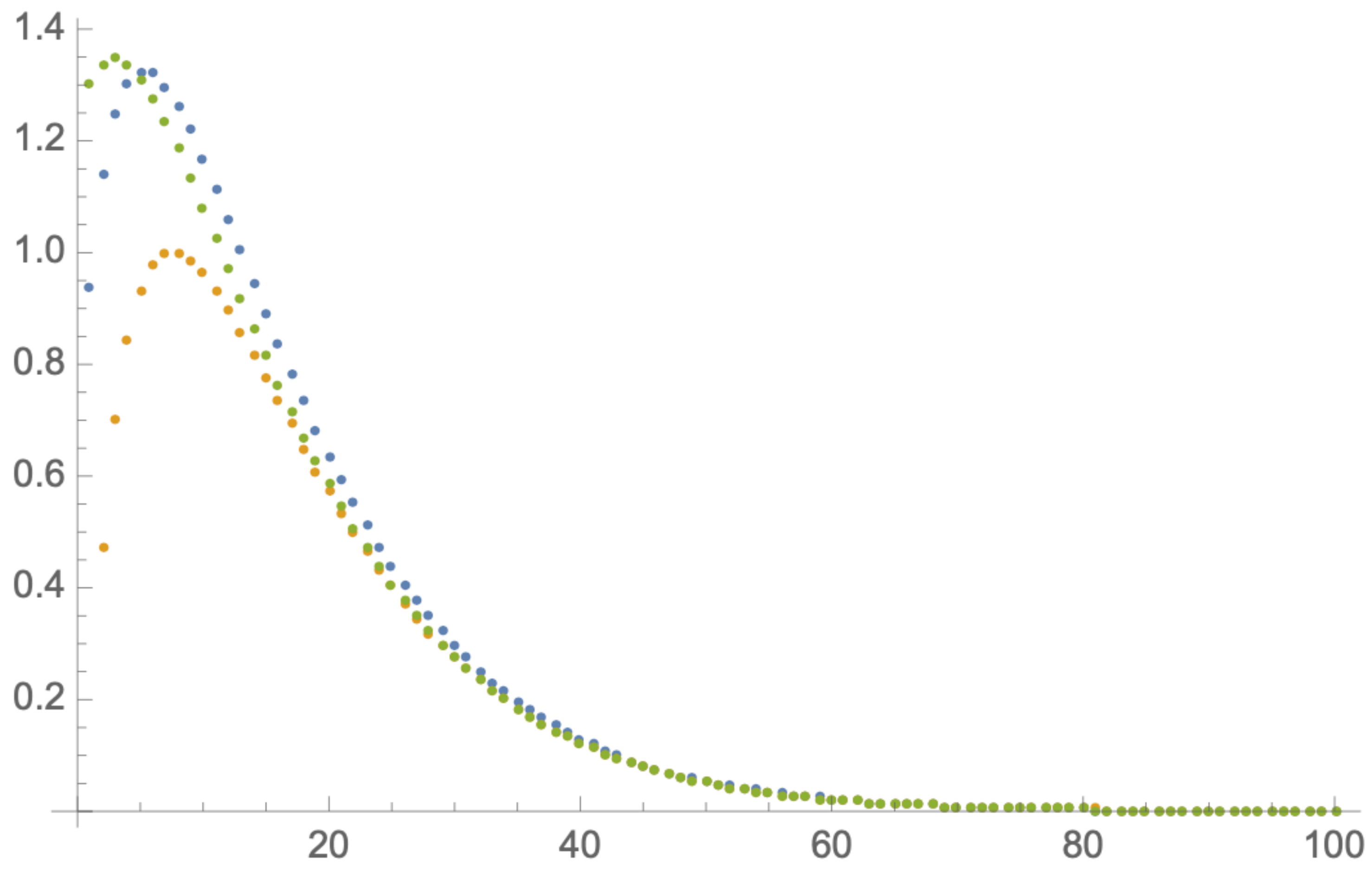


Figure 3: Entropy of the radiation field evolving from initial fermionic (orange), bosonic (green), and distinguishable (blue) initial states based on $\psi_0(0) \otimes \psi_{-1}(0)$.

Given our Green functions, it is straightforward in principle to set up and investigate our “serious toy model”.

Note that the ideal remnant object is governed by the quasinormal modes!

A most intesting possibility - also for experiments - is to make parts of quantum radiation fields interfere, by keeping the detectors pure and later "detecting the (quantum state of) the detectors".

(5) Encoding Motion

Spreading Made Simple-ish

We define models that can accommodate motion and barriers to motion through:

$$\begin{aligned}\Omega|k,R\rangle &= \alpha(k)|k+1,R\rangle + \beta(k)|k-1,L\rangle \\ \Omega|k,L\rangle &= -\beta(k)^*|k+1,R\rangle + \alpha(k)^*|k-1,L\rangle\end{aligned}$$

When $\alpha(k) = 1$ we have an obvious interpretation in terms of left-moving and right-moving states of motion. More general functional forms for $\alpha(k), \beta(k)$ allow us to set up mesas for barrier penetration problems, double mesas for tunneling problems, and so forth.

By doubling the unit cell - encoding direction of motion as a position variable! -

$$
\begin{aligned}
|n, R\rangle &\equiv |2n\rangle \\
|n, L\rangle &\equiv |2n+1\rangle
\end{aligned}
$$

we turn the evolution rule into a unitary matrix operation

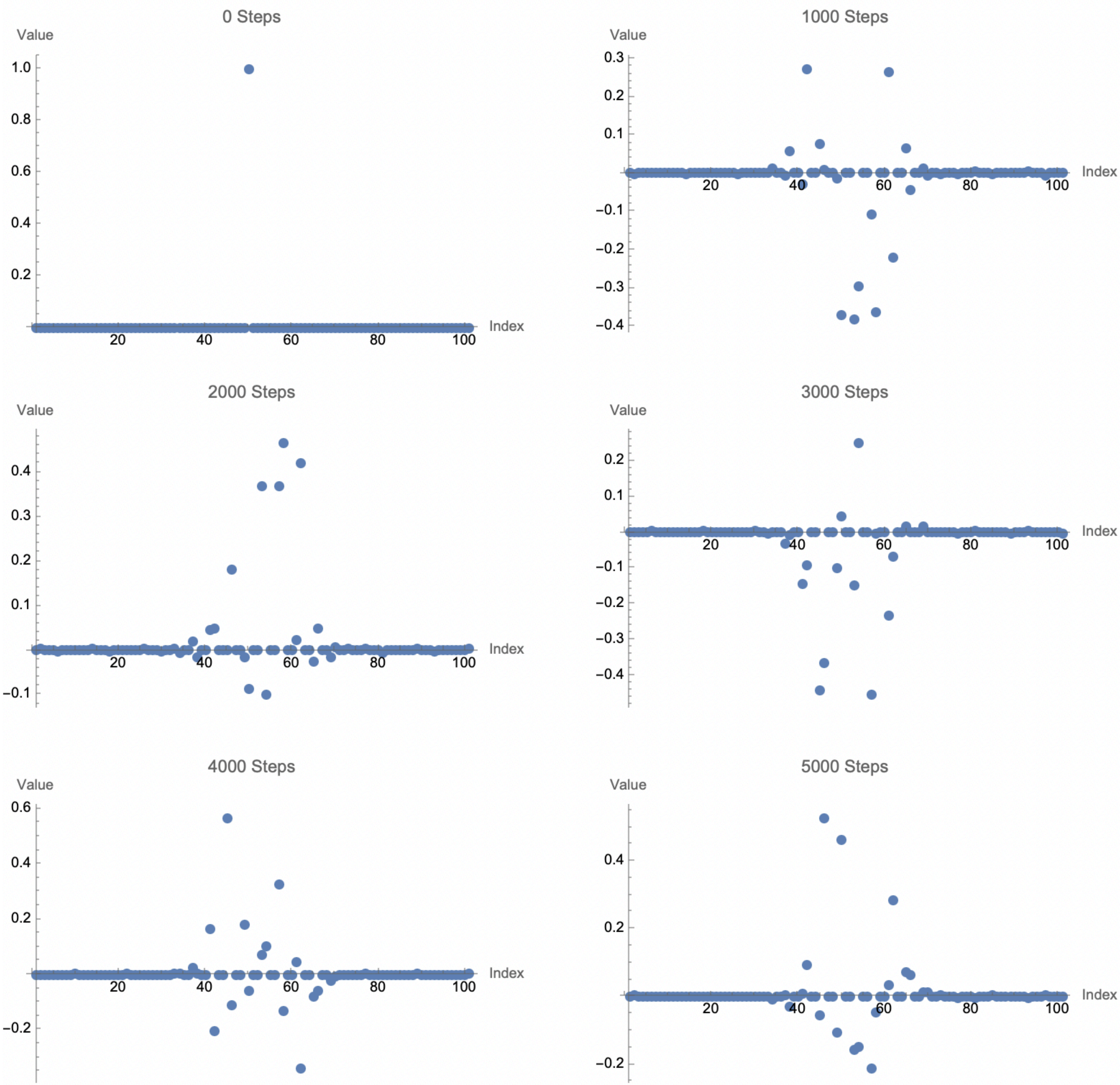
$$
\begin{aligned}
\Omega|2k\rangle &= \alpha(k)|2k+2\rangle + \beta(k)|2k-1\rangle \\
\Omega|2k+1\rangle &= -\beta(k)^*|2k+2\rangle + \alpha(k)^*|2k-1\rangle
\end{aligned}
$$

The associated amplitude evolution is

$$
\begin{aligned}
a(2k, t+1) &= \alpha(k-1)a(2k-2, t) - \beta(k-1)^* a(2k-1, t) \\
a(2k+1, t+1) &= \beta(k+1)a(2k+2, t) + \alpha(k+1)^* a(2k+3, t)
\end{aligned}
$$

These are the equations we can use to set up mesa and double mesa problems in a form that is very tractable numerically, allowing complete tracking of space-time behavior.

Here is a well simulation:


0 Steps
Value
1.0
0.8
0.6
0.4
0.2
Index
20
40
60
80
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Value
0.3
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−0.2
−0.3
−0.4
Index
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40
60
80
100
2000 Steps
Value
0.4
0.3
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0.1
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Index
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3000 Steps
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0.2
0.1
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−0.2
−0.3
−0.4
Index
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40
60
80
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4000 Steps
Value
0.6
0.4
0.2
−0.2
Index
20
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100
5000 Steps
Value
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0.2
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Index
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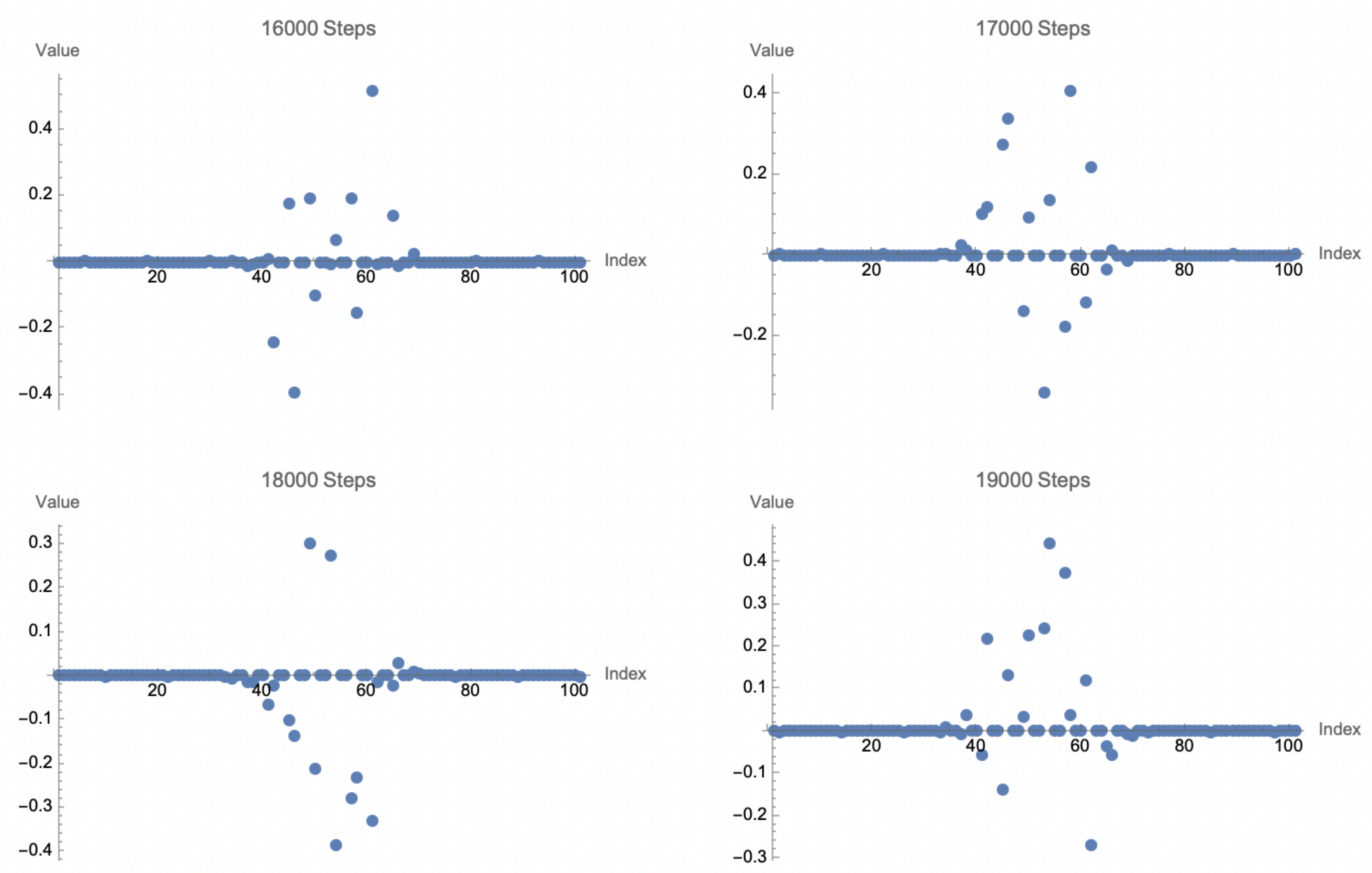


```
sumsOfSquaresList
```

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Out[]= {{1, 0}, {0.717143, 0.0110749}, {0.714658, 0.00632496}, {0.691595, 0.0224914}, {0.669931, 0.0374848}, {0.690849, 0.0101604}, {0.671085, 0.0237343},
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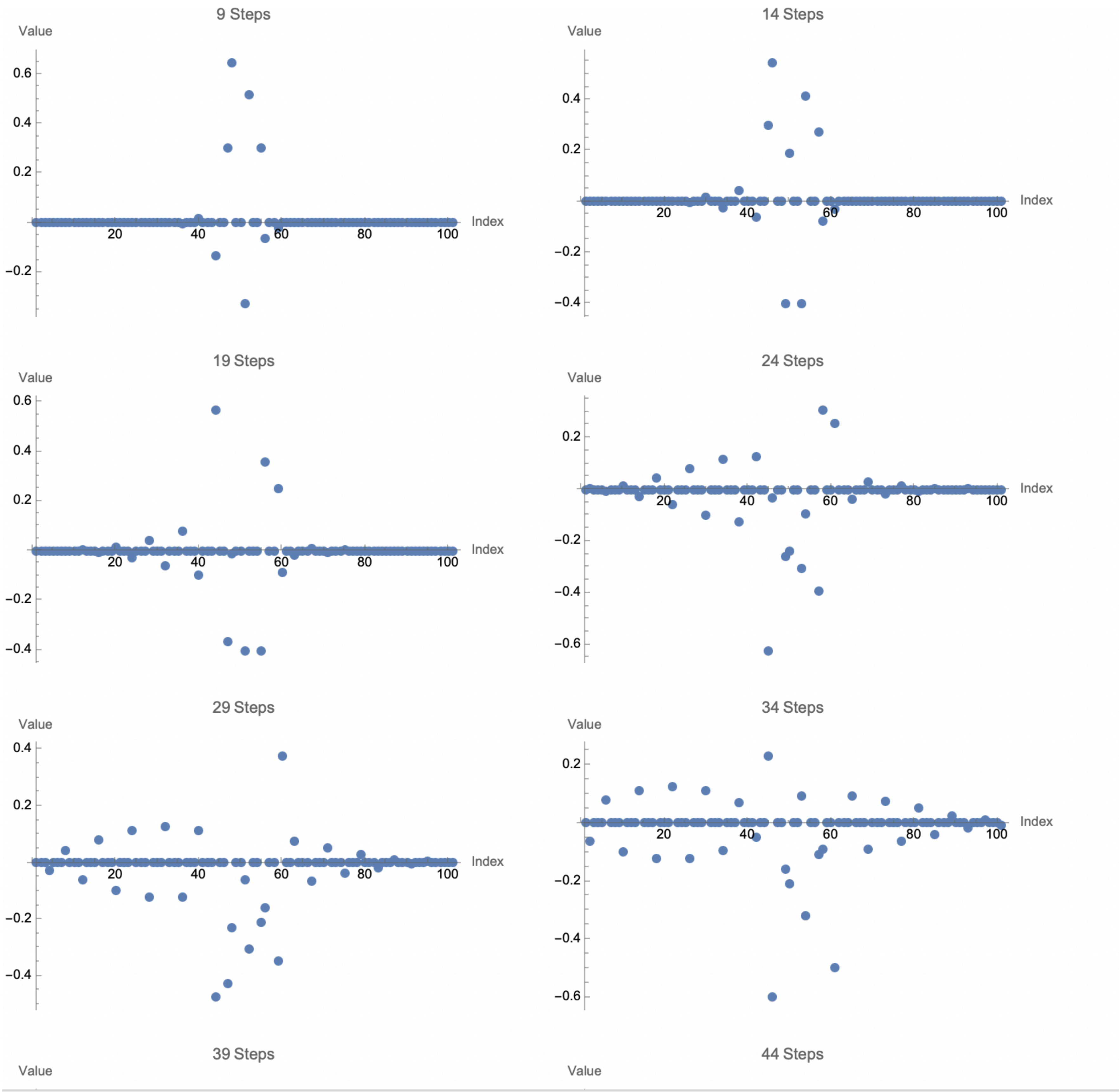
```

Stasis

Here is a mesa simulation:


9 Steps
Value
Index
14 Steps
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19 Steps
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Index
24 Steps
Value
Index
29 Steps
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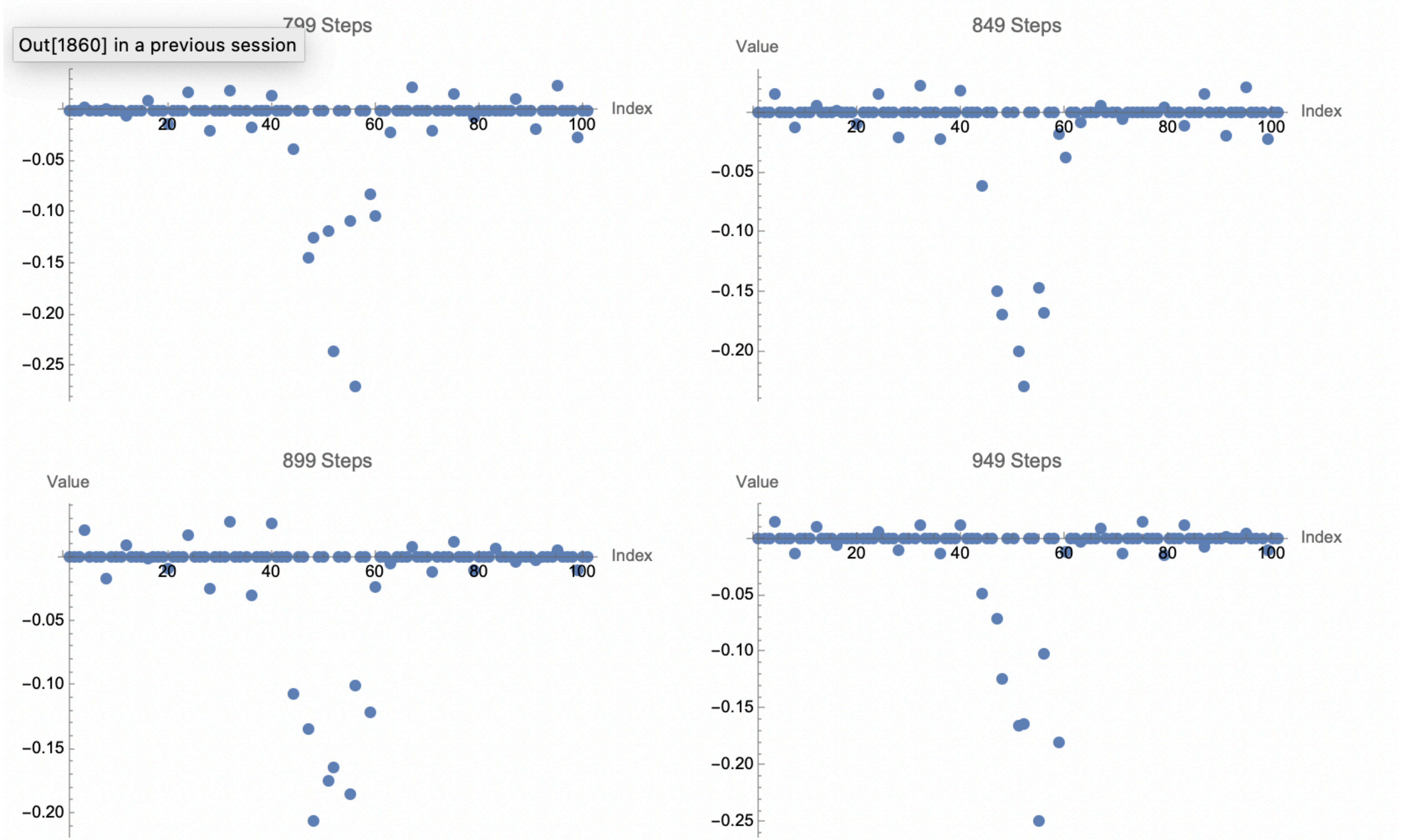
Index
34 Steps
Value
Index
39 Steps
Value
44 Steps
Value


Out[1860] in a previous session
799 Steps
Index
20
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80
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−0.15
−0.20
−0.25
849 Steps
Value
Index
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−0.05
−0.10
−0.15
−0.20
899 Steps
Value
Index
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80
100
−0.05
−0.10
−0.15
−0.20
949 Steps
Value
Index
20
40
60
80
100
−0.05
−0.10
−0.15
−0.20
−0.25

```
sumsOfSquaresList
```

```
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```

Steady leakage

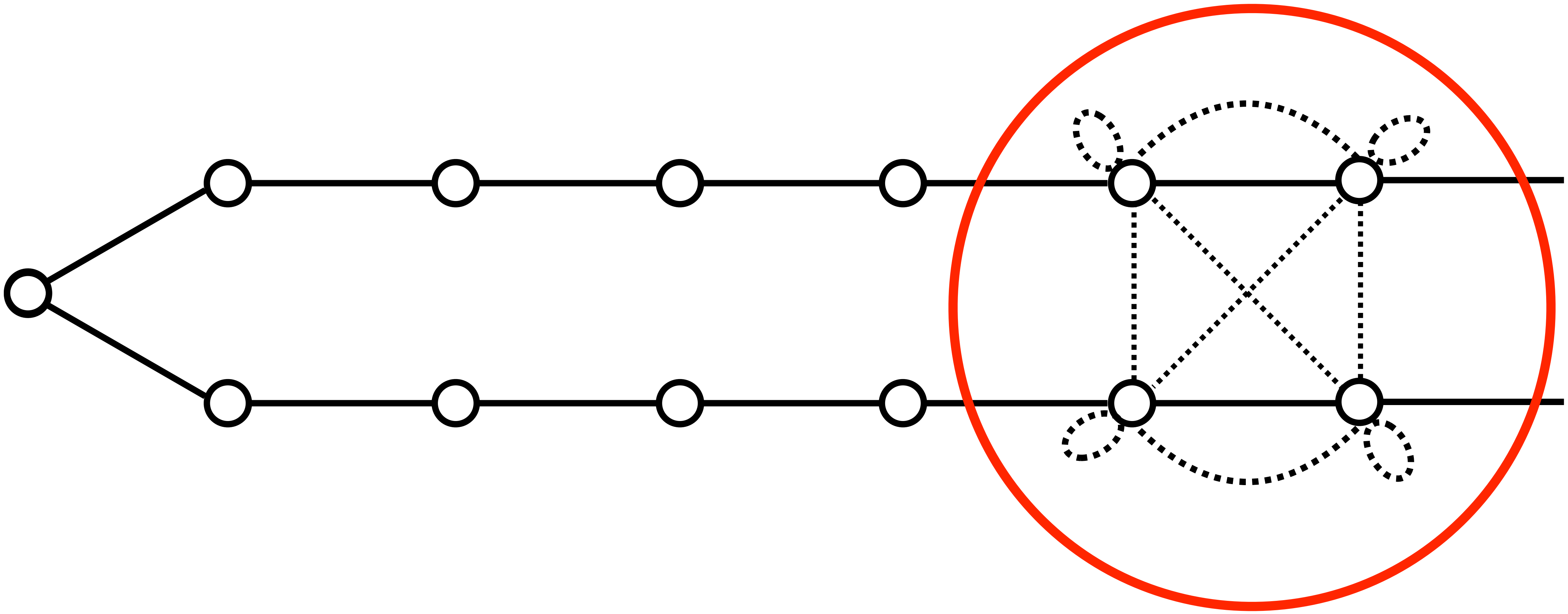
This work is very much in progress.

A plausible guiding hypothesis: In trappy situations the general excitation settles, after some non-universal transients, into a set of slowly decaying quasi-normal modes (approximate bound states).

Causality might appear as a cutoff, as in the simple model. The would resolve the “no phase” paradox.

(6) Dynamical Space-Times

Suggested Explorations



Scissor and paste

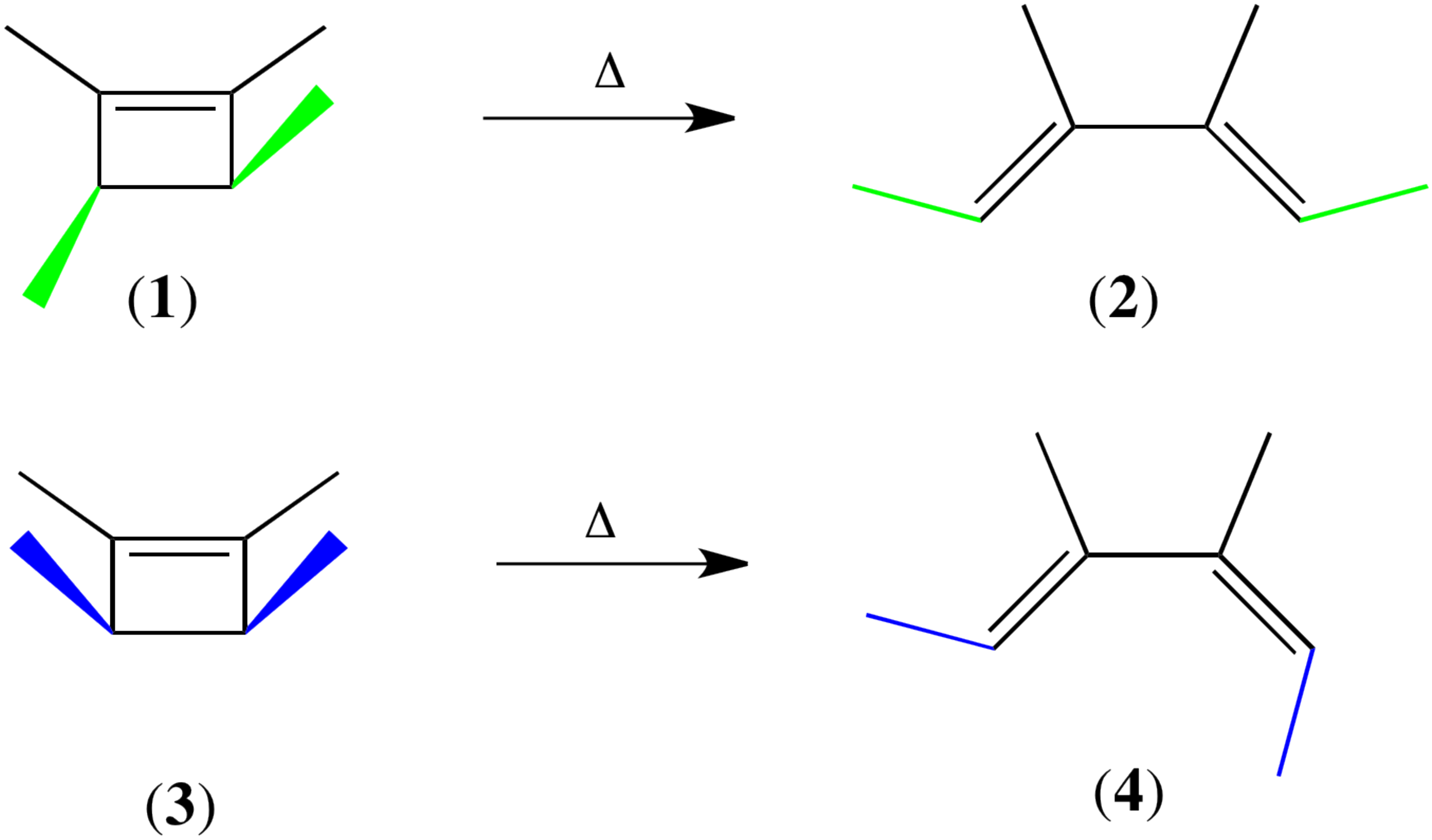
Shift - or motion!

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

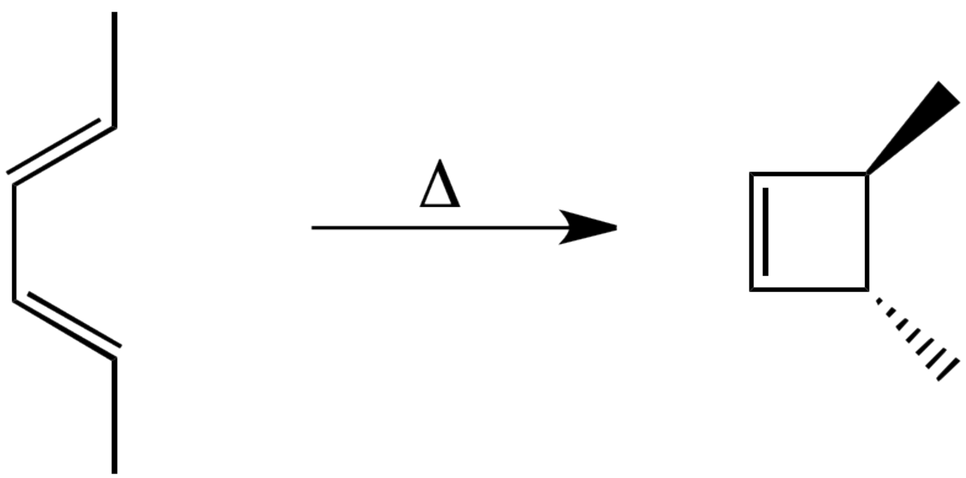
We can make the * 4x4 unitary time-dependent, or even move it around, stochastically or "lawfully" (autonomously or externally controlled). Particles will propagate, and waves will spread, accordingly.

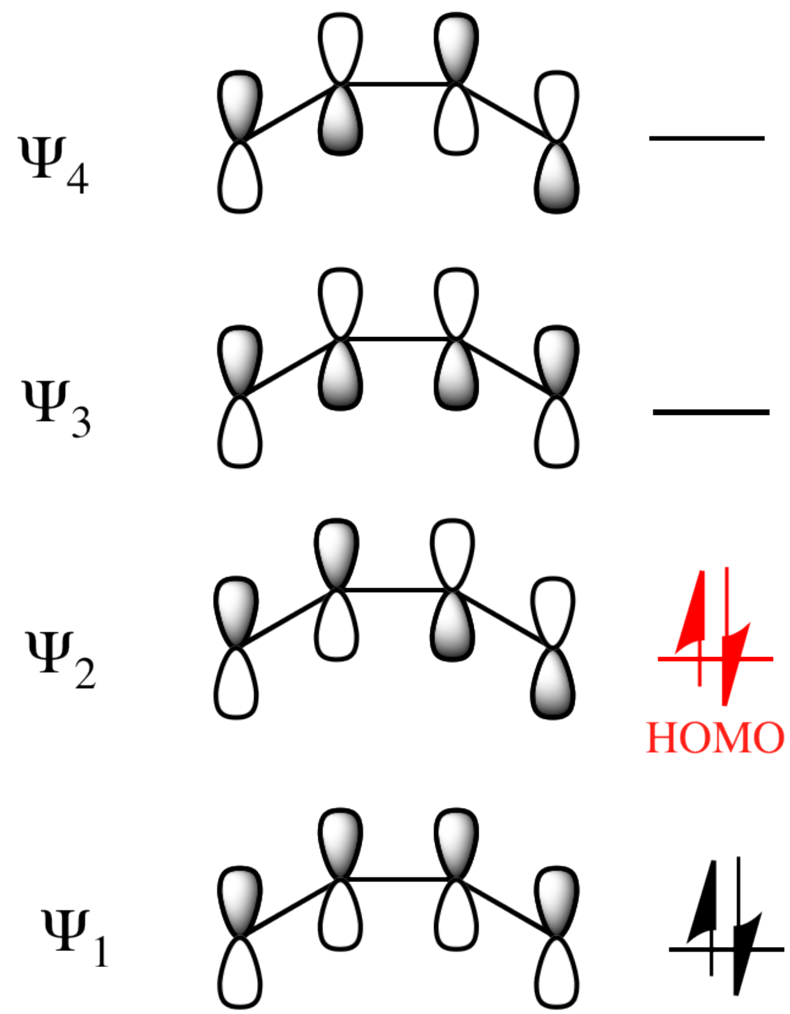
All this happens within the kinematic framework of quantum-mechanical unitary time evolution.

I expect that we can made nice caricatures of several kinds of interesting physical systems:

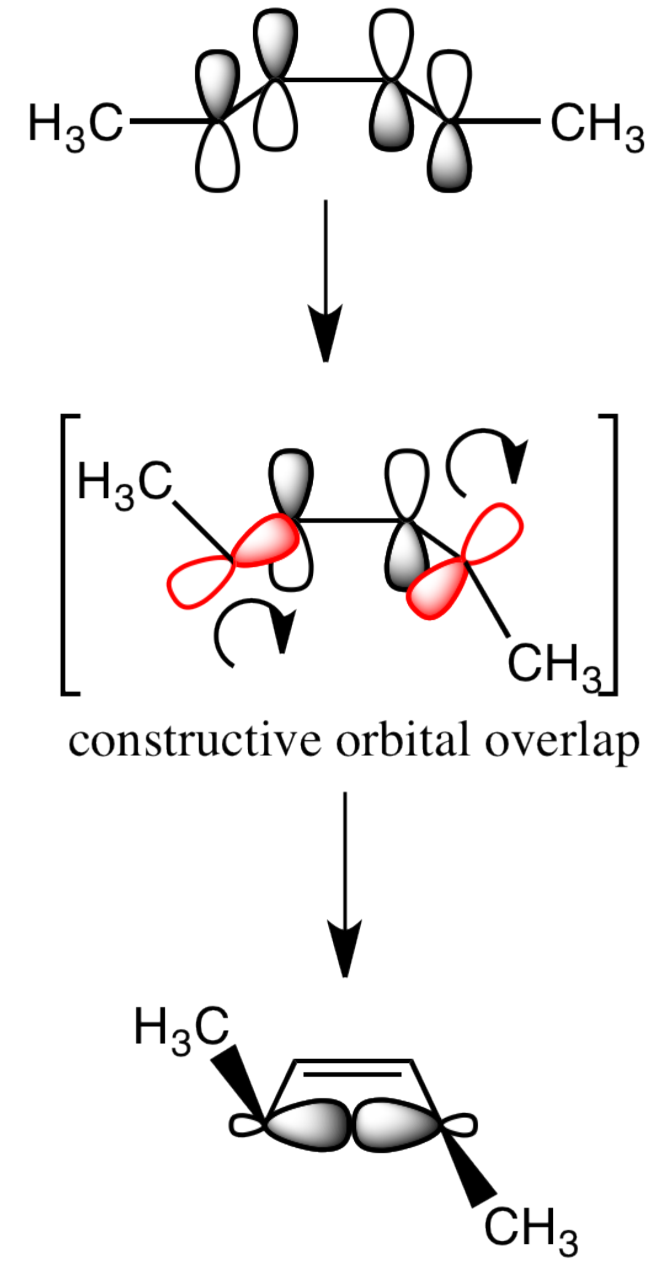


Woodward-Hoffman Rules (and Their Violation)

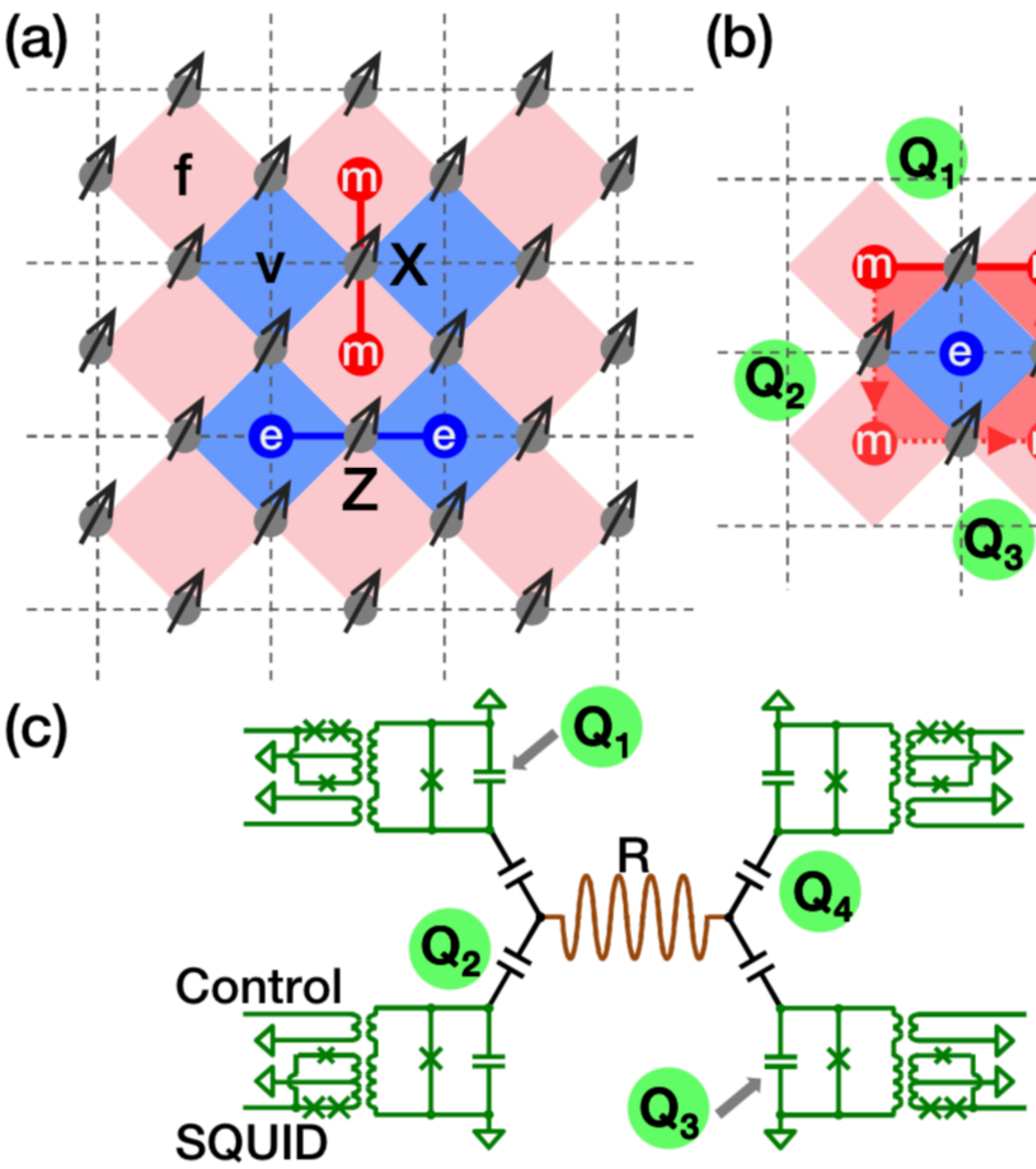


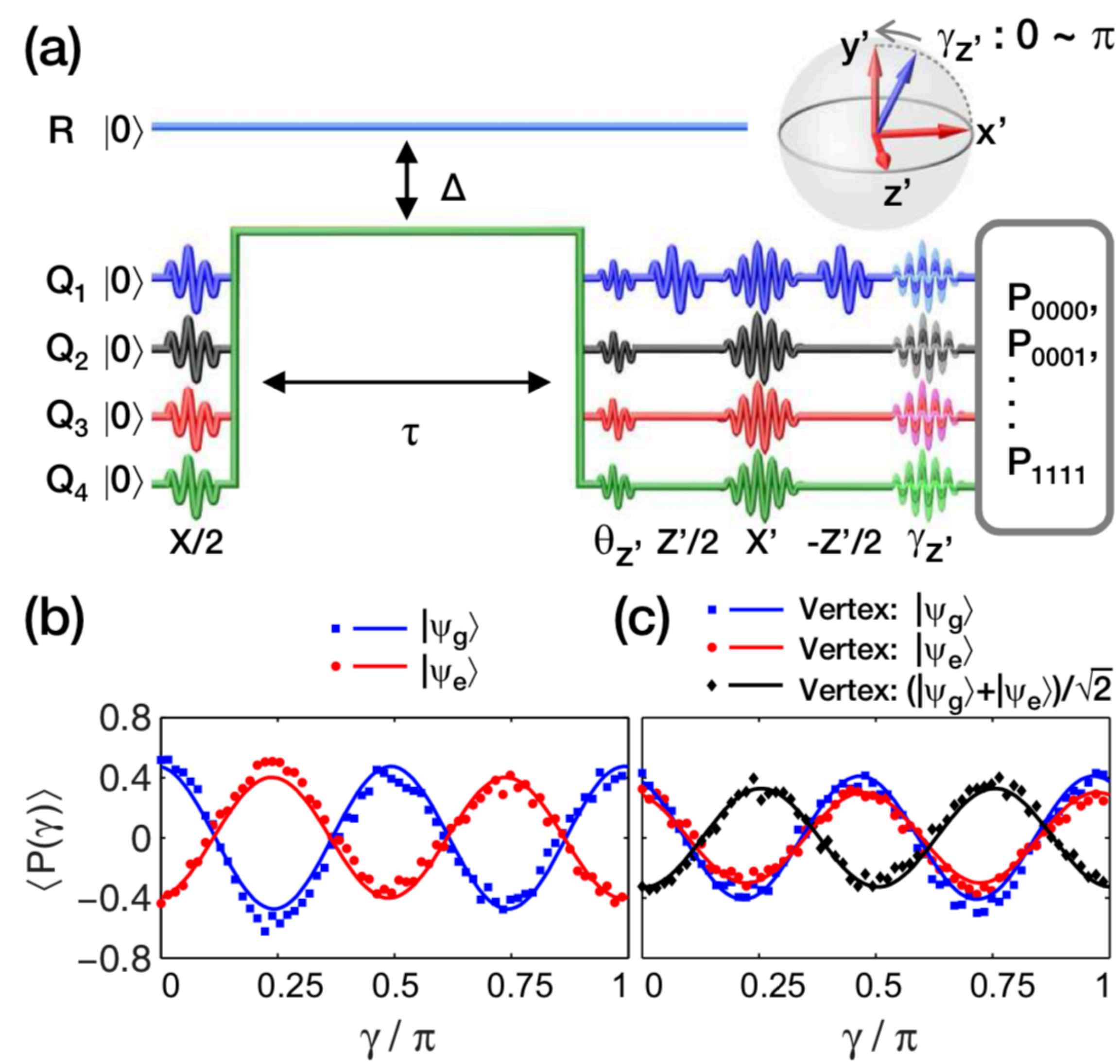


π-system molecualr orbitals



symmetry allowed product





Charge & Flux

- & (unitary) Switches!

Formation of a "baby universe"