The p-spin model

Energy: $E = -\sum_{i_1<i_2<\cdots<i_p} J_{i_1 i_2 \cdots i_p} S_{i_1} S_{i_2} \cdots S_{i_p} - \sum_i h_i S_i$

spherical constraint: $\sum_i S_i^2 = N$

$J_{i_1 i_2 \cdots i_p}$ independent and Gaussian with zero mean and variance $\frac{J^2 p!}{2N^{p-1}}$

Langevin dynamics: $\dot{S}_i = -rS_i + \sum_{j_2<j_3<\cdots<j_p} J_{i,i_2 \cdots i_p} S_{j_2} S_{j_3} \cdots S_{j_p} + h_i(t) + \xi_i(t)$

We want to calculate the response function $G(t-t') \equiv G_{ii}(t-t') = \frac{\langle \delta S_i(t) \rangle}{\delta h_j(t')}$

where the average is both thermal (over noise) and over disorder (the J_{ij}s).

The equations of motion (p=3)

Define $S_i^0(t)$ by $(\partial_t + r)S_i^0(t) = h_i(t) + \xi_(t)$,i.e.

$$S_i^{(0)}(t) = \int_{t'} G_0(t-t')[h_i(t') + \xi_i(t)]$$ with $G_0(t) = \Theta(t)\mathrm{e}^{-rt}$

Equation of motion:

$$S_i(t) = S_i^0(t) + \int_{t'} G_0(t-t') \sum_{\langle jk \rangle} J_{ijk} S_j(t') S_k(t') = S_i^0(t) + \int_{t'} G_0(t-t') \frac{1}{2} \sum_{jk} J_{ijk} S_j(t') S_k(t')$$

iterate once:

$$S_i^{(1)}(t) = S_i^{(0)}(t) + \frac{1}{2}\int_{t'} G_0(t-t') \sum_{jk} J_{ijk} S_j^{(0)}(t') S_k^{(0)}(t')$$

$$= \int_{t'} G_0(t-t')[h_i(t') + \xi_i(t')] + \frac{1}{2}\int_{t_1} G_0(t-t_1) \sum_{jk} J_{ijk} \int_{t_2} G_0(t_1-t_2)[h_j(t_2) + \xi_j(t_2)] \int_{t_3} G_0(t_1-t_3)[h_k(t_3) + \xi_k(t_3)]$$

iterate twice:

$$S_i^{(2)}(t) = S_i^{(0)}(t) + \frac{1}{2}\int_{t_1} G_0(t-t_1) \sum_{jk} J_{ijk} \left[S_j^{(0)}(t_1) + \frac{1}{2}\int_{t_2} G_0(t_1-t_2) \sum_{lm} J_{jlm} S_l^{(0)}(t_2) S_m^{(0)}(t_2) \right] \left[S_k^{(0)}(t_1) + \frac{1}{2}\int_{t_3} G_0(t_1-t_3) \sum_{pq} J_{kpq} S_p^{(0)}(t_3) S_q^{(0)}(t_3) \right]$$

Averaging over J and ξ: just doing the algebra:

Averaging over the J's:

$$S_i^{(2)}(t) = S_i^{(0)}(t) + \frac{1}{2}\int_{t_1} G_0(t-t_1)\sum_{jk} J_{ijk}\left[S_j^{(0)}(t_1) + \frac{1}{2}\int_{t_2} G_0(t_1-t_2)\sum_{lm} J_{jlm}S_l^{(0)}(t_2)S_m^{(0)}(t_2)\right]\left[S_k^{(0)}(t_1) + \frac{1}{2}\int_{t_3} G_0(t_1-t_3)\sum_{pq} J_{kpq}S_p^{(0)}(t_3)S_q^{(0)}(t_3)\right]$$

To survive the averaging, either J_{jlm} or J_{kpq} has to be equal to J_{ijk}.

There are 4 such pairs of factors, and the average of the product of the pair of J's in each is $3J^2/N^2$, so summing over j and k yields

$$S_i^{(2)}(t) = S_i^{(0)}(t) + \int_{t_1} G_0(t-t_1)\frac{3J^2}{N^2}\sum_{jk}\left[S_j^{(0)}(t_1)\int_{t_2} G_0(t_1-t_2)S_i^{(0)}(t_2)S_j^{(0)}(t_2)\right]$$

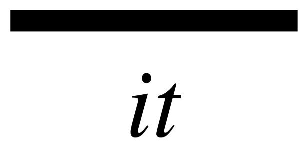
All $S^{(0)}$s in the last term here are proportional to the (independent) noises ξ at their sites, so the only pair average that survives is

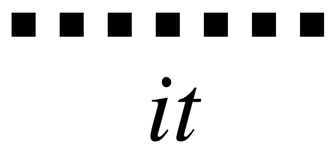
$$\langle S_j^{(0)}(t_1)S_j^{(0)}(t_2)\rangle = \int dt_1' dt_2' G_0(t_1-t_1')G_0(t_2-t_2')\cdot 2T\delta(t_1'-t_2') = C_0(t_1-t_2)$$

We want the response function, which is the derivative of the averaged $S_i^{(2)}(t)$ with respect to $h_i(t')$:

$$\frac{\delta\langle S_i^{(2)}(t)\rangle}{\delta h_i(t')} = G_0(t-t') + 3J^2\int_{t_1} G_0(t-t_1)\int_{t_2} G_0(t_1-t_2)C_0(t_1-t_2)G_0(t_2-t')$$

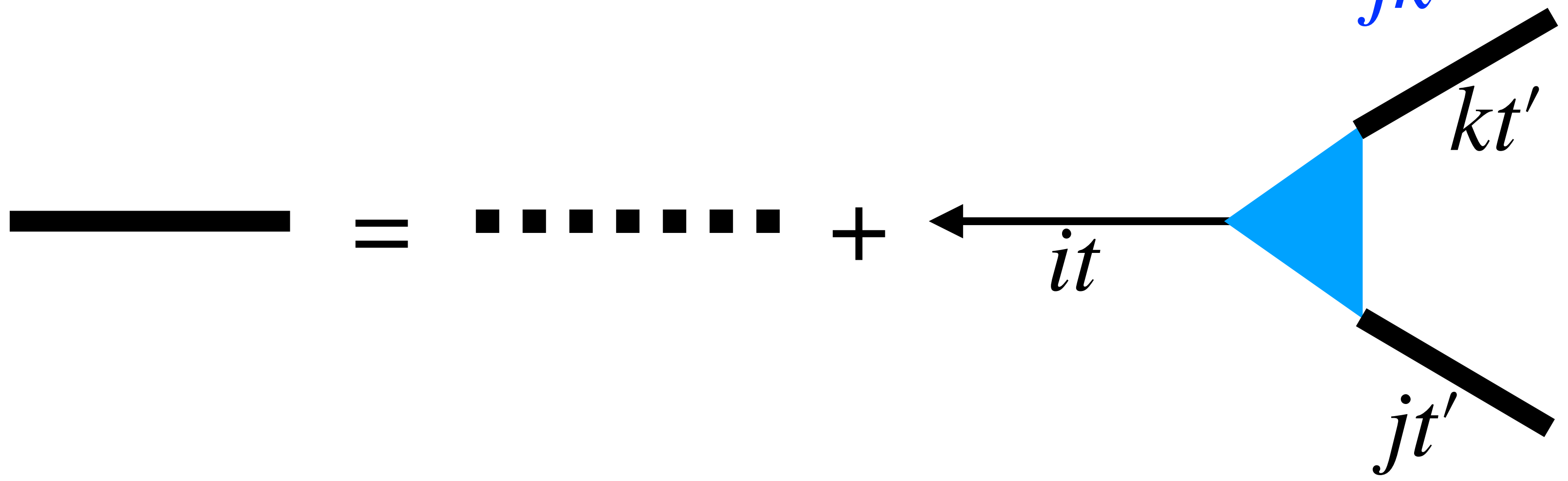
Easier with diagrams:

Represent $S_i(t)$ by a thick solid line 

and $S_i^0(t)$ by a thick dotted line 

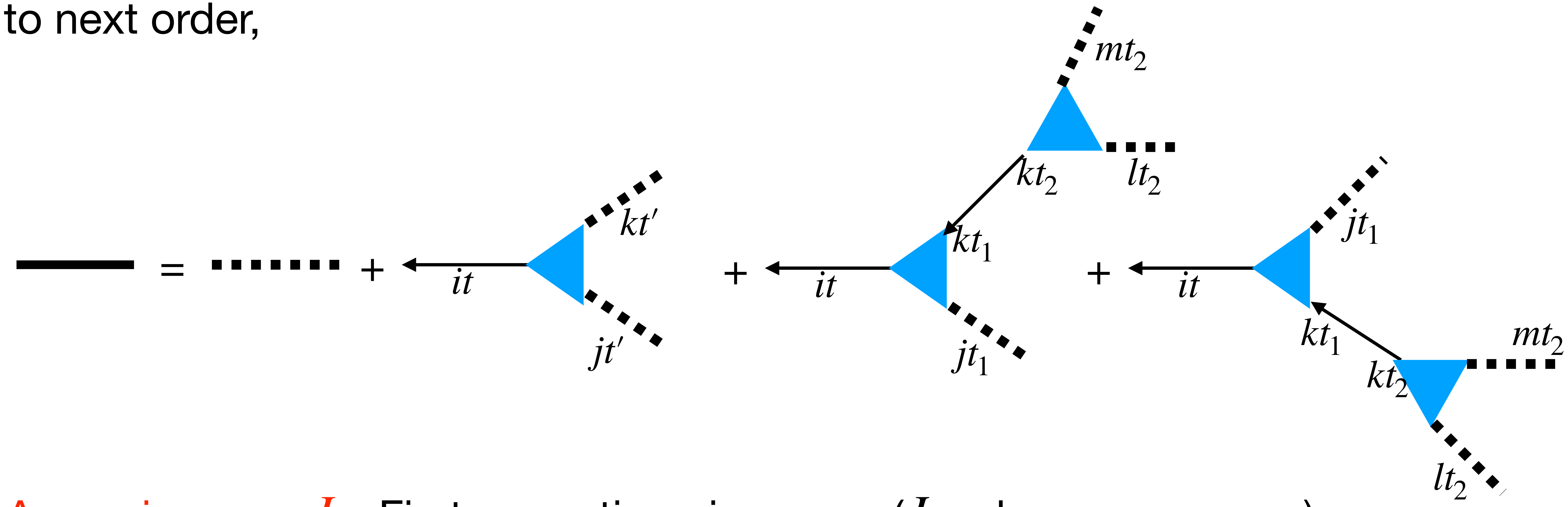
Represent $G_0(t - t')$ by a thin directed line from t' to t: 

Equation of motion: $S_i(t) = S_i^0 + \frac{1}{2}\int_{t'} G_0(t - t') \sum_{jk} J_{ijk} S_j(t') S_k(t')$



Iterate:

to next order,



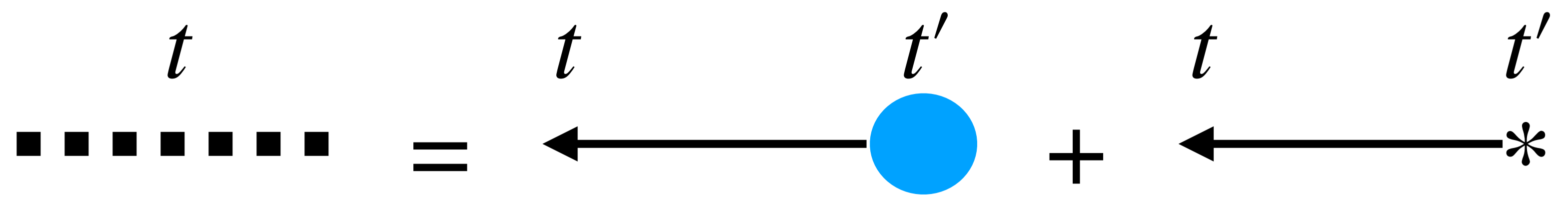
Averaging over Js: First correction gives zero (J_{ijk}s have zero mean)

in 2nd and 3rd (which are the same), require $\langle ij \rangle = \langle ml \rangle$

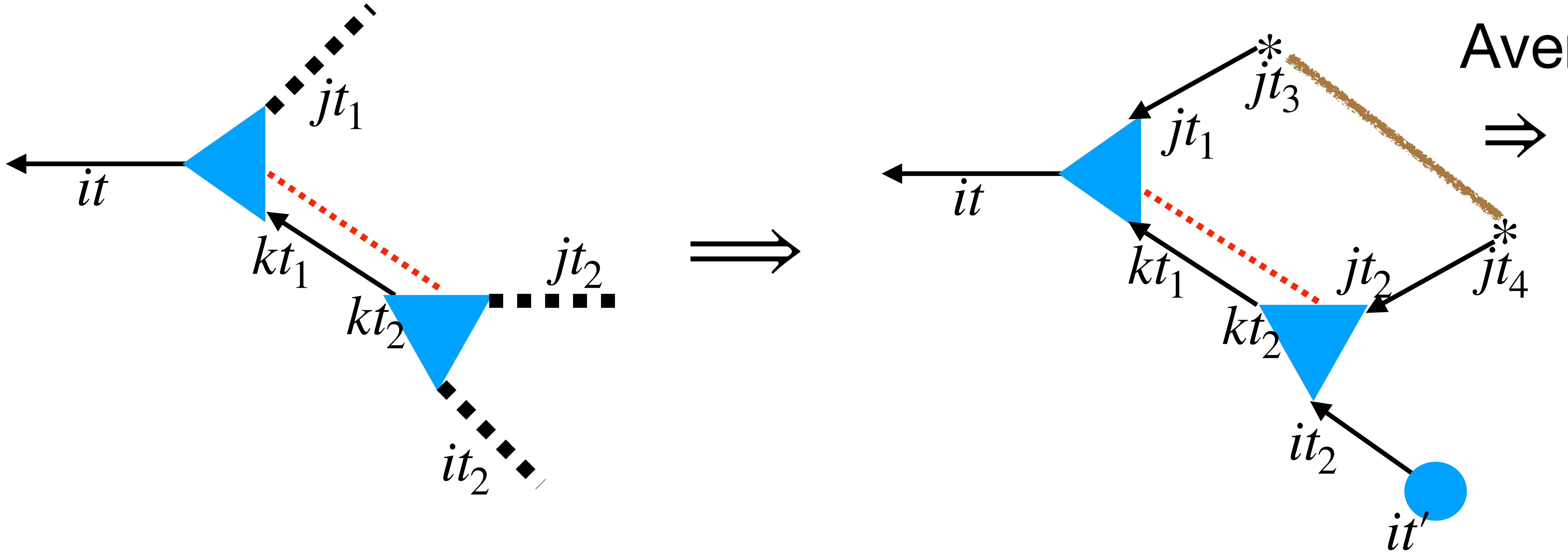
(different J_{ijk}s are independent)

averaging over noise:

$$S_i^0(t) = \int_{t'} G_0(t - t')[h(t') + \xi(t')],$$



where the dot means $h(t')$ and the asterisk means $\xi(t')$



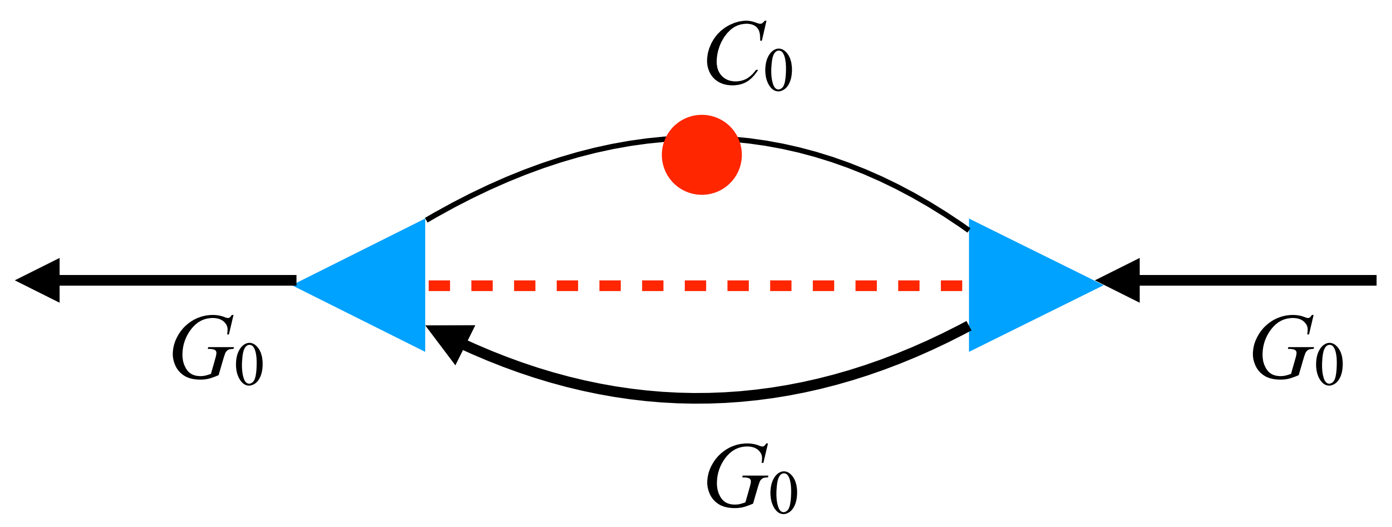
Averaging over noise

$\Rightarrow 2T\delta(t_3 - t_4)$

or, finally,

Using $C_0(\omega) = G_0(\omega) \cdot 2T \cdot G(-\omega)$ or $C_0(t_1 - t_2) = 2T \int dt_1 G_0(t_1 - t_3) G_0(t_2 - t_3)$

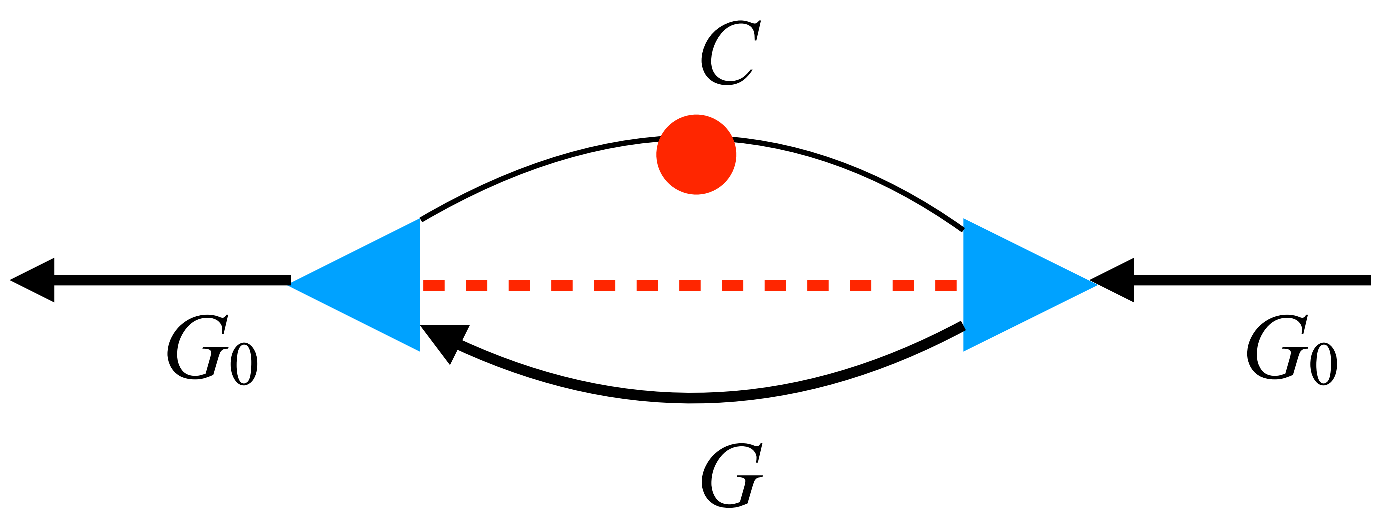
the graph becomes

$G_0 \Sigma_0 G_0 =$ 

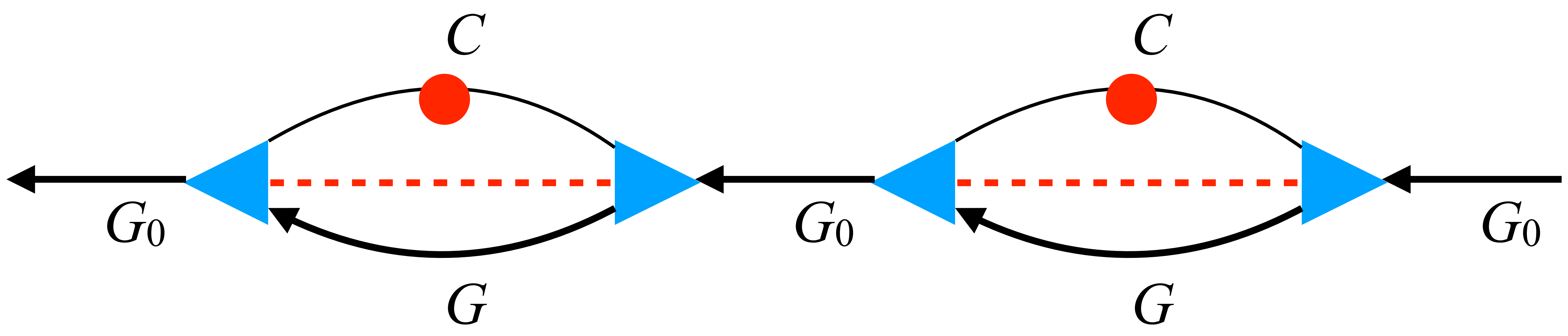
Summing all the terms:

These insertions can also be made on the internal lines:

$$G_0 \Sigma G_0 =$$



and they can be repeated arbitrarily many times:



etc.

Finally,

These are the only graphs that survive the averaging over J and ξ (for $N \to \infty$) !

so (Dyson equation) $$G(\omega) = G_0(\omega) + G_0(\omega)\Sigma(\omega)G(\omega) \implies G(\omega) = \frac{1}{-\mathrm{i}\omega + r - \Sigma(\omega)}$$

with $$\Sigma(\omega) = 3J^2 \int \frac{d\omega'}{2\pi} G(\omega - \omega')C(\omega')$$

or, in time domain, $$G(t-t') = G_0(t-t') + \int dt_1 dt_2 G_0(t-t_1)\Sigma(t_1-t_2)G(t_2-t')$$

with $$\Sigma(t-t') = 3J^2 G(t-t')C(t-t')$$

General p:

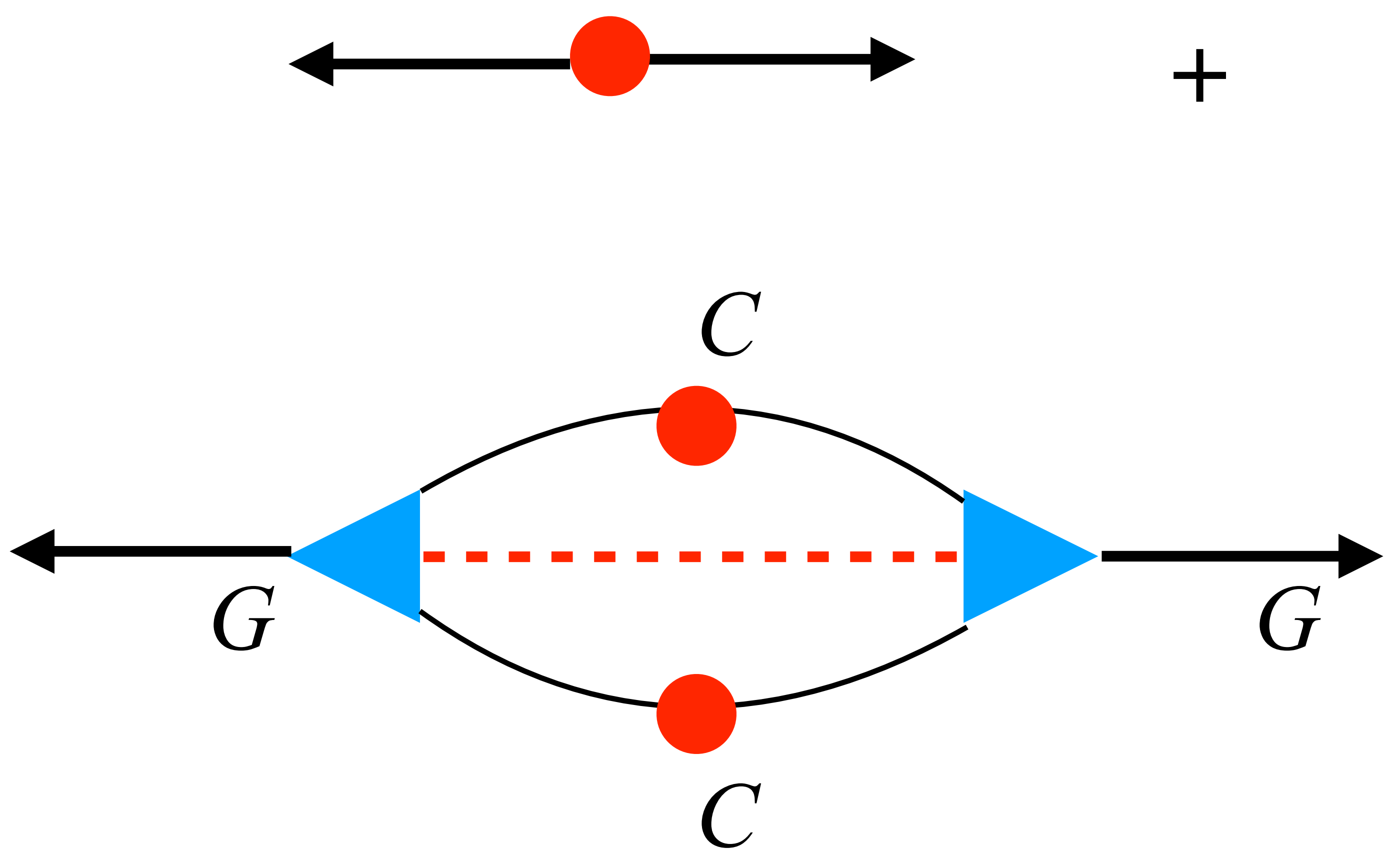
Just add extra factors of C:

$$\Sigma(t-t') = \frac{p(p-1)}{2} J^2 C^{p-2}(t-t') G(t-t')$$

but what is C?

We could get it from the fluctuation-dissipation theorem, but here it is in diagrams, (not proved here):

$p = 3$:



or (general p): $C(t-t') = 2T\int dt_1 G(t-t_1)G(t'-t_1) + \frac{pJ^2}{2}\int dt_1 dt_2 G(t-t_1)C^{p-1}(t_1-t_2)G(t'-t_2)$

Note: different factor from that in Σ

Summary:

As for the SK model, after averaging the problem is reduced exactly to a single-site self-consistent problem with a a retarded self-interaction and an effective (non-white) noise. But now the self-interaction is

$$\Sigma(t-t') = \frac{p(p-1)J^2}{2} G(t-t') C^{p-2}(t-t')$$

and the noise variance is

$$2T\delta(t-t') + \frac{pJ^2}{2} C^{p-1}(t-t')$$

Looking for a spin glass state:

take $t \to \infty, \;\; t' \to -\infty$ in the equation for C, assuming it has a constant piece:

$$q = \frac{pJ^2}{2} q^{p-1} \int dt_1 G(t - t_1) \cdot \int dt_2 G(t' - t_2) = \frac{pJ^2}{2} q^{p-1} (G(\omega = 0))^2$$

static response function measures fluctuations:

$$G(\omega = 0) = \frac{\langle (S_i - \langle S_i \rangle)^2 \rangle}{T} = \frac{1 - q}{T}$$

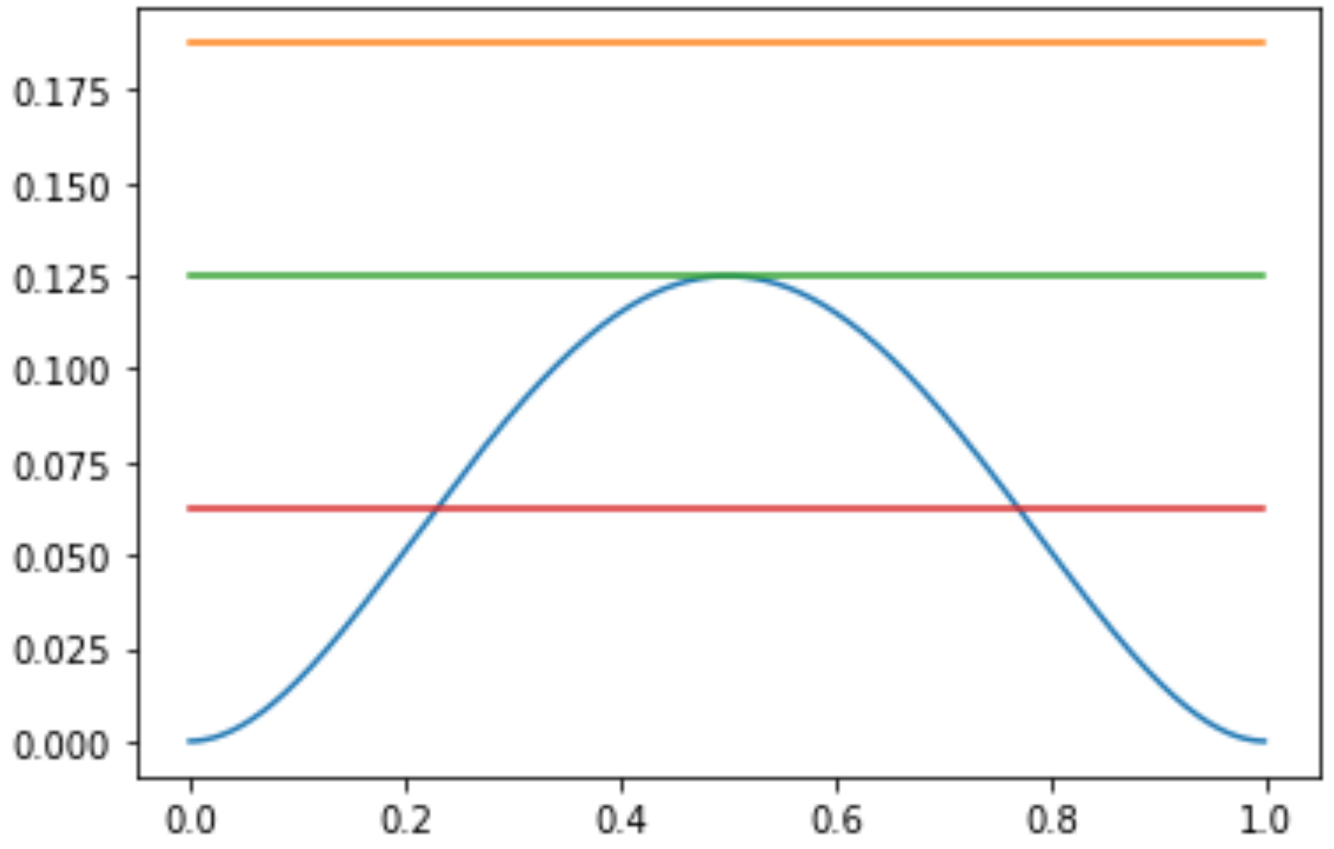
so we have

$$q = \frac{pJ^2}{2T^2} q^{p-1} (1 - q)^2$$

$p = 4, \;\; J = 1$:

$$2q^2(1 - q)^2 = T^2$$

(or $q = 0$)



$T > T_g$

$T = T_g$

$T < T_g$

Stability analysis

To analyse stability, look at the low-frequency limit of $G(\omega)$, as we did for the SK model:

$$\tau(\omega) \equiv \mathrm{i}\frac{\partial G^{-1}(\omega)}{\partial \omega} = 1 - \mathrm{i}\frac{\partial \Sigma(\omega)}{\partial \omega}$$

$\Sigma \propto C^{p-2}G$ has a part proportional to q^{p-2}:

$$\Sigma(\omega) = \Sigma_0(\omega) + \Sigma_1(\omega) = \frac{p(p-1)J^2}{2}q^{p-2}G(\omega) + \Sigma_1(\omega)$$

But this is just like the calculation we did for the SK model, where we had

$$\Sigma(\omega) = J^2 G(\omega) + \Sigma_1(\omega)$$

(for a different Σ_1, but the Σ_1s don't matter except for a constant factor) so we have

$$\tau(\omega) = \frac{1 - \mathrm{i}\partial\Sigma_1/\partial\omega}{1 - \frac{p(p-1)J^2}{2}q^{p-2}G^2(\omega)}$$

marginal stability condition

The stability limit is when

$$1 - \frac{p(p-1)J^2}{2T^2} q^{p-2}(1-q)^2 \to 0$$

But from the equation we got for q,

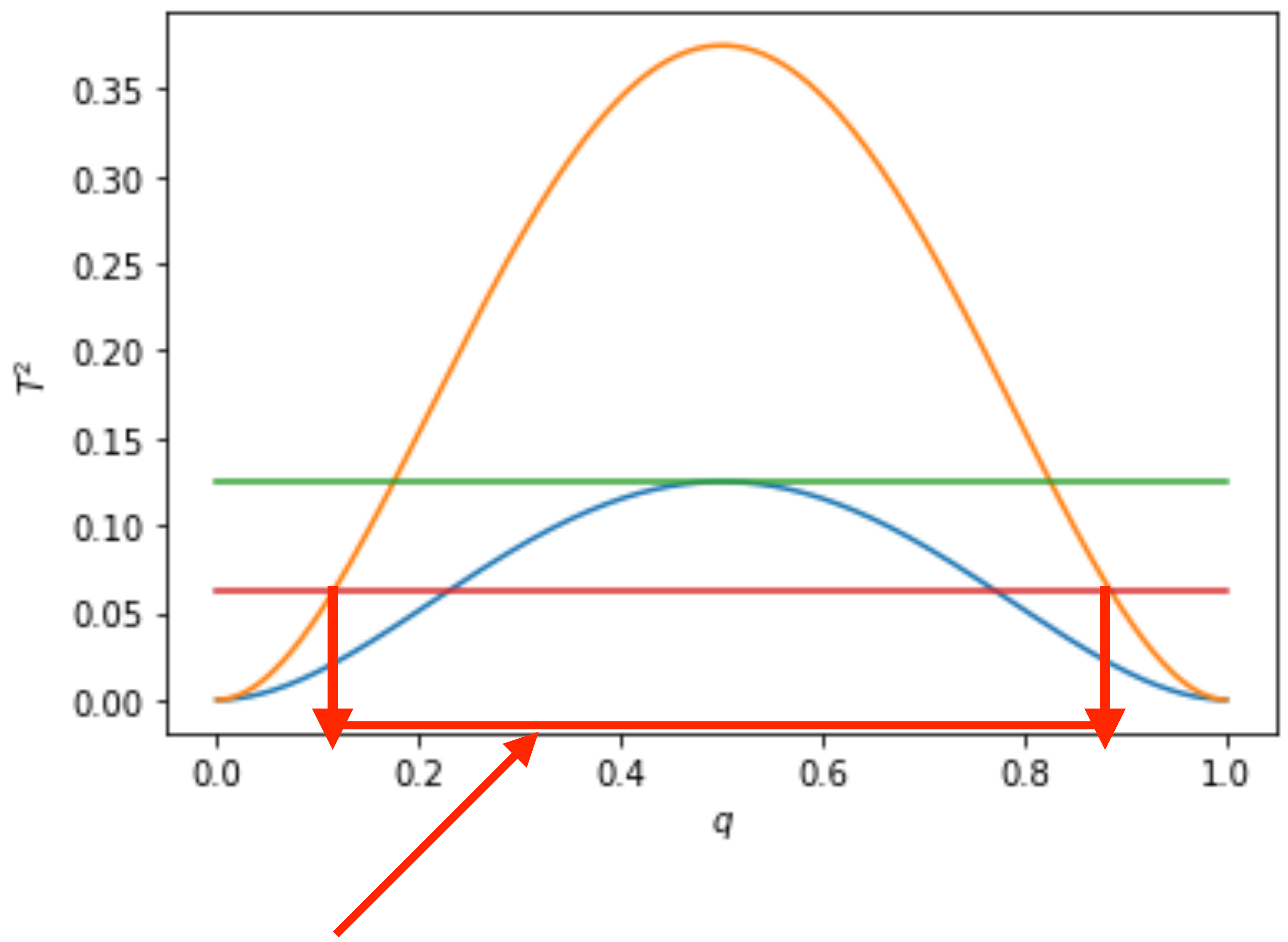
$$q = \frac{pJ^2}{2T^2} q^{p-1}(1-q)^2$$

so

$$1 - \frac{p(p-1)J^2}{2T^2} q^{p-2}(1-q)^2 = 1 - (p-1) = 2 - p < 0$$

i.e., the solution we found for q is always unstable.

(for $p = 4$:)



Solutions in this range forbidden at this T

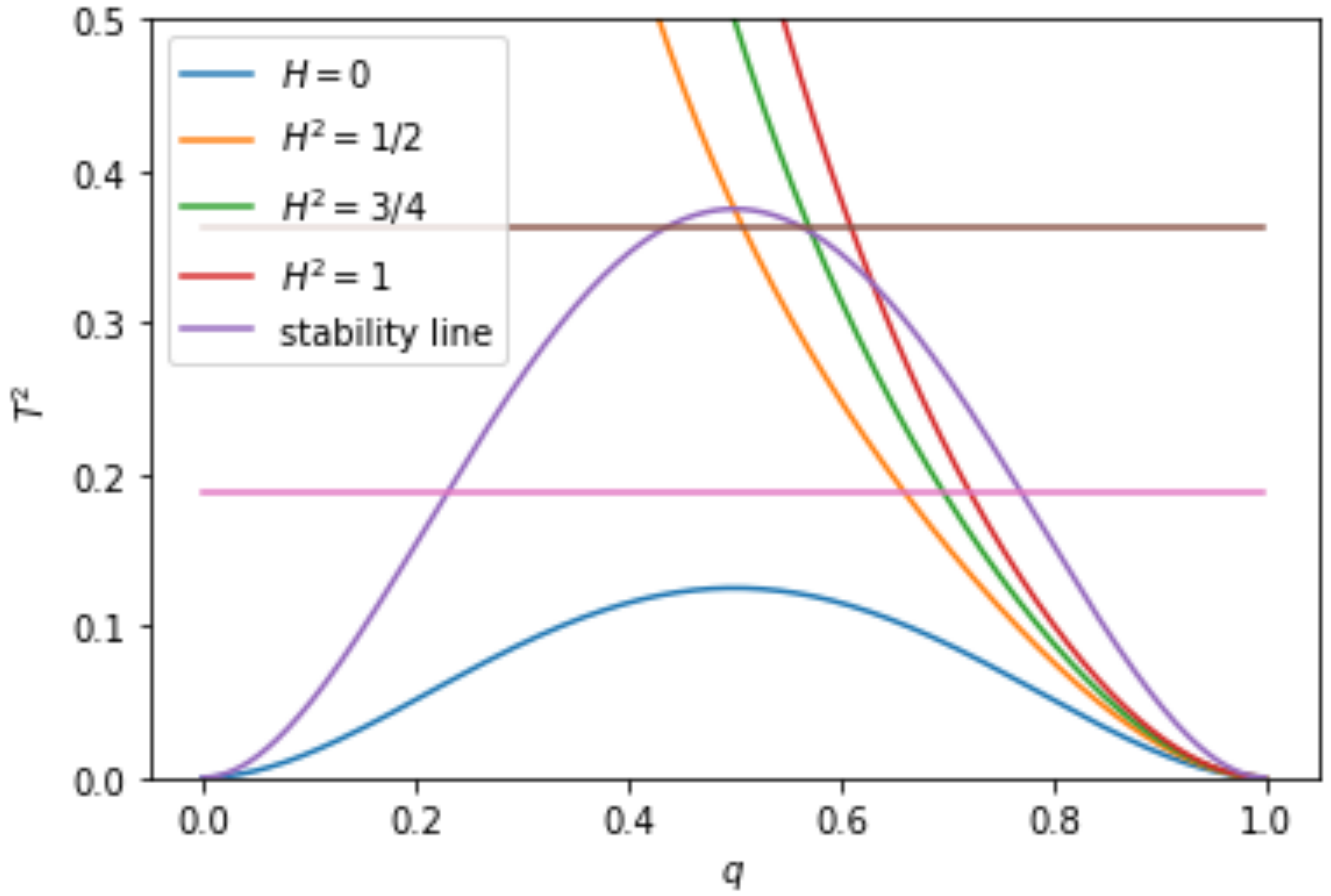
Add an external field:

Equation for q gets a new term: $$q = \frac{pJ^2}{2T^2} q^{p-1}(1-q)^2 + H^2 \left(\frac{(1-q)}{T} \right)^2$$

$p = 4,\ \ J = 1$:

$$2q^2(1-q)^2 + \frac{H^2(1-q)^2}{2q} = T^2$$

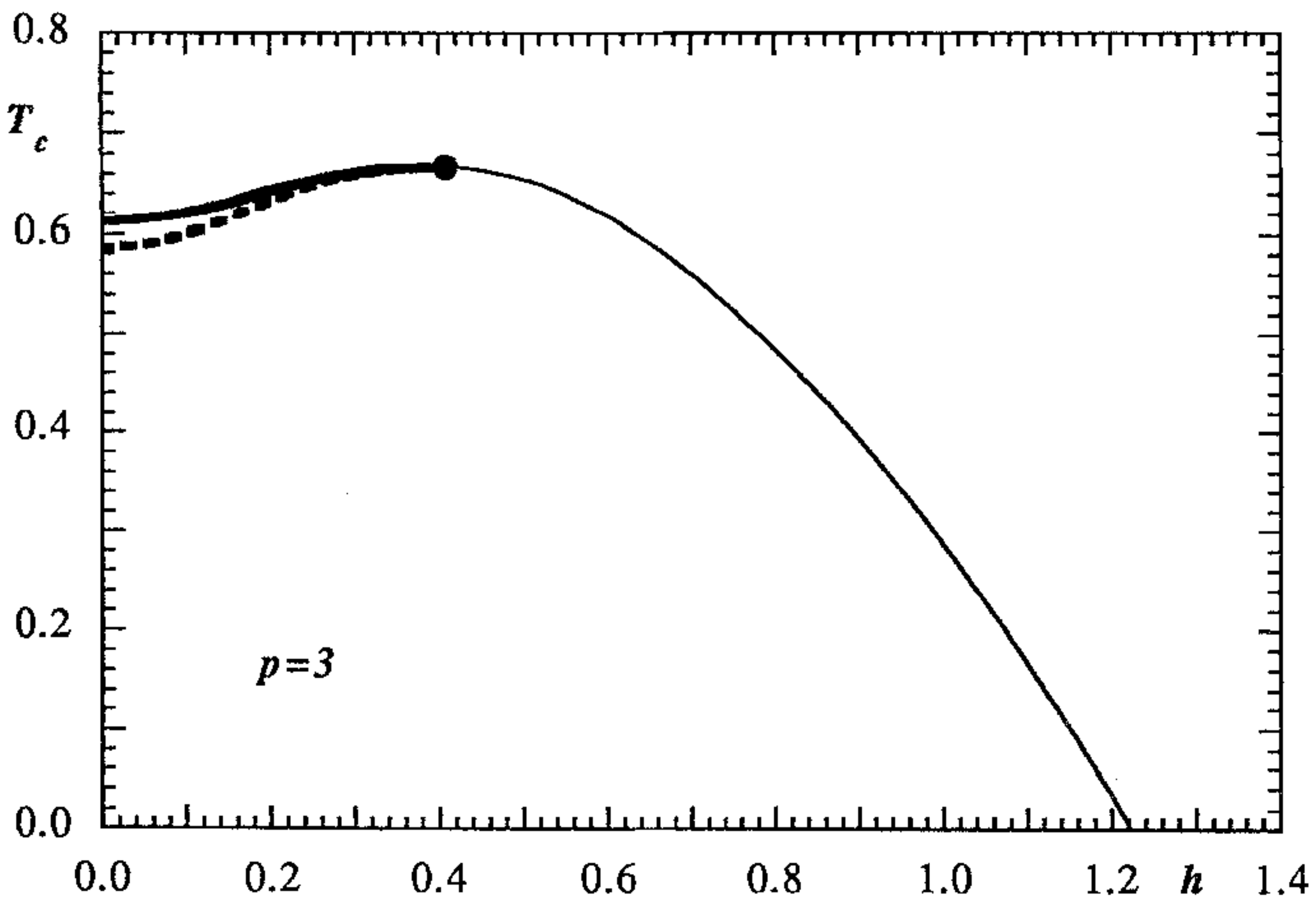
Can get stable solutions for big enough H



"AT line"

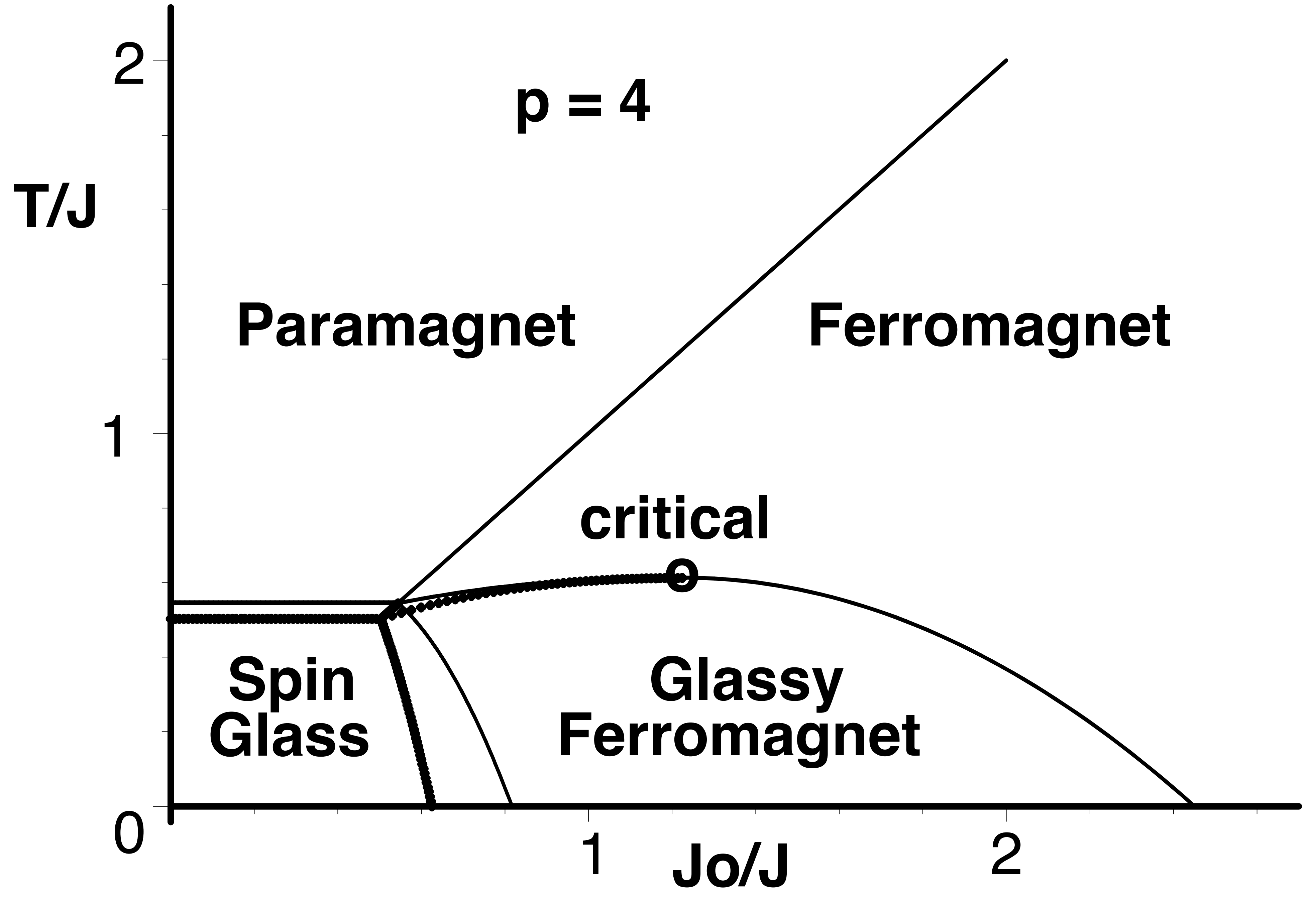
Can plot limit of stable solutions in H-T plane:
(Analog for this problem of the Almeida-Thouless line we saw for the SK model, note very different shape)

$p = 3:$



Crisanti, Horner & Sommers, 1993

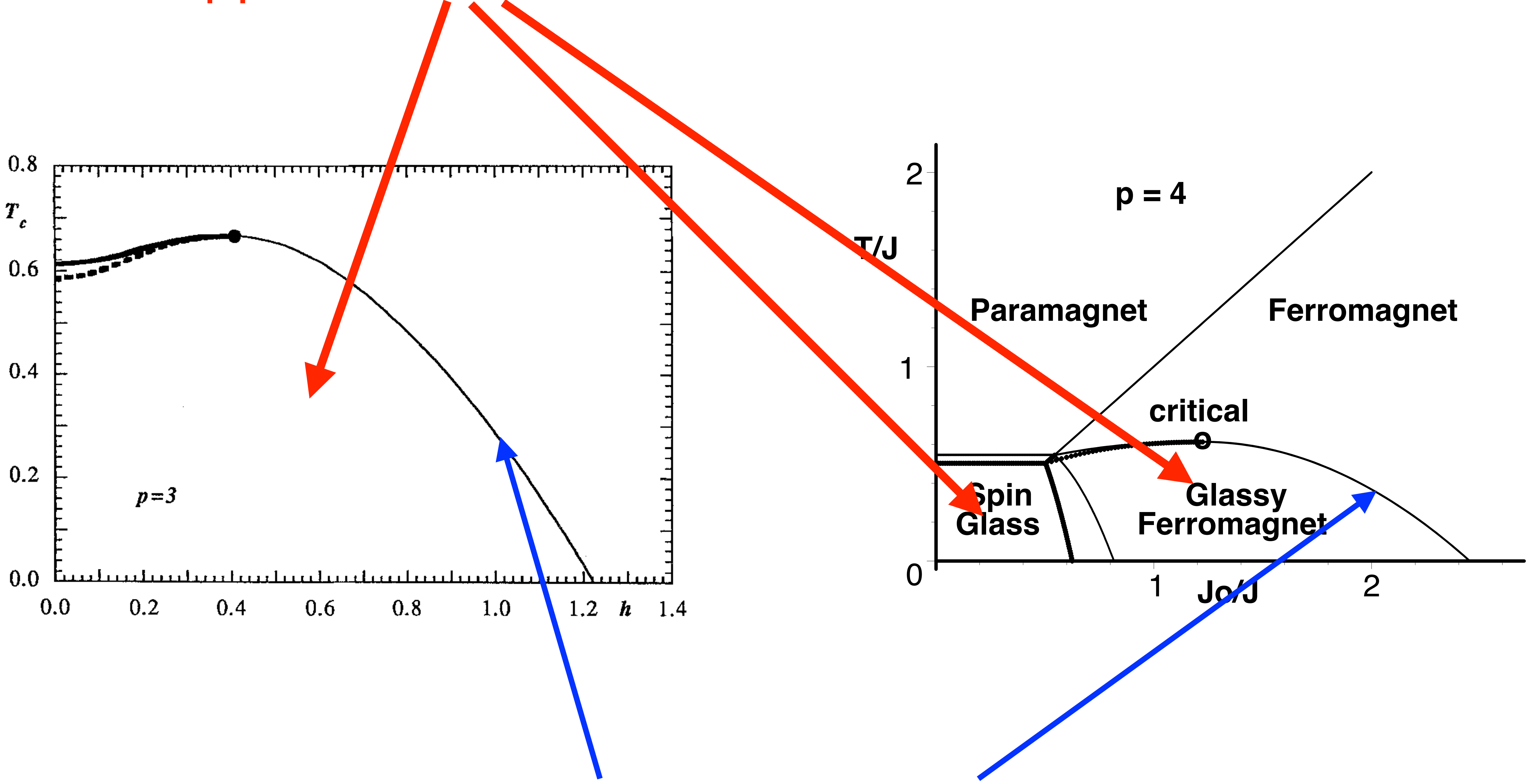
With a ferromagnetic interaction:



JH, D Sherrington and T Niewenhuizen, 1999

Spin glass phase?

What happens here?



Critical slowing down like in SK model approaching these “AT lines” from above

Aging solution

(Cugliandolo and Kurchan, 1993)

Summary: System starts at $t = 0$ in a random configuration. Consider dependence of $G(t, t')$ and $C(t, t')$ on t', the "age" of the system in two time ranges:

(1) $t - t' \ll t'$: expect stationarity, FDT, i.e., just the theory so far.

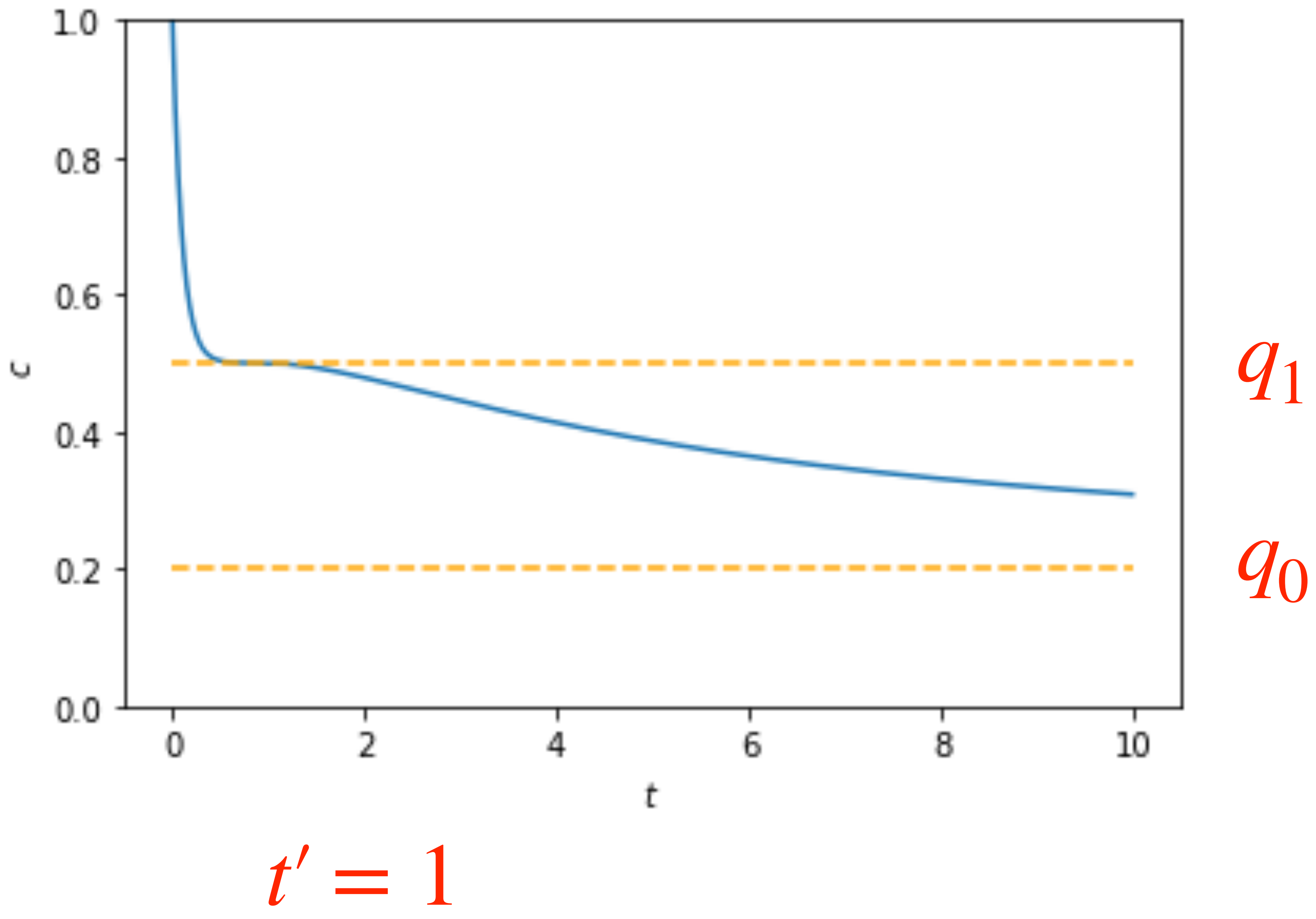
(2) $t - t' = \mathrm{O}(t')$: not stationary, expect dependence on t/t': $C(t, t') = \mathscr{C}(t'/t)$

What about G? $\quad \partial_{t'} C(t, t') = \dfrac{1}{t} \partial_{t'} \mathscr{C}(t'/t)$

so if we define $\quad G(t, t') = \dfrac{1}{t} \mathscr{G}(t'/t)$, the FDT would be $\mathscr{C}'(t'/t) = T\mathscr{G}(t'/t)$

The miracle: If assume a modified FDT, $\quad x\mathscr{C}'(t'/t) = T\mathscr{G}(t'/t), \quad x < 1$,
The equations for $\mathscr{C}$ and $\mathscr{G}$ simplify to a single equation (just as the short-time equations did with the standard FDT. (Use marginal stability condition to fix x.)

Generic shape of $C(t)$



For $H = J_0 = 0$,
$q_0 = 0$