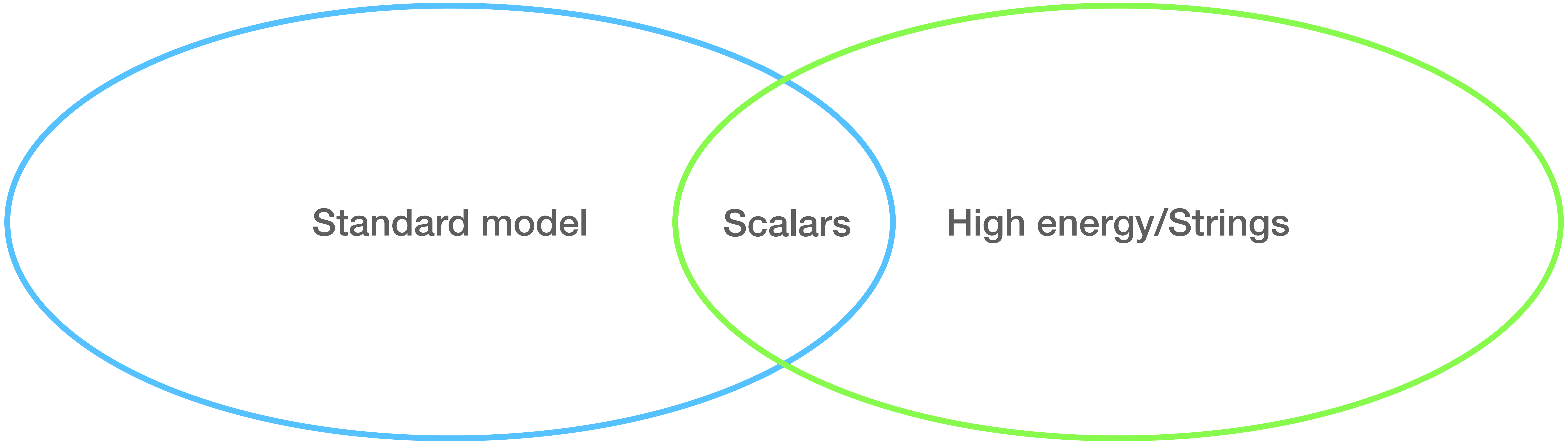
Quantum scalar fields in D=4

What have we missed?

Bo Sundborg, 29th Oct 2024 — 33rd Nordic Network Meeting on “Strings, Fields and Branes”

What is the role of scalars?
Who ordered them?



Problems, puzzles and potential

Problems

Aizenman and Duminil-Copin, ’21
Fröhlich ’82

- Proper (not EFT) scalar $\lambda\phi^4$ theory has been proven to be trivial in D = 4.
- The coupling λ diverges in the UV (Landau pole).
- Scalar theories are UV sensitive in EFT — effective field theory.

Problems

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Yet, scalars are essential in the Standard Model and in string theory.

Puzzles

- Renormalisation group (RG) flow appears to be a gradient flow.
- This seem to hold up to 6 loops. **Pannell and Stergiou ’24**
- Nobody seems to know why.
- It has recently been claimed (again) that for many, *N*, scalars, $\lambda\phi^4$ theory is *not* trivial. **Romatschke ’23**

Puzzles

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Pannell and Stergiou '24

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Romatschke '23

We are missing something about scalars or about QFT.

Potential

- A non-perturbative RG may address triviality (and UV Landau poles).
  - Is large N (number of scalars) limits the right path?
  - Does QFT in $D = 4 - \epsilon$ indicate a difference between small N and large N?
- Complex CFTs may evade UV sensitivity.
- RG flow geometry may explain the gradient flow.

Potential

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- Complex CFTs may evade UV sensitivity.
- RG flow geometry may explain the gradient flow.

We can learn something.

RG flows and scalars

Technical note:

- For simplicity, I will focus on classically scale invariant theories.

- This involves fine tuning, or fitting into a bigger framework with conformal invariance, for example.

Renormalisation overview

Action $S(\lambda_I) \to S(\lambda_I(\mu))$

λ_I are couplings.

Perturbative sums diverge due to integrals over scale.

Typically μ is replaced with a dynamical scale Λ.

For example, $\lambda(\Lambda_c) = c$.

The definition of the theory without Λ was too naive.

Integrals over scales are not a problem if we study scale dependence

Technical challenge:
Introduce a scale without ruining Poincaré and gauge symmetries…

Renormalisation: physical scale dependence

Scale invariant theories

The scale μ is introduced for technical reasons, but drops out in observables.

Non-scale invariant theories

The scale μ still drops out of observables, but other scales Λ_c characterise the theory.

$\lambda_I(\mu)$ are not directly observable.

Correlators or amplitudes which depend on $\lambda_I(\mu)$, can probe its scale dependence indirectly.

The number of fields is important

Multi-scalar $\lambda\phi^4$ theory

Multi-scalar λϕ⁴ theory in D = 4 - ε

$$\mathscr{L} = \frac{1}{2}\partial_m\phi_i\partial^m\phi_i - \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \qquad i = 1,\ldots,n$$

$D = 4 - \epsilon$

$$\beta_{ijkl} = \frac{d}{dt}\lambda_{ijkl} = \left(-\varepsilon\lambda_{ijkl}\right) + B\left(\lambda_{ijmn}\lambda_{mnkl} + 2 \text{ permutations }\right)$$

One loop
beta function

$D = 4 - \epsilon?$

Mainly for $d = 3$ phase transitions and general QFT…

Rychkov and Stergiou '19

Osborn and Stergiou '18
Osborn and Stergiou '21
Herzog, Jepsen, Osborn, Oz '24

Brézin, Le Guillou and Zinn-Justin '74
Wallace and Zia '74
Michel '84

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$t = \ln(\mu/\mu_0)$ is the "renormalisation time". t decreases from UV to IR.

μ is the RG scale and μ_0 a reference scale.

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We will take $B = \dfrac{1}{16\pi^2} \to 1.$

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Interactions generate one loop Feynman diagrams

$$\mathscr{L} = \frac{1}{2}\partial_m\phi_i\partial^m\phi_i - \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \qquad i = 1,\ldots,n$$

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One loop
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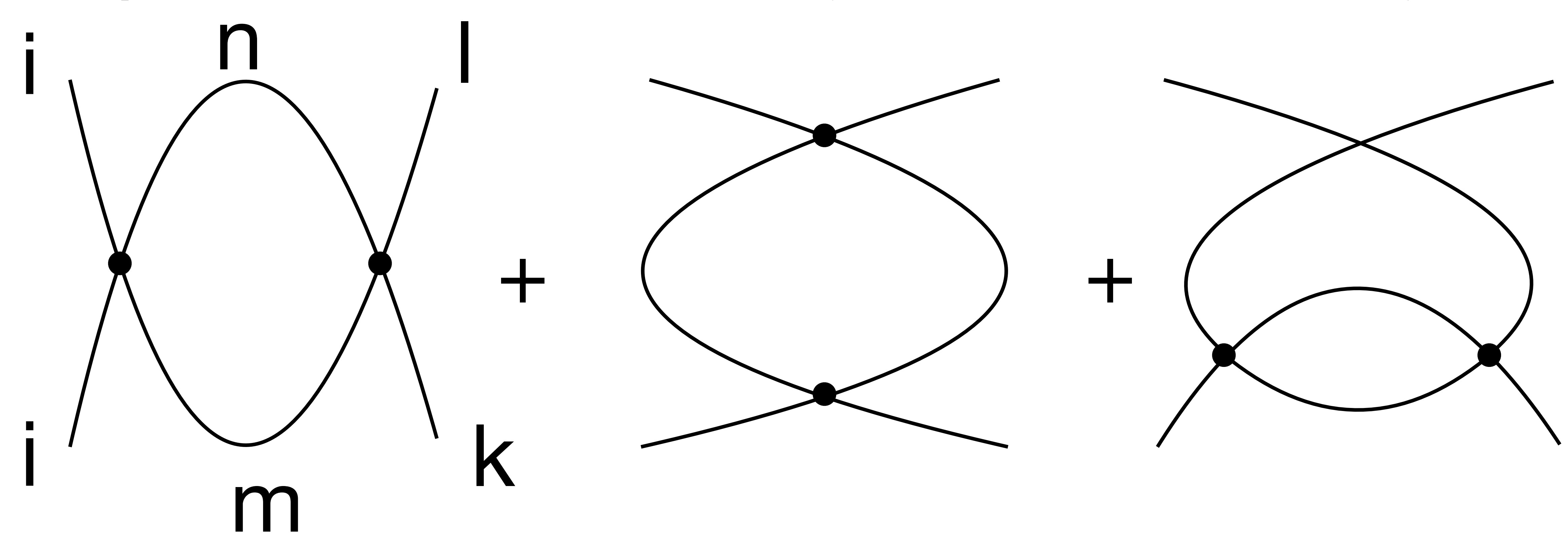
Classical scale dependence in $D = 4 - \epsilon$



Single-scalar intermission

One field intermission, n = 1

Example: scale symmetry breaking

$\mathscr{L} = \frac{1}{2}\partial_m\phi\partial^m\phi - \frac{1}{4!}\lambda\phi^4$

$D = 4$

$$\beta = \frac{d\lambda}{dt} = 3\lambda^2$$

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One field intermission, n = 1

Example: Landau pole

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One loop
beta function

$t = \ln(\mu/\mu_0)$ is the "renormalisation time". t decreases from UV to IR.

μ is the RG scale and μ_0 a reference scale.

$$-\frac{1}{3\lambda} = t - t_0 \qquad \lambda = \frac{1}{3(t_0 - t)}$$

A Landau pole
at $t = t_0 = t_{LP}$

$$\lambda(\mu) = \frac{1}{3\ln\frac{\mu_{LP}}{\mu}}$$

One field intermission, n = 1

Dimensional transmutation and Landau poles

- The RG equation $\frac{d\lambda}{dt} = \beta(\lambda)$.

- Solved at one loop by $\lambda(\mu)$.

- λ varies continuously from 0 to ∞.
  Define a physical scale Λ_c by

- Λ_1 is the scale where $\lambda = 1$, which is a bit arbitrary but appears physical.

$$\lambda(\mu) = \frac{1}{3\ln\frac{\mu_{LP}}{\mu}}$$

$$c = \lambda(\Lambda_c) \qquad \Lambda_c = e^{-1/3c}\mu_{LP}$$

Dimensional transmutation and Landau poles

Summary

We used the one loop approximation $\beta^{(1)}(\lambda)$ and found

- Dimensional transmutation: A physical scale Λ_1 in spite of scale invariance.
- Landau pole: a divergence of λ at a slightly higher scale.

How to view this?

- Dimensional transmutation is under control in perturbation theory.
- The Landau pole signals divergent coupling and one loop is not enough…
  - There are no signs of stable zeros of β which would stop the running to ∞…

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Beyond perturbation theory

Multi-scalar $\lambda\phi^4$ stability

There are two concepts of stability:

1. The stability of the critical point $\phi_i = 0$ in the potential $V_\lambda(\phi)$.

2. The stability of the RG flow determined by $\beta(\lambda)$ in the space of couplings λ_{ijkl}.

The different concepts are related by values of λ_{ijkl}.

Classical stability and the RG flow

Rychkov and Stergiou '19

$$\mathscr{L} = \frac{1}{2}\partial_m\phi_i\partial^m\phi_i - \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \quad i = 1,\ldots,n$$

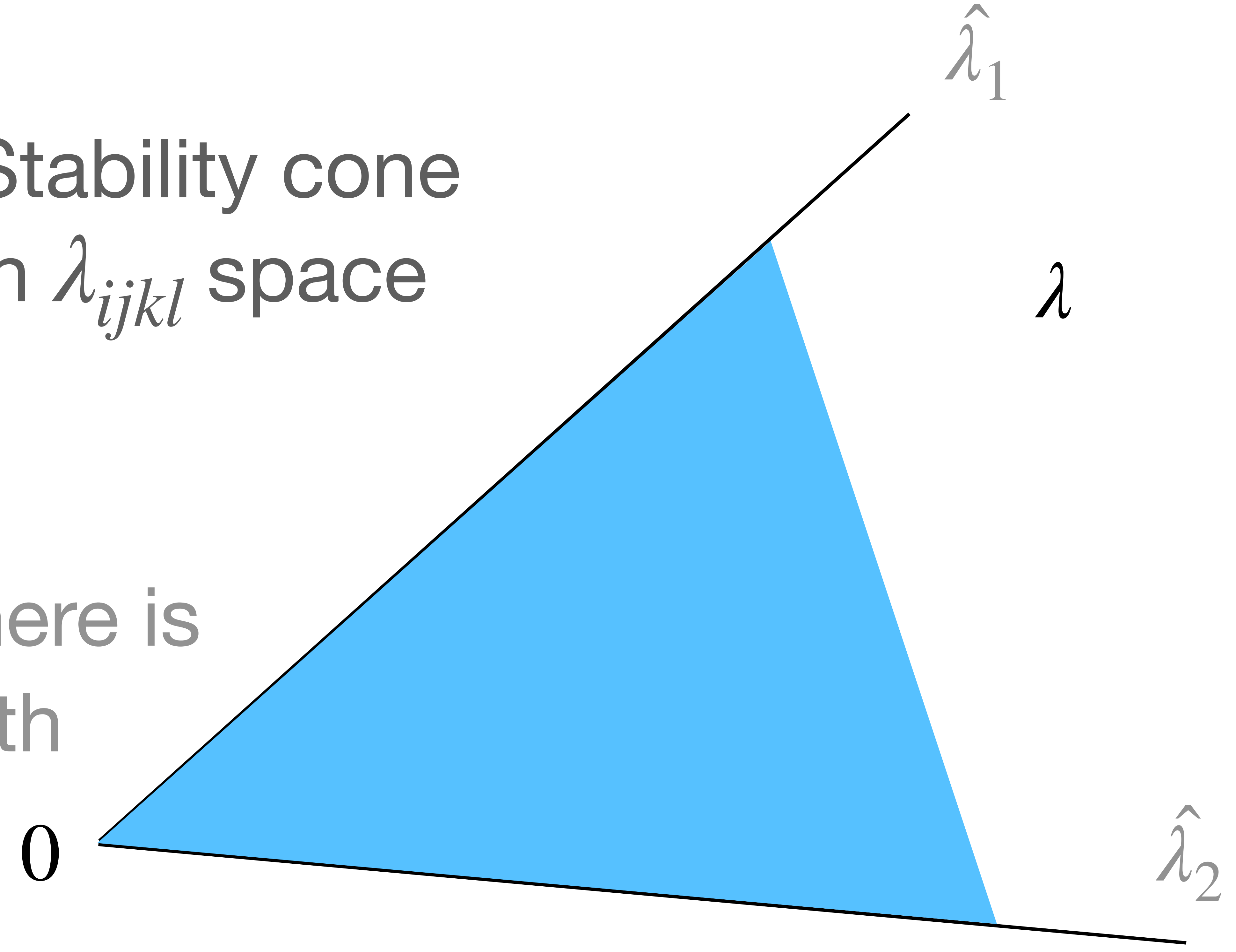
Classical stability at λ_{ijkl} :

$$V(\phi) = \lambda(\phi) = \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \geq 0$$

For all ϕ_i.

For a boundary $\hat{\lambda}$ there is a flat direction $\hat{\phi}$ with $\hat{\lambda}(\hat{\phi}) = 0$

Stability cone in λ_{ijkl} space



Fluctuation driven first order phase transition

Rychkov and Stergiou '19

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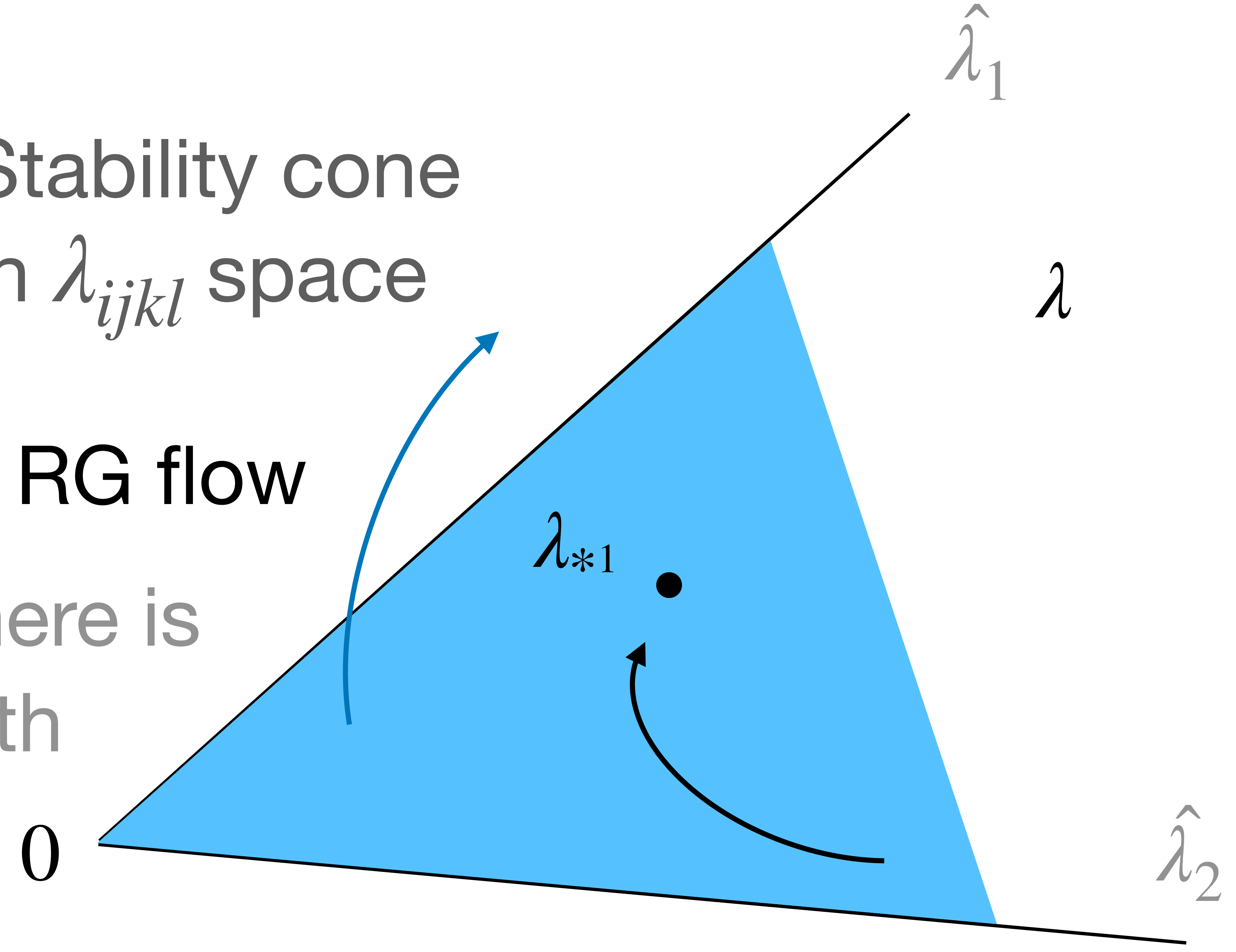

"Fluctuation driven first order phase transition" if stabilised

Coleman-Weinberg mechanism via Gildener-Weinberg in D = 4.

RG flow does not enter stability cone

Rychkov and Stergiou '19

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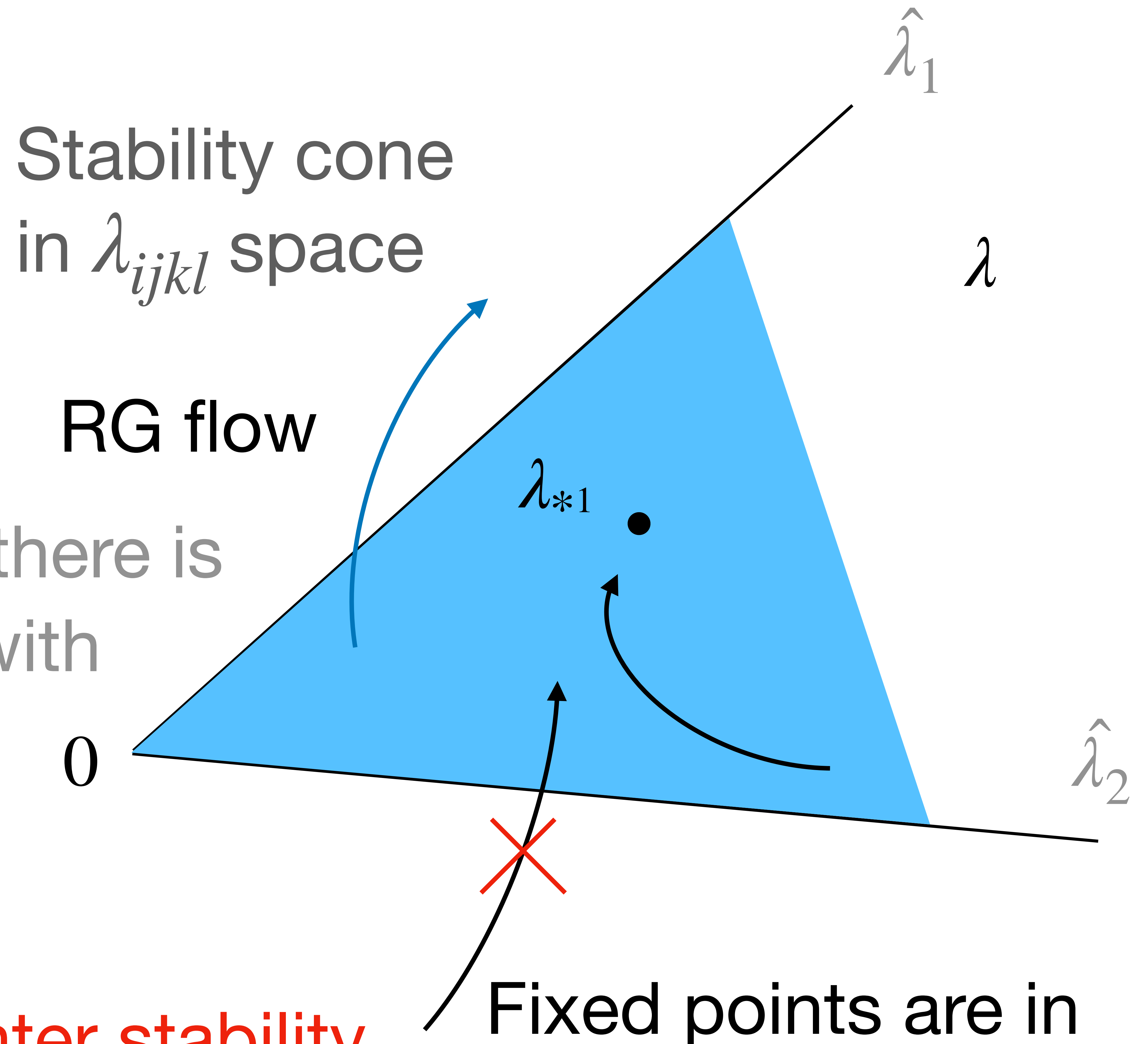

RG flow does not enter stability cone. Proof for $D < 4$.

Fixed points are in the stability cone or its boundary.

The RG flow equation

- Why the fixed point equations are different in $D = 4$ and $D < 4$ ($\epsilon > 0$):

$$\frac{d}{d(-t)}\lambda(\bar{\phi}) = \varepsilon\lambda(\bar{\phi}) - 3V_{ij}V_{ij} \leqslant \varepsilon\lambda(\bar{\phi})$$

$$V_{ij} = \lambda_{ijmn}\bar{\phi}_m\bar{\phi}_n$$

- Dominated by $\lambda(\phi)$ for $\epsilon > 0$, and by $V_{ij}(\phi)$ for $D = 4$.

Comparing D = 4 and D = 4 - ε

Purely scalar $\lambda\phi^4$ theories

- In D = 4 there is classical scale invariance, and there is only a trivial fixed point in perturbation theory.
- Dimensional transmutation yields non-trivial vevs of scalars for RG trajectories reaching the boundary of the stability cone. Vevs in the almost flat directions.

- In D = 4 - ε classical scale symmetry is broken by ε giving non-trivial fixed points. Fixed points bring back scale invariance.
- Fluctuation driven first order phase transitions result from RG flows leaving the stability cone.

Coleman and Weinberg ’73 **Gildener and Weinberg ’76** **Rychkov and Stergiou ’19**

The number of fields is important

Symmetries in multi-scalar theories

Hierarchies of symmetric flows

Rychkov and Stergiou ’19
Michel ’84

$\mathscr{L} = \frac{1}{2}\partial_m\phi_i\partial^m\phi_i - \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \;\; i = 1,\ldots,n$

Consider $O(n)$ transformations in a subgroup $G \subset O(n)$.

Suppose G preserves λ_{ijkl}. Then G also preserves the beta function at λ_{ijkl} and the *flow remains in the space of G invariant* λ_{ijkl}.

A hierarchy of subgroups $G \subset O(n)$ yields a hierarchy of symmetric flows.
Transformations in $O(n)$ not preserving λ_{ijkl} map it to an equivalent λ'_{ijkl}.

Fixed points are characterised by their symmetry groups $G \subset O(n)$.

Symmetries of fixed points, invariants

Rychkov and Stergiou '19
Michel '84

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Subgroups $G \subset O(n)$ may be characterised by their invariants, eg quadratic invariants $A_{ij}\phi_i\phi_j$ and quartic invariants $B_{ijkl}\phi_i\phi_j\phi_k\phi_l$.

The invariant tensors of a given rank form a linear space.

The number of independent four-tensors, I_4, gives the dimension of a G invariant RG flow.

The number of independent two-tensors, I_2, measure the degree of fine tuning required in the action.

Classes of symmetries of fixed points

Rychkov and Stergiou ’19

The subgroups of $O(n)$ depend sensitively on n.

Infinite classes have been studied.

Table 1: Summary of examples of fully interacting fixed points given in text.

Name	N	G	I_4	I_2
$O(N)$	$N \geqslant 1$	$O(N)$	1	1
cubic	$N \geqslant 3$	$(\mathbb{Z}_2)^N \rtimes S_N$	2	1
tetrahedral	$N \geqslant 4$	$S_{N+1} \times \mathbb{Z}_2$	2	1
bifundamental	$N = mn$ $(m,n \geqslant 2, R_{mn} \geqslant 0)$	$O(m) \times O(n)/\mathbb{Z}_2$	2	1
“MN"	$N = mn$ $(m,n \geqslant 2, m \neq 4)$	$O(m)^n \rtimes S_n$	2	1
tetragonal	$N = 2n \geqslant 4$	$(D_8)^n \rtimes S_n$	3	1
Michel	$N = r_1 \cdots r_k$	$G_{r_1 \ldots r_k}$	$k+1$	1
biconical[7]	$N = m_1 + m_2$	$O(m_1) \times O(m_2)$	3	2

I_4 gives the dimension of a G invariant RG flow.

I_2 measures the fine tuning required in the action.

Classes of symmetries: universality

Rychkov and Stergiou ’19

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We can *represent any symmetry acting linearly* on real scalars in $O(n)$.

Example: A subgroup of $O(m) \times O(m) \subset O(m^2)$ transforms m^2 scalars as matrices.

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Rychkov and Stergiou '19

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I_4 gives the dimension of a G invariant RG flow.

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RS classes of symmetries: maximal symmetry

Rychkov and Stergiou '19

The maximal subgroup of $O(n)$ is $O(n)$.

$$\mathscr{L} = \frac{1}{2}\partial_m\phi_i\partial^m\phi_i - V(\phi)$$

$$V(\phi) = \frac{1}{2}m^2\phi_i\phi_i + \frac{1}{4!}\lambda(\phi_i\phi_i)^2$$

$$\lambda_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{ij} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\lambda$$

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$I_4 = 1$ gives a 1-dimensional G invariant RG flow.

$I_2 = 1$ requires fine tuning the coefficient of $\phi_i\phi_i$.

Summary, so far

- The RG flow encodes the change of the action with scale.
- The classically stable potentials $V_\lambda(\phi)$ lie in a "stability cone".
- Couplings may flow out of the stability cone in the IR.
- In $D < 4$, no RG flows enter the stability cone.
- All fixed points are inside the stability cone, or on its boundary.
- The RG flow is organised hierarchically by symmetry subgroups.

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Fixed point structure in multi-scalar theories

Fixed points

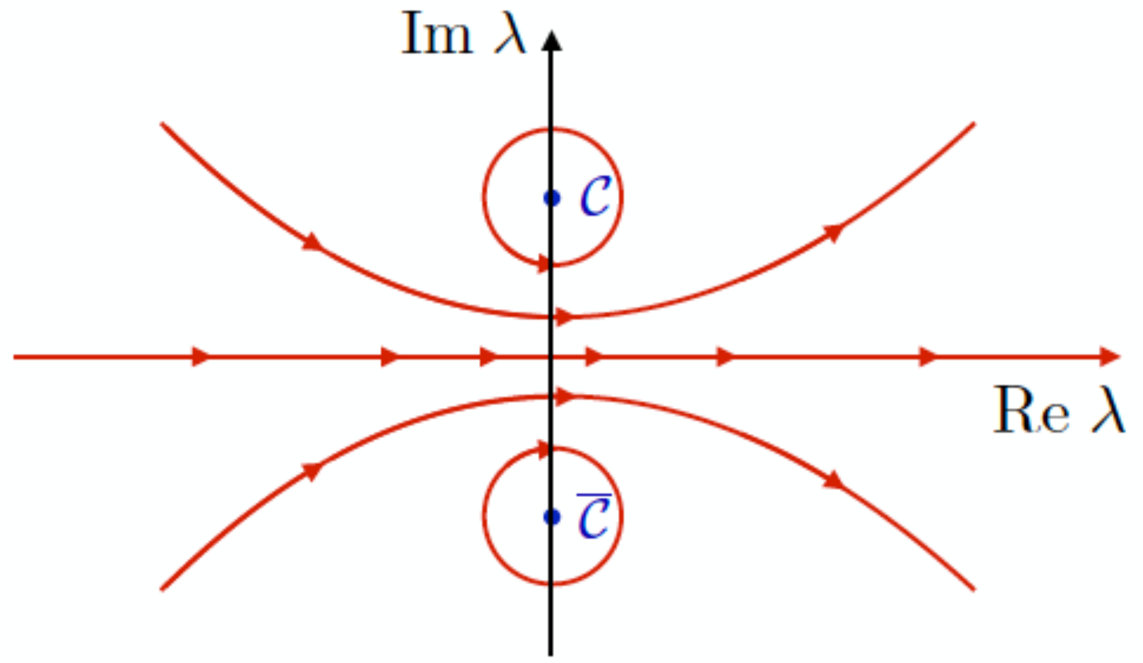
- Real fixed points are scale invariants
- The corresponding theories are CFTs
- Complex conjugate pairs of fixed points represent complex CFTs
- Real couplings between the complex fixed points evolve slowly, “walking” rather than running.
  - A large scale ratio corresponds to a $\delta\lambda$ of order unity
  - Complex fixed points come with hierarchies of scales!

Gorbenko, Rychkov and Zan ’18

Fixed points can collide

and move out into the complex plane

- $\beta(\lambda) = \dfrac{d\lambda}{dt} = -y - \lambda^2 + O\left(\lambda^3\right)$



- $\Lambda_{\mathrm{UV}}/\Lambda_{\mathrm{IR}} \sim e^{\Delta t} \sim \exp(\pi/\sqrt{y})$ for $|\lambda| \sim 1$

- A natural large hierarchy is generated from a small y.

Bounds for real fixed points

Rychkov and Stergiou '19

Fixed points λ_* in $D = 4 - \epsilon$, governed by roots of second order polynomial:

- Fixed points $\lambda_{*ijkl}\lambda_{*ijkl} \leq \frac{\epsilon^2}{8} n$
- Lower bound $A_* \geq -\frac{\epsilon^3}{48} n$
- Bounds are saturated when two fixed points coincide. There is then a marginal operator.

Extremal CFTs

Rychkov and Stergiou '19

Consider an *extremal* fixed point CFT *saturating the bounds.* Since A always decreases towards the IR, no flow away from this fixed point reaches another fixed point. Deformation by relevant operators makes no difference:

If any flow leaves, it goes outside the stability cone or to strong coupling.

Fixed points λ_* in $D = 4 - \epsilon$ are governed by roots of second order polynomial.

- The bound on the roots is saturated at an extremal location of the roots.
- How can roots of a second order polynomial be extreme? They coincide.
- Polynomial algebra yields a direction of coincidence and *a marginal operator*.

Extremal CFTs, *n* and D

Rychkov and Stergiou '19

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If any flow leaves, it goes outside the stability cone or to strong coupling.

- Extremal fixed points are reasonable guesses for the vacuum of a theory, if the vacuum is determined by one-loop effects.
- For $n < 4$ the general form of the bounds cannot be saturated. For $n < 4$, the extremal CFTs are not maximally symmetric.
- Perhaps we can learn about $D = 4$ limit vacua by taking limits of extremal fixed points? For $n > 4$, we would then expect non-trivially broken symmetry.

A potential for the flow

Gradient flow at one loop

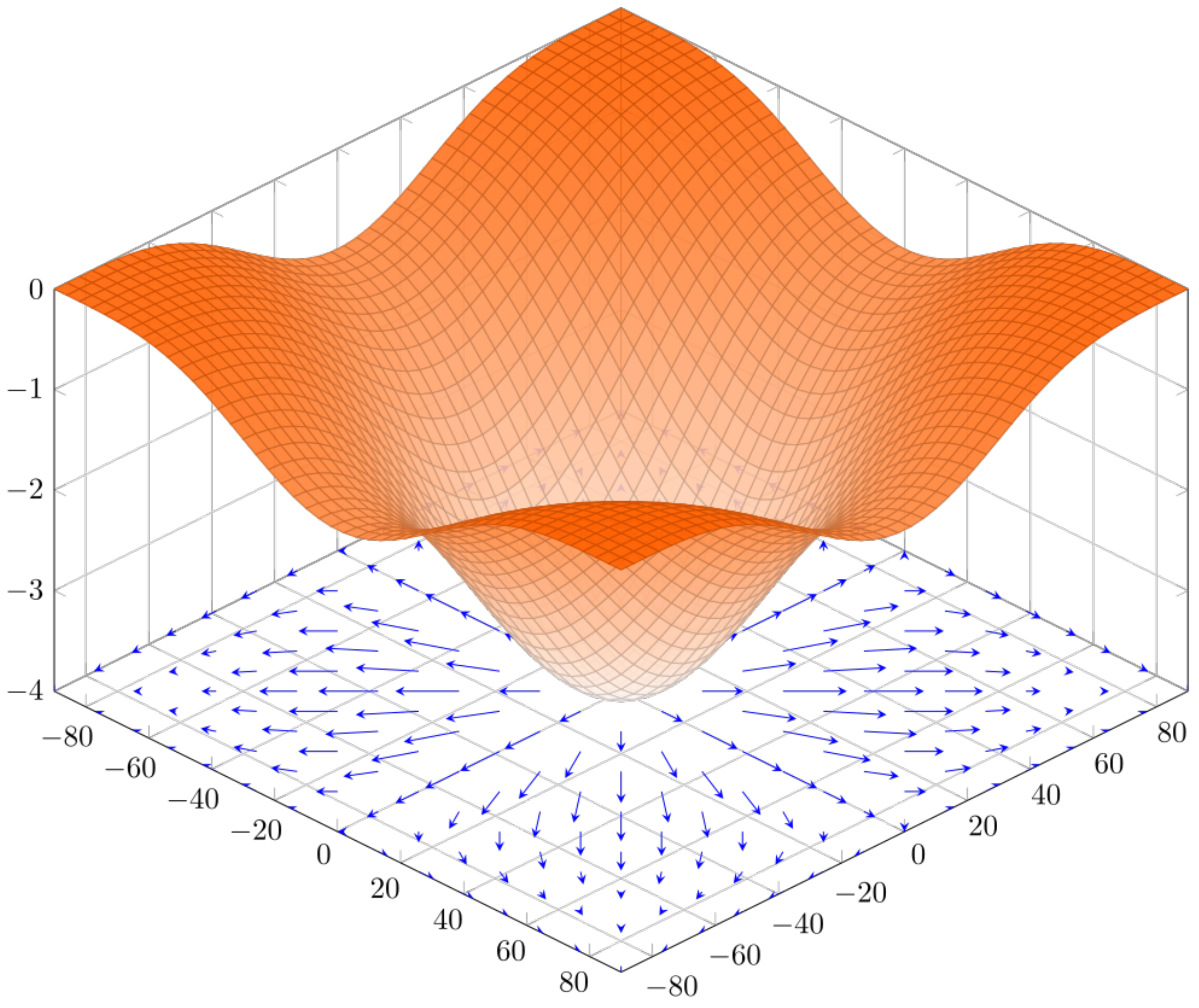
Rychkov and Stergiou '19

Wallace and Zia '74,'75

- Beta function is gradient of $A(\lambda)$: $\delta A(\lambda) = \beta_{ijkl}(\lambda)\delta\lambda_{ijkl}$
- Fixed points λ_* with scale invariance: $0 = \beta_{ijkl}(\lambda_*) = \frac{\delta}{\delta\lambda_{ijkl}} A(\lambda)\Big|_{\lambda=\lambda_*}$

The existence of $A(\lambda)$ to this order, demonstrates monotonicity of RG flow. The RG flow is always in the gradient direction. A decreases in the IR.

$$\frac{d\lambda_{ijkl}}{dt} = \beta_{ijkl}(\lambda) = \frac{\delta A}{\delta\lambda_{ijkl}}$$



Pannell and Stergiou '24

"The $\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l$ RG flow seems to be a gradient flow to six loops."

It stops being a gradient flow at 6 loops or L loops.

It is a gradient flow to all orders.

RG flows and scalars comments

- The RG flow is a gradient flow: “potential” A changes monotonously.
  - To finite loop order or all loops?
- Couplings may flow out of the stability cone in the IR.
  - Radiative corrections, Coleman-Weinberg, Gildener-Weinberg
- In $D = 4 - \epsilon$, no RG flows enter the stability cone.
- The RG flow is organised hierarchically by symmetry subgroups.

Revisiting D = 4 and D = 4 - ε

Purely scalar $\lambda\phi^4$ theories

- In D = 4 there is classical scale invariance, and there is only a trivial fixed point in perturbation theory.
- Dimensional transmutation yields non-trivial vevs of scalars for RG trajectories reaching the boundary of the stability cone. Vevs in the almost flat directions.
- There may be symmetric vacua, eg in O(N) model, with RG flow of couplings.
- Gildener-Weinberg vacua do not have maximal symmetry.

- In D = 4 - ε classical scale symmetry is broken by ε giving non-trivial fixed points. Fixed points bring back scale invariance.
- Fluctuation driven first order phase transitions result from RG flows leaving the stability cone.
- Choosing explicitly symmetric RG flows is consistent.
- Extremal fixed points do not have maximal symmetry.

Coleman and Weinberg ’73 **Gildener and Weinberg ’76** **Rychkov and Stergiou ’19**

Large N and strong coupling

Large N methods are advertised as non-perturbative.

- What does this mean?
- Is it relevant for scalar $\lambda\phi^4$ theory and its Landau pole?
- Is strong coupling in the UV worse than strong coupling in the IR?

Triviality at large *N*?

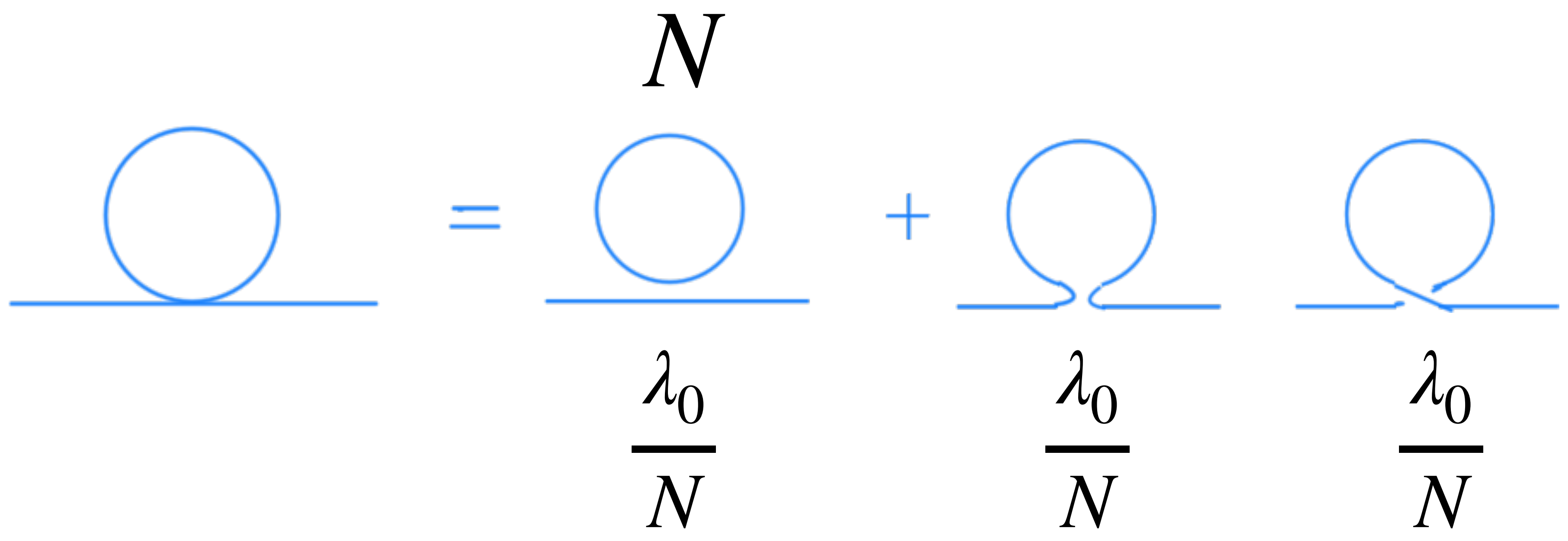
Are scalar $\lambda\phi^4$ theories non-trivial for large enough *n*? Romatschke '23

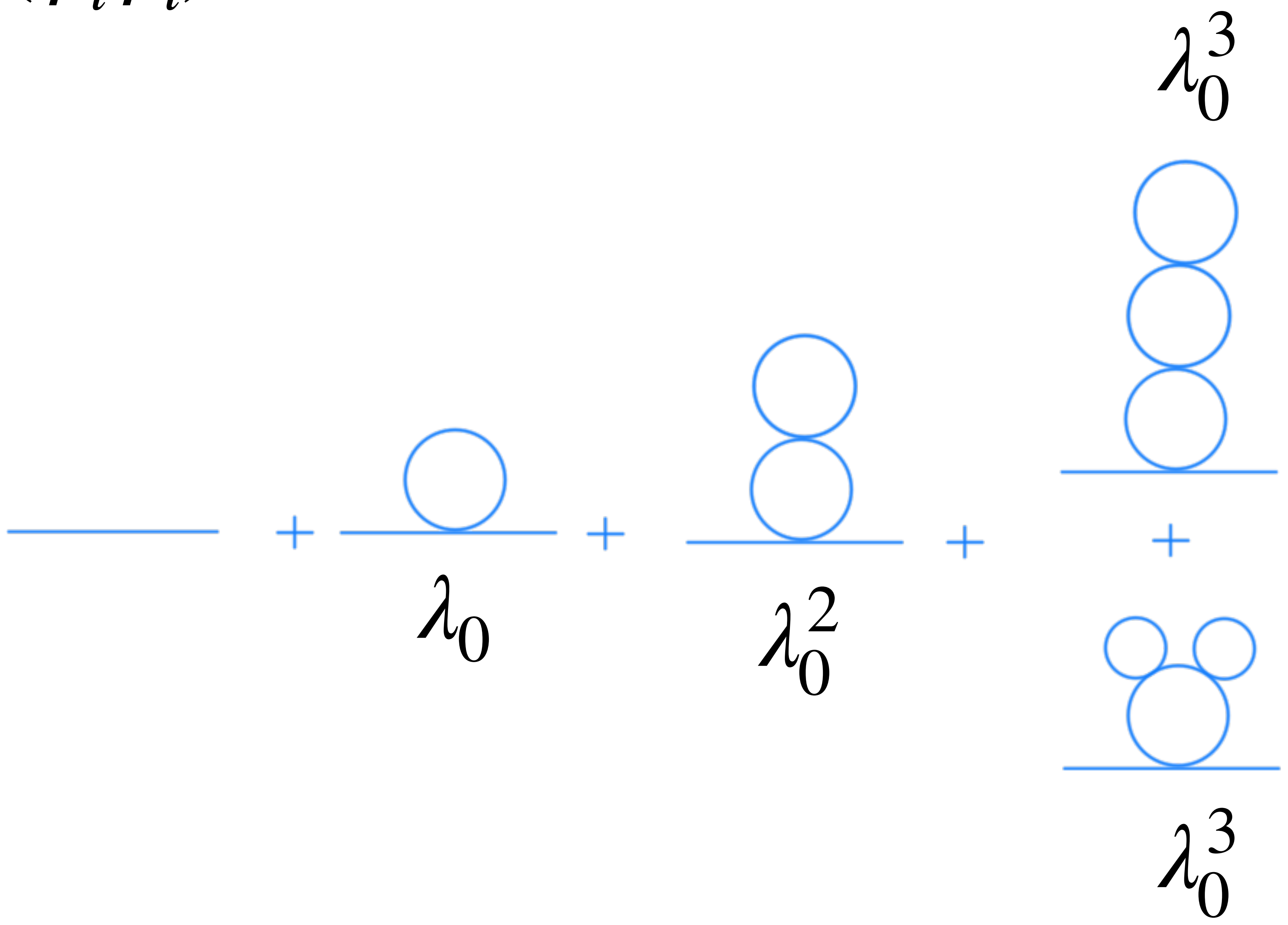
- Romatschke questions the general triviality of $\lambda\phi^4$ — proven only for $n = 1, 2$.
- A lot at stake! Perhaps scalar QFT are well defined.
- We have seen that the properties of scalar QFT change with *n.*
    - To deal with Landau pole non-perturbative methods are needed
    - To avoid the triviality take *n* large.
    - Large *n* methods are claimed to be non-perturbative…

The symmetric O(N) model diagrams

- Consider the maximally symmetric O(n) model, the O(N) model, in $D = 4$.

$$Z = \int \mathscr{D}\phi_i e^{-S_E} \qquad \mathscr{L}_E = \frac{1}{2}\partial_m \phi_i \partial^m \phi_i + \frac{1}{2} m_0^2 \phi_i \phi_i + \frac{\lambda_0}{N}(\phi_i \phi_i)^2$$





The O(N) model Hubbard-Stratonovic

Romatschke '23

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$$e^{-\int dx \frac{\lambda_0}{N}(\phi_i\phi_i)^2} = \int D\zeta e^{-\int dx\left[\frac{i}{2}\zeta\phi_i\phi_i + \frac{\zeta^2 N}{16\lambda_0}\right]}$$

Do the Gaussian ϕ integral: $Z = \int \mathscr{D}\zeta e^{-NA} \qquad A = \frac{1}{2}\mathrm{Tr}\ln\left[-\partial^2 + m_0^2 + i\zeta\right] + \int dx \frac{\zeta^2}{16\lambda_0}$

Vary ζ for saddle point of $z^* = i\zeta$: $\qquad 0 = \frac{1}{2}\int d^dk \frac{1}{k^2 + m_0^2 + z^*} - \frac{z^*}{8\lambda_0}$

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The saddle picks out the leading large N result

Do the Gaussian ϕ integral: $Z = \int \mathscr{D}\zeta e^{-NA}$ $\quad A = \frac{1}{2}\mathrm{Tr}\ln\left[-\partial^2 + m_0^2 + i\zeta\right] + \int dx \frac{\zeta^2}{16\lambda_0}$

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Note the ϕ_i propagator $M^2 = m_0^2 + z^*$

The O(N) model cutoff regularisation

Romatschke '23

UV cutoff Λ_{UV} in $D = 4$

$$0 = \frac{1}{2}\int d^dk \frac{1}{k^2 + m_0^2 + z^*} - \frac{z^*}{8\lambda_0}$$

$$\frac{z^*}{\lambda_0} = \frac{1}{(2\pi)^2}\left[\Lambda_{\rm UV}^2 + \left(m_0^2 + z^*\right)\ln\frac{m_0^2 + z^*}{\Lambda_{\rm UV}^2}\right]$$

Using the physical ϕ_i mass combination $M^2 = m_0^2 + z^*$:

$$\frac{m_0^2 + z^*}{\lambda_0} = \frac{1}{(2\pi)^2}\left(m_0^2 + z^*\right)\ln\frac{m_0^2 + z^*}{\Lambda_{\rm UV}^2} + \frac{m_0^2}{\lambda_0} + \frac{1}{(2\pi)^2}\Lambda_{\rm UV}^2$$

The O(N) model cutoff renormalisation

Romatschke '23
Abbott, Kang and Schnitzer '76

Using physical ϕ_i mass:

$$\frac{M^2}{\lambda_0} = \frac{1}{(2\pi)^2} M^2 \ln \frac{M^2}{\Lambda_{\rm UV}^2} + \frac{m_0^2}{\lambda_0} + \frac{1}{(2\pi)^2}\Lambda_{\rm UV}^2$$

Defining renormalised λ_R, m_R

$$M^2 \frac{1}{\lambda_0} = \frac{1}{(2\pi)^2} M^2 \left(\ln \frac{M^2}{\mu^2} + \ln \frac{\mu^2}{\Lambda_{\rm UV}^2} \right) + \frac{m_0^2}{\lambda_0} + \frac{1}{(2\pi)^2}\Lambda_{\rm UV}^2$$

$$M^2 \left(\frac{1}{\lambda_R} + \frac{1}{(2\pi)^2} \ln \frac{\mu^2}{\Lambda_{\rm UV}^2} \right) = \frac{1}{(2\pi)^2} M^2 \left(\ln \frac{M^2}{\mu^2} + \ln \frac{\mu^2}{\Lambda_{\rm UV}^2} \right) + \frac{m_R^2}{\lambda_R}$$

Yielding the saddle condition

$$\frac{M^2}{\lambda_R} = \frac{1}{(2\pi)^2} M^2 \ln \frac{M^2}{\mu^2} + \frac{m_R^2}{\lambda_R}$$

The O(N) model renormalisation cont'd

Romatschke '23
Abbott, Kang and Schnitzer '76

The renormalisation conditions:

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_R} + \frac{1}{(2\pi)^2} \ln \frac{\mu^2}{\Lambda_{\rm UV}^2}$$

$$\frac{m_R^2}{\lambda_R} = \frac{m_0^2}{\lambda_0} + \frac{1}{(2\pi)^2}\Lambda_{\rm UV}^2$$

For the critical theory with $m_R = 0$
the saddle condition is

$$\frac{M^2}{\lambda_R} = \frac{1}{(2\pi)^2} M^2 \ln \frac{M^2}{\mu^2}$$

with solutions

$$M^2 = 0,\ M^2 = \mu^2 e^{(2\pi)^2/\lambda_R}$$

The O(N) model renormalisation cont'd

Romatschke '23
Abbott, Kang and Schnitzer '76

The renormalisation conditions:

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with solutions

$$M^2 = 0, \; M^2 = \mu^2 e^{(2\pi)^2/\lambda_R}$$

Truly non-perturbative.

Defining a cut-off independent theory

The O(N) model removing cutoff?

Are scalar λφ⁴ theories non-trivial for large enough *n*?

Romatschke '23
Abbott, Kang and Schnitzer '76

The running coupling is similar to that of simple $\lambda\phi^4$ theory, $\beta = \lambda_R^2/(2\pi)^2$, but now *the leading result is non-perturbative.*

$$\lambda_R = \frac{1}{\frac{1}{\lambda_0} + \frac{1}{(2\pi)^2}\ln\frac{\Lambda_{UV}^2}{\mu^2}}$$

RG theory does not tell us how to remove the cutoff. What is λ_0?

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RG theory does not tell us how to remove the cutoff.

Suppose we fix the UV coupling to a finite λ_{ref} and then take $\Lambda_{UV} \to \infty$.

$$\lambda_R(\mu = \Lambda_{UV}) = \lambda_0 = \lambda_{ref}$$

$$\lambda_R(\mu = Q) \to 0$$

We seem to find $\lambda_R(Q) \to 0$ for any finite Q.

This argument is non-perturbative and consistent with quantum triviality.

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RG theory does not tell us how to remove the cutoff. What is λ_0?

Suppose we ask that $\lambda_R(Q)$ is independent of large Λ_{UV}. Then there is a compensating term in λ_0.

$\lambda_R(\mu = Q)$ is fixed

$$\lambda_0 = \frac{(2\pi)^2}{\ln\frac{\Lambda_{LP}^2}{\Lambda_{UV}^2}}$$

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We find a Landau pole: $\lambda_R(Q) = \dfrac{(2\pi)^2}{\ln \frac{\Lambda_{LP}^2}{Q^2}}$

Removing the cutoff we get a trivial theory OR a Landau pole and $\lambda_R(Q) < 0$ for $Q > \Lambda_{LP}$.

The O(N) model summary

- The leading large N result sums diagrams of all orders in λ_0.
- The leading large N result is non-perturbative.
- Due to the simplicity of the $O(N)$ model the non-perturbative beta function has the same form as the one loop beta function of simple $\lambda\phi^4$ model!
  - One solution has a Landau pole (now to take seriously) and negative coupling in the deep UV.

Romatschke '23, Abbott, Kang and Schnitzer '76,…

  - Another solution is trivial.

Challenges for the Landau pole O(N) model

- Negative coupling in the deep UV suggests instability.
  - Does it really?
- Is negative coupling related to PT-symmetry replacing Hermiticity?? Is such a framework required?
- Landau pole in non-perturbative $\lambda_R(\mu)$ suggests divergence in observables.
  - Does it really?
- There may be controlled phase transitions, or instabilities, in the finite temperature $O(N)$ model…

Potential takeaways

Potential

- A non-perturbative RG may address triviality (and UV Landau poles).
  - Is large N (number of scalars) limits the right path?
  - Does QFT in $D = 4 - \epsilon$ indicate a difference between small N and large N?
- Complex CFTs may evade UV sensitivity.
- RG flow geometry may explain the gradient flow.

We can learn something.

Potential

- A non-perturbative RG may address triviality (and UV Landau poles).
  - Is large N (number of scalars) limits the right path? **Flodgren '24** **Flodgren and Sundborg '23, '24**
  - Does QFT in $D = 4 - \epsilon$ indicate a difference between small N and large N?
- Complex CFTs may evade UV sensitivity.
- RG flow geometry may explain the gradient flow. **Guan and Sundborg '25?**

We can learn something.

Thank you!