Review talk at Nordic meeting:
Different perspectives on $T\bar{T}$ theory.

$T\bar{T}$ Definition.

[Zamolodchikov Smirnov '16, Cavaglia et al '16]

$T\bar{T}$ theory = 2D theory obtained from irrelevant deformation of 2D seed theory (QFT w/ transl. inv.)

↳ defined through flow eq. for the action

$$\frac{d}{d\lambda} S^{(\lambda)}_{T\bar{T}} = \frac{1}{2} \int d^2x \sqrt{-\gamma}\, \mathcal{O}^{(\lambda)}_{T\bar{T}}$$

$$\text{w/ } \mathcal{O}_{T\bar{T}} = T_{\mu\nu} T^{\mu\nu} - (T_\mu{}^\mu)^2 = -2 \det T_\mu{}^\nu$$

Reason for defining ito flow eq. (= PT def.) :

$$S^{(d\lambda)}_{T\bar{T}} = \underbrace{S^{(0)}}_{\text{seed theory}} + d\lambda \int d^2x \sqrt{-\gamma}\, \mathcal{O}^{(0)}_{T\bar{T}}$$

↳ seed theory

↳ $T^{(0)}_{++} T^{(0)}_{--}$ in case of seed CFT, or "$T\bar{T}$"

↳ to be perturbed w/ $\mathcal{O}^{(d\lambda)}_{T\bar{T}}$ to flow further in λ

* irrelevant : λ has dimension L^2
  $\mathcal{O}_{T\bar{T}}$ has mass dimension $4 > 2$

  ⇝ changed UV prop. & non-locality

* solvable : E spectrum

interesting !

≠ perspectives / equivalent definitions

$$S^{(0)} = S_{CFT} = S_0(\varphi\,;\gamma_{\mu\nu})$$

TS perspective — $T\bar{T}$ as new theory on old background $\quad S_{T\bar{T}}(\varphi\,;\gamma_{\mu\nu})$

WS perspective — $T\bar{T}$ as old theory on new background $\quad S_0(\varphi, g_{\mu\nu})$ + dyn th for $g_{\mu\nu}$

deformation ~ turning on gravity in CFT

* non-locality

Cardy's random geometry, mgr, NG string, JT' dilaton gravity

[Cardy '18, '15] [Tolley '19] [McGough Mezei Verlinde '16, Cavaglia et al '16] [Dubovsky et al '17, '18]

[NC Kruthoff Verlinde '19, $c \neq 24$]

All of form CFT coupled to gravity, all 2D.

$dL \qquad m^2 = \frac{1}{L} \qquad \alpha' = L \qquad$ c.c. $\Lambda = \frac{2}{L}$

common origin:

mgr - St (Stückelbergized massive gravity)

w/ diff gauge inv.

[Tolley '19]

≠ ways of fixing diff gauge inv. leads to ≠ descriptions.

PT, others nPT: vielbein and metric formalisms

$T\bar{T}$ holo $T\bar{T}$

Cardy's random geometry , mgr , NG string , JT' dilaton gravity

common origin :

mgr - St (Stückelberized massive gravity)

PLAN :

1) $T\bar{T}$ interpretation of mgr : most clear, its action satisfying $T\bar{T}$ flow eq.

2) Stückelberize mgr to mgr-St , and its $T\bar{T}$ interpretation

3) Show each description arises from common mgr-St origin

- w/ limited but some technical detail
- Upshot : X and $\partial\phi$ take on interpretation of diff. Stückelberg fields

4) Holo $T\bar{T}$ (3D)

mgr $\rightarrow$ $T\bar{T}$

Vielbein notation: $g_{\mu\nu} = \underbrace{\eta_{ab}\, e^a_\mu e^b_\nu}$

↳ vielbein as "sqrt of metric"

↳ work to switch ST and local Lorentz indices $V^\mu = V^a e^\mu_a$

mgr = 2D 'ghost-free' massive gravity theory or dRGT model (de Rham Gabadadze Tolley)

$$S_{mgr}(e,\varphi\, ; f, \lambda) = \underbrace{m^2}_{\equiv \frac{1}{\lambda}} \int d^2x \, \frac{1}{2} \hat{\varepsilon}^{\mu\nu} \hat{\varepsilon}_{ab} (e^a_\mu - f^a_\mu)(e^b_\nu - f^b_\nu) + S_0(e,\varphi)$$

$$= S_\lambda(e\, ; f, \lambda) + S_0(e,\varphi)$$

Levi-Civita symbols $\hat{\varepsilon}$, parameter m or λ, dyn. f. : φ and e, non-dyn : f

2 metrics : $\gamma_{\mu\nu} = \eta_{ab} f^a_\mu f^b_\nu$ fixed reference metric

$g_{\mu\nu} = \eta_{ab}\, e^a_\mu e^b_\nu$ dynamical metric (in mgr)

$\delta e \rightarrow$ relation between $e, f, \lambda, \varphi \rightarrow e_*(\varphi, f, \lambda)$

$$\det T^a_\mu = \det T^{(0)a}_\mu$$

$$e^a_\mu - f^a_\mu = \underbrace{- \lambda\, e\, \hat{\varepsilon}_{\mu\nu} \hat{\varepsilon}^{ab}\, T^{(0)\nu}_b}_{\partial_\mu \alpha^a(\lambda, T^{(0)})}$$

or short $e - f = d\alpha(\lambda, T^{(0)})$

(mgr $\rightsquigarrow$ $T\bar{T}$ ctd)

$$S_{mgr}\left(e_*(\varphi, f, \lambda)\ ,\varphi\ ;\ f, \lambda \right) \quad = \quad S_{T\bar{T}}(\varphi\ ;\ f, \lambda)$$

$$\frac{d}{d\lambda} S_{mgr}\left(e_*, \varphi\ ;\ f, \lambda\right) = -\int d^2x \ \det \ T_\mu^a(e, f, \lambda)\Big|_{e_*}$$

$$\text{w/} \quad T_a^\mu := \frac{1}{f}\frac{\delta S_{mgr}}{\delta f_\mu^a}$$

$$\frac{d}{d\lambda} S_{T\bar{T}}\ (\varphi\ ;\ f, \lambda) \ = \ -\int d^2x \ \det \ T_\mu^a\ (\varphi, f, \lambda)$$

$$\text{w/} \quad T_a^\mu := \frac{1}{f}\frac{\delta S_{T\bar{T}}}{\delta f_\mu^a}$$

Follows straightforward, using that the variation of the action vanishes on e_*.

mgr - st.

$S_{mgr}(e,\varphi\ ;\ f,\ell)$ would be symm. in exchanging e and f, if both were dynamical fields

$$S_{aux}(e,f,\varphi\ ;\ \ell) = \frac{1}{\ell}\int d^2x\ \frac{1}{2}\hat{\varepsilon}^{\mu\nu}\hat{\varepsilon}_{ab}(e_\mu^a - f_\mu^a)(e_\nu^b - f_\nu^b) + S_0(e,\varphi)$$

↳ diff invariant! (broken to Isom(f) in S_{mgr})

Write $f_\mu^a = \partial_\mu X^A\ F_A{}^a$

as a way to immediately impose that f_μ^a transforms in the right way under diffs $x \to x'$.

$X^A(x)$: scalar fields (no WS spacetime indices)

$F_A{}^a(X)$: new reference vielbein, w/ new TS spacetime indices $A = 1, 2$

$$S^{st}_{mgr}(e,X,\varphi\ ;\ F,\ell) \equiv S_{aux}(e,\partial X\, F,\varphi\ ;\ \ell)$$

$$S_{mgr} \underset{f=\partial X F}{\rightsquigarrow} S^{st}_{mgr}$$ STÜCKELBERIZATION

e_μ^a $\qquad$ $(e_\mu^a\ ,\ X^A)$

↳ (DIFF) STÜCKELBERG FIELDS

Stückelberized description of theory has more dof but also more gauge freedom.

III $\qquad$ III $\quad$ II

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mgr-ft $\rightsquigarrow$ $T\bar{T}$

First, note : X^A have the interpretation of COORDINATES for the fixed reference metric .

$$f_\mu^a(x) = \frac{\partial X^A}{\partial x^\mu} F_A{}^a(X)$$

$$\underbrace{\gamma_{\mu\nu}(x)}_{\eta_{ab} f_\mu^a f_\nu^b} dx^\mu dx^\nu = \underbrace{\hat{\gamma}_{AB}(X)}_{\eta_{ab} F_A{}^a F_B{}^b} dX^A dX^B$$

Then again a matter of showing the flow eq. is satisfied:

$$\frac{d}{d\lambda} S^{ft}_{mgr}(e_*, X_*, \varphi ; F, \lambda)\Big|_{\substack{\text{gauge fixed} \\ \text{\& solved for grav. dof}}} = -\int d^2x \; \det T_\mu{}^a(e, X, F, \lambda)\Big|_*$$

$$= -\int d^2X \; \det T_A{}^a(e, X, F, \lambda)\Big|_*$$

$$= -\int d^2X \; \det T_A{}^a(\varphi, F, \lambda)$$

$$S^{ft}_{mgr}(e_*, X_*, \varphi ; F, \lambda)\Big|_{\substack{\text{gauge fixed} \\ \text{\& solved for grav. dof}}} = S_{T\bar{T}}(\varphi ; F, \lambda)$$

w/ $T\bar{T}$ on a ST w/ TS coord. X^A

Let's now discuss in detail how to obtain the LHS.

mgr - st : gauge fixing and solving for e, X.

Mink. TS , $F = \delta$ (T$\bar{T}$ on flat Mink.)

$$ds^2_{TS} = \gamma_{\mu\nu}\, dx^\mu dx^\nu = \eta_{ab}\, dX^a\, dX^b$$

δe in mgr $\rightsquigarrow$ $e^a - f^a = d\alpha^a(\lambda, T^{(0)})$ $\quad (e^a = e_\mu^{\ a} dx^\mu)$

$$e_\mu^{\ a} - f_\mu^{\ a} = \partial_\mu \alpha^a \equiv -\lambda\, e\, \hat{\varepsilon}_{\mu\nu}\, \hat{\varepsilon}^{ab}\, T_b^{(0)\,\nu}$$

δe in mgr - st $\rightsquigarrow$ $e^a - dX^a = d\alpha^a(\lambda, T^{(0)})$ $\quad (f = dX$ for $F = \delta)$

δX $\rightsquigarrow$ $de^a = 0$ flatness of WS metric

$$\Rightarrow \quad e^a = dZ^a$$

$$ds^2_{WS} = g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{ab}\, dZ^a\, dZ^b$$

$$\Rightarrow \quad dZ^a - dX^a = d\alpha^a$$

Diff inv.
||

$$dz^a - dX^a = d\alpha^a \qquad (\delta e \text{ EOM})$$

diff inv.
||

↳ "$s = 0$"
gauge choice $X = x$, $F = f$: mgr - gt → mgr
||

sol to δe eqs $e(T^{(0)})$ MASSIVE GRAVITY DESCRIPTION

Or, δe eqs become $dz = dx + d\alpha(\lambda, T^{(0)})$

s.t. $ds^2_{WS} = dz^+ dz^- = \left(dx^+ + d\alpha^+(\lambda, T^{(0)})\right)\left(dx^- + d\alpha^-(\lambda, T^{(0)})\right)$

CFT matter $T^{(0)}$ backreacted to metric

CARDY 'random geometry' DESCRIPTION : dynamical WS interpretation
ito field-dependent diff $\alpha(\lambda, T^{(0)})$

↳ "$s = -1$"
gauge choice $z = x$, $e = \delta$ or $e_+{}^+ = e_-{}^- = 1$, $e_+{}^- = e_-{}^+ = 0$
||

δe eqs become $dX = dx - d\alpha(\lambda, T^{(0)})$

or
$$\boxed{\begin{cases} \partial_- X^- = 1 \\ \partial_+ X^+ = 1 \\ \partial_- X^+ = -\lambda\, T_{--}^{(0)} \\ \partial_+ X^- = -\lambda\, T_{++}^{(0)} \end{cases}}$$

POLYAKOV / NG DESCRIPTION

and JT' DESCRIPTION

↳ more general
$$X = x + s\,\alpha$$
$$z = x + (s+1)\,\alpha$$
any s

Cardy's random geometry

Exploit stress tensor def. $\delta S = \int \sqrt{-g}\, T^{ij}\, \delta g_{ij}$ to interpret a deformation of a CFT involving its stress tensor components as coupling the CFT to a fluctuating metric w/ 'dynamical coord.' $\tilde{w}$

$$S_{CFT} \text{ in } ds^2 = dw\, d\bar{w}$$

$$\downarrow$$

$$S_{QFT} = S_{CFT} + \text{deformation}(T)$$

$$\overset{\text{interpret}}{=} S_{CFT} + \delta S_{CFT}\Big|_{\text{under } \delta g \text{ of CFT metric}}$$

$$= S_{CFT} \text{ in new metric } g + \delta g \text{ or } ds^2 = d\tilde{w}\, d\tilde{\bar{w}}$$

w/ $d\tilde{w} = dw + \delta g\, d\bar{w}$, $d\tilde{\bar{w}} = d\bar{w} + \delta g\, dw$

Polyakov / Nambu-Goto (NG) description of $T\bar{T}$

$$S = \frac{1}{4\pi\alpha'}\int d^2x\,\sqrt{-g}\,\left(g^{\mu\nu}\,\hat{\gamma}_{AB}\,\partial_\mu X^A\partial_\nu X^B + \varepsilon^{\mu\nu}\, i\, B_{AB}\,\partial_\mu X^A\partial_\nu X^B\right)$$
$$+ S_0(g_{\mu\nu},\varphi)$$

w/ $\frac{1}{4\pi\alpha'} = \frac{1}{\mathcal{L}}$, $\hat{\gamma} = \eta$, $B_{AB} = B\,\varepsilon_{AB}$ w/ $B = -1$

describes $T\bar{T}$: match of spectrum (B important)
[NC Kruthoff Verlinde '19]

w/ diff $\times$ Weyl gauge inv. $\rightsquigarrow$ conf. gauge $g = \eta$

$$S = \frac{1}{\mathcal{L}}\int d^2x\;\partial_- X^+\,\partial_+ X^- \qquad + S_0$$

$\delta X \rightsquigarrow \partial_+\partial_- X^A = 0 \quad \Rightarrow \quad X^\pm = x^\pm(x^\pm) + \tilde{x}^\pm(x^\mp)$

$\delta g \rightsquigarrow \frac{1}{\mathcal{L}}\left(\partial_\mu X^A\partial_\nu X_A - \frac{1}{2}g_{\mu\nu}\,g^{\alpha\beta}\partial_\alpha X^A\partial_\beta X_A\right) + T^{(0)}_{\mu\nu} = 0$

VIRASORO CONDITION ($\rightsquigarrow$ NG)

Residual gauge inv. under $x^\pm \to \tilde{x}^\pm(x^\pm)$

↳ temporal gauge
$$\boxed{\begin{aligned}\partial_+ X^+ &= 1\\ \partial_- X^- &= 1\end{aligned}}$$

Remaining δg Virasoro condition
$$\boxed{\begin{aligned}\partial_- X^+ &= -\mathcal{L}\,T^{(0)}_{--}\\ \partial_+ X^- &= -\mathcal{L}\,T^{(0)}_{++}\end{aligned}}$$

$$\begin{cases} \partial_- X^- = 1 \\ \partial_+ X^+ = 1 \\ \partial_- X^+ = -\mathcal{L}\, T^{(0)}_{--} \\ \partial_+ X^- = -\mathcal{L}\, T^{(0)}_{++} \end{cases}$$

Match of Polyakov / NG description w/ $s=-1$ mgr-st.

At level of action, Tolley shows S can be obtained from integrating out Lorentz and conformal Stückelberg fields from mgr-st.

JT' description of $T\bar{T}$

$$S = \int d^2x \left(\Phi R + \frac{2}{\mathcal{L}} \right) \sqrt{-g} \quad + S_0(g_{\mu\nu}, \varphi)$$

$\delta\Phi \quad \rightsquigarrow \quad R = 0 \qquad$ flat WS metric $\quad ds^2 = e^{2\rho} dx^+ dx^-$

$\partial_+ \partial_- \rho = 0 \qquad\qquad = dX^+(x^+)\, dX^-(x^-)$

Fix diff gauge inv. by $X^\pm = x^\pm$ (this is $Z^a = x^a$ in mgr-st)

$$\delta g_{\substack{+- \\ ++ \\ --}} \rightarrow \begin{cases} \partial_+ \partial_- \Phi = -\frac{1}{2\mathcal{L}} \\ -\partial_\pm^2 \Phi = \frac{1}{2} T^{(0)}_{\substack{++ \\ --}} \end{cases} \quad \text{or} \quad \boxed{\begin{cases} \partial_\pm X^\pm = 1 \\ \partial_\pm X^\mp = -\mathcal{L}\, T^{(0)}_{\substack{++ \\ --}} \end{cases}}$$

by defining $X^\pm = 2\mathcal{L}\, \partial_\mp \Phi$

Match of JT' description w/ NG and $s=-1$ mgr-st

Polyakov fields X or $\partial\Phi$ of JT' dilaton are diff. Stückelberg fields.

At level of action, Dubovsky et al show $S_{JT'} = S^{st}_{mgr}$ + tot. der.

<u>Holo $T\bar{T}$</u> [Freidel '08, McGough et al '16]

$$Z_{T\bar{T}} = \int \mathcal{D}e\, \mathcal{D}\varphi \; e^{i S_{\text{fugr}} (e, \varphi \,;\, f, \lambda)}$$

↳ solves the $T\bar{T}$ flow eq. for the path integral $\int \mathcal{D}\varphi\, e^{i S_{T\bar{T}}}$

↳ solves the radial WdW eq. in AdS_3 QG

$\hat{H}\,\Psi = 0$ for WdW wavefunction Ψ

if $\lambda = 8\pi G\, \ell$

$T\bar{T}$ coupling — grav. cst. — ↳ AdS scale

$$Z_{T\bar{T}} = \Psi$$

In metric formalism

$$Z_{T\bar{T}}[\gamma_{\mu\nu}] = \int \mathcal{D}\varphi \; e^{i S_{T\bar{T}} [\varphi ; \gamma_{\mu\nu}]}$$

$$\Psi[\gamma_{\mu\nu}] = \int_{g_{\text{bulk}}\big|_{\Sigma(r_c)} = \gamma_{\mu\nu}} \mathcal{D}g_{\text{bulk}} \; e^{i S_{\text{grav}}[g_{\text{bulk}}]}$$

CUT-OFF HOLOGRAPHY

$T\bar{T}^{(\lambda)}$ theory on background $\gamma_{\mu\nu}$ is holo. dual to AdS_3 gravity w/ Dirichlet b.c. (TS) $g_{\text{bulk}}\big|_{\Sigma(r_c)} = \gamma_{\mu\nu}$ imposed at a finite radial distance $r_c(\lambda)$ into the bulk.

"$T\bar{T}$ deformation pushes the CFT into the bulk."

[Guica Monten '19]:

'Mirage of cut-off holography' from asymptotic mixed b. c. (WS)

Holo $T\bar{T}$ outlook.

Beyond AdS / CFT holo . [Silverstein et al, ...]

$T\bar{T}$ as tool towards WdW QG formulation of AdS / CFT .

[Araujo - Regado Khan Wall '22 "Cauchy slice holo"]

[Blacker NC Hergueta Ning '24]

↳ $Z_{T\bar{T}}$ solves Schrödinger evolution eq. in volume time

Thank you !