Yangian symmetric conformal Feynman integrals and GKZ-systems

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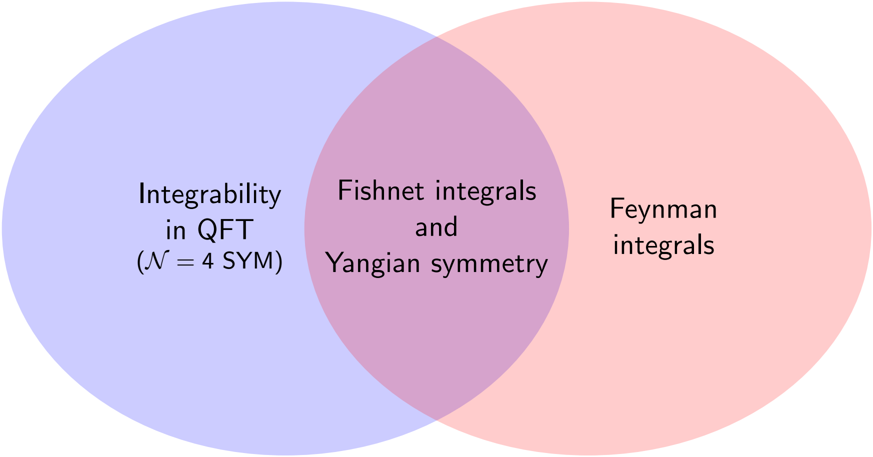
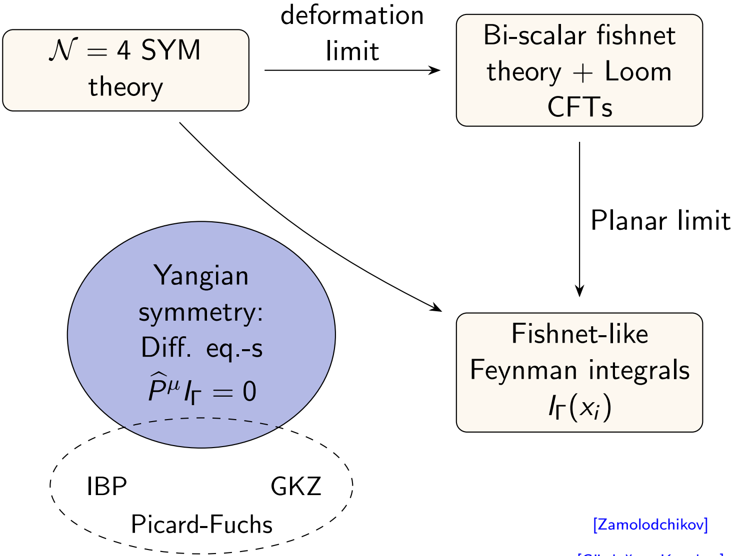
33rd Nordic Meeting on "Strings, Fields and Branes"

Nordita, 2024

Based on 2304.04654 with V.Kazakov, F. Levkovich-Maslyuk
&
to appear with F. Levkovich-Maslyuk

Background


Integrability
in QFT
($\mathcal{N} = 4$ SYM)
Fishnet integrals
and
Yangian symmetry
Feynman
integrals



[Zamolodchikov]

[Gürdoğan, Kazakov]

[Chicherin, Kazakov, Loebbert, Müller, Zhong]

What is Yangian invariance for Feynman integrals?

For a graph Γ we have:

$$I_\Gamma\left(D, \Delta | x\right) = \int \prod_{k \in \text{internal}} d^D x_k \prod_{\langle i,j \rangle} \frac{1}{\left(x_i - x_j\right)^{2\Delta_{ij}}}$$

Yangian invariance

- Conformal $\mathfrak{so}(D,2)$ symmetry: $P^\mu, D, L^{\mu\nu}, K^\mu$ - holds for graphs with the right vertex valency
- Additional "level-one" symmetry :

$$\widehat{P}^\mu I_\Gamma(x) = 0$$

with

$$\widehat{P}^\mu = -\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu} + g^{\mu\nu}D_j)P_{k,\nu} - (j \leftrightarrow k)] + \sum_j s_j P_j^\mu =$$

$$= \frac{1}{2}\sum_{j<k}\left(\delta^{\mu\alpha}\delta^{\lambda\nu} - \delta^{\nu\alpha}\delta^{\mu\lambda} - \delta^{\mu\nu}\delta^{\alpha\lambda}\right)(x_j - x_k)^\alpha \frac{\partial^2}{\partial x_j^\lambda \partial x_k^\nu} +$$

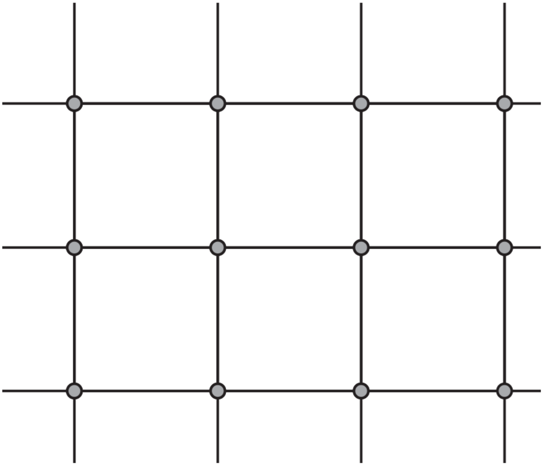
$$+ \sum_j s_j \frac{\partial}{\partial x_j^\mu}$$

The parameters s_j depend on the graph.

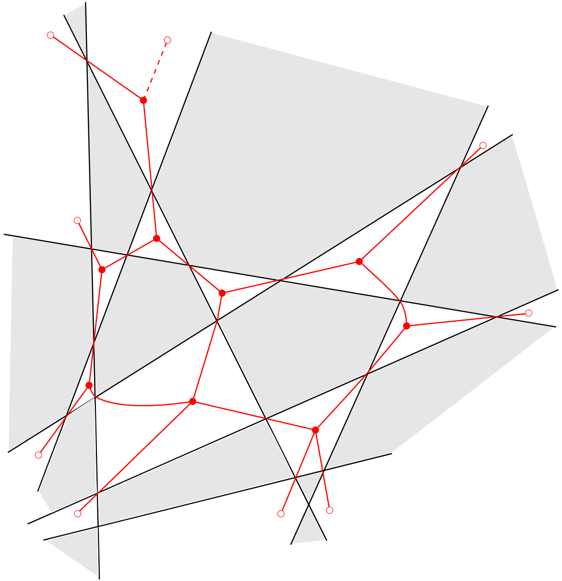
What kind of graph can be Yangian invariant?

[2304.04654, V.Kazakov, F.Levkovich-Maslyuk,V.M.]

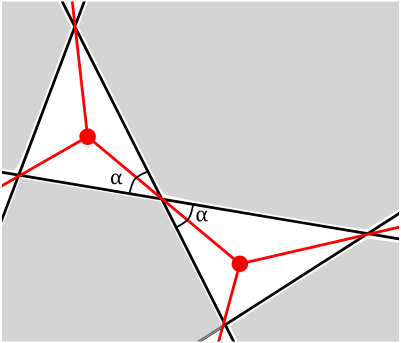
Fishnets



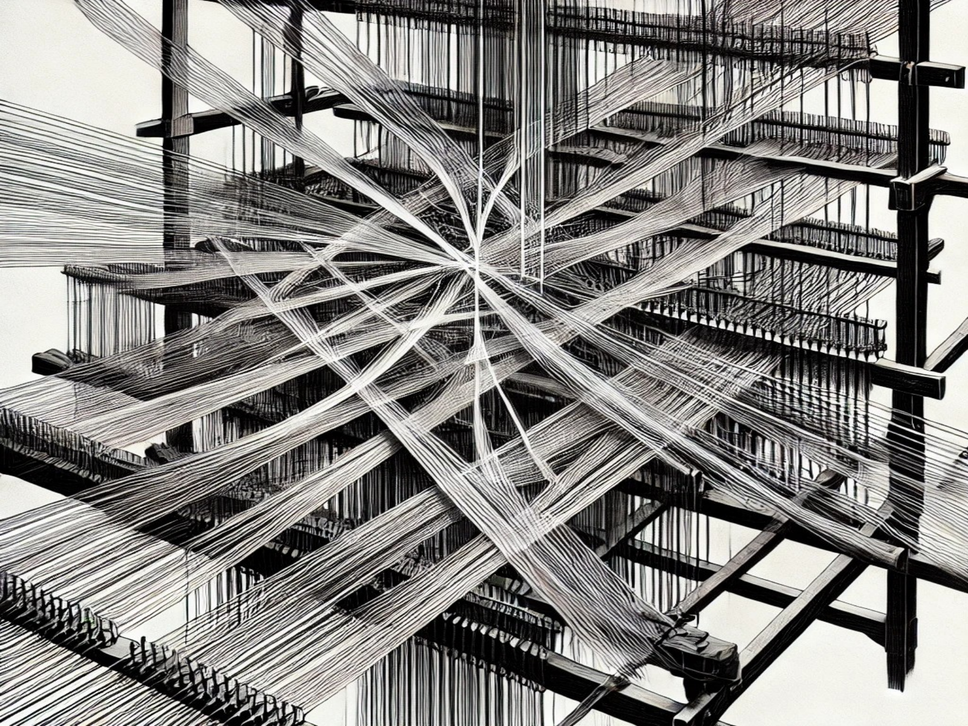
Loom graphs



Graph drawn on a Baxter lattice



Fragment of a Feynman graph. $\frac{1}{(x_1 - x_2)^{2\Delta}} \quad \longleftrightarrow \quad \Delta = D\,\frac{\pi-\alpha}{2\pi}$.



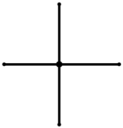
What does Yangian symmetry do for us?

- Conformal symmetry $\Rightarrow$ dependence only on conformal cross ratios
- The Yangian

$$\widehat{P}^{\mu} = \sum_{i<k} \frac{x_{ik}^{\mu}}{x_{ik}^{2}} \mathrm{PDE}_{ik}$$

- From examples - Yangian symmetry completely fixes the integral! "Bootstrap" the integral from symmetries [F.Loebbert, D.Müller, H.Münkler]
- Problem: only possible to do for simple 4-point cross (and with hard work for some 6-point graph)

Example: The cross



- Following [F.Loebbert, D.Müller, H.Münkler]

$$I_+(x) = \int \frac{d^D x_0}{x_{10}^{2\Delta_1} x_{20}^{2\Delta_2} x_{30}^{2\Delta_3} x_{40}^{2\Delta_4}} =$$

$$= x_{14}^{2\Delta_2+2\Delta_3-D} x_{13}^{2\Delta_4-D} x_{34}^{-2\Delta_3-2\Delta_4+D} x_{24}^{-2\Delta_2} I_+^{(0)}(u,v)$$

- Cross ratios are chosen as:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Out of $4 \cdot 3/2 = 6\ \mathrm{PDE}_{ik}$ only 2 are independent

$$0 = \big(\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + (u\partial_u + v\partial_v)^2 - u\partial_u^2 - \gamma\partial_u\big)\, I_+^{(0)}(u,v)$$

$$0 = \big(\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + (u\partial_u + v\partial_v)^2 - v\partial_v^2 - \gamma'\partial_v\big)\, I_+^{(0)}(u,v).$$

- 4-dim solution space - Appel F_4 functions
- + symmetries + boundary conditions = exact expression for the integral.

We have the symmetry, but can we actually solve the equations?

Our main result:

PDE_{ik} $\Longleftrightarrow$ GKZ equations

Gelfand-Kapranov-Zelevinsky systems
[1989]
Good & well-known systems of d.e.:
finite-dim space of solutions, constructed algorithmically

with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i(t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory. A separate paper will be devoted to these integrals.

In essence, Yangian symmetry is equivalent to:

$$L_{iklj} = \frac{\partial^2}{\partial(x_{ik}^2)\partial(x_{lj}^2)} - \frac{\partial^2}{\partial(x_{il}^2)\partial(x_{kj}^2)}$$

[A.Pal,K.Ray]

$$x_i^\mu \longrightarrow x_{ij}^2 \longrightarrow u, v, \ldots$$

GKZ systems:

$\mathcal{A}$ + $\vec{b}$ =

$n \times N$ matrix — vector $\mathbb{C}^n$

$$\prod_{\ell_i>0} \partial_{z_i}^{\ell_i} - \prod_{\ell_i<0} \partial_{z_i}^{-\ell_i}\,, \ell \in \ker(\mathcal{A})$$

$$\sum_j \mathcal{A}_{ij} z_j \frac{\partial}{\partial z_j} - b_i$$

system for a function of n-variables z_i

Triangulations, bases, . . .

$z_k = x_{ij}^2$

$\mathcal{A}, \vec{b}$ - special

Solutions:

$$\sum_{u\in \ker \mathcal{A}} \frac{1}{\prod_{i=1}^{N} \Gamma(\gamma_i + u_i + 1)} z_i^{u_i+\gamma_i}$$

$$\mathcal{A}\gamma = b^T$$

Conclusion & Speculations

- The Yangian system is equivalent to a GKZ systems with a very special type of toric matrices $\mathcal{A}$.

  Note: no enlargement of variable space is needed - the GKZ variables are exactly the Poincare invariants of coordinates.

- We know how to solve Yangian constraint for any number of external points.

Essentially a dream scenario for F.I. calculus

Differential equations organize into a symmetry algebra and are completely and analytically solvable, by mapping to GKZ

Conclusion & Speculations

- There is hope that these GKZ systems can be solved in general (for any number of points) - we could compute all loop integrals in the fishnet (fishnet-like) theory? Complete perturbative solution?

- GKZ systems famously appear in mirror symmetry, for CY periods. Fishnets in $D = 2$ are known to be CY periods [Duhr,Klemm,Loebbert,Nega,Porkert] . Are our results the proper D-extension?