Elliptic integrable models and their spectra from the superconformal indices

Anton Nedelin

King's College London

Nordic Strings Meeting 2024

NORDITA 29-31 October

based on works with S.Razamat, H.-C. Kim and B.Nazzal

2407.08776, 2305.09718, 2106.08335

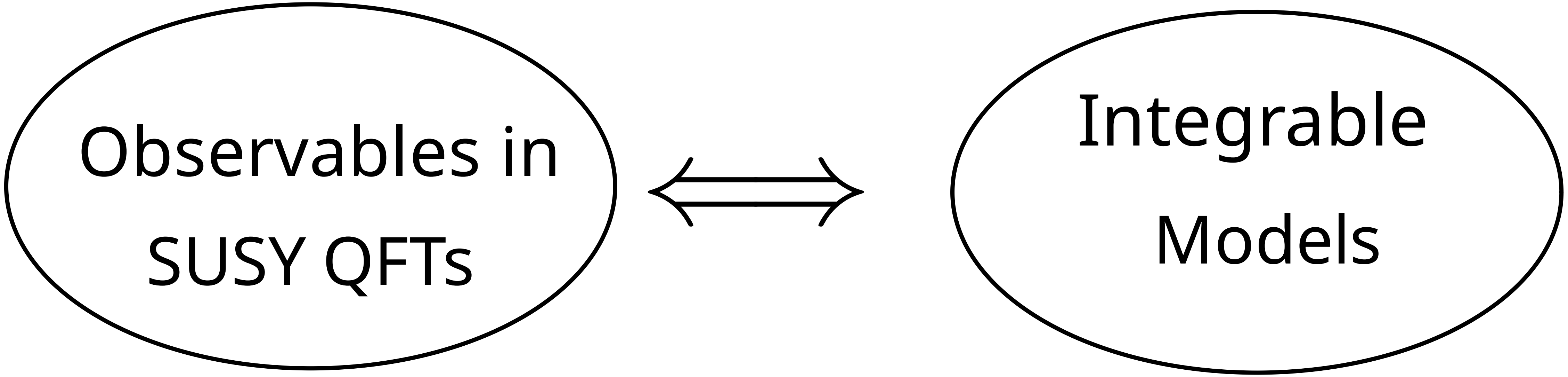
Strings Fields & Branes 33

NORDIC NETWORK MEETING

KING'S College LONDON

General Setting

Main idea is in the relation

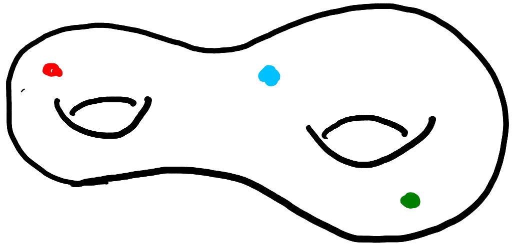


In the spirit of BPS/CFT (Nekrasov-Shatashvili et al.)

Particular Setting

6d SCFT

⇓ compactification on the punctured Riemann surface



Allows one to write index for any, even non-Lagrangian compactification (index bootstrap)

Superconformal index (SCI) of 4d IR theory

6d (2,0) theory

⟹

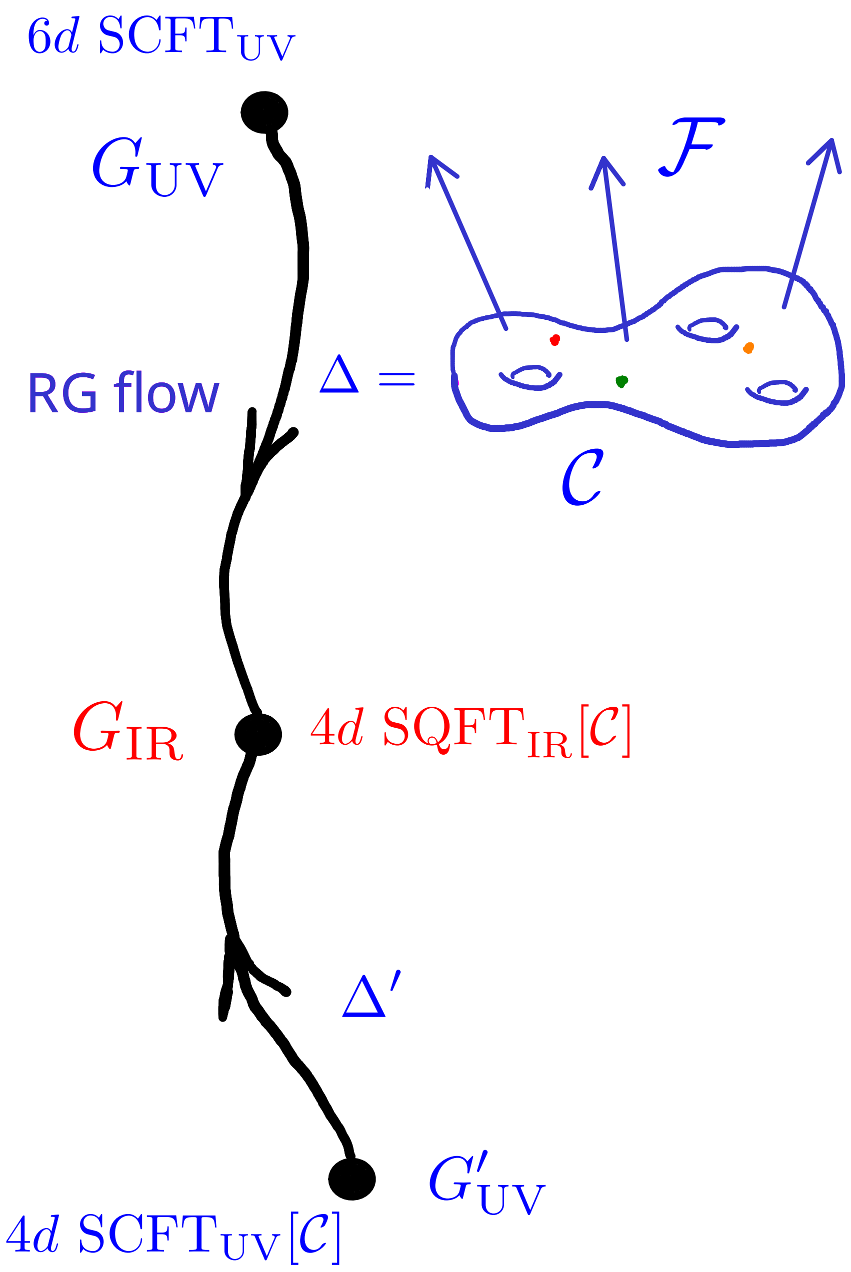
Introduce codim.-2 defect in 4d

Integrable analytic finite-difference operator $A\Delta O$

Ruijsenaars-Schneider model

Gaiotto, Rastelli, Razamat 1207.3577;

$4d\ \mathcal{N}=1$ SQFTs from $6d$ SCFTs



- Generalities: Razamat, Sabag, Sela, Zafrir 2203.06880 and many more

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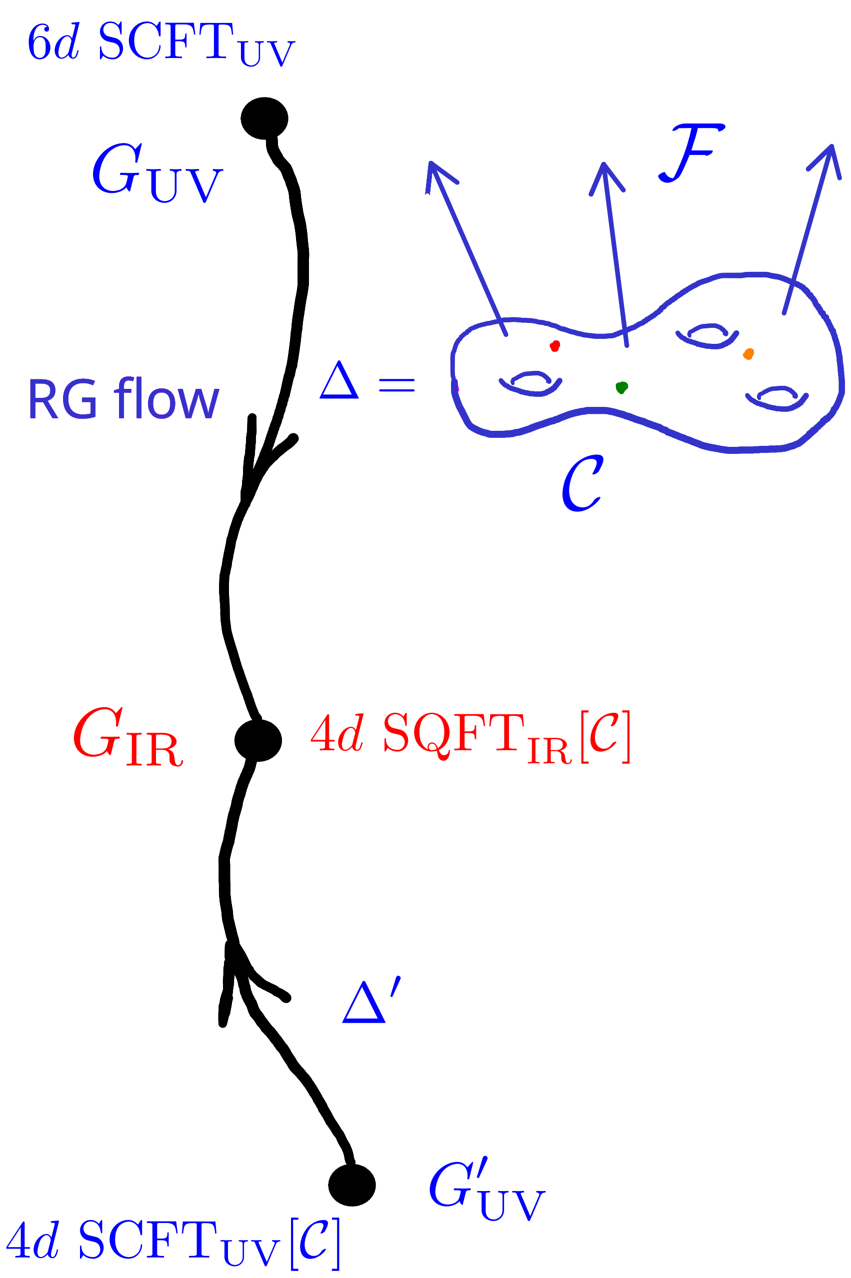

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  We assume 6d SCFT has 5d lagrangian QFT compactification.

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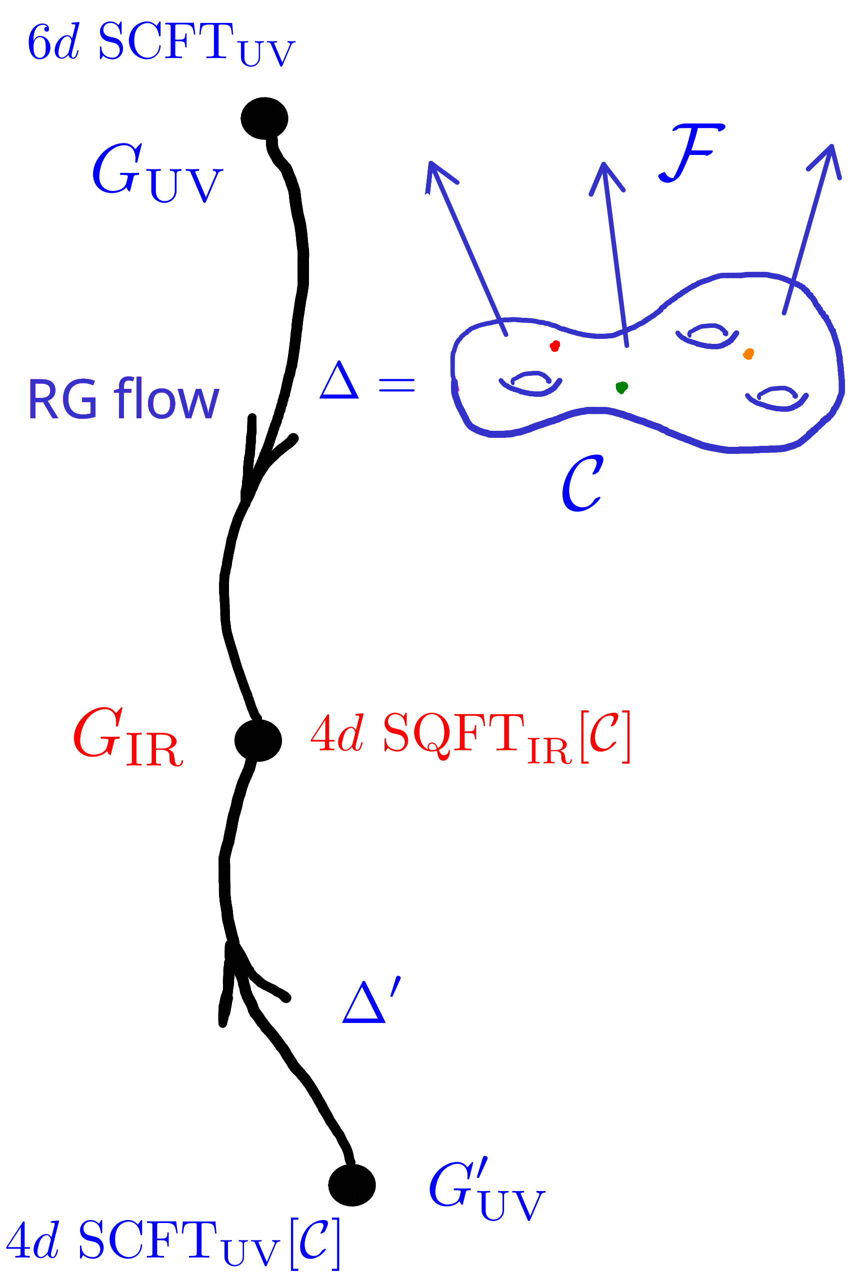

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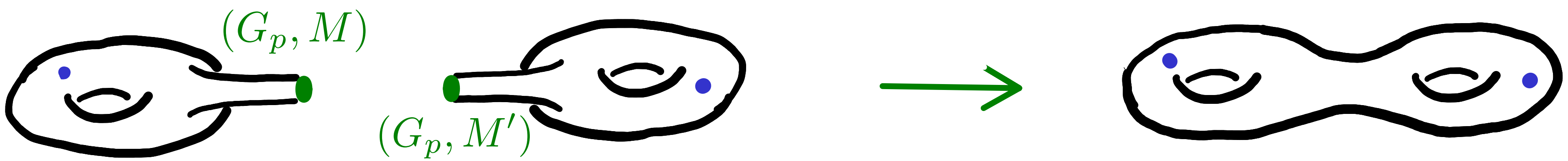
- Gluing two punctures:
  1) Identify moment maps of two punctures $W \sim M \cdot M'$
  2) Gauge puncture global symmetry G_p



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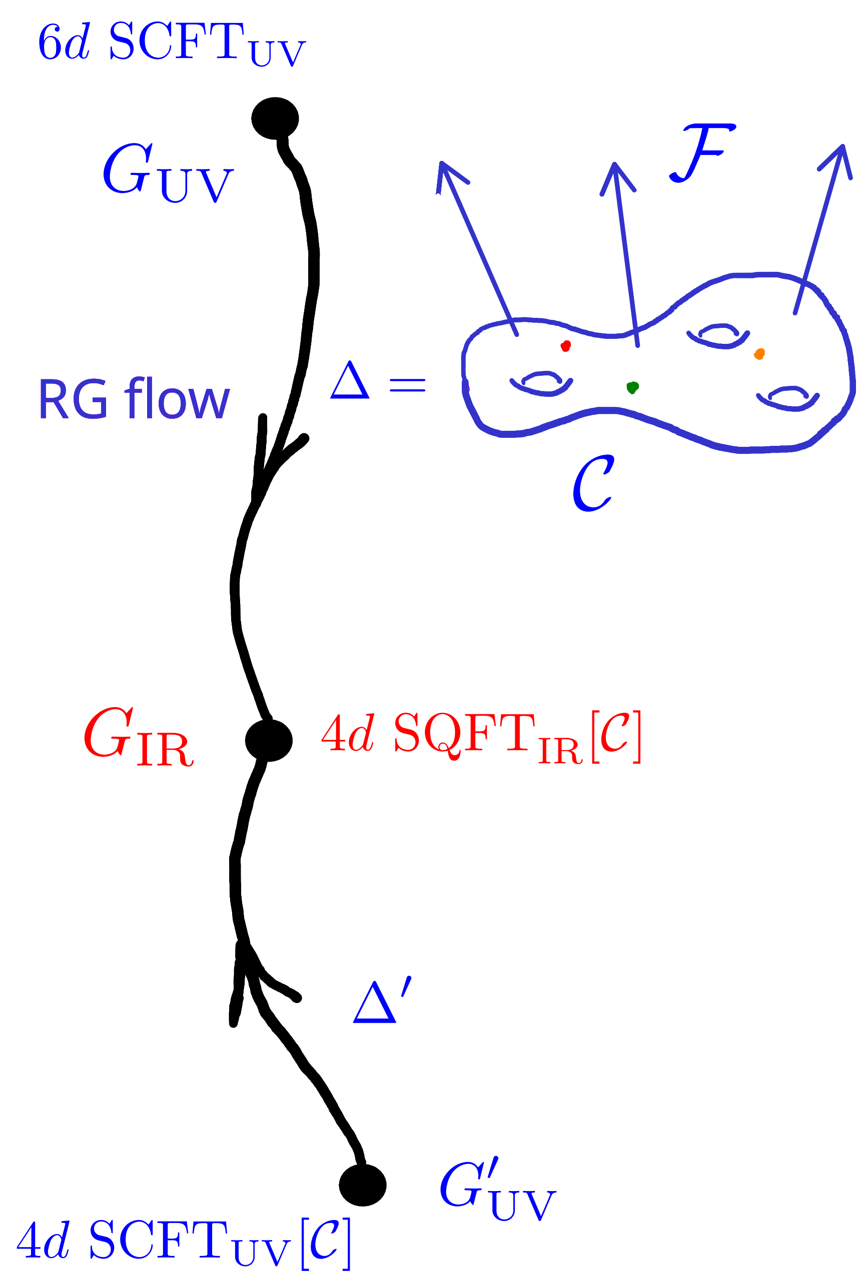

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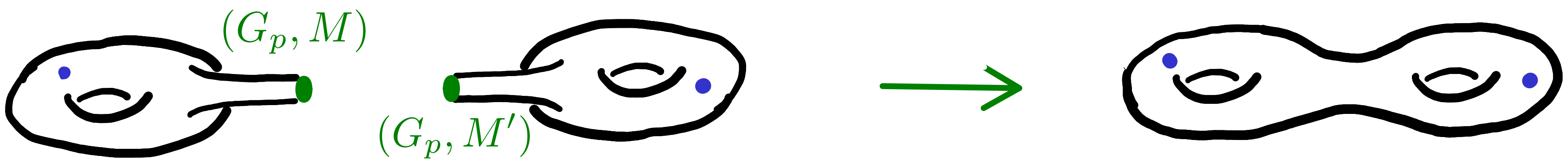

- Punctures can be closed partially or completely by giving VEV $\langle \partial_+^k \partial_-^l M_i \rangle \neq 0$

  Global symmetry is broken to a subgroup

Superconformal Index

- Given $4d$ $\mathrm{SQFT}_{\mathrm{IR}}[\mathcal{C}]$ put it on $S^3 \times \mathbb{R}$ and compute superconformal index

$$\mathcal{I}(p,q,\{u\}) = \mathrm{tr}_{S^3}\left[(-1)^F q^{j_2-j_1+\frac{1}{2}R} p^{j_2+j_1+\frac{1}{2}R} \prod_{i=1}^{G_{4d}} u_i^{Q_i}\right]$$

- $(j_1, j_2) \rightarrow$ charges under Cartans of $Spin(4)$, corresponding fugacities are p and q .
- $R \rightarrow$ charges of $U(1)$ R-symmetry
- $Q_i \rightarrow$ charges of the Cartan of $4d$ global symmetry G_{4d}
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G_{4d} → Cartans of $6d$ global symmetry G_{6d}

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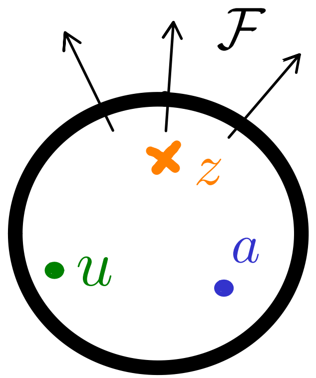
- Gluing and closing punctures have simple interpretations in terms of operations on the index.

Integrable Operators from Indices



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  (II) 4d trinion is Lagrangian theory.

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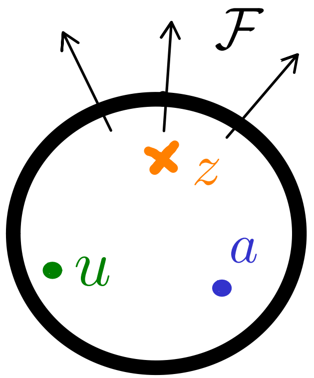

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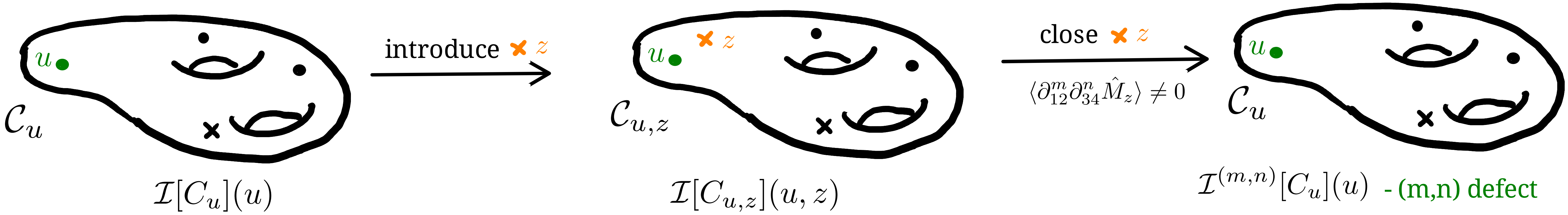
- **Main idea:** introduce defect into theory by closing puncture.



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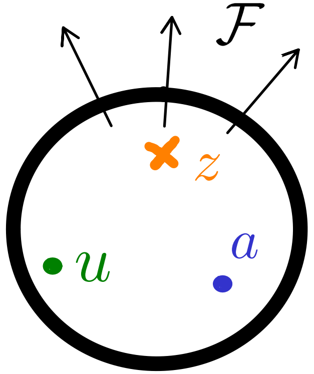

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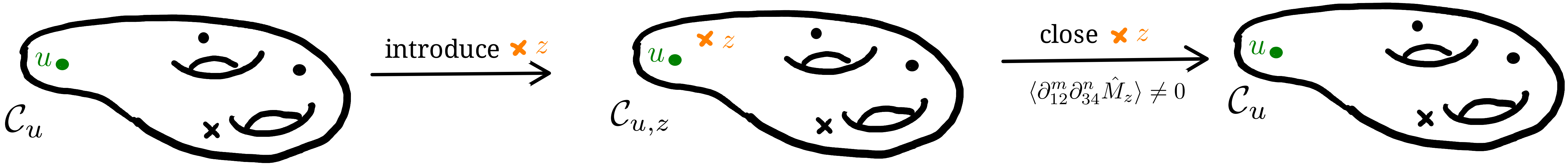

$\mathcal{I}[C_u](u)$ $\mathcal{I}[C_{u,z}](u,z)$ $\mathcal{I}^{(m,n)}[C_u](u)$ - (m,n) defect

- Index with (m,n) defect:

$\mathcal{H}_u^{(m,n)}$ - tower of finite difference operators $(A\Delta Os)$

$$\mathcal{I}_{IR}^{(m,n)} \sim \mathrm{Res}_{z\to p^m q^n U_{\hat{M}_z}} \mathcal{I}\left[\mathcal{C}_{u,z}\right](u,z) \quad \Longrightarrow \quad \boxed{\mathcal{I}_{IR}^{(m,n)} \sim \mathcal{H}_u^{(m,n)} \cdot \mathcal{I}\left[\mathcal{C}_{g,s}[u],\mathcal{F}\right]}$$

Gaiotto, Rastelli, Razamat 1207.3577; Nazzal, AN, Razamat 2106.08335;

Properties of Operators

- Two properties follow by construction from 4d S-duality:

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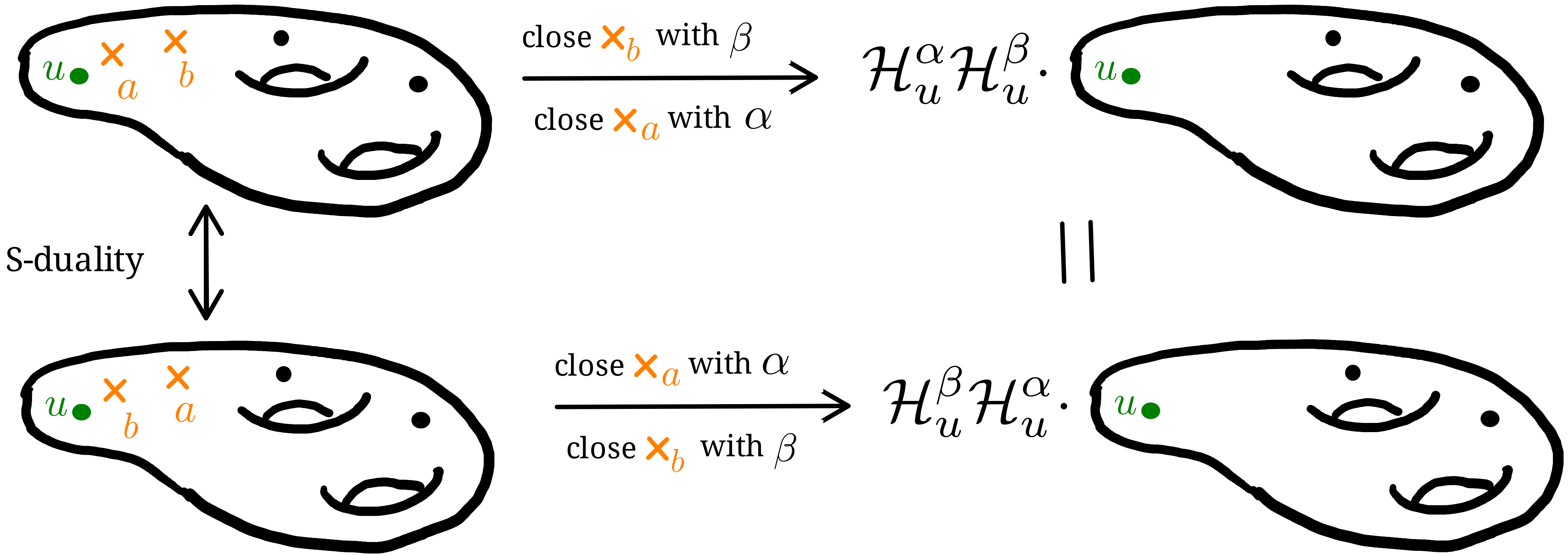
(I) $A\Delta Os$ all commute with each other

$$\left[\mathcal{H}_u^\alpha, \mathcal{H}_u^\beta\right] = 0$$

α, β - labels of operators including (m,n) and more (puncuture closure) $\Longrightarrow$ integrability?



Properties of Operators

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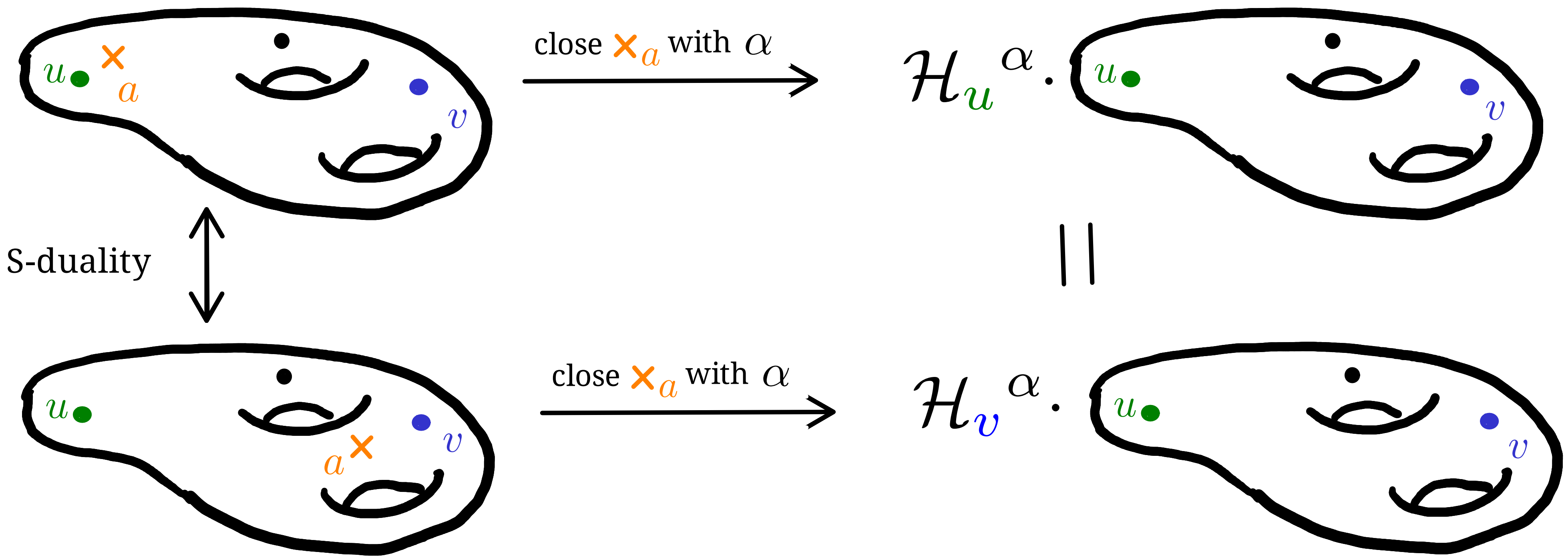
(II) Indices of 4d $\mathcal{N}=1$ theories obtained in certain compactifications are Kernel Functions of the corresponding operators:

$$\mathcal{H}_u^{\alpha} \cdot \mathcal{I}_{g,s}[u,v,...] = \mathcal{H}_v^{\alpha} \cdot \mathcal{I}_{g,s}[u,v,...] = ...$$



Indices from Spectrum

- Assume spectrum of $\mathcal{H}_x^\alpha$ is known: $\boxed{\mathcal{H}_{\mathbf{x}}^\alpha \cdot \psi_\lambda(\mathbf{x}) = E_{\alpha,\lambda}\psi_\lambda(\mathbf{x})}$

  λ - eigenstate label. Depends on Hamiltonian (integer, partition etc.)

  Eigenfunctions ψ_λ are orthogonal w.r.t. gluing measure $\Delta(x)$: $\oint d\mathbf{x}\ \Delta(\mathbf{x})\psi_\lambda(\mathbf{x})\psi_{\lambda'}(\mathbf{x}^{-1}) = \delta_{\lambda,\lambda'}$

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- Natural ansatz for Kernel functions of $\mathcal{H}^{\alpha}_{x}$

$$\boxed{\mathcal{I}(\{\mathbf{x}_j\}) = \sum_{\lambda} C_{\lambda} \prod_{j=1}^{s} \psi_{\lambda}(\mathbf{x}_j)}$$

Compactification on Riemmann surface with s punctures of the same type.

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Compactification on Riemann surface with s punctures of the same type.

Gaiotto, Rastelli, Razamat 1207.3577;

- Assume λ have natural ordering so we can enumerate them: $\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$

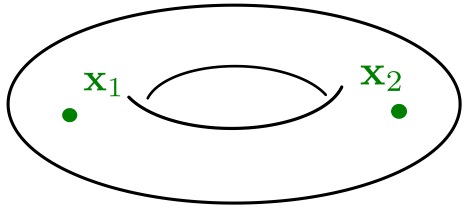
- Simplest example, two-punctured surface $\mathcal{I}(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{\infty} C_{\lambda_i}\psi_{\lambda_i}(\mathbf{x})\psi_{\lambda_i}(\mathbf{y})$

  C_{λ_i} depends on details of compactification (fluxes, genus)

Spectrum from Indices

Nazzal, AN, Razamat 2305.09718

- Start with two-punctured surface with some non-zero flux and/or non-zero genus



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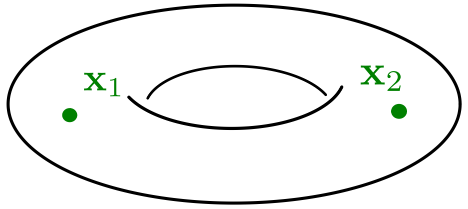

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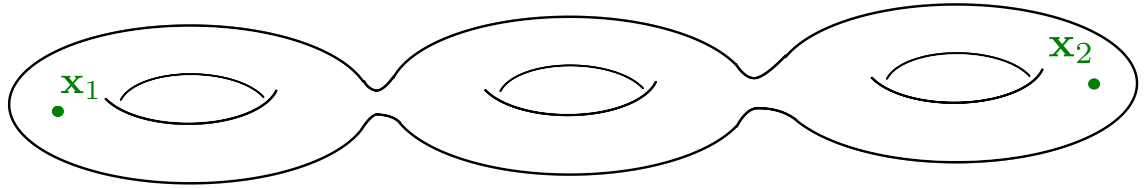

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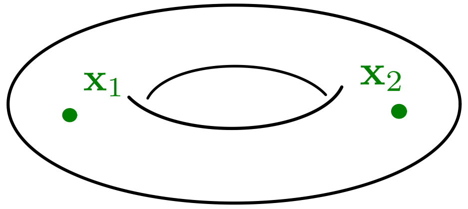

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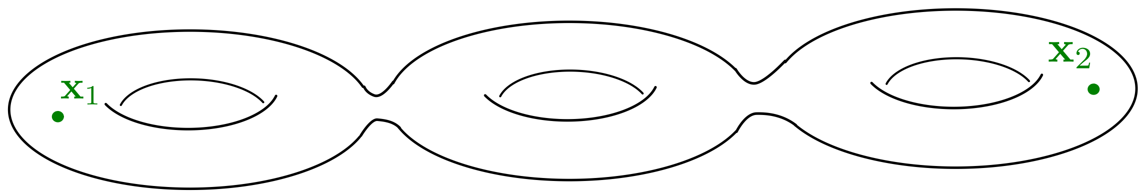

- Consider series in $y = pq$ parameter assuming C_{λ_i} are ordered:

$$C_{\lambda_0} = O(1),\quad C_{\lambda_1} = O\left(y^{n_1}\right),\, n_1 > 0 \quad C_{\lambda_2} = O\left(y^{n_2}\right),\, n_2 > n_1\,, \ldots.$$

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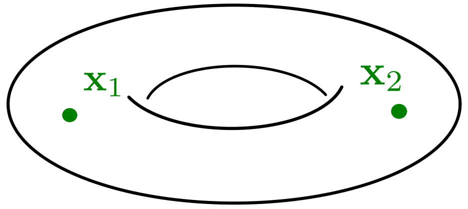

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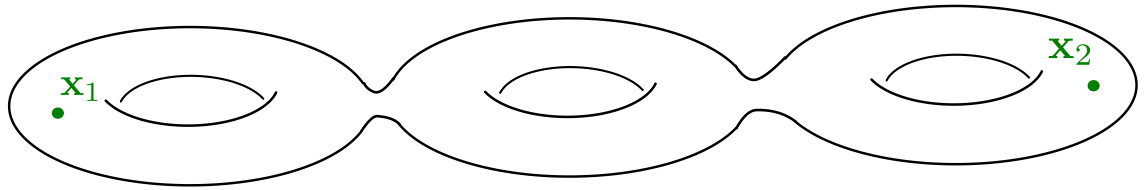

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In practice: take finite n, results are valid up to a fixed order in $y = pq$

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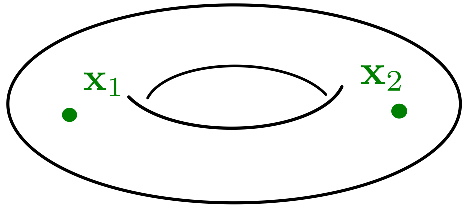

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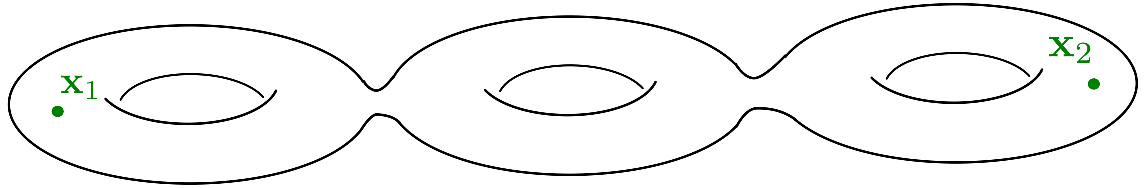

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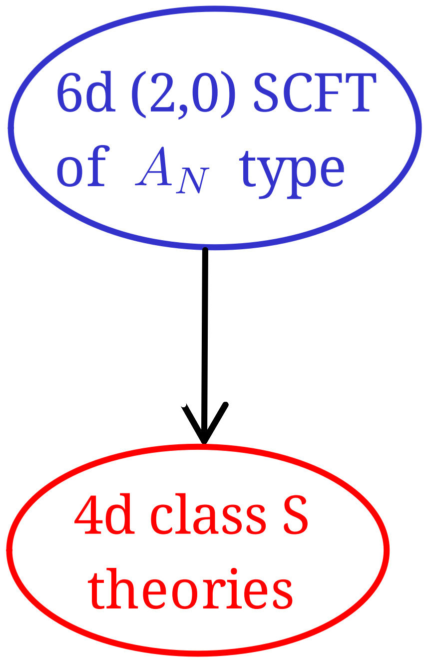
- Other C_{λ_i} and $\psi_{\lambda_i}(\mathbf{x})$ can be found similarly using results for the lower states.

Example : Ruijsenaars – Schneider Model



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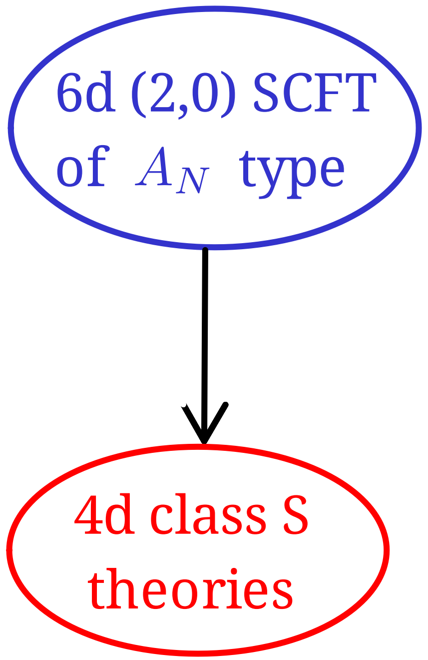

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- Trinion theory is free bi-fundamental hypermultiplet:

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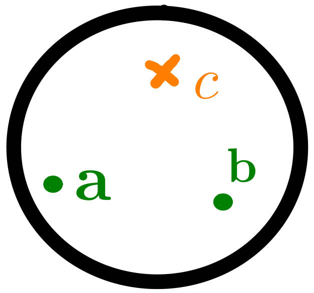

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4d class S theories

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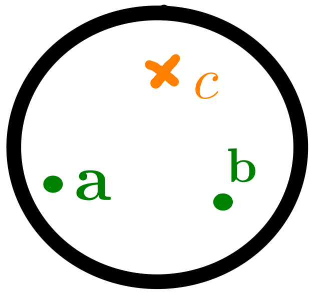

- Associated operators: A_N Ruijsenaars-Schneider system

Gaiotto, Rastelli, Razamat 1207.3577;
Ruijsenaars and Schneider '86;

A_1 case, only one operator.
(N independent operators in general)

$$\mathcal{H}_{A_1} \cdot \psi(x) = \frac{\theta_p\left(tx^2\right)}{\theta_p\left(x^2\right)}\psi(q^{1/2}x) + \frac{\theta_p\left(tx^{-2}\right)}{\theta_p\left(x^{-2}\right)}\psi(q^{-1/2}x)$$

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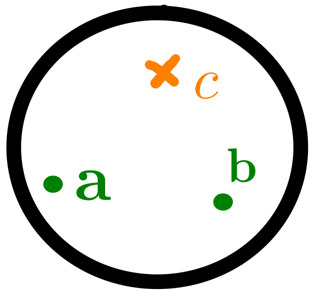

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- Associated operators: A_N Ruijsenaars-Schneider system Gaiotto, Rastelli, Razamat 1207.3577; Ruijsenaars and Schneider '86;

A_1 case, only one operator. (N independent operators in general)

$$\mathcal{H}_{A_1} \cdot \psi(x) = \frac{\theta_p\left(tx^2\right)}{\theta_p\left(x^2\right)}\psi(q^{1/2}x) + \frac{\theta_p\left(tx^{-2}\right)}{\theta_p\left(x^{-2}\right)}\psi(q^{-1/2}x)$$

- Macdonald limit: $p \to 0$ spectrum is known. Eigenfunctions are Macdonald Polynomials

Used for index calculations. Gadde, Rastelli, Razamat, Yan 1110.3740;

Example : Ruijsenaars – Schneider Model

6d (2,0) SCFT of A_N type

↓

4d class S theories

- Punctures with A_N symmetry (5d compactification is $\mathcal{N}=2$ SU(N+1) SYM).
- Trinion theory is free bi-fundamental hypermultiplet:

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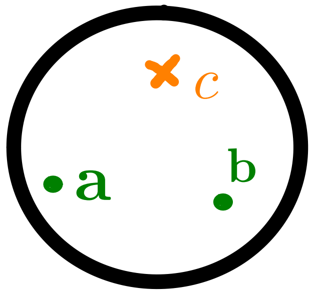

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Used for index calculations. Gadde, Rastelli, Razamat, Yan 1110.3740;

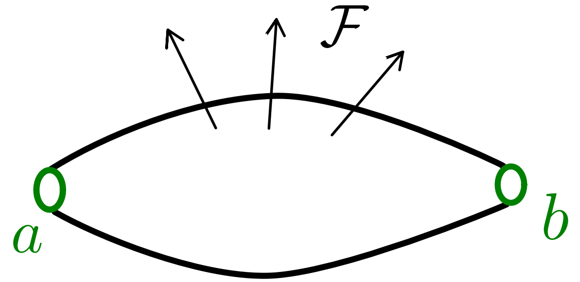
- In general form spectrum is not known yet!

A_1 RS Spectrum from Indices

Kim, AN, Razamat 2407.08776; Nazzal, AN, Razamat 2305.09718;

- Staring point: two-punucutured sphere (tube) with flux

$$\mathcal{I}_2 = \Gamma\left((pq)^{1/4} t^{1/2} a^{\pm 1} b^{\pm 1}\right)$$



- Gluing measure:

$$\Delta(x) = \frac{(p;p)_\infty (q;q)_\infty}{2} \frac{\Gamma\left(\sqrt{pq}\, t^{-1} x^{\pm 2}\right)\Gamma\left(\sqrt{pq}\, t^{-1}\right)}{\Gamma\left(x^{\pm 2}\right)}$$

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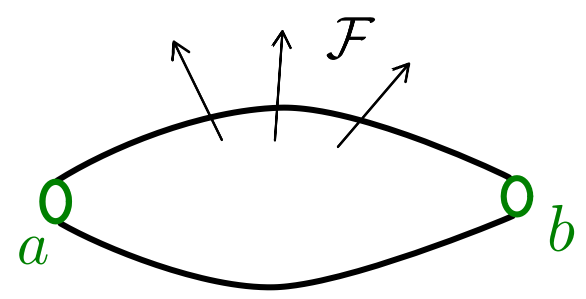

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- Resulting eigenfunctions:

$$\psi_0(x) = 1 + \sqrt{pq}(t + t^{-1}(x^2 + x^{-2})) + pq\left[\frac{1}{2}(t^2 + t^{-2}) - 2 + t^{-2}(x^4 + x^{-4})\right] + \ldots$$

$$\psi_1(x) = (x + x^{-1})\left[1 + \sqrt{pq}\left(\frac{t}{2} - \frac{1}{2t} + t^{-1}(x^2 + x^{-2})\right) + \frac{p+q}{2} + \frac{3}{8}(p^2 + q^2) + \ldots\right]$$

Expansion is up to $(pq)^{\frac{7}{2}}$ for $\psi_0(x)$ and up to $(pq)^{\frac{3}{2}}$ for $\psi_1(x)$

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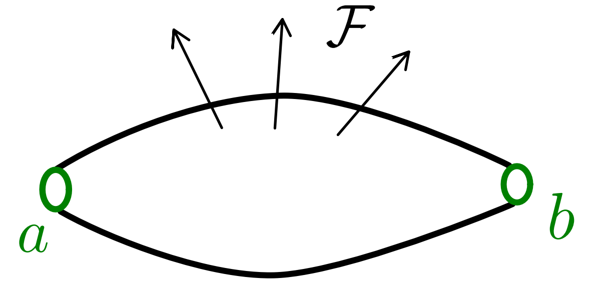

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Expansion is up to $(pq)^{\frac{7}{2}}$ for $\psi_0(x)$ and up to $(pq)^{\frac{3}{2}}$ for $\psi_1(x)$

- Eigenvalues:

$$\mathcal{H}_{A_1} \cdot \psi_n(x) = E_n \psi_n(x)$$

Substituting eigenfunctions

$$E_0 = 1 - p + (t + \frac{1}{t})\sqrt{pq} - pq + (t + \frac{1}{t}) p\sqrt{pq} - p^2 + \ldots$$

More Results

- 6d (2,0) SCFT of A_N type $\Longrightarrow$ A_N Ruijsenaars-Schneider model

  ground and first excited states for N = 1,2; Kim, AN, Razamat 2407.08776;

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- 6d E - string theory $\Longrightarrow$ BC_1 van Diejen model Nazzal, Razamat 1801.00960; Nazzal, AN, Razamat 2106.08335;

  ground state for certain values of parameters Nazzal, AN, Razamat 2305.09718;

  Kim, AN, Razamat (in progress)

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- 6d minimal SCFTs $\Longrightarrow$ novel A_2 and A_3 models Razamat 1808.09509; Ruijsenaars 2003.11353;

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- Minimal (D,D) conformal matter $\Longrightarrow$ novel A_N and C_2 generalizations of van Diejen.
  nothing so far Nazzal, AN, Razamat 2106.08335; Nazzal, AN 2305.09718;

Conclusions and Outlook

- We proposed a new method for deriving the perturbative spectrum of a large class of relativistic elliptic integrable models.

- The method has been tested on various models: Rujsenaars-Schneider, van Diejen and some novel models derived in 6d compactifications.

  In all cases at least ground state wavefunction and energy were derived.

- For the cases of A_1 and A_2 Ruijsenaars-Schneider model results were compared with the alternative approach of ramified instantons claculations. (not in this talk!) Kim, AN, Razamat 2407.08776;

- Many more things to be considered (see previous slide).

Thank you !