Detecting local topology via the spectral localizer

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Chern numbers in $d = 2$

Short-range Hamiltonian H on $\ell^2(\mathbb{Z}^2, \mathbb{C}^L)$, Fermi $P = \chi(H \leq E)$

For periodic system: Bloch-Floquet theory

$$\mathrm{Ch}(P) = 2\pi i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \mathrm{Tr}\big(P_k[\partial_{k_1} P_k, \partial_{k_2} P_k]\big) = \frac{i}{2\pi} \int \mathrm{Tr}(PdPdP) \in \mathbb{Z}$$

Noncommutative analog for random $H = (H_\omega)_{\omega \in \Omega}$ using positions

$$\begin{aligned} \mathrm{Ch}(P) &= 2\pi i\, \mathbb{E}\, \mathrm{Tr}\big(\langle 0|P\,\big[[X_1, P], [X_2, P]\big]|0\rangle\big) = 2\pi i\, \mathcal{T}(PdPdP) \\ &= 2\pi i\, \mathbb{E}\, \mathrm{Tr}\big(\langle 0|[PX_1P, PX_2P]|0\rangle\big) \qquad (\text{averaged local marker}) \end{aligned}$$

Index theorem (Connes, Bellissard 1980's, *et al.* early 1990's)

$E \in \Delta \subset \mathbb{R}$ Anderson localized. Then almost surely

$$\mathrm{Ch}(P) = \mathrm{Ind}(PFP) \in \mathbb{Z} \qquad , \qquad F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$$

and $\mu \in \Delta \mapsto \mathrm{Ch}(P)$ constant

Numerical computation of Chern number

Periodic system: implementation of k-integral, twisted BC

disordered system: compute P from H (costly), then above formula

Topological photonic crystals: 100's of bands, not feasible

Spectral localizer: (Loring 2015) gap at $E = 0$, (dual) Dirac trap

$$L_\kappa = \begin{pmatrix} -H & \kappa(X_1 - iX_2) \\ \kappa(X_1 + iX_2) & H \end{pmatrix}$$

Selfadjoint $L_\kappa = (L_\kappa)^*$ with compact resolvent. **Fact:** gap at 0

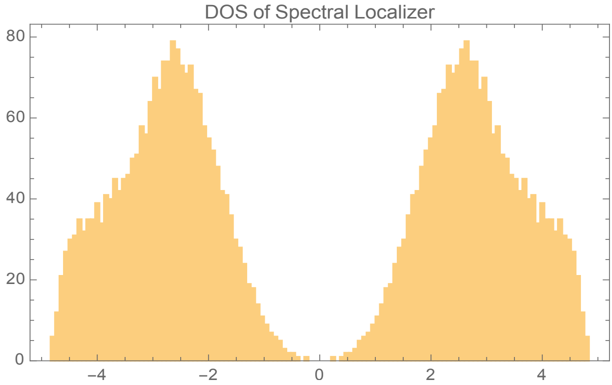
$L_{\kappa,\rho}$ finite volume restriction to $[-\rho, \rho]^2$. For κ small and ρ large:

$$\mathrm{Ch}(P) = \frac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})$$

Computation: only LDL necessary for Sig! **No spectral calculus!**

Implementation for dirty $p + ip$ superconductor

Density of states (DOS) of the localizer for $\kappa = 0.1$ and $\rho = 20$



Looks harmless, however, note gap at 0

Spectral asymmetry $= -2 \; = \#$ positive $- \; \#$ negative eigenvalues

Finite volume computation of Chern numbers

Theorem (with Loring 2017, 2020)

Let $g = \|(H-\mu)^{-1}\|^{-1}$ be gap of homogeneous H. Suppose

$$\frac{2\,g}{\rho} \;<\; \kappa \;<\; \frac{g^3}{12\,\|H\|\,\|[X_1+iX_2,H]\|}$$

Then $L_{\kappa,\rho}$ has (topological protection) gap $\mu_{\kappa,\rho} \geq \frac{g}{2}$ at 0 and

$$\mathrm{Ch}(P) \;=\; \frac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})$$

If H "differentiable", conditions always OK for κ small and ρ large

Homogeneous model: typically $\kappa \approx 0.1$, $\rho \approx 20$ sufficient

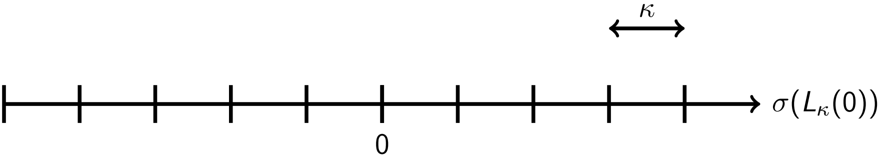
Proof: K-theory of fuzzy spheres or spectral flow

Also: other dimensions, strong & weak, $\mathbb{Z}_2$'s, & other things

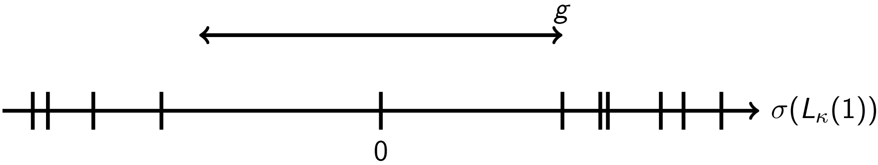
Intuition: H topological mass term added to Dirac

$$L_\kappa(\lambda) \;=\; \begin{pmatrix} -\lambda\, H & \kappa\,(X_1 - iX_2) \\ \kappa\,(X_1 + iX_2) & \lambda\, H \end{pmatrix} \qquad , \qquad \lambda \geq 0$$

Spectrum for $\lambda = 0$ symmetric and with space quanta κ



Spectrum for $\lambda = 1$: less regular, central gap open and asymmetry



Spectral asymmetry determined by low-lying spectrum (finite vol!)

Spectral flow proof (for odd index pairings)

Using $\mathrm{Sf} = \mathrm{Ind}$ for phase $U = A|A|^{-1}$ and $\Pi = \chi(D > 0)$ Hardy:

$$
\begin{aligned}
\mathrm{Ch}_d(A) &= \mathrm{Ind}(\Pi A \Pi + \mathbf{1} - \Pi) \;=\; \mathrm{Ind}(\Pi U \Pi + \mathbf{1} - \Pi) \\
&= \mathrm{Sf}(U^* D U, D) \;=\; \mathrm{Sf}(\kappa\, U^* D U, \kappa\, D) \\
&= \mathrm{Sf}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{Sf}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & \mathbf{1} \\ \mathbf{1} & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{Sf}\left(\begin{pmatrix} \kappa\, U^* D U & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{Sf}\left(\begin{pmatrix} \kappa\, D & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right)
\end{aligned}
$$

Now localize and use $\mathrm{Sf} = \frac{1}{2}\,\mathrm{Sig\text{-}Diff}$ on paths of s.-a. matrices $\square$

Local nature of L_κ in space and in energy

Shift localizer to energy E ($e.g.$ through mobility gap or band)

and in space to $x = (x_1, x_2)$ ($e.g.$ through interface)

$$L_{\kappa,\rho}(E,x) \;=\; \begin{pmatrix} -(H_\rho - E) & \kappa\big((X_1 - x_1) - i(X_2 - x_2)\big) \\ \kappa\big((X_1 - x_1) + i(X_2 - x_2)\big) & H_\rho - E \end{pmatrix}$$

Here H_ρ either Dirichlet or periodic boundary condition

N.B.: for large X_j, H_ρ and its edge states dominated

Intuition: low lying spectrum depends on phase space point (x, E)

But: bound on topological protection depends on global quantities

(global gap g and operator norms $\|H\|$ and $\|[X, H]\|$)

So: no stability of μ under large perturbations far out

Three crucial improvements of stability criterion:

local energy gap, relative operator norms, optimized constants

Preparations: local gap and tapering estimate

Local gap of H

ρ-local gap $g_\rho(H,x)$ is largest g such that

$$(H^2)_{B_\rho(x)} \;\geq\; g^2\,\mathbf{1}_\rho(x)$$

N.B.: $(H^2)_{B_\rho(x)} \neq (H_{B_\rho(x)})^2$, so no edge states

Obvious: locality, $g_\rho(H,x)$ decreasing in ρ, global gap criterion

Tapering Function $F:\mathbb{R}\to[0,1]$

Even, C^1, with $F(y)=0$ for $|y|\geq 1$ and $F(y)=1$ for $|y|\leq\frac{1}{2}$

Given $\rho>0$, set $F_\rho(y)=F(\frac{y}{\rho})$

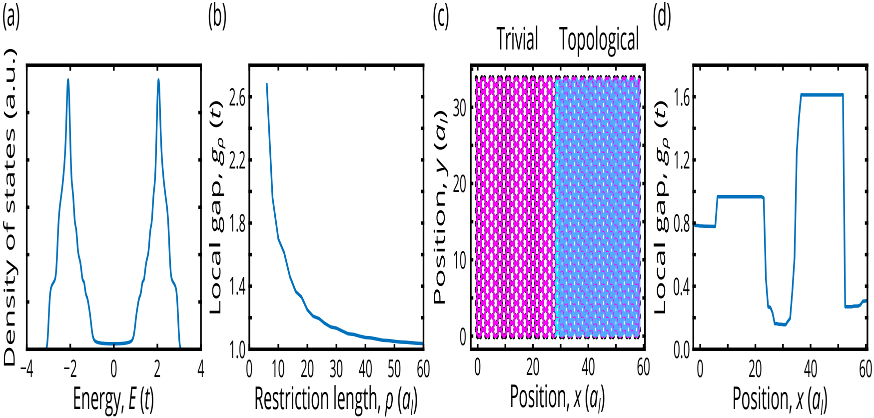
Tapering estimate with constant C_F

$$\|[F_\rho(X-x),H]\| \;\leq\; \frac{C_F}{\rho}\,\|[X,H]\|$$

Numerical illustration of local gap

(a) and (b): Haldane model on 80×80 sites, x center of sample

(c) and (d): massive graphene/Haldane heterostructure, $\rho = 12$

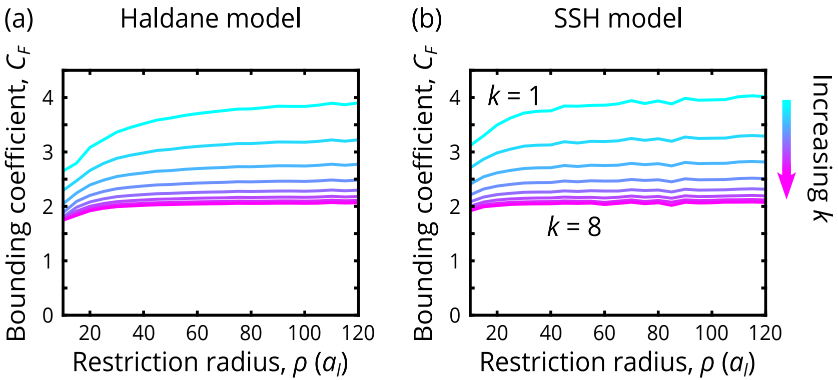


Constant in tapering estimate

Bratteli-Robinson: $C_F = \|\widehat{F'}\|_{L^1}$ Construction:

$$F(x) = \varphi(2x+2) - \varphi(2x-1) \text{ with } \varphi(x) = \frac{1}{\int_0^1 dy\, \phi(y)} \int_0^x dy\, \phi(y)$$

Optimizing $\phi_k(x) = \exp(-2^k \frac{1}{x(1-x)})$ for $x \in [0,1]$ gives $C_F = 2$



Improved local criterion for topological protection

Theorem (with Cerjan, arbitrary even dimension d)

Let $a, b \geq 0$ with $1 - a - b^2 > 0$ and set $c^2 = \frac{a}{1-a-b^2}$ as well as

$$R_\kappa = \left(i\,\mathbf{1} + \frac{c\,\kappa}{g_\rho} D(x)\right)^{-1} \quad , \quad D(x) = \sum_{j=1}^{d} \gamma_j (X_j - x)$$

Suppose finite volume criterion

$$\frac{2\,g_\rho}{\rho} < \kappa \leq \frac{g_\rho^3}{\frac{1}{1-a-b^2}\left(C_F \,\|HR_\kappa\| + g_\rho\right)\|[D(x), H]R_\kappa\|}$$

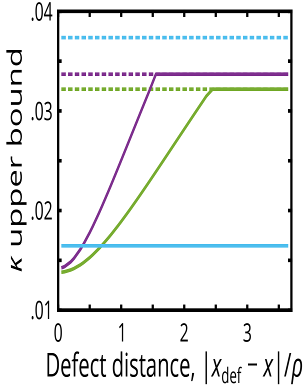
Then for $\rho' \geq \rho$ signature constant & localizer gap satisfies

$$\mu_{\kappa,\rho'}(H, x) \geq b\,g_\rho(H, x)$$

Before: $a = 0$, $b = \frac{1}{2}$ with $g \leq g_\rho$, still better constant $\frac{4}{3} C_F \approx \frac{8}{3}$

Improvement for perturbation $H = H_{\mathrm{Hal}} + W$

Support of large perturbation W centered at x_{def}; dashed without



$$\frac{g_\rho^3}{\frac{1}{1-a-b^2}\left(C_F\,\|HR\| + g_\rho\right)\|[D(x),H]R\|} \qquad c^2 = \frac{a}{1-a-b^2}$$

$$R = \left(\imath\mathbf{1} + \tfrac{c\,\kappa}{g_\rho}D(x)\right)^{-1} \quad a = \tfrac{3}{20} \quad b = \tfrac{1}{2}$$

$$R = \left(\imath\mathbf{1} + \tfrac{2c}{\rho}D(x)\right)^{-1} \quad a = \tfrac{3}{20} \quad b = \tfrac{1}{2}$$

$$R = -\imath\mathbf{1} \quad a = 0 \quad b = \tfrac{1}{2}$$

Upper bound on κ much weaker. Improved locality property!

With x_{def} small or without perturbation better to use $a = 0$

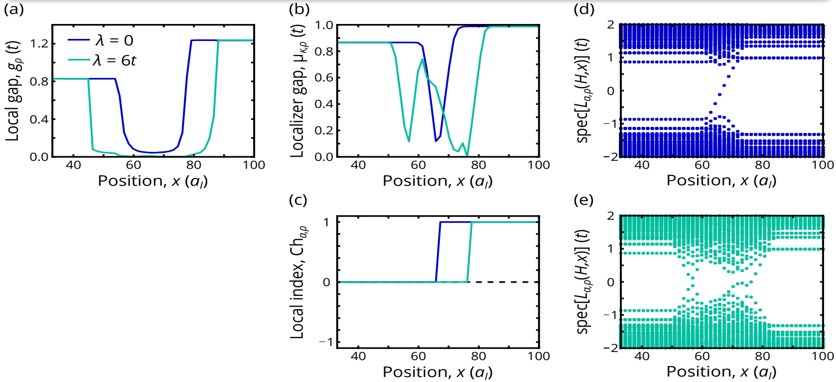
Stability of spectral flow for $H = H_{\text{Hetero}} + \lambda W$

Now W placed on interface of heterostructure as before

Local gaps change when x crosses support of W. However:

Proposition

Spectral flow of $x \mapsto L_{\kappa,\rho}(E,x)$ is stable



Technical elements of proof

Let $x = 0$, $E = 0$ and set $F_\rho = F_\rho(|D|)$ and $R = R_\kappa$:

$$
\begin{aligned}
(L_{\kappa,\rho})^2 \;&=\; \pi_\rho\,(\kappa D + H\sigma_3)\,\mathbf{1}_\rho\,(\kappa D + H\sigma_3)\,\pi_\rho^* \\
&= \;\kappa^2\,\pi_\rho D^2 \pi_\rho^* + \pi_\rho H \mathbf{1}_\rho H \pi_\rho^* + \kappa\,\pi_\rho [D,H]\sigma_3 \pi_\rho^* \\
&\geq \;\kappa^2\,\pi_\rho D^2 \pi_\rho^* + \pi_\rho H F_\rho^2 H \pi_\rho^* + \kappa\,\pi_\rho [D,H]\sigma_3 \pi_\rho^* \\
&= \;\kappa^2\,\pi_\rho D^2 \pi_\rho^* + \pi_\rho F_\rho H^2 F_\rho \pi_\rho^* + \pi_\rho\big([H,F_\rho]F_\rho H + \text{h.c.} + \kappa\,[D,H]\sigma_3\big)\pi_\rho^* \\
&\geq \;(1-a)\kappa^2\,\pi_\rho D(\mathbf{1} - F_\rho^2) D \pi_\rho^* + a\,\kappa^2\,\pi_\rho D^2 \pi_\rho^* + g_\rho^2\,F_\rho^2 + \pi_\rho B \pi_\rho^* \\
&\geq \;(1-a)\,g_\rho^2\,(\mathbf{1} - F_\rho^2) + a\,\kappa^2\,\pi_\rho D^2 \pi_\rho^* + g_\rho^2\,F_\rho^2 + \pi_\rho B \pi_\rho^* \\
&\geq \;(1-a)\,g_\rho^2 \mathbf{1}_\rho + a\,\kappa^2\,\pi_\rho D^2 \pi_\rho^* + \pi_\rho B \pi_\rho^* \\
&= \;b^2\,g_\rho^2 \mathbf{1}_\rho + (1-a-b^2)\,g_\rho^2\,\pi_\rho (R^*)^{-1}\Big(\mathbf{1} + \tfrac{1}{1-a-b^2}\,\tfrac{1}{g_\rho^2}\,R^* B R\Big) R^{-1} \pi_\rho^*
\end{aligned}
$$

Hypothesis readily implies following bound, assuring the claim:

$$
\|R^* B R\| \;\leq\; (1-a-b^2) g_\rho^2
$$

Spectral localizer in mobility gap regime

Add Anderson-type disorder to topological model (Haldane)

Search for good choice of κ, ρ. Due to Poisson statistics

$$\#\{E_j \in (-\delta,\delta)\ :\ \text{localization center}\ \in B_\rho(0)\}\ \approx\ \delta\,\rho^d\,\tfrac{d\mathcal{N}}{dE}(0)$$

where $\frac{d\mathcal{N}}{dE}(0)$ is DOS at energy $E = 0$ (choice of reference here)
Expected value of gap is roughly smallest δ for which r.h.s. is 1:

$$\mathbb{E}(g_\rho)\ \approx\ \frac{1}{\rho^d\ \frac{d\mathcal{N}}{dE}(0)}$$

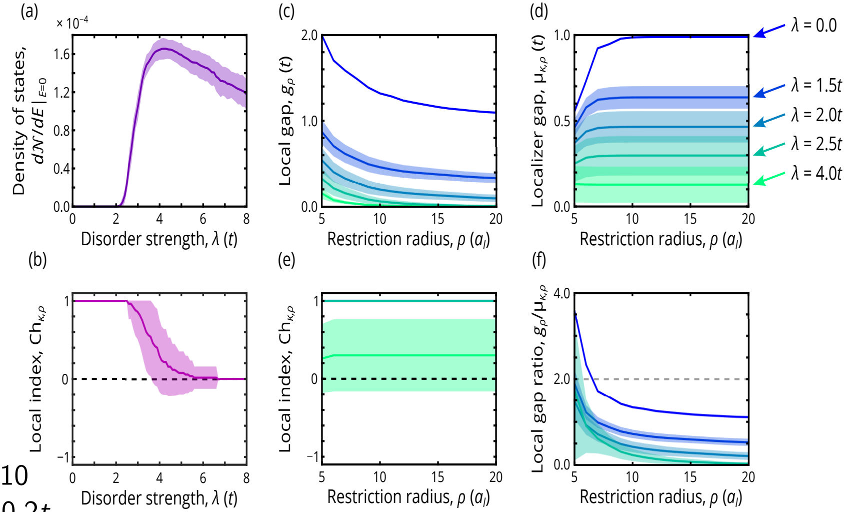
By the deterministic criterion, gap hence "often" open for choice

$$\kappa \approx \frac{\mathbb{E}(g_\rho)}{\rho}\qquad,\qquad \rho\ \leq\ \Big(C_F\,\|[H\|\,\|[D,H]\|\,\tfrac{d\mathcal{N}}{dE}(0)\Big)^{-\frac{1}{2d-1}}$$

Useful only for small DOS, but numerics show wider applicability:

Numerics for disordered Haldane (50 realizations)

$$H(\lambda) = H_{\mathrm{Hal}}(t) + \lambda \sum_{n\in\Gamma} v_n |n\rangle\langle n| \quad \text{with } (v_n)_{n\in\Gamma} \text{ i.i.d. in } [-\tfrac{1}{2}, \tfrac{1}{2}]$$



$\rho = 10$

$\kappa = 0.2t$

Modifications and extensions

Locality criteria and locality properties transpose to:

- odd Chern (winding) numbers with odd spectral localizer
- $\mathbb{Z}_2$ via skew localizer (skew-symmetric) (with Doll, 2021)
- spin Chern number via twisted localizer (with Doll, 2020)
- non-hermitian localizer (with Cerjan, Koekenbier, 2023)
- higher order topology ? (with Cerjan, Loring, 2024)
- fragile topology ? (Lee, Wong, et. al. 2025)

Further modifications of the spectral localizer:

- weak winding numbers $\notin \mathbb{Z}$ in semimetals (with Stoiber, 2021)
- Weyl/Dirac point count with low-lying spec. (with Stoiber, 2022)
- length of Fermi surface in metals (Franca, Grushin, 2023)
- topology in non-linear regime (Wong *et al.*, 2024)
- periodic spectral localizer (with Doll, Loring, 2025) • • •

Modification: odd spectral localizer for odd d

Chiral Hamiltonian with (mobility) gap at 0

$$H = -JHJ = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad , \quad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

Also approximate chirality $\|H + JHJ\| < 2g$ is actually sufficient

Odd Chern numbers (higher winding numbers)

$$\mathrm{Ch}_{\{1,\ldots,d\}}(A) = \frac{i(i\pi)^{\frac{d-1}{2}}}{d!!} \sum_{\sigma \in S_d} (-1)^{\sigma} \, \mathbb{E}\,\mathrm{Tr}\Big(\langle 0| \prod_{j=1}^{d} (A^{-1} i[X_{\sigma_j}, A]) |0\rangle\Big)$$

Build odd spectral localizer from (dual) Dirac $D = \sum_{j=1}^{d} \gamma_j X_j$,
then under same condition on κ and ρ with bounded $[A, D]$:

$$\boxed{L_\kappa = \begin{pmatrix} \kappa D & A^* \\ A & -\kappa D \end{pmatrix}} \quad \Longrightarrow \quad \boxed{\mathrm{Ch}_{\{1,\ldots,d\}}(A) = \frac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})}$$

$\mathbb{Z}_2$-invariants via skew localizer

Works for all 16 AZ-classes with strong $\mathbb{Z}_2$ index

Focus: $d = 2$ and odd TRS $I^*\overline{H}I = H$ with $I = i\sigma_2$ (QSHE)

Fredholm $T = PFP$ satisfies $I^* T^t I = T$ and thus well-defined

$$\mathrm{Ind}_2(T) \; = \; \dim(\mathrm{Ker}(T)) \bmod 2 \; \in \; \mathbb{Z}_2$$

Real skew localizer from $\Re(H) = \frac{1}{2}(H + \overline{H})$ and $\Im(H) = \frac{1}{2i}(H - \overline{H})$

$$L_\kappa \; = \; \begin{pmatrix} \Im(H) + \kappa X_1 I & \Re(H) I + \kappa X_2 \\ I\, \Re(H) - \kappa X_2 & \Im(H) - \kappa X_1 I \end{pmatrix} \; = \; \overline{L_\kappa} \; = \; -(L_\kappa)^*$$

Theorem (with Doll, under same local criteria)

$$\mathrm{Ind}_2(PFP) \; = \; \mathrm{sgn}(\mathrm{Pf}(L_{\kappa,\rho}))$$

For 8 of 16 cases, skew localizer is off-diagonal & only det needed

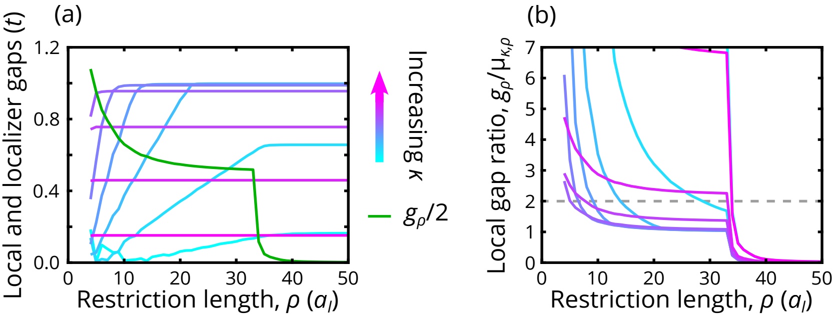
Local local gaps in heterostructure (as above)

Fixed x center of topological phase; at $\rho = 33$ touching of interface

Plot of local gap g_ρ (green) and localizer gap $\mu_{\kappa,\rho}$ for various

$\kappa = 0.005,\ 0.02,\ 0.05,\ 0.1,\ 0.2,\ 0.5,\ 1,\ 1.5,\ 2$ (cyan to magenta)

Optimal (minimal) choice of ρ for given κ: when flat attained



Theoretical bound $\frac{g_\rho}{\mu_{\kappa,\rho}} \leq \frac{1}{b} = 2$ (not covered by older result)