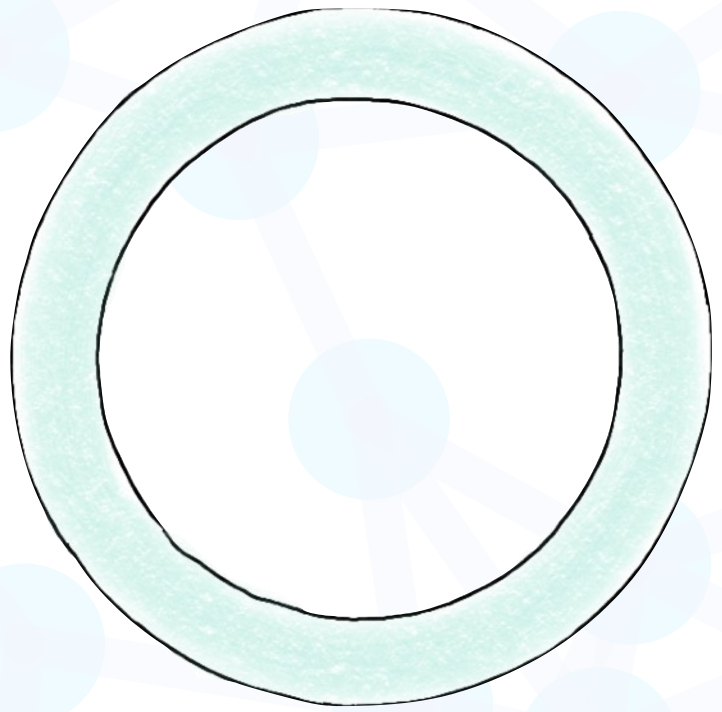
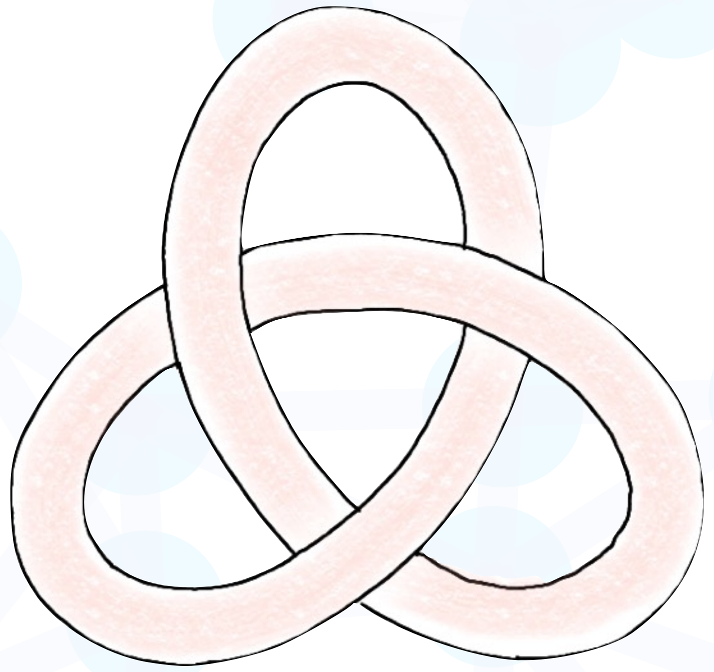
Local Topological Markers

Characterising topology of disordered and interacting matter in odd dimensions

Julia D. Hannukainen





with: Miguel F. Martínez, Jens H. Bardarson, and Thomas Klein Kvorning

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Phys. Rev. Lett. **129**, 277601 (2022)
Phys. Rev. Research. **6**, L032045 (2024)

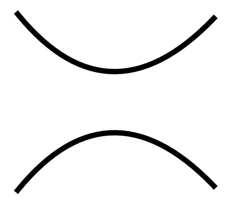
The local Chern marker and its odd dimensional analogues

Key object: $\rho_{ij} = \langle \Phi | c_i^\dagger c_j | \Phi \rangle$

Characterise states

$|\Phi\rangle$

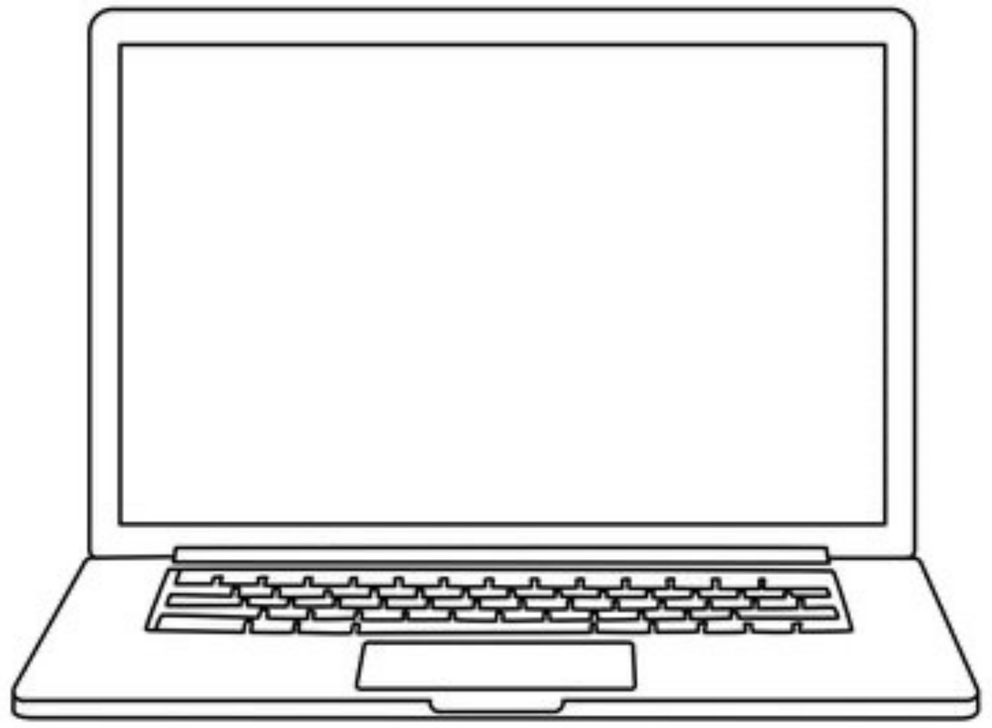
Spectral gap not necessary



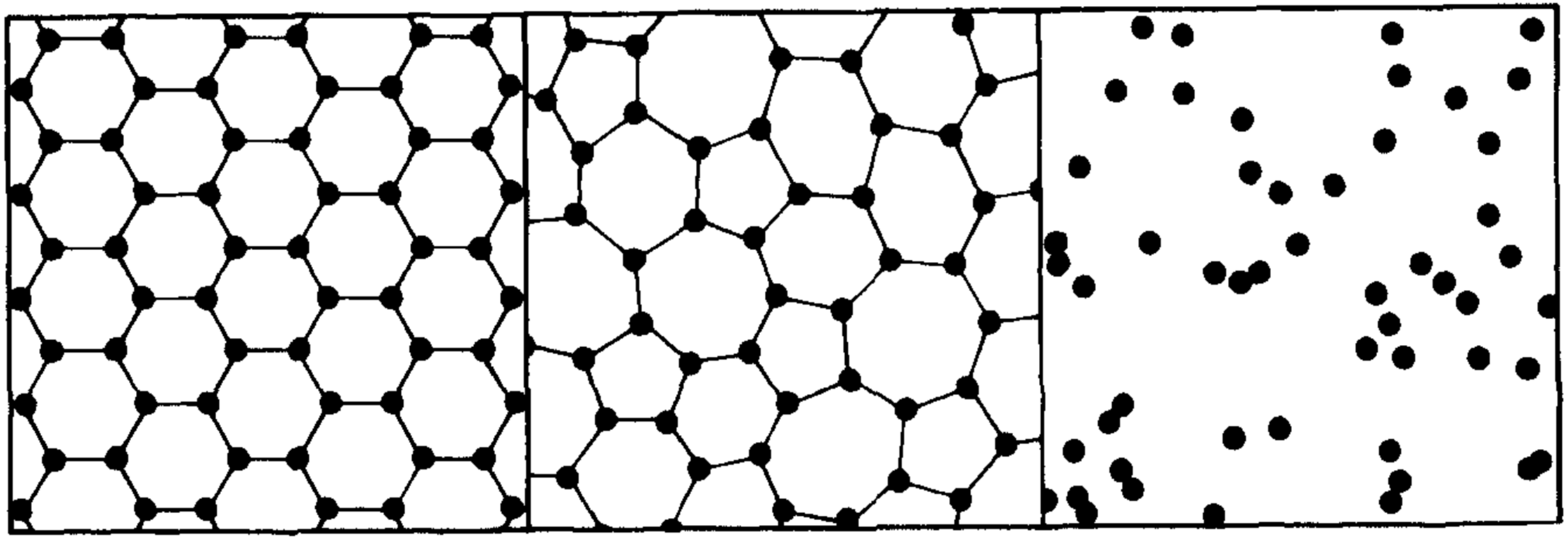
Interacting states

$|\Phi_{\text{int}}\rangle$

$$\nu(\mathbf{r}) = -\frac{8\pi i}{3} \sum_{\alpha} \varepsilon^{ijk} [\rho S X_i \rho X_j \rho X_k \rho]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}$$



Amorphous materials are defined by their lack of long-range order



Zallen, (1998)

Why topological amorphous matter?

'Nearly all materials if cooled fast enough and far enough can be prepared amorphous'

Topological Amorphous matter

New properties?

Zallen, (1998)
M. G.Vergniory et al., Nature. 566(7745), 480–485 (2019)
L. Elcoro, et al. Nat Commun 12, 5965 (2021)
P, Corbae, JDH et al **142** EPL (2023)

Topologically equivalent insulators

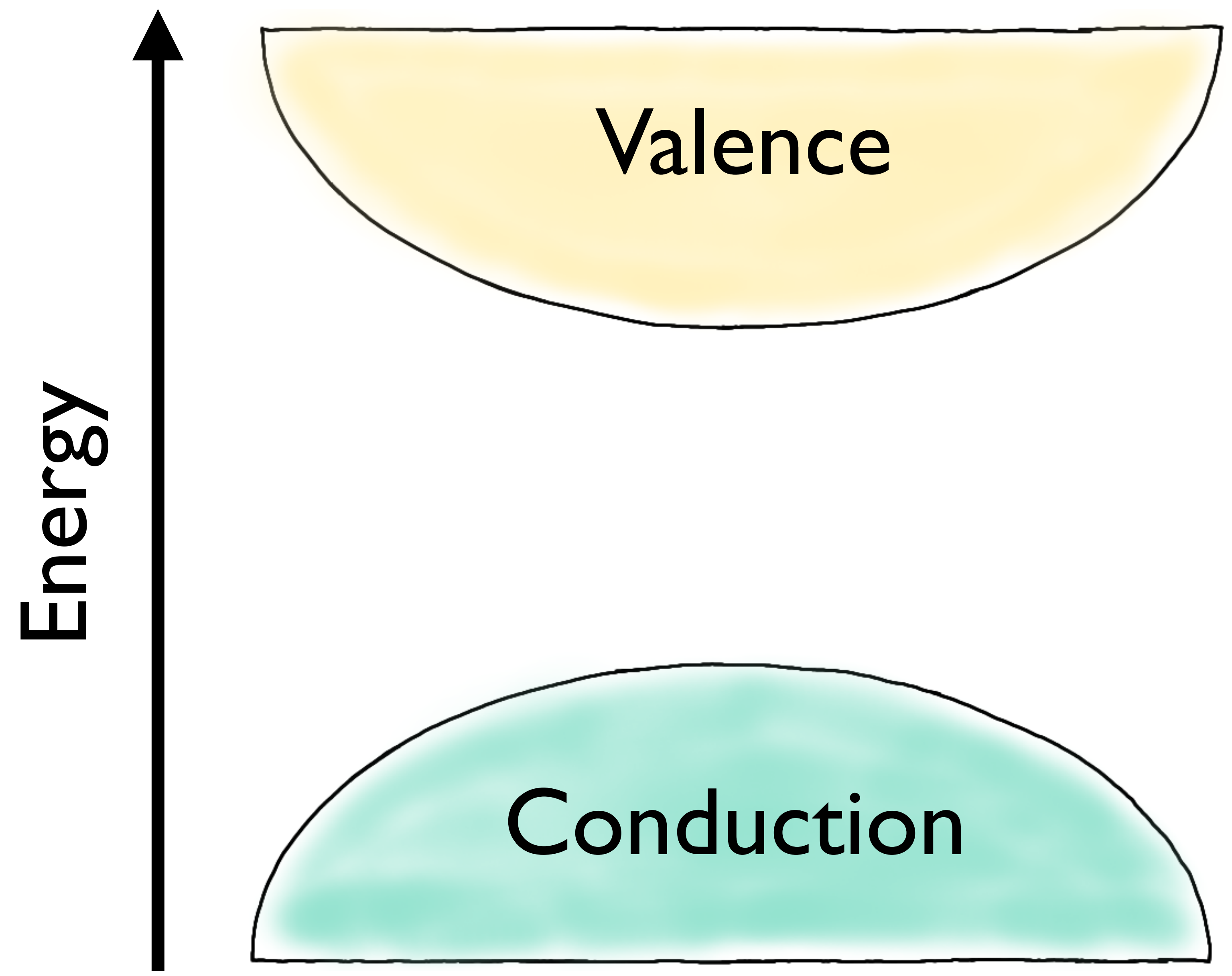
$$\mathcal{H}_1(\mathbf{k}) \sim \mathcal{H}_2(\mathbf{k})$$

Keep gap open

Preserve symmetries

$\mathcal{T}, \mathcal{P}, \mathcal{S}$

Momentum space invariants



Hasan, M. Z. and Kane, C. L. Rev. Mod. Phys. 82 (4) (2010)

Translation invariance is not a requirement for topology

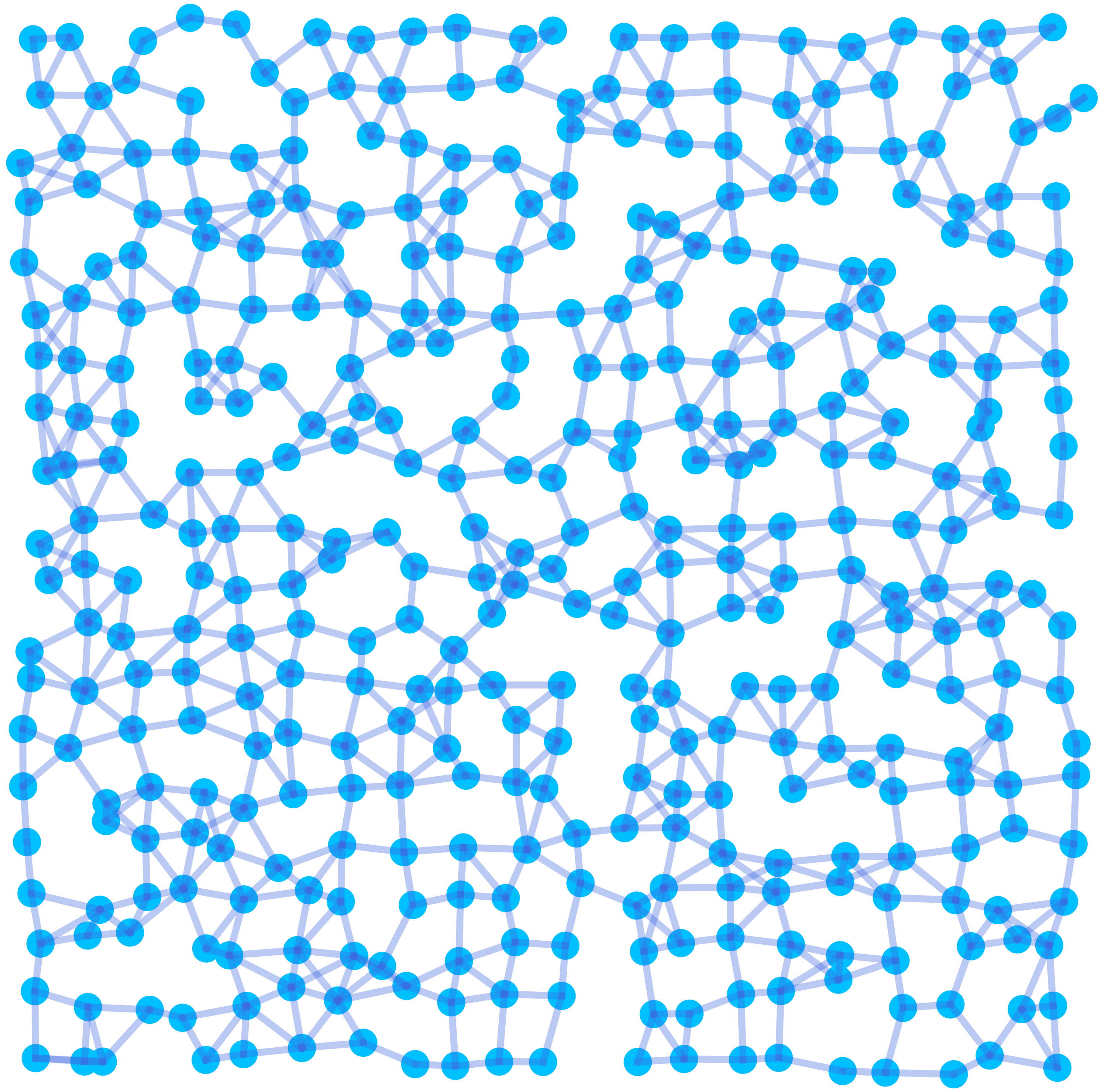
Free fermion state

$$|\Phi\rangle \sim |\Psi\rangle$$

If: $$|\Phi\rangle = U|\Psi\rangle$$

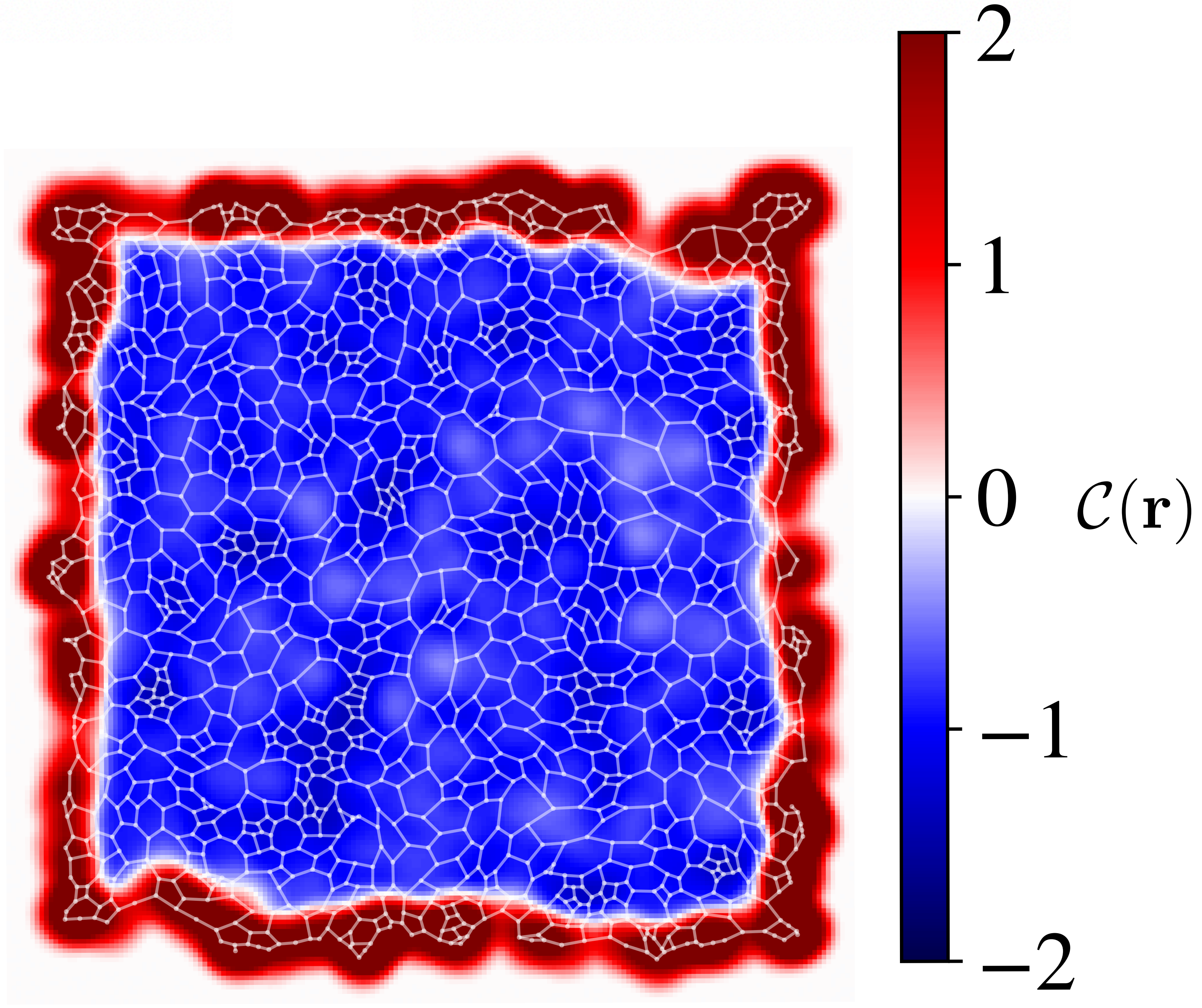
$$[U, O] = 0$$

$$O \in \{\mathcal{T}, \mathcal{P}, \mathcal{S}\}$$



X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010)

Local topological markers are real space characteristics of topology



Amorphous Chern insulator

$$\mathcal{C} = -\frac{1}{\pi}\sum_{n\in occ.}\int_{BZ} \mathrm{d}\mathbf{k}\langle \partial_{k_x} u_{n\mathbf{k}} | \partial_{k_y} u_{n\mathbf{k}}\rangle$$

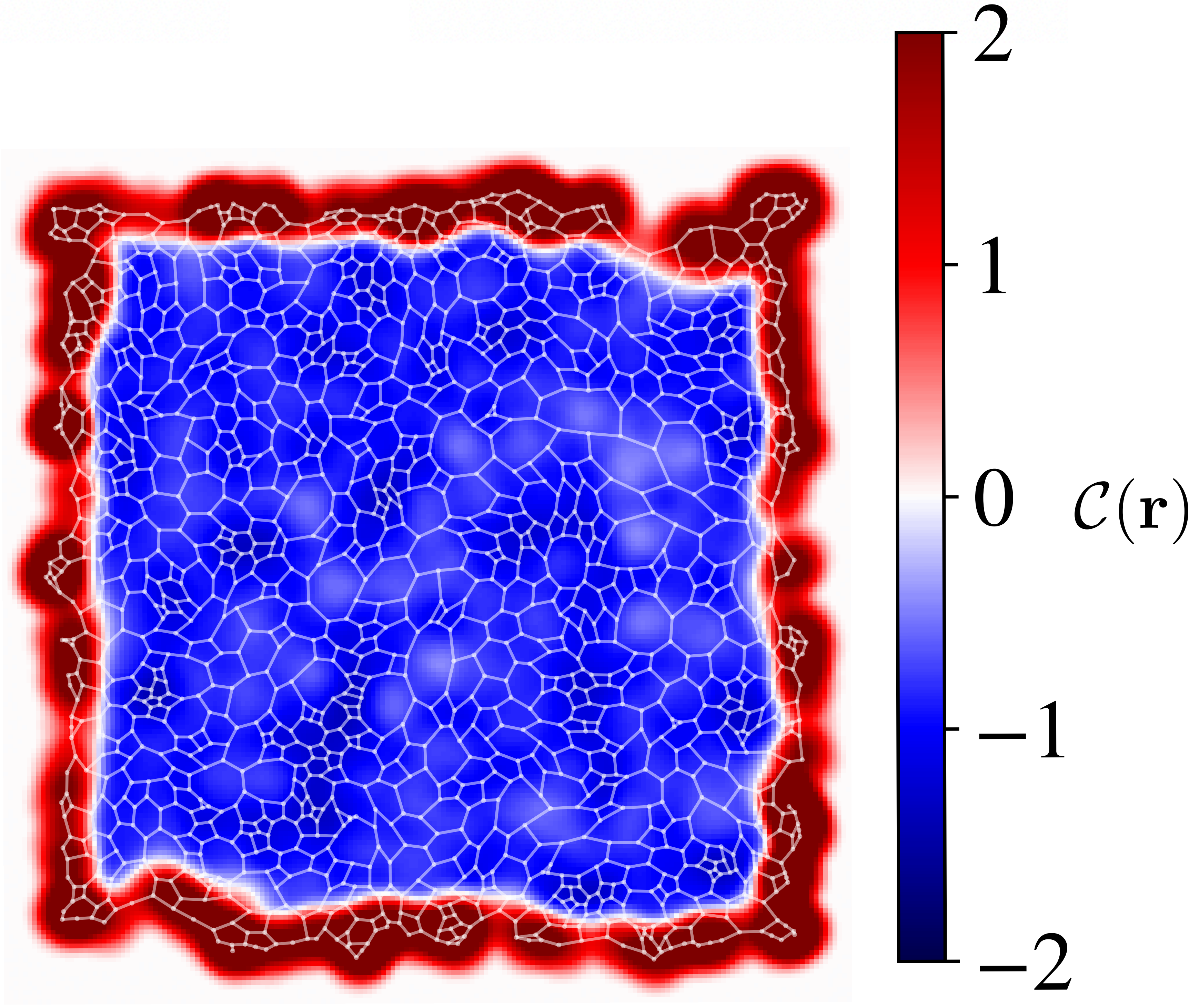
Fourier transform

$$\mathcal{C}(\mathbf{r}) = 2\pi \ \mathrm{Im}\ \langle \mathbf{r}|[\hat{Q}\hat{x}, \hat{P}\hat{y}]|\mathbf{r}\rangle$$

$$\hat{P} = \sum_{n\in occ.} |n\rangle\langle n| \qquad \hat{Q} = 1 - \hat{P}$$

R. Bianco, R. Resta Phys. Rev. B 84, 241106(R) (2011)
Figure from: Q. Marsal, D.Varjas, A.G. Grushin Proc. Natl. Acad. Sci. U.S.A. 117, 30260 (2020)

Quantisation by averaging over the bulk



Amorphous Chern insulator

Quantised invariant:

In practice:

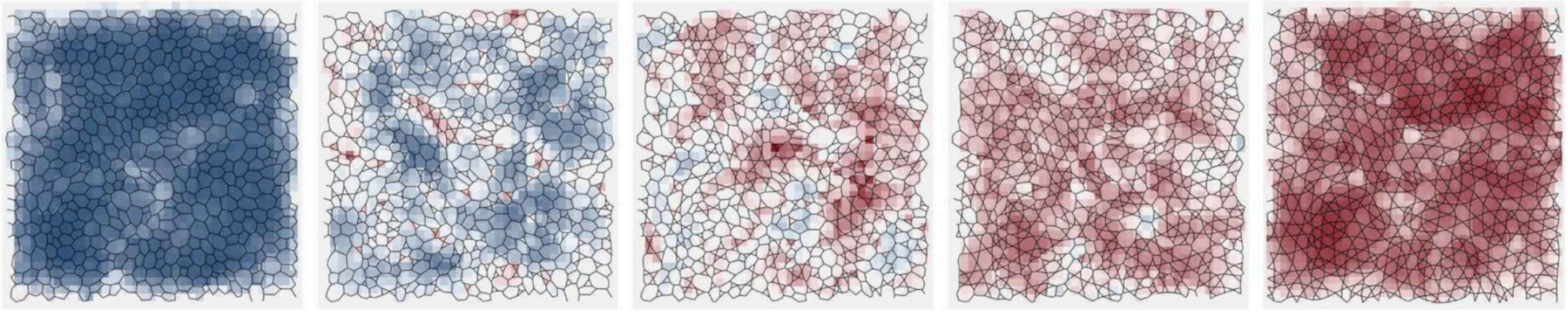
$$\mathcal{C} = \sum_{\mathbf{r} \in \text{bulk}} \frac{\mathcal{C}(\mathbf{r})}{\text{Area}}$$

Philosophy:

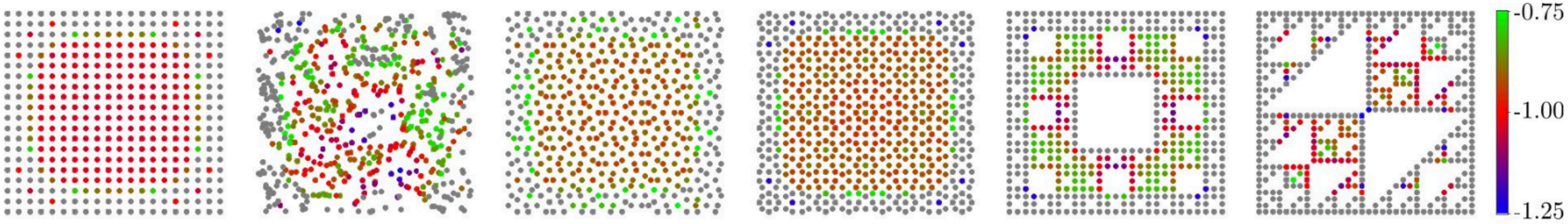
Coarse grained translation invariant lattice

R. Bianco, R. Resta Phys. Rev. B 84, 241106(R) (2011)
Figure from: Q. Marsal, D.Varjas, A.G. Grushin Proc. Natl. Acad. Sci. U.S.A. 117, 30260 (2020)

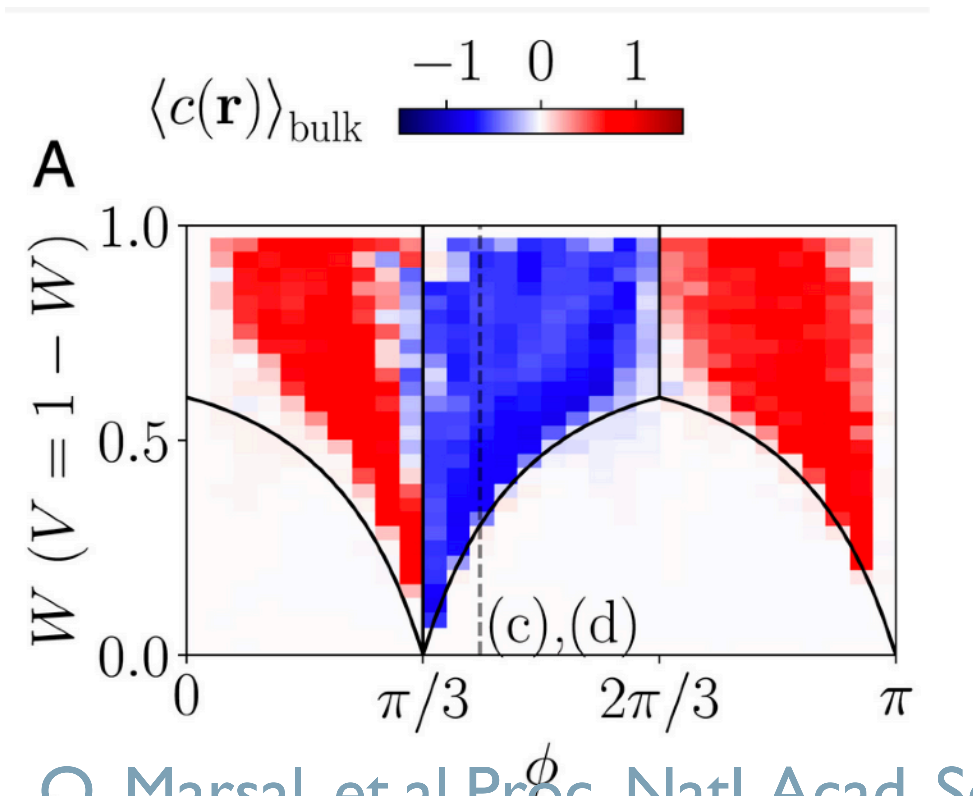
Chern markers are popular in the literature



N. P. Mitchell, et.al., Nat. Phys. 14 (2018)

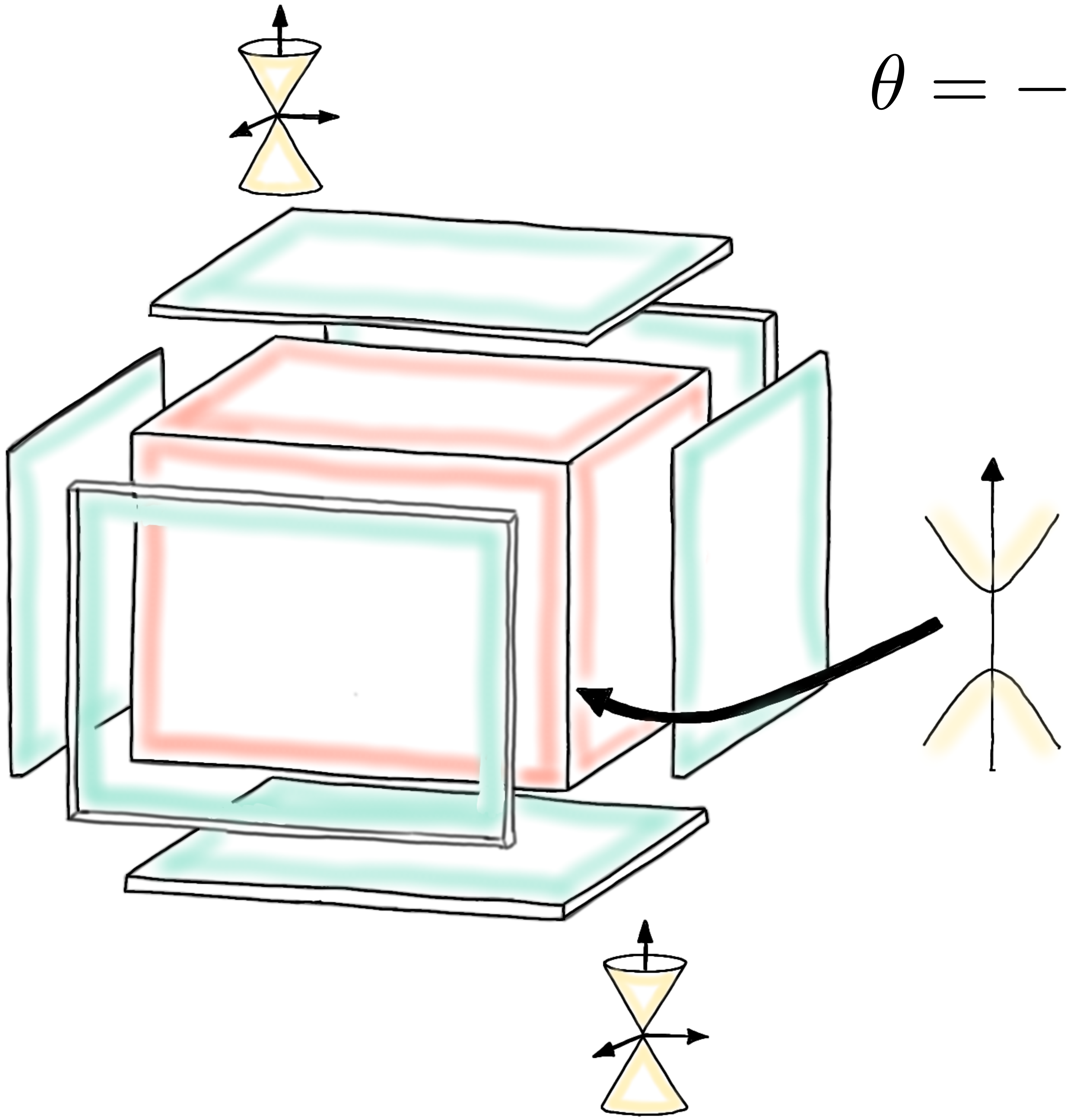


S. Manna, et.al. Phys. Rev. B **109**, 174512 (2024)



Q. Marsal, et.al.Proc. Natl.Acad. Sci. U.S.A. **117**, 30260 (2020)

Why can we not do the same in three dimensions?



$$\theta = -\frac{1}{4\pi}\int_{\mathrm{BZ}} d^3k\, \varepsilon^{ijk}\,\mathrm{Tr}\left(\mathcal{A}_\mathrm{i}\partial_\mathrm{j}\mathcal{A}_\mathrm{k} - \frac{2\mathrm{i}}{3}\mathcal{A}_\mathrm{i}\mathcal{A}_\mathrm{j}\mathcal{A}_\mathrm{k}\right)$$

Need a bit more vocabulary:

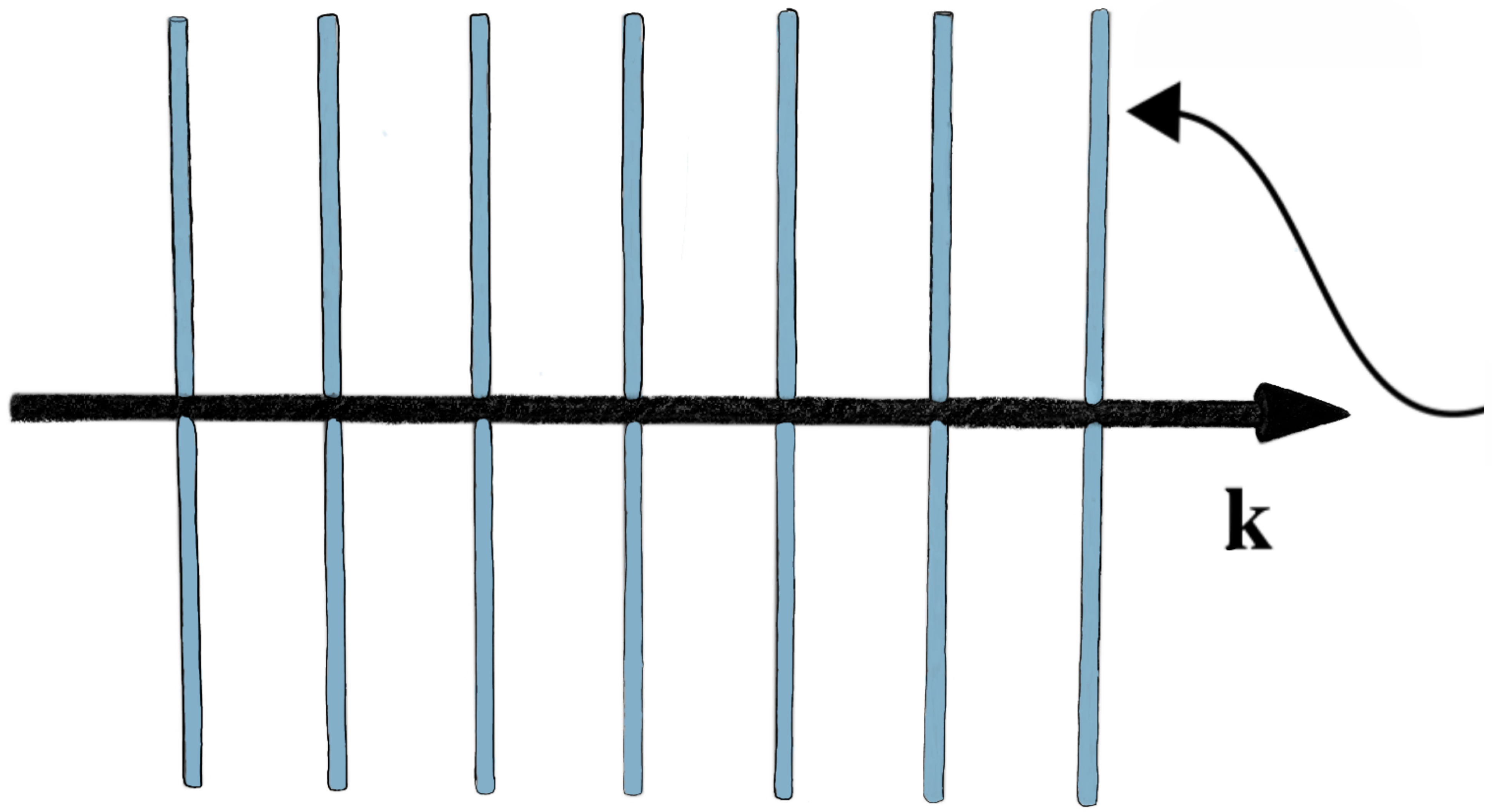
Vector bundles and bundle invariants.

Keep the coarse grained picture in mind!

X-L. Qi, T. L. Hughes, S. raghu, S-C. Zhang, Phys. Rev. Lett. **102**, 187001 (2009)
A. M. Essin, J. E. Moore, D. Vanderbilt, Phys. Rev. Lett. **102**, 146805 (2009)

Classification of vector bundles

Translation invariance



$$\text{fiber} = \mathrm{Im}[\rho(\mathbf{k})]$$

$$\|$$

complex
vector space

Smooth

$$\rho = \sum_{\substack{\lambda\in\text{filled}\\ \text{bands}}} |\lambda\rangle\langle\lambda|$$

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009)

The local Chern marker as the Fourier transform of the Chern character

Chern number: $$\mathcal{C}_n = \int_{\mathrm{BZ}} ch_n \qquad D = 2n$$

Chern character: $$ch_n = \frac{1}{(2\pi i)^n} \mathrm{Tr}(\overbrace{\mathcal{F} \wedge \cdots \wedge \mathcal{F}}^{n \text{ times}}) \qquad \mathcal{F}_{\mu\nu} = -\rho[(\partial_\mu \rho), (\partial_\nu \rho)]\rho,$$

$$ch_n = \frac{1}{(2\pi i)^n} \frac{1}{n!} \varepsilon^{i_1, \ldots, i_{2n}} \mathrm{Tr}\left(\rho(\mathbf{k})\partial_{k_{i_1}} \rho(\mathbf{k}) \ldots \rho(\mathbf{k})\partial_{k_{i_{2n}}} \rho(\mathbf{k})\right)$$

Fourier transform $\longrightarrow$ $$\boxed{\mathcal{C}(\mathbf{r}) = \sum_\alpha \frac{\varepsilon^{i_1, \ldots, i_D} [\rho X_{i_1} \rho X_{i_2} \cdots X_{i_D} \rho]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}}{(D/2)!/(2\pi i)^{D/2}}}$$

S. Ryu *et al*, *New J. Phys.* **12** 065010 (2010)

The single particle density matrix

$$|\Phi\rangle = \prod_i c_i^\dagger |0\rangle$$

$$\rho_{ij} = \langle \Phi | c_i^\dagger c_j | \Phi \rangle$$

$$\rho = \sum_{\substack{\lambda \in \text{filled} \\ \text{bands}}} |\lambda\rangle\langle\lambda|$$

$$|\Phi\rangle \sim |\Psi\rangle$$

If: $|\Phi\rangle = U|\Psi\rangle$

$$[U, O] = 0$$

$$O \in \{T, C, S\}$$

ρ_Φ and ρ_Ψ adiabatically connected

gapped

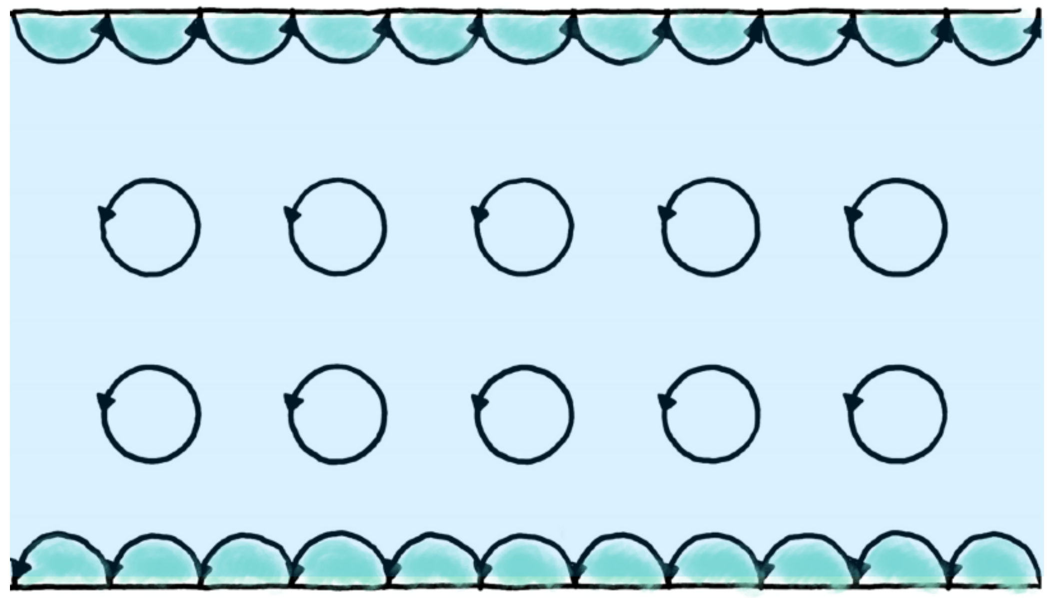
preserving symmetries

O. Penrose and L. Onsager, Physs. Rev. 104, 576 (1956)
S. Bera et al, Phys. Rev. Lett. 115, 046603 (2015)
X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010)

The Chern marker is easy to use

$$\mathcal{C}(\mathbf{r}) = 2\pi i \sum_{\alpha} \varepsilon^{1,2}[\rho X_1 \rho X_2 \rho]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}$$

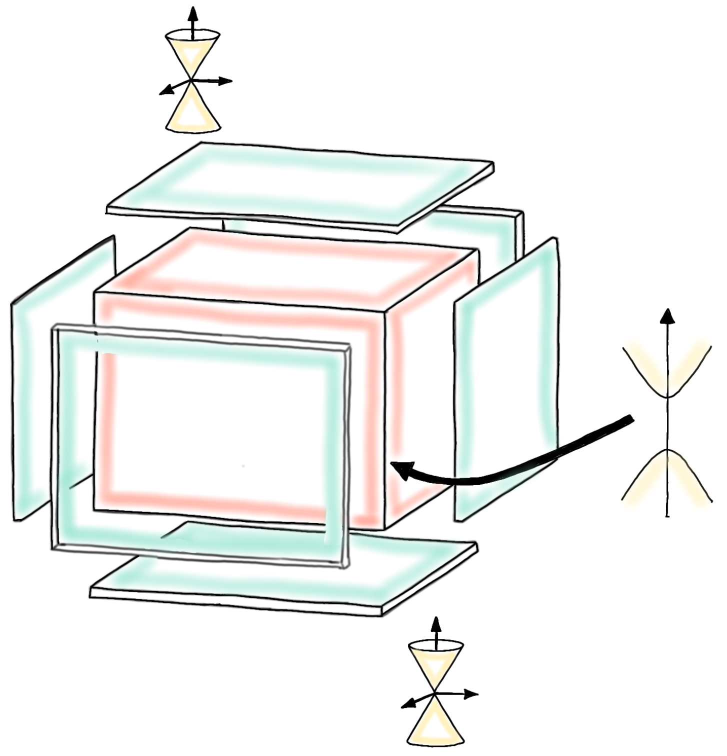
$$\rho = \langle \psi_0 | c_i^{\dagger} c_j | \psi_0 \rangle$$



$\Longrightarrow$ Characterisation of state

$\Longrightarrow$ Spectral gap not necessary

Task: Express odd dimensional invariant as a Chern character



Odd dimensions

Chern-Simons invariant:

$$\mathcal{CS}(B) = \int_{BZ} cs_n \mod (1)$$

→

Chiral winding number

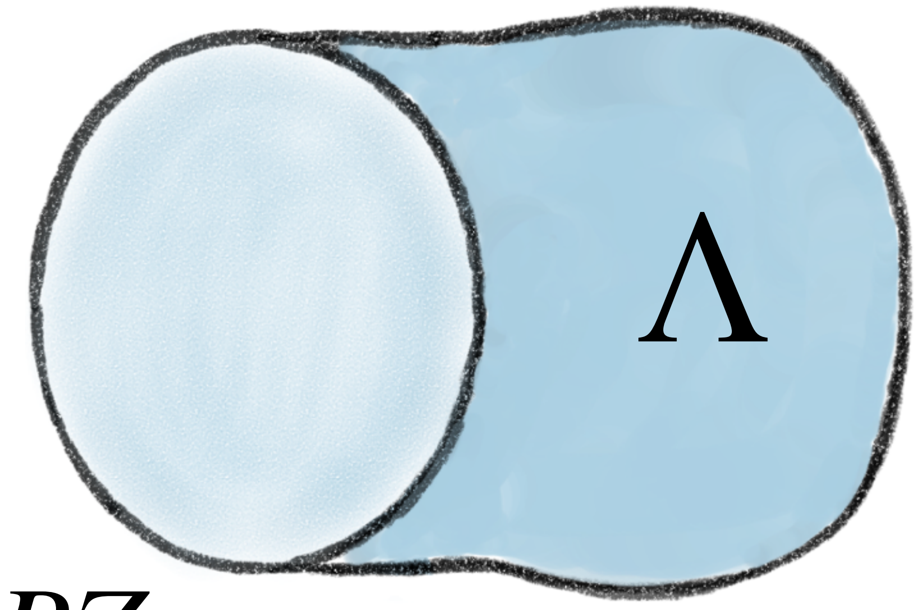
$\mathbb{Z}_2$ invariant

not a function of ρ alone

In terms of the Chern character:

$$\mathcal{CS}(B) = \int_{\partial\Lambda} cs_n = \int_{\Lambda} ch_n$$

BZ not a boundary!

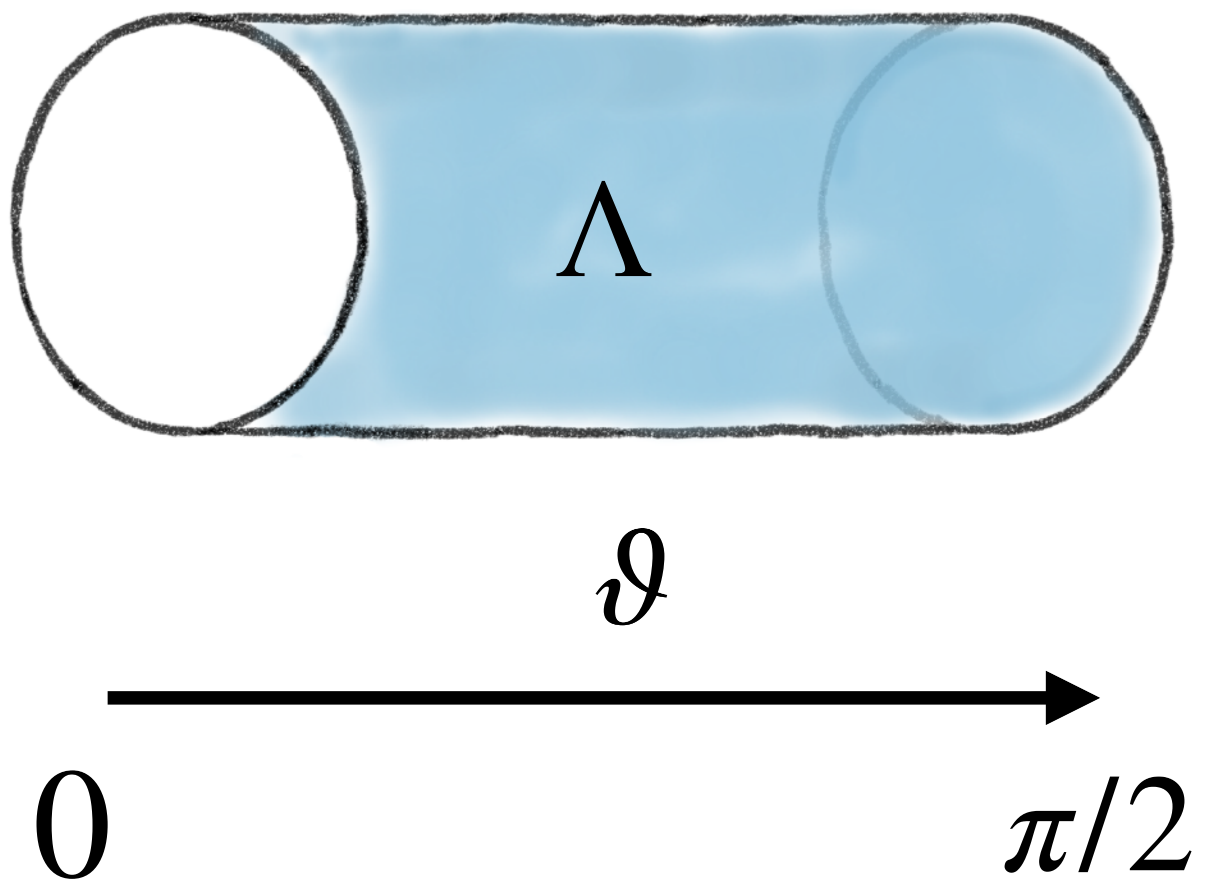


$\partial\Lambda = BZ$

S. Ryu *et al*, *New J. Phys.* **12** 065010 (2010)

Introduce ϑ which acts as an additional dimension

Extend the base space: $BZ \to BZ \times \vartheta \Big|_0^{\pi/2}$

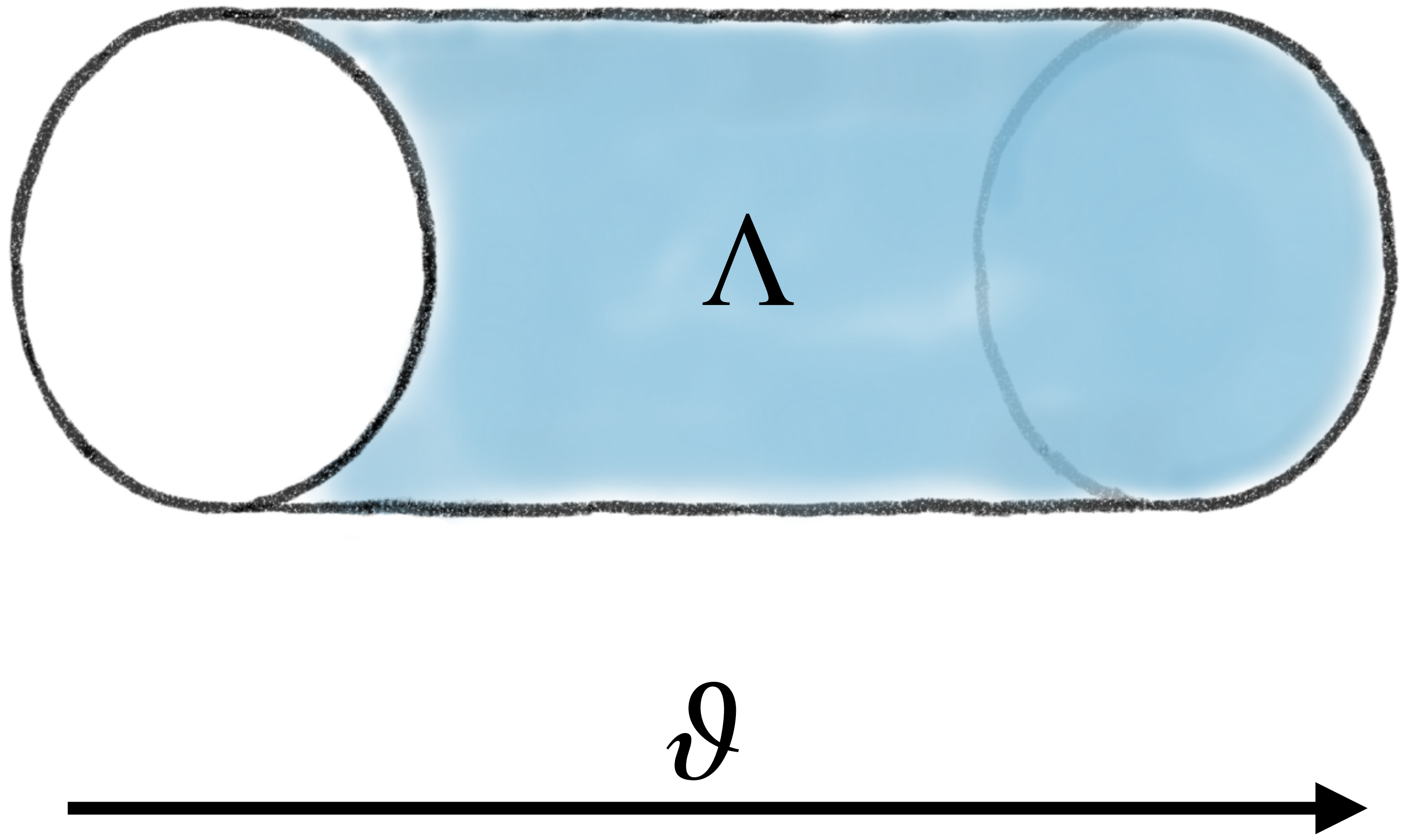


Family of projectors: P_ϑ

$$\int_0^{\pi/2} \mathrm{d}\vartheta \int_{\mathrm{BZ}} \mathrm{d^D k\ ch_n} = \mathcal{CS}(\mathrm{B}_{\vartheta=\pi/2}) - \mathcal{CS}(\mathrm{B}_{\vartheta=0})$$

JDH, M. Martinez, J.H. Bardarson, T. Klein Kvorning Phys. Rev. Lett. **129**, 277601 (2022)

Choose your boundaries wisely



$$\int_0^{\pi/2} d\vartheta \int_{BZ} d^D k \; ch_n \; = \; \mathcal{CS}(B_\rho) - \underbrace{\mathcal{CS}(B_{\text{trivial}})}_{=0}$$

trivial state | topological state

$P_{\vartheta=0}$ | $P_{\vartheta=\frac{\pi}{2}} = \rho$

JDH, M. Martinez, J.H. Bardarson, T. Klein Kvorning Phys. Rev. Lett. **129**, 277601 (2022)

The Chern-Simons invariant gives rise to two types of invariants

$$\mathcal{CS}(B_\rho)$$

Chiral winding number

Chiral constraint

$$\nu = 2\mathcal{CS}(B_\rho)$$

$\mathbb{Z}_2$ invariant

No chiral constraint, but $\mathcal{T}$ or $\mathcal{P}$

$$\nu_{\mathrm{cs}} = 2\mathcal{CS}(B_\rho) \mod 2$$

S. Ryu *et al*, *New J. Phys.* **12** 065010 (2010)

The local chiral marker is the real space equivalent of the chiral winding number

$$\{\rho, S\} = S,\ S^2 = 1$$

$$ch_n = \frac{1}{(2\pi i)^{(D+1)/2}} \frac{1}{((D+1)/2)!} \varepsilon^{\vartheta, i_1, \ldots, i_D} \mathrm{Tr}[\mathcal{P}(\mathbf{k})\partial_\vartheta \mathcal{P}(\mathbf{k})\partial_{k_{i_1}} \cdots \mathcal{P}(\mathbf{k})\partial_{k_{i_D}} \mathcal{P}(\mathbf{k})]$$

Fourier transform

Local chiral marker

$$\nu(\mathbf{r}) = 2i \sum_\alpha \int_0^{\pi/2} \mathrm{d}\vartheta \frac{\varepsilon^{i_0, \ldots, i_D} [P_\vartheta X_{i_0} P_\vartheta \ldots X_{i_D} P_\vartheta]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}}{[(D+1)/2]!/(2\pi i)^{(D-1)/2}}$$

where: $X_0 = i\partial_\vartheta$

JDH, M. Martinez, J.H. Bardarson, T. Klein Kvorning Phys. Rev. Lett. **129**, 277601 (2022)

An analytic expression for P_ϑ

$$\{\rho, S\} = S,\ S^2 = 1$$

$$P_\vartheta = \frac{1}{2}\left[1 - \sin(\vartheta)\big(1 - 2\rho\big) - \cos(\vartheta)S\right]$$

$$P_0 = \frac{1}{2}\Big(1 - S\Big)$$

$$\{P_\vartheta, S\} = S - \cos(\vartheta)$$

$P_{\vartheta=0}$ $\qquad$ $P_{\vartheta=\frac{\pi}{2}} = \rho$

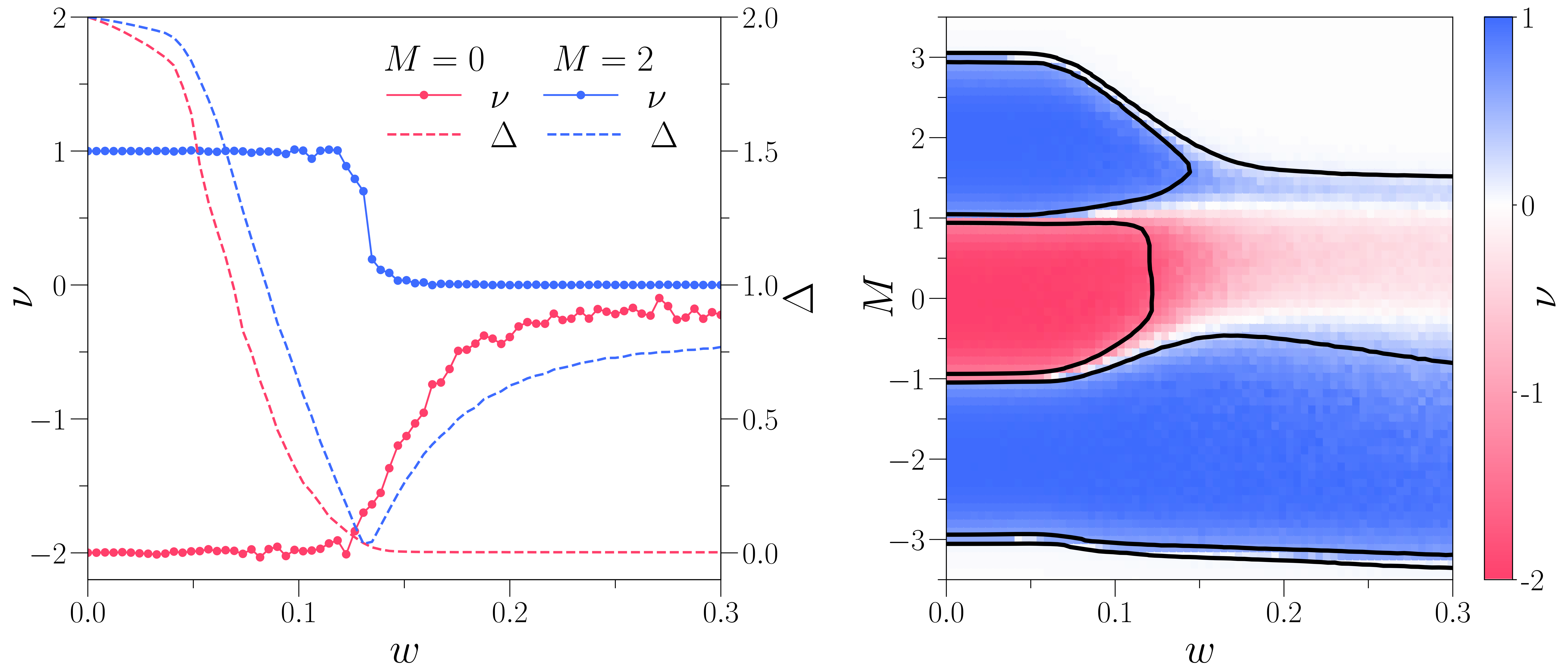
trivial $\qquad$ no constraint $\qquad$ S : chiral

$$\boxed{\nu(\mathbf{r}) = -\frac{8\pi i}{3}\sum_\alpha \varepsilon^{ijk}\left[\rho S X_i \rho X_j \rho X_k \rho\right]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}}$$

JDH, M. Martinez, J.H. Bardarson, T. Klein Kvorning Phys. Rev. Lett. **129**, 277601 (2022)

The three dimensional amorphous topological superconductor

$\mathcal{T}, \mathcal{P}, \mathcal{S}$



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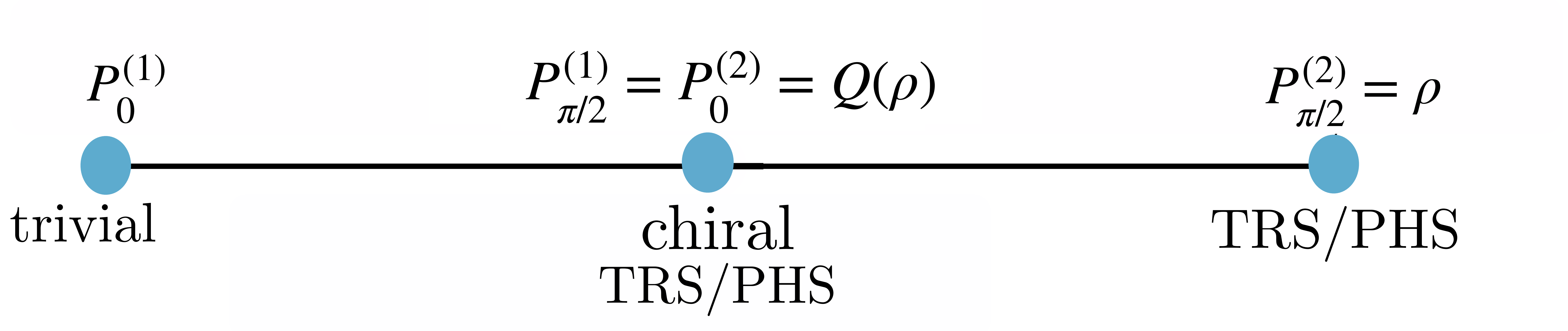
The $\mathbb{Z}_2$ invariant Chern-Simons marker for odd dimensions without chiral symmetry

No chiral constraint $\implies$ P_ϑ complicated



$$\mathcal{CS}(B_\rho) - \mathcal{CS}(B_Q) = 0 \quad \longrightarrow \quad P_\vartheta = \frac{1}{2}\left[1 - \sin(\vartheta)(1-2Q) - \cos(\vartheta) S_Q\right]$$

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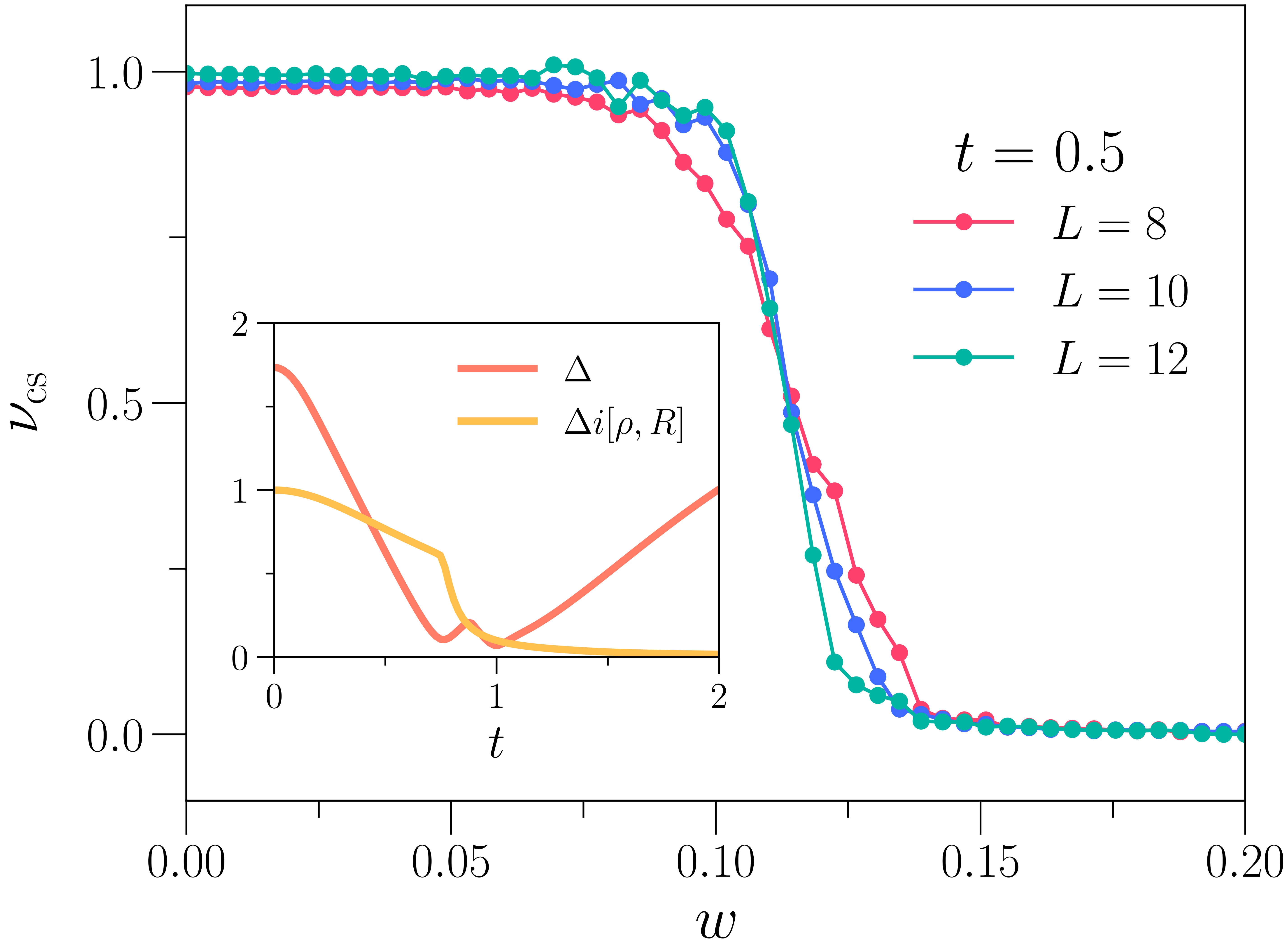
The three dimensional amorphous topological insulator



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The chiral and Chern-Simons markers characterise 4/5 classes in each odd dimension

Cartan label	T	P	S	d=1	d=2	d=3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	+1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	0
BDI	+1	+1	+1	$\mathbb{Z}$	0	0
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	+1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	+1	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$2\mathbb{Z}$	0
CI	+1	-1	0	0	0	$2\mathbb{Z}$

Cartan, E., Bull. Soc. Math. France 54, 214 (1926),
Altland, A., Zirnbauer, M.R., Phys. Rev. B 55, 1142 (1997).

Topological phases of interacting states

Interactions

$|\Phi_{\rm int}\rangle \neq \prod_\alpha c_\alpha^\dagger|0\rangle$

$(\rho_{\rm int})_{ij} = \langle\Phi_{\rm int}|c_i^\dagger c_j|\Phi_{\rm int}\rangle$

not a projector

$|\Phi_{\rm int}\rangle \sim |\Psi_{\rm slater}\rangle$

If: $\rho_{\rm int}$ gapped

$\rho_{\rm int}$ adiabatically connected to $\rho_{\rm slater}$

If: $|\Phi_{\rm int}\rangle = U|\Psi_{\rm slater}\rangle$

$[U, O] = 0$

O. Penrose and L. Onsager, Physs. Rev. 104, 576 (1956)
S. Bera et al, Phys. Rev. Lett. 115, 046603 (2015)
X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010)

The Ising Majorana model can host topological MBL phases

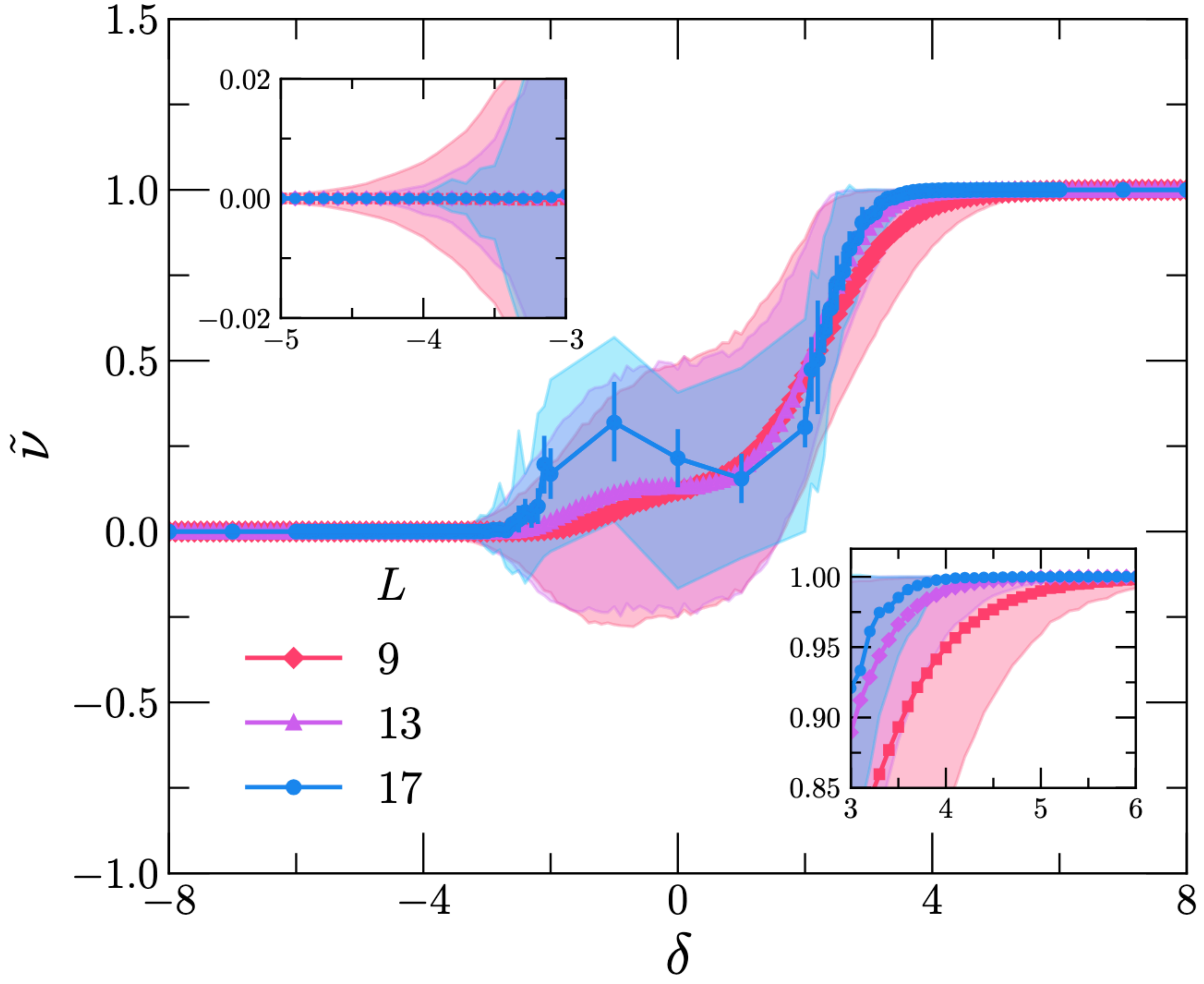
$$H = \sum_j \left(-it_j\gamma_j\gamma_{j+1} + g\gamma_j\gamma_{j+1}\gamma_{j+2}\gamma_{j+3}\right) \qquad \mathcal{T}, \mathcal{P}, \mathcal{S}$$

$$\gamma_{2j} = i(c_j - c_j^\dagger) \qquad \gamma_{2j-1} = c_j + c_j^\dagger$$

$g = 0.5$

$t_{2j} \in [0, e^{\delta/2}]$

$t_{2j-1} \in [0, e^{-\delta/2}]$



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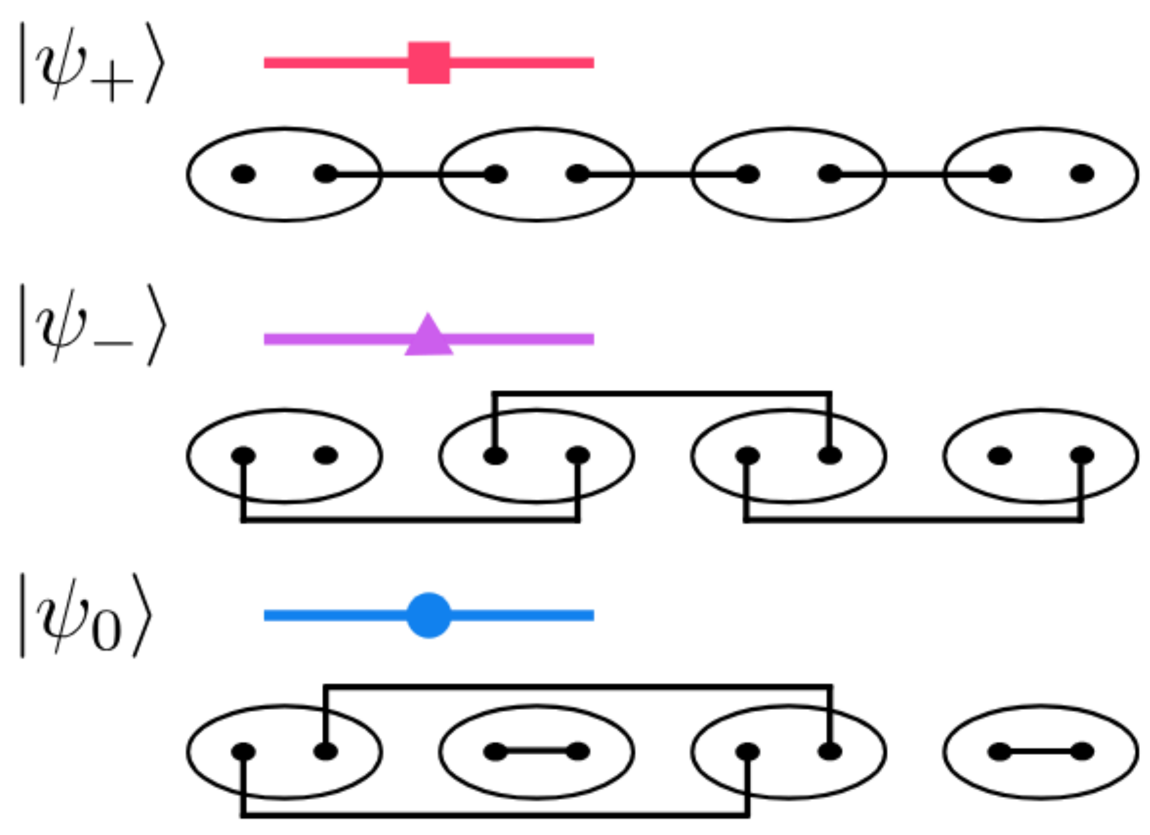
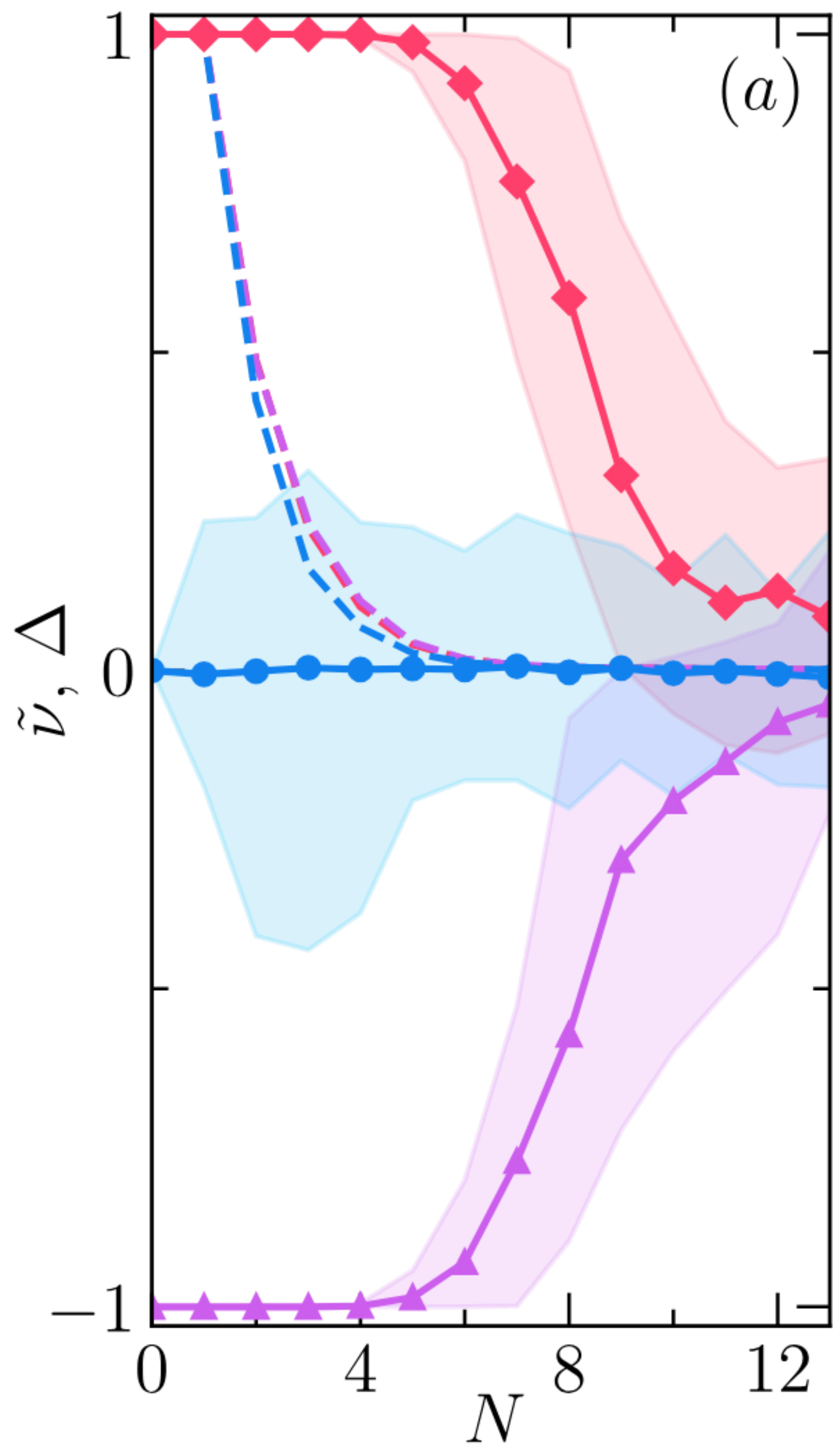
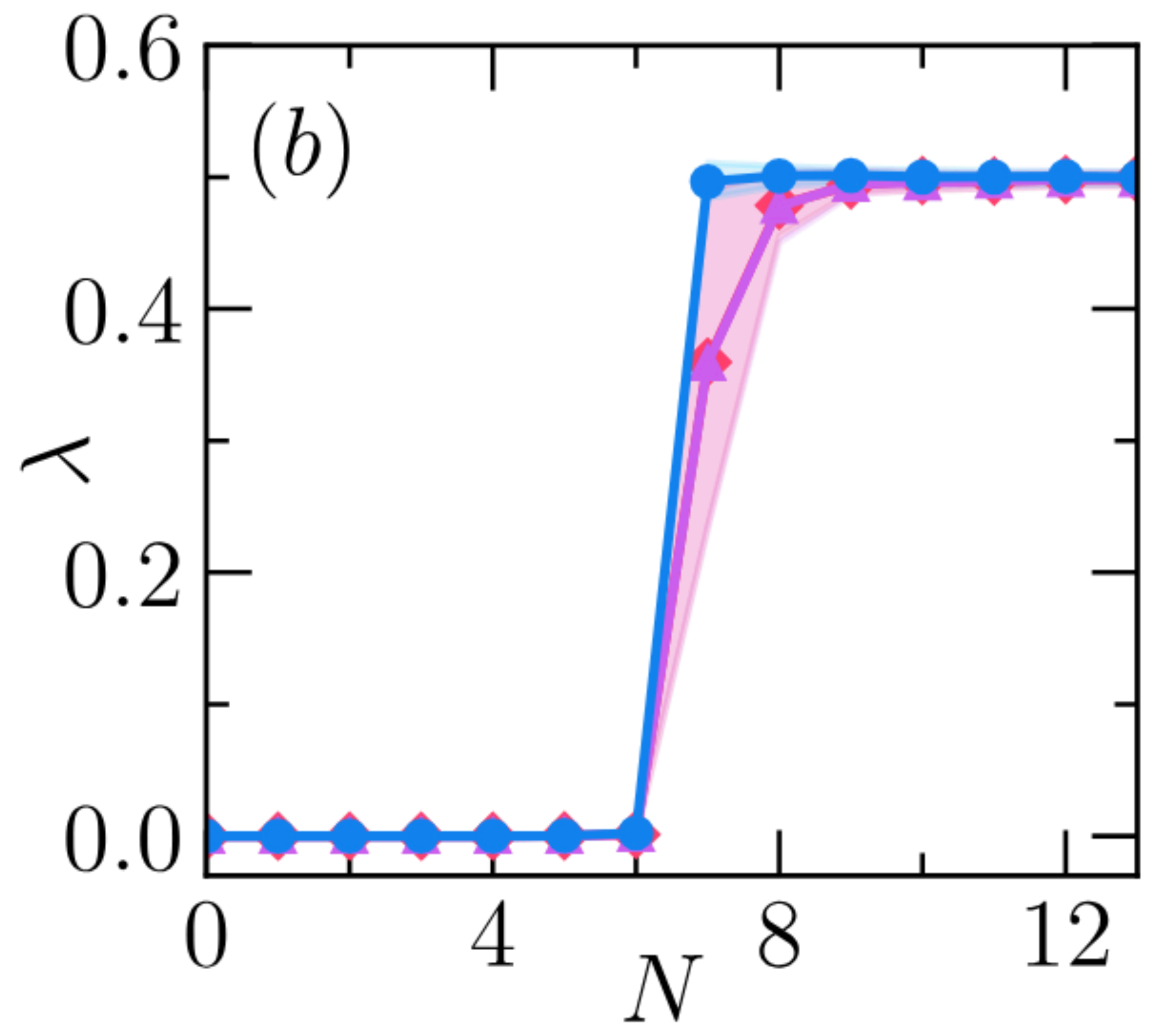
Topology of random Circuit states



$L = 26$



Entanglement spectrum



$$\lambda_\alpha = \frac{\varepsilon_\alpha - \varepsilon_{\alpha+1}}{\varepsilon_\alpha - \varepsilon_{\alpha+2}}$$

$\lambda_\alpha = 0$: Degenerate

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Summary

Dimensional reduction: Chern character $\Longrightarrow$ local chiral and Chern-Simons markers

Requirement: The single-particle density matrix is gapped

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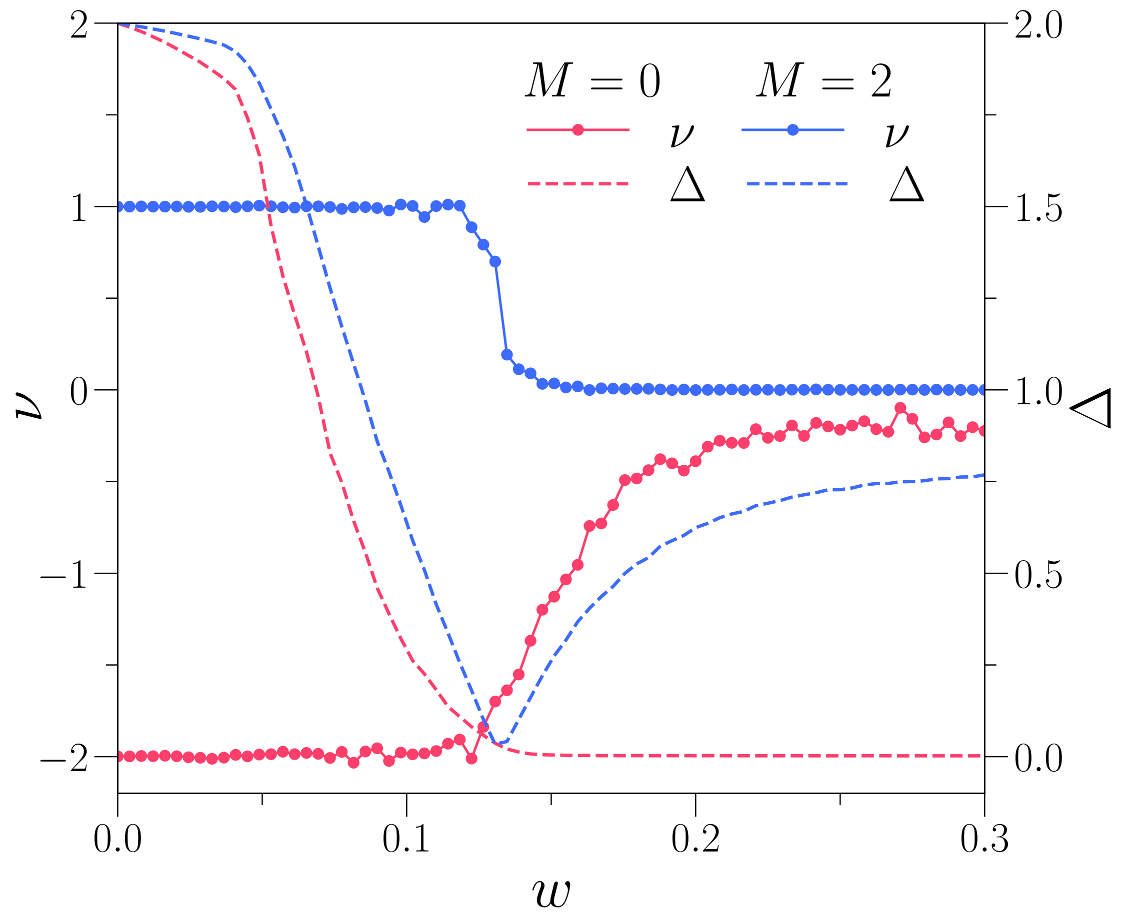

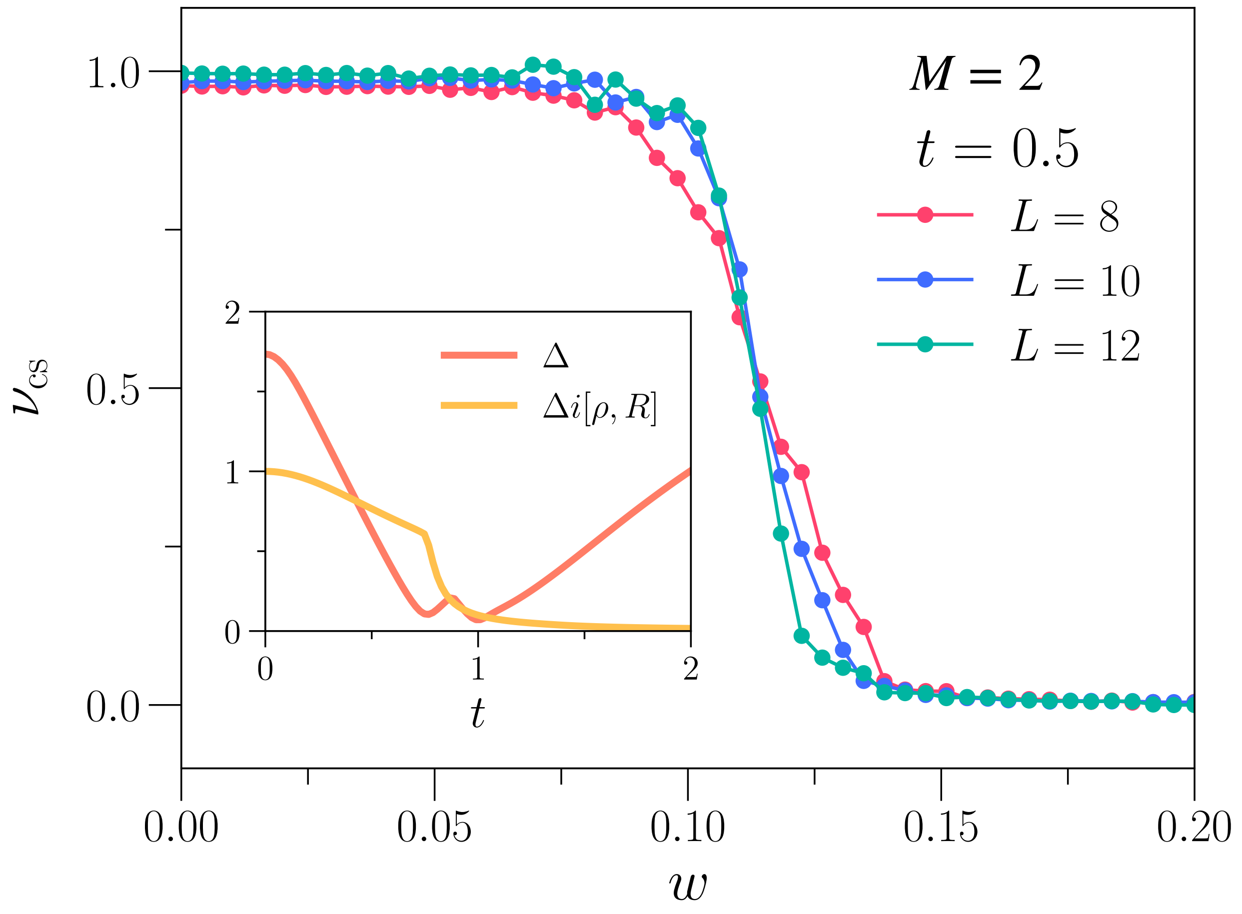

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