

# Quantifiable local topological protection in quantum materials

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# Almost commuting operators and physics

According to von Neumann (in 1927)

*quantum mechanics attributes to the quantities  $q_k$  and  $p_k$  the well-known operators  $\mathbf{Q}_k = q_k$  and  $\mathbf{P}_k = \frac{\hbar}{i} \frac{\partial}{\partial q_k}$ , whose lack of commutability [...] corresponds to the lack of simultaneous measurability of these quantities. We now assume that two other, commuting, operators  $\mathbf{P}_k^0, \mathbf{Q}_k^0$  exist whose difference from  $\mathbf{P}_k$  (respectively,  $\mathbf{Q}_k$ ) is so small that its size [...] does not significantly exceed the value  $\sim \hbar$  required by the uncertainty relation.*

and also (in 1932)

*the macroscopic procedure consists of replacing all possible operators  $A, B, C, \dots$ , which as a rule do not commute with each other, by other operators  $A^0, B^0, C^0, \dots$ , (of which these are functions to within a certain approximation) which do commute with each other.*

Such approximations are *not* always possible, which is why we have Chern insulators.

# Almost commuting matrices and $K$ -theory

In basis  $j_1, \dots, j_{s+1}, \dots, j_n$ , define

$$Z_s j_k i = \delta_{k,i} j_k; \quad L_s j_k i = \delta_{k,i} (j_k + 1)$$

for some  $1 = j_1 < j_2 < \dots < j_n = 1, \frac{2}{n} + \frac{2}{n} = 1$ .

Now set

$$X_s = \frac{1}{2}(L_s^Y + L_s); \quad Y_s = \frac{i}{2}(L_s^Y - L_s):$$

These almost commute and generate a “fuzzy sphere” since

$$\| [X_s, Y_s] \| \neq 0; \quad \| [X_s, Z_s] \| \neq 0; \quad \| [Y_s, Z_s] \| \neq 0 \text{ and } X_s^2 + Y_s^2 + Z_s^2 = 1 \neq 0:$$

Man-Duen Choi (in 1988) showed

$$X_s^2 + Y_s^2 + Z_s^2 = 1$$

has two more positive than negative eigenvalues. He proved that  $X_s^0, Y_s^0, Z_s^0$  commuting with each other implies

$$X_s^0 = X_s + Y_s^0, \quad Y_s^0 = Y_s + Z_s^0, \quad Z_s^0 = Z_s + 1 - s:$$

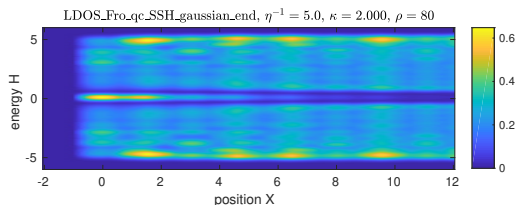
# Almost commuting matrices and 1D physics

Theorem (Lin, 1995).  $\delta > 0, \rho > 0$  for the following.

If  $H$  and  $X$  are  $n$ -by- $n$  Hermitian matrices with  $\|kHX - XHk\| < \delta$  then there are  $H^0$  and  $X^0$  commuting with  $kH - H^0k < \rho$  and  $kX - X^0k < \rho$ .

Hastings (2008) – This tells us a gapped, finite, 1D system in Class A is “Wannierizable” / close to an atomic limit.

True in all 10 AZ symmetry classes if one allows zero-modes (follows from extensions of Lin’s theorem by David Herrera, 2024 and others).



LDOS at the end of a quasiperiodic variation of an SSH chain. From paper on windowed LDOS, joint with Fu and Watson.

## States approximately local in position and energy

For a system (non-interacting) with open boundaries, the key observables are  $H$  for energy and position observables  $X_1; \dots; X_d$ .

Eigenstates  $\psi_1$  and  $\psi_2$  of  $H$  at slightly different energies might be delocalized in position while

$$\psi = \frac{1}{\sqrt{2}} \psi_1 + \frac{1}{\sqrt{2}} \psi_2$$

is well-localized. The LDOS can miss this.

For Hermitian observable  $A$  we have

$$\langle \psi | A | \psi \rangle - \langle \psi | A \rangle^2 = \langle \psi | A^2 | \psi \rangle - \langle \psi | A \rangle^2 = \langle \psi | (A - \langle A \rangle)^2 | \psi \rangle :$$

*A joint approximate eigenvector* for  $X_1; \dots; X_d; H$  is a state that is approximately localized in energy and position.

# The quadratic local gap

We define a local gap at position  $\mathbf{x}$  and energy  $E$  as

$$Q_{\mathbf{x};E}(X_1; \dots; X_d; H) = \min_{k=1}^d \left( \sum_{j=1}^d |X_j - x_j|^2 + k(H - E)^2 \right)^{\frac{1}{2}}$$

and this equals the square root of the smallest eigenvalue of

$$Q_{\mathbf{x};E}(X_1; \dots; X_d; H) = \sqrt{\sum_{j=1}^d |X_j - x_j|^2 + (H - E)^2}$$

We call the function

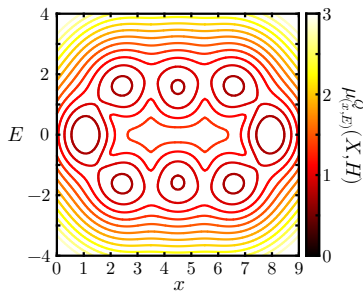
$$(\mathbf{x}; E) \mapsto Q_{\mathbf{x};E}(X_1; \dots; X_d; H)$$

the *quadratic pseudospectrum* of  $(X_1; \dots; X_d; H)$ , or the *quadratic local gap*

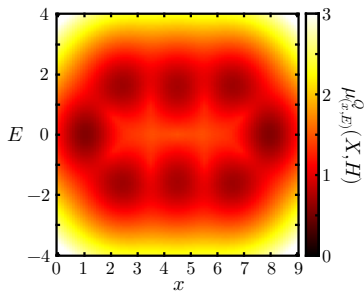
Since  $Q_{\mathbf{x};E}(X_1; \dots; X_d; H)$  will generally be sparse, this is practical to compute numerically.

# The Quadratic local gap in 1D

a) Level sets



b) Image



Shown is the quadratic pseudospectrum for a very short topologically non-trivial SSH system.

In practice, replace  $X_j$  by  $X_j$  to somewhat align position units with energy units. Generally a large range of  $\epsilon$  gives consistent results.

We seek a “square root” of  $Q_{x;E}(X_1; \dots; X_d; H)$  that detects  $K$ -theory.

# The Clifford local gap

Pick Hermitian matrices  $X_j$  that anticommute and so that  $X_j^2 = I$ . The spectral localizer is

$$L_{\mathbf{x};E}(X_1; \dots; X_d; H) = \sum_{j=1}^d (X_j - x_j I) + (H - EI) \quad d+1:$$

As long as  $H$  is local, so  $[H; X_j]$  are small, we find

$$(L_{\mathbf{x};E}(X_1; \dots; X_d; H))^2 = (Q_{\mathbf{x};E}(X_1; \dots; X_d; H)) \quad I:$$

We call the function

$$(\mathbf{x}; E) \mapsto \min_{\mathbf{x};E}(X_1; \dots; X_d; H)$$

the *Clifford pseudospectrum* of  $(X_1; \dots; X_d; H)$  where

$$\min_{\mathbf{x};E}(X_1; \dots; X_d; H)$$

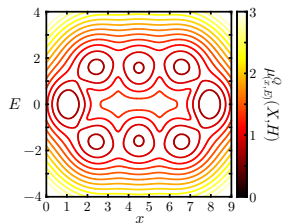
equals minimum absolute value of an eigenvalue of  $L_{\mathbf{x};E}(X_1; \dots; X_d; H)$ .

Almost as good at finding well-localized states.

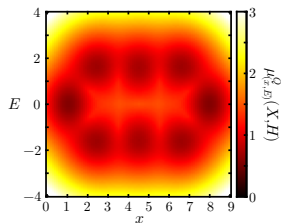
Also detects local topology.

# Both local gaps for an SSH chain

a) Level sets

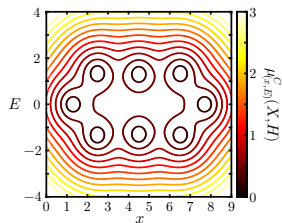


b) Image

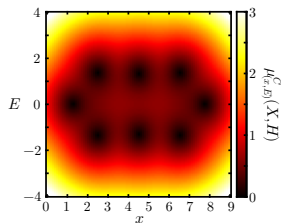


The quadratic pseudospectrum.

a) Level sets



b) Image



The Clifford pseudospectrum.

# Local topology in Chern insulators

When  $d = 2$  we generally use  $x_1 = x, x_2 = y, x_3 = z$ .

In class A, the invariant is

$$\frac{1}{2} \text{Sig}(L_{x,y,E}(X; Y; H))$$

where the signature is the number of eigenvalues that are positive minus the number that are negative.

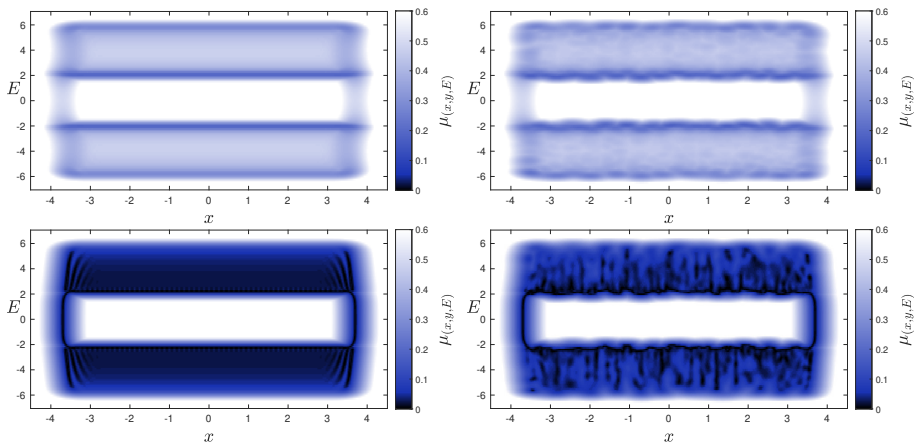
The signature can change only when  $L_{x,y,E}(X; Y; H)$  becomes singular. When  $\int_{x,y,E}^C(X_1; \dots; X_d; H)$  is large a large change in  $H$  will be required to change the local index.

If  $XH = HX$  and  $HY = YH$  we are in “an atomic limit” and we find

$$\frac{1}{2} \text{Sig}(L_{x,y,E}(X; Y; H)) = 0.$$

If signature is zero then there is a path  $(X_t; Y_t; H_t)$ , almost commuting at every  $t$ , from  $(X_t; Y_t; H_t)$  to an atomic limit. Can be a long path.

# Both local gaps of a Chern insulator



Shown are  $y = 0$  slices. Units are adjusted, replacing  $X$  and  $Y$  by  $X$  and  $Y$ .

Top is quadratic, bottom is Clifford. Left is clean. Right is with disorder.

If  $\mathcal{U}$  implements Chiral-symmetry then we have  $X = X$  and  $H = H$ . Here

$$L_{X;E}(X; H) = (X \quad xI) \quad x + (H \quad EI) \quad y$$

Shifting  $H$  to  $H + EI$  ruins symmetry, unless  $E = 0$ .

At  $E = 0$  the local invariant is

$$\frac{1}{2} \text{Sig}(((X \quad xI) + iH) \quad ) :$$

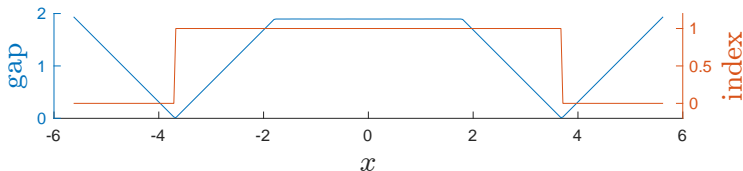
Notice  $((X \quad xI) + iH)$  is a block out of the spectral localizer

$$L_{X;0}(X; H) = \begin{pmatrix} 0 & (X \quad xI) \\ (X \quad xI) + iH & 0 \end{pmatrix} \quad iH \quad :$$

# Local measure of strength of topological protection

Studying materials, only need to track  $L_\lambda(X; Y; H)$  as  $\lambda$  moves on a line.

Need only compute the index formula (K-theory) and the eigenvalue closest to zero (local gap).



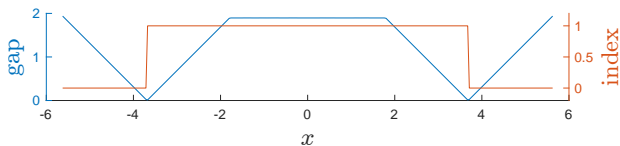
No disorder.

The localizer gap increases (up to a point) as  $(x; y)$  moves away from the edge of the system.

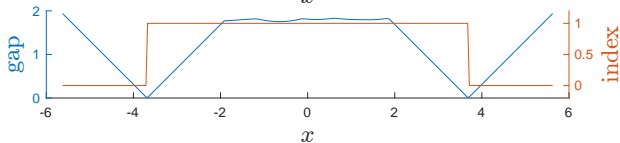
Small disorder can move the edge state a little.

Moving or eliminating the edge state takes a larger perturbation.

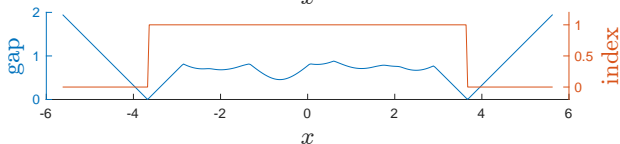
# Adding disorder



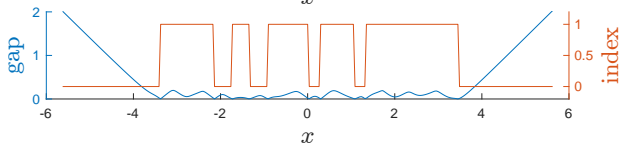
Disorder 0% of size of bulk gap.



Disorder 12% of size of bulk gap.



Disorder 60% of size of bulk gap.



Disorder 120% of size of bulk gap.

## Dimension cross-over in class D

Consider a Shiba lattice, in class D. Atop the surface of a superconductor, place pattern of magnetic atoms.

## Dimension cross-over in class D

Consider a Shiba lattice, in class D. Atop the surface of a superconductor, place pattern of magnetic atoms.

For line in  $x$ -direction, use as local invariant

$$\text{sign}(\det((X - xI) + iH))$$

and

$$\frac{1}{2} \text{Sig}(L_{x;y;E}(X; Y; H))$$

for a disk or square.

## Dimension cross-over in class D

Consider a Shiba lattice, in class D. Atop the surface of a superconductor, place pattern of magnetic atoms.

For line in  $x$ -direction, use as local invariant

$$\text{sign}(\det((X - xI) + iH))$$

and

$$\frac{1}{2} \text{Sig}(L_{x,y;\epsilon}(X; Y; H))$$

for a disk or square.

For other shapes, we can use both.

This is on the Arxiv, joint with Cerjan and Rodriguez-Vega.

From a disk to a line of adatoms.

Spectrum of 1D and 2D localizers as  $x$ -position varies.

2D local invariant and local gap.

1D local invariant and local gap.

## Quantum spin Hall systems

For a class All system in 2D, we assume  $T^2 = -I$  and that  $X, Y$  and  $H$  all commute with  $T$ . Typically  $T = U K$  where  $U = I \otimes i \sigma_y$ . For fixed  $x, y$  and  $H$ ,

$$L_{x;y;E}(X; Y; Z)$$

is in class D in 0D for the time-reversal operator

$$(I \otimes i \sigma_y \otimes i \sigma_y) K$$

which is, in some basis, just  $K$  (complex conjugation). Theory says there is a unitary  $Q$  so that

$$Q(L_{x;y;E}(X; Y; Z)) Q^\dagger$$

is anti-symmetric. Then we compute Fermionic parity, here a local  $\mathbb{Z}_2$  index,

$$\text{Sign Pf } Q(L_{x;y;E}(X; Y; Z)) Q^\dagger :$$

The Pfaffian can only change sign by going through zero, so only when the spectral localizer becomes singular.

## A crafty unitary

Suppose  $T = U K$  and  $U^2 = I$ . We seek a unitary  $Q$  so that

$$Q K Q^\dagger = T:$$

Since  $Q K Q^\dagger = Q Q^\dagger K$  what is required is  $Q Q^\dagger = U$ .

In the case of

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

we can use

$$Q = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

Thus  $Q L_{X;Y;E}(X;Y;Z) Q^\dagger$  is skew-symmetric and purely imaginary. Where there is a local gap, the Pfaffian of this is well-defined and takes a real, non-zero value.

## Calculating these invariants

Good software to compute the sign of a Pfaffian is in development. Expect to compute this efficiently for Hilbert space dimension  $N$  up to about 1 million on a powerful workstation.

The other local indices, in any dimension, any AZ class, end up with

$\text{sign } \mathcal{E} !$  use sparse LDLT factorization

$\text{sign } \det(\mathcal{E}) !$  use sparse LU factorization

$\text{sign } \text{Pf}(\mathcal{E}) !$  stuck with dense Hessenberg factorization, for a little longer.

Matlab code for all these are in an online supplement to my 2015 paper. In all cases,  $\mathcal{E}$  is built from the full spectral localizer.

Avoid computing all the eigenvalues.

## When can we link these invariant to established invariants?

Schulz-Baldes, with me or Nora Doll, proved roughly the following.

If in finite-volume system is gapped, and we use  $X_j$  for small  $j$ , and if we truncate to a large radius  $R$ , then at the center the local index of  $(X; Y; H)$  agrees with the established bulk index of  $(X; Y; H)$ .

To know how big we need  $R$ , and how small  $j$ , we need an estimate on the spectral gap of  $H$ , i.e.  $g = \kappa H^{-1} \kappa^{-1}$ .

In practice, we can use smaller  $R$  and a wider range of  $j$  than theorems indicate.

## Truncating in finite-area topological insulator

No gap in  $\text{spec}(H_\Lambda)$  (due to spectral pollution / interesting edge states) so this is no help in computing  $\text{spec}(H)$ .

For correct  $\epsilon$  and  $\delta$ , the local gap will approximate the distance of  $\epsilon$  to the spectrum of  $H$ .

The Hilbert space is  $\ell^2(V) \subset \mathbb{C}^2$  where  $G = (E; V)$  is the graph based on an Ammann-Beenker tiling.

Define  $H$  as an infinite matrix with  $H_j = t_z$  associated with each vertex and a hopping term for each edge

$$H_{jk} = t_z \left[ \frac{i}{2} x \cos(\theta_{jk}) + \frac{i}{2} y \sin(\theta_{jk}) \right]$$

Estimate on distance from to the spectrum of  $H$ :

## $p_x + ip_y$ tight-binding model, a bulk and an edge state

These methods work by finding approximate eigenvalues of  $H$  that avoid the edges.

Better methods in finite-dimensional numerical linear algebra have been developed since this work. See Hege, Moscolari and Teufel, 2025.

# Some places where the spectral localizer finds local topology

Models of photonic Quasicrystals, not just tight-binding (with Wong and Cerjan).

Topological semimetals (Stoiber and Schulz-Baldes)

Topological metals, if we find a local gap (with Cerjan)

Fragile topology given  $C_2T$  symmetry (with Lee, Wong, Vaidya, Cerjan).

Metal-insulator heterostructures

With modifications, pseudospectral methods work with periodic boundary conditions.

The LDOS cannot distinguish topological and trivial metals.

The Clifford pseudospectrum shows a clear edge effect in the topological case. (Hue shows the local index).

From paper written with Cerjan.

From a paper with Stephan Wong  
and Alexander Cerjan.

(a) A photonic crystal with Perfectly Matched Layer to simulate light lost to free space.

(b) Bandstructure.

(c) Real part of local density of states.

(d) Localizer gap data, at  $xed E^1$

(e) Flow of eigenvalues of spectral localizer { crossing indicate change in local topology.

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Dixon, Kahlil Y., et al. "Classifying Topology in Photonic Heterostructures with Gapless Environments." *Physical Review Letters*, 131:21 (2023), 213801.

## Patterned semiconductor

(a) (b) Patterned etching in semiconductor.

(c) (d) As magnetic field increases we see the Landau levels.

(e) (f) Local index in model matches well.

Next slide: Hofstadter butterfly.

Simulated data <sup>2</sup>. Local K-theory illustrated in color. Not a simple tight-binding model, but a discretized version of the single-particle Hamiltonian:

$$H = \frac{1}{2m} (i\hbar \nabla + eA(x))^2 + V(x) + \frac{\hbar g}{2} s_z B$$

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<sup>2</sup>C. Spataru, W. Pan, A. Cerjan, PRL, 2025.

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# The End