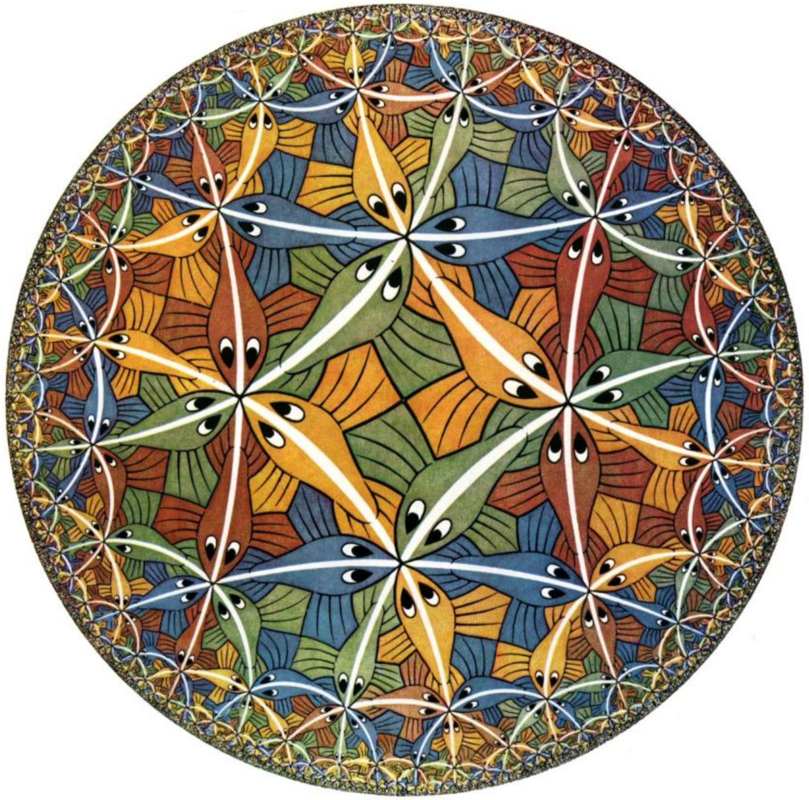
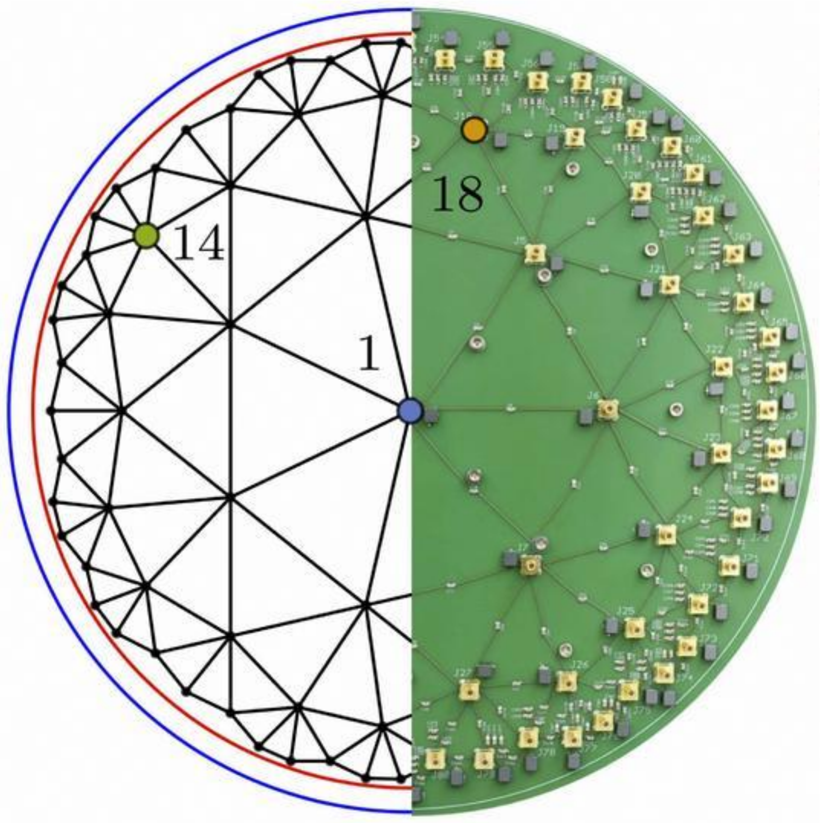
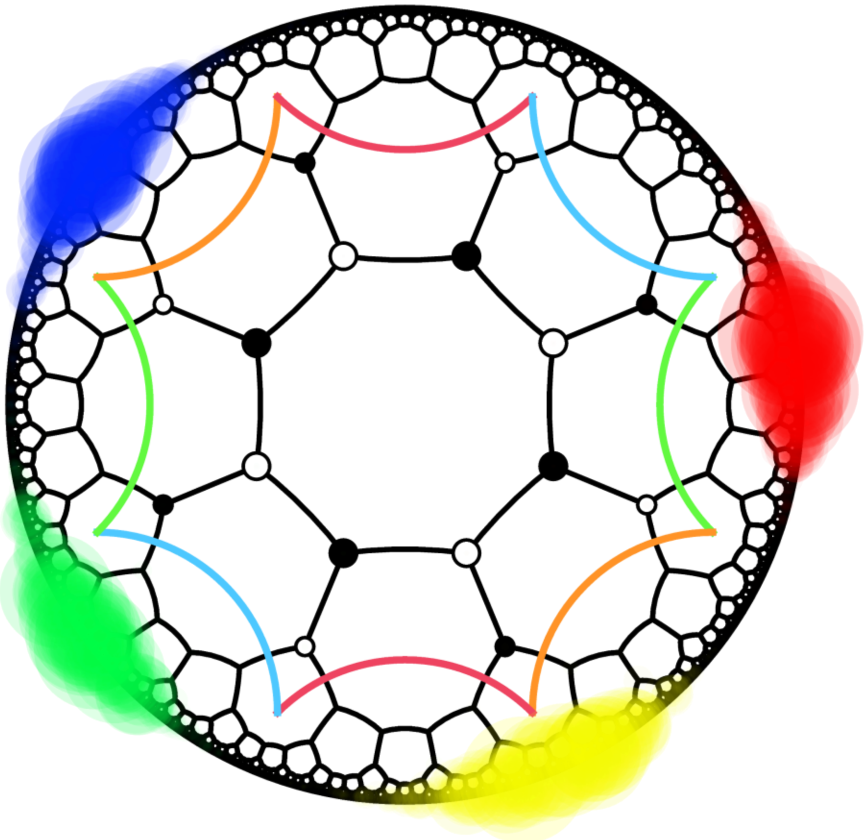
# Unraveling spectra and topology of hyperbolic lattices





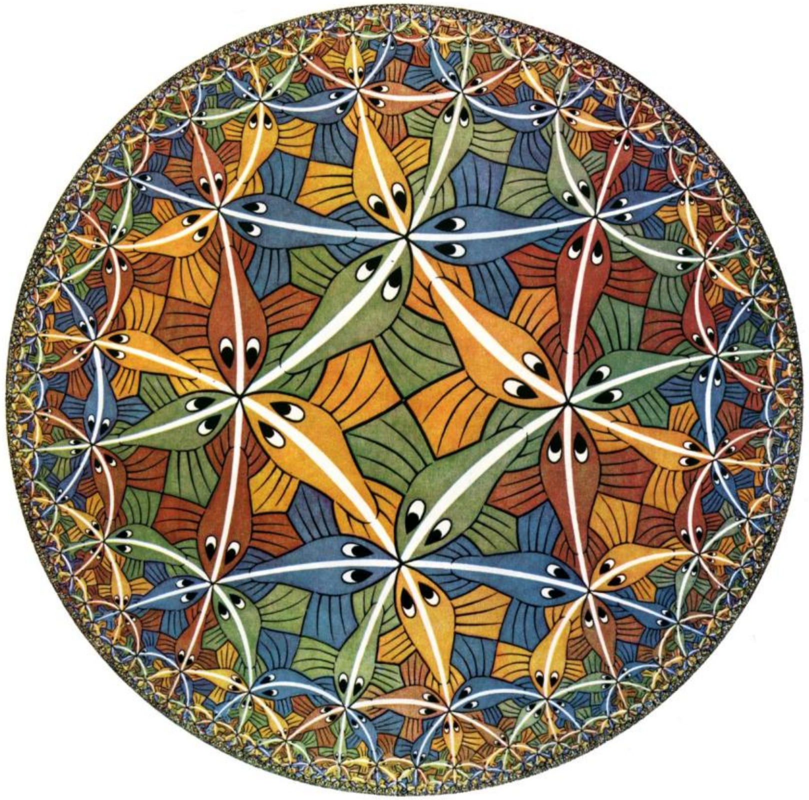
**Tomáš Bzdušek**

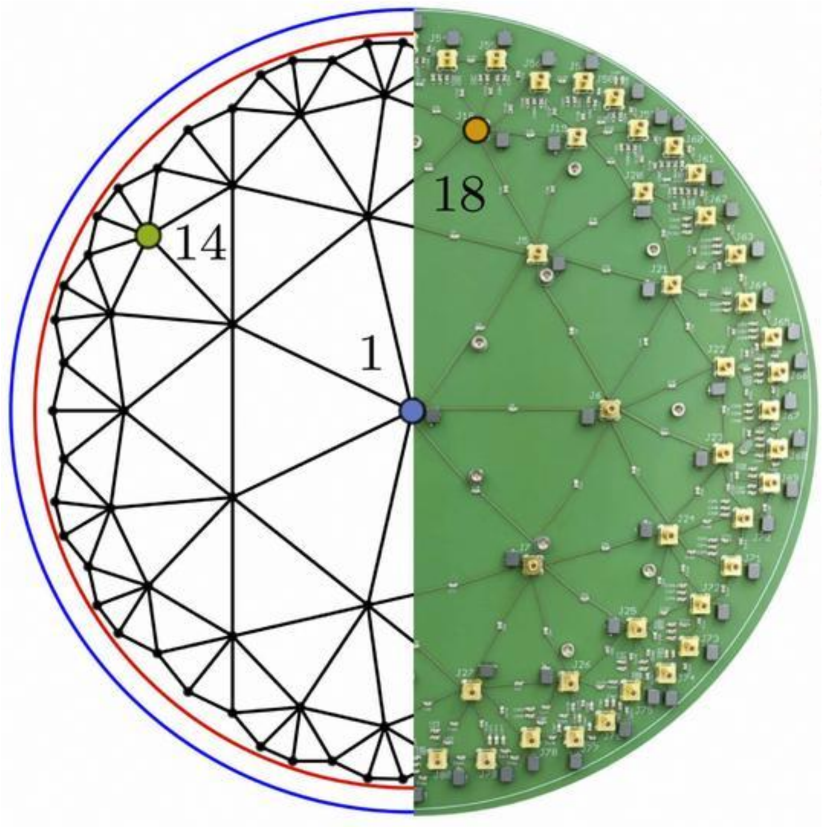
Stockholm, 26 March 2025

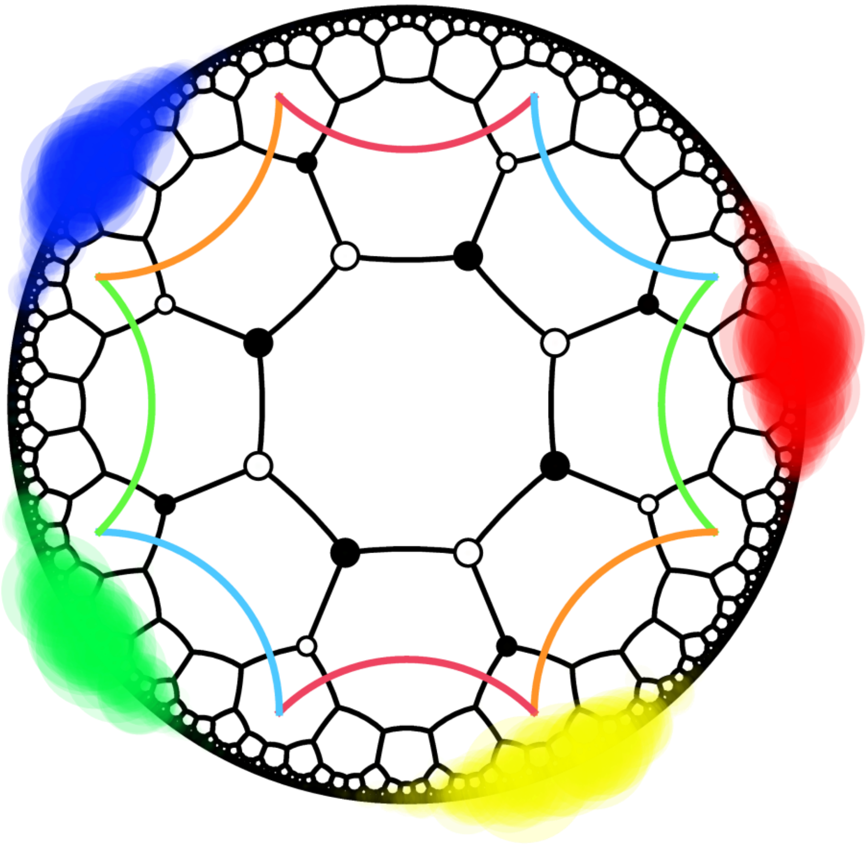
University of Zurich[UZH]

Swiss National Science Foundation

# Negatively curved space:
# in art, experiment, and theory





# Many thanks to our collaborators!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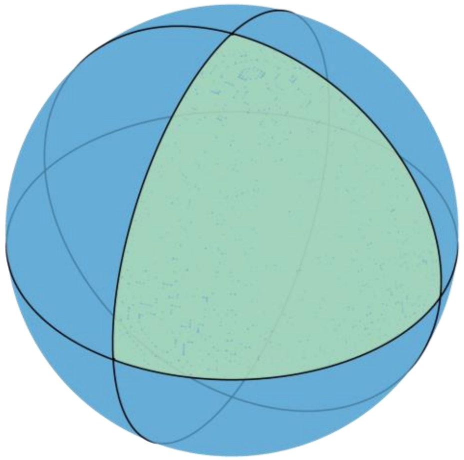
Lavi Upreti
Martin Greiter
Tobias Hofmann
Tobias Helbig
Tobias Kießling
Stefan Imhof
Hauke Brand

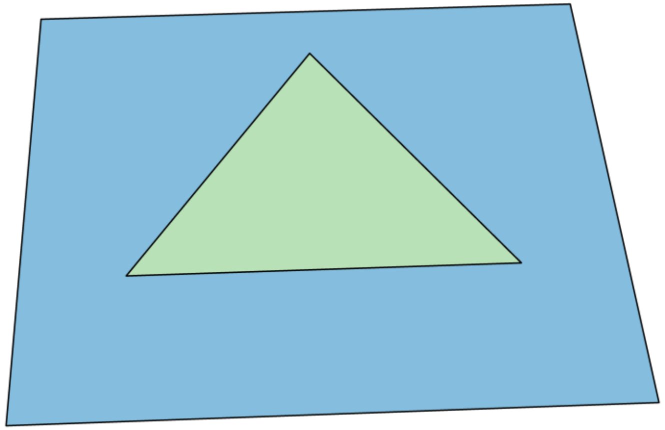
# Curved spaces

Sphere, $K > 0$
(constant curvature)



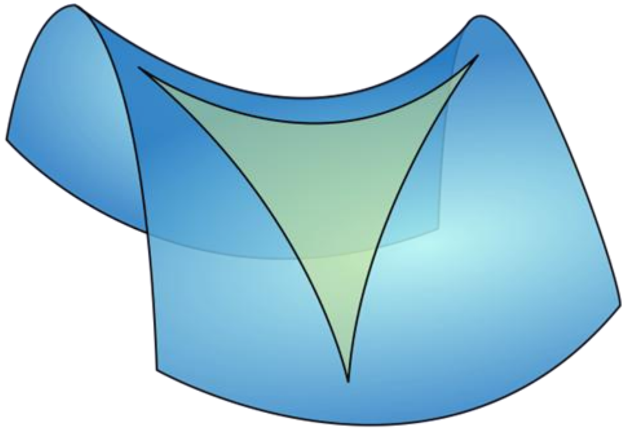
$$\alpha + \beta + \gamma > \pi$$

Euclidean plane, $K = 0$
(constant curvature)



$$\alpha + \beta + \gamma = \pi$$

Saddle, $K < 0$
non-constant curvature

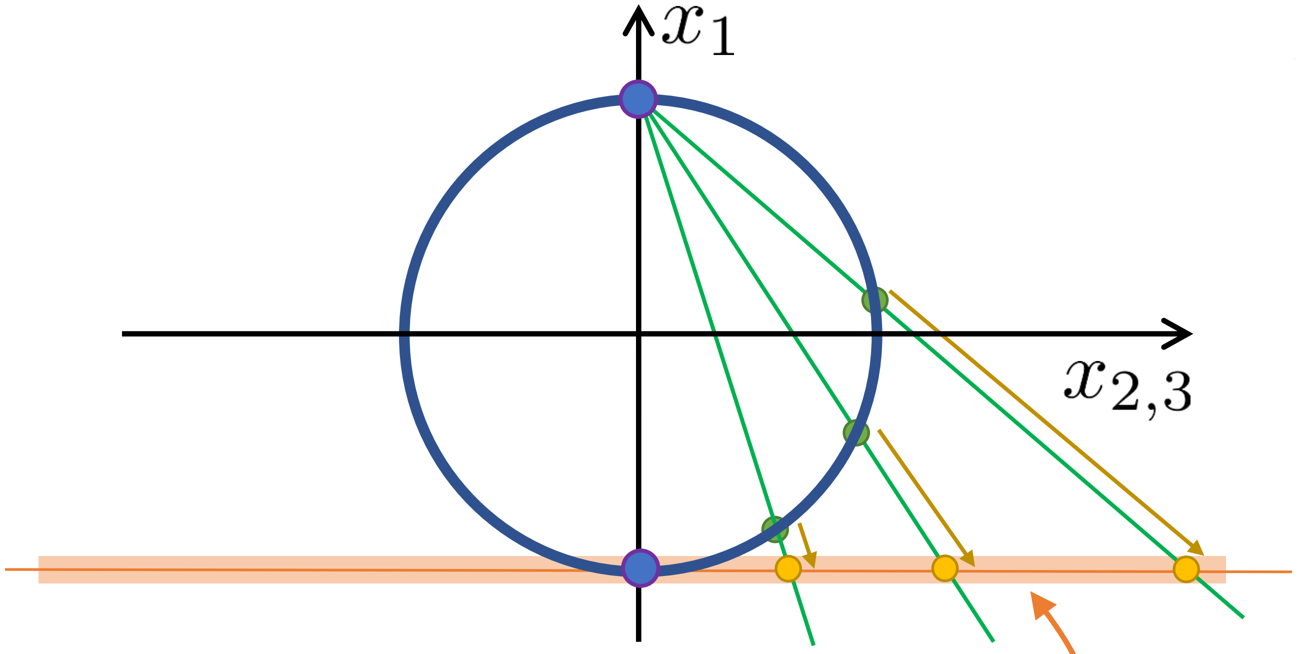


$$\alpha + \beta + \gamma < \pi$$

# Hyperbolic plane – sheet of *constant* negative curvature

**Sphere:** points of constant Euclidean distance from the origin

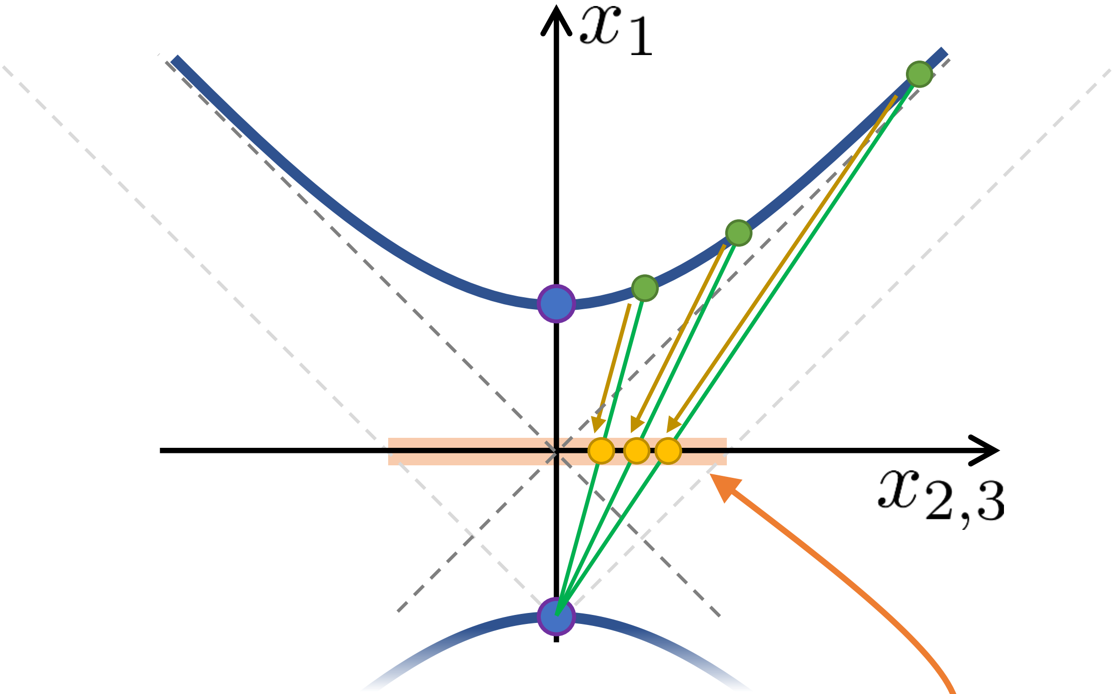
$$\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 | + x_1^2 + x_2^2 + x_3^2 = 1\}$$




Stereographic projection into the $\mathbb{R}^2$ plane.

**Hyperbolic plane:** points of constant Minkowski distance from the origin

$$\mathbb{H}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 | + x_1^2 - x_2^2 - x_3^2 = 1\}$$




Stereographic projection into the "Poincaré disk".

# Regular "{*p*,*q*}" lattices

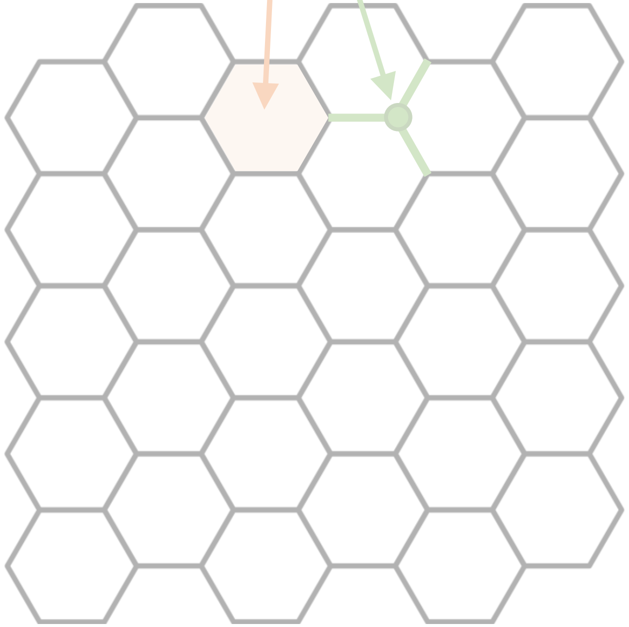
*Schläfli symbol*

**Euclidean** "{6,3}" tessellation

polygons

vertices

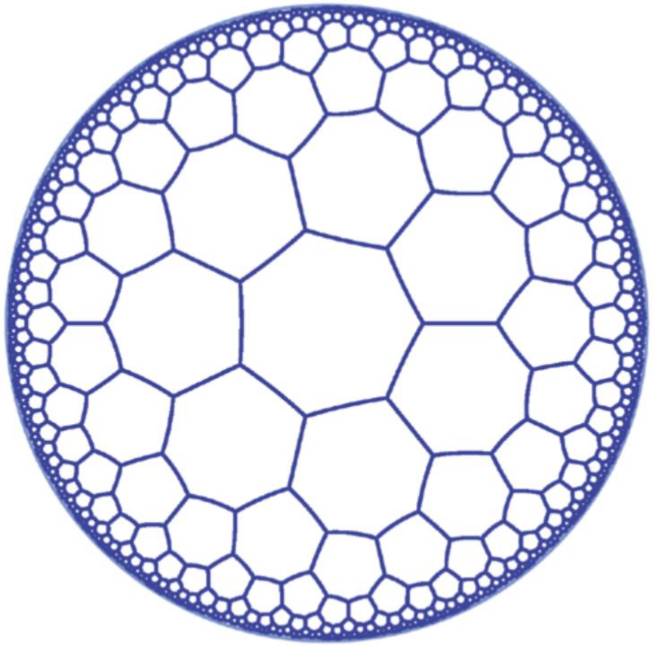
**Spherical** "{5,3}" tessellation

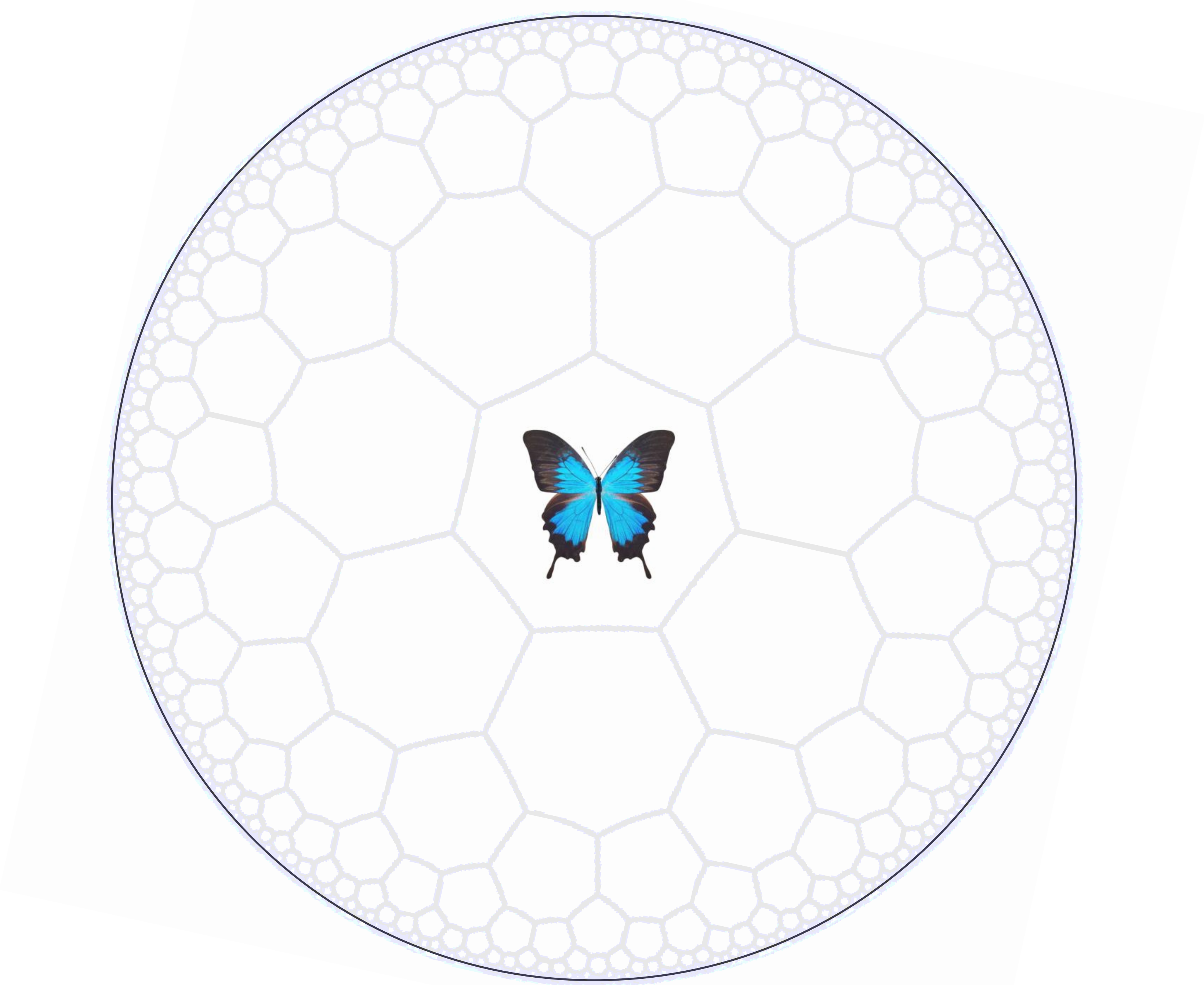


**Hyperbolic** "{7,3}" tessellation

[...]

Stereogr. proj.





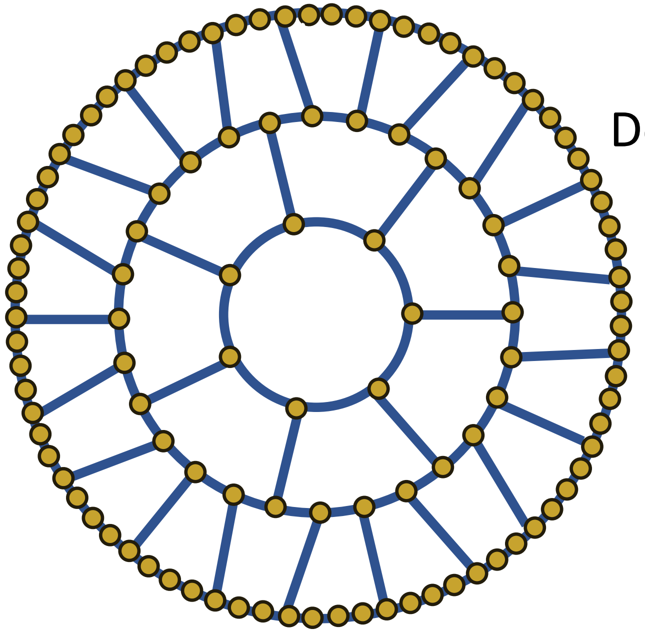
**Generate your own hyperbolic tiling!** – http://www.malinc.se/m/ImageTiling.php

# Coupling along lattice edges – only the graph matters

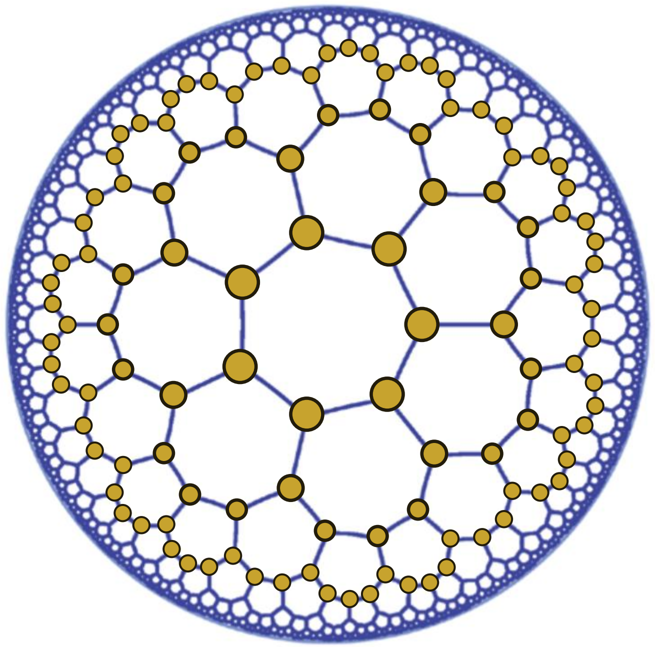
**Hyperbolic** “{7,3}” tessellation

[...]

Stereogr. proj.



Deform the graph *while respecting the coupling strength* on each bond

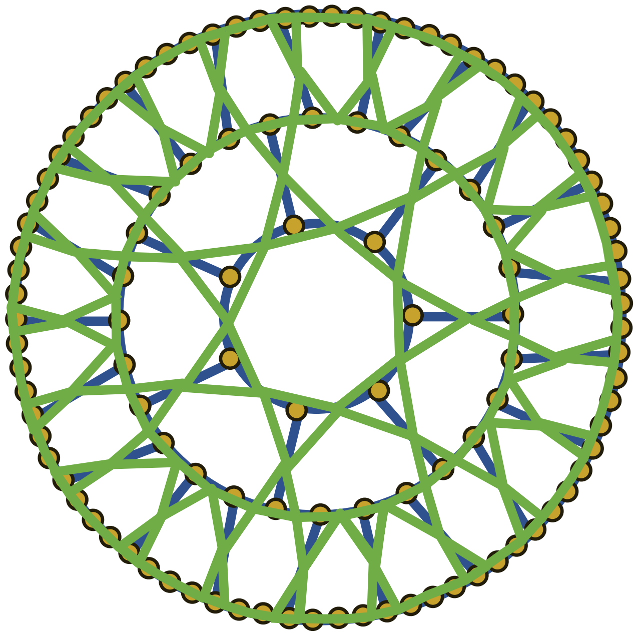


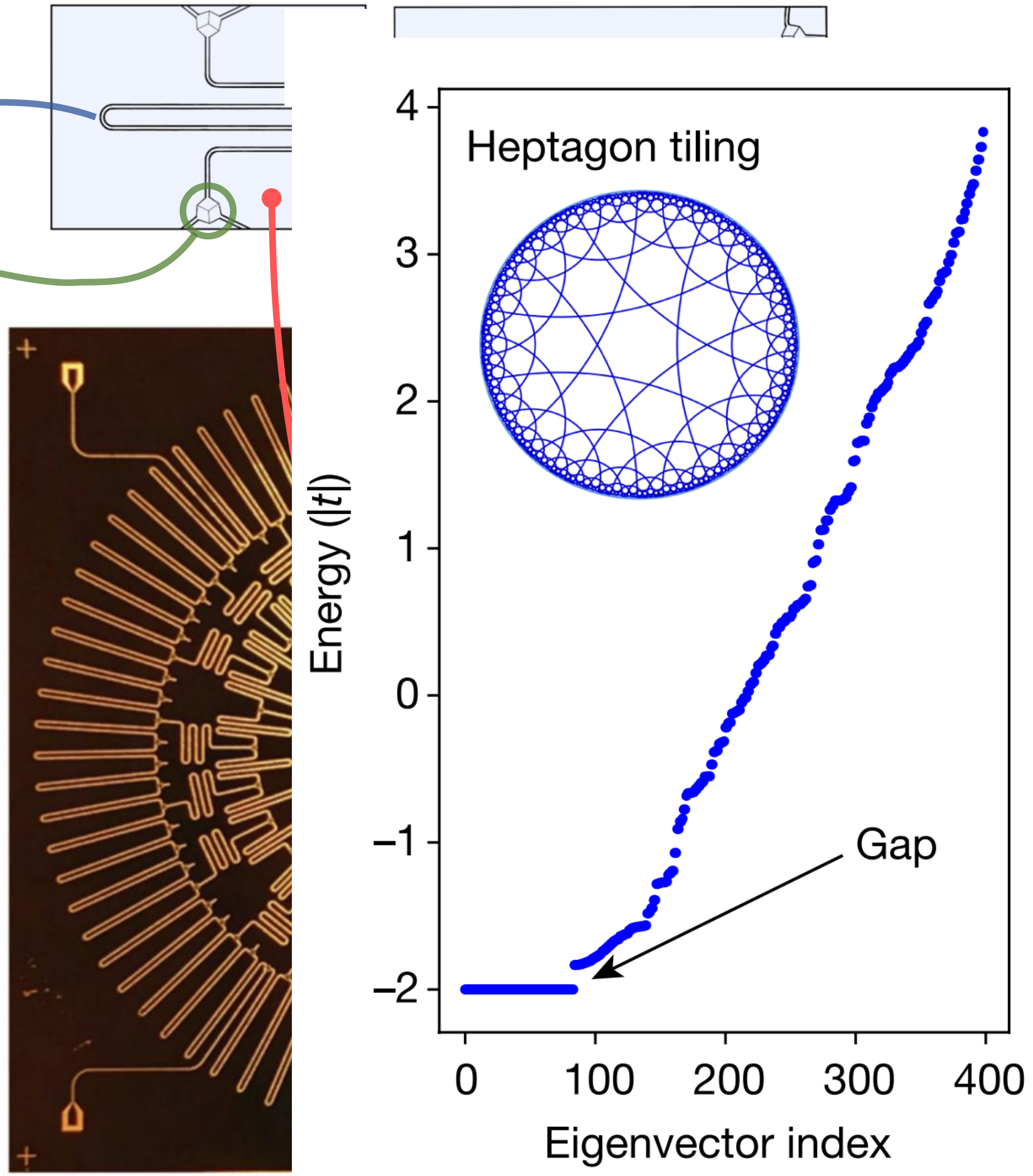
# Hyperbolic lattice in circuit QED

Degrees of freedom on edges,

coupled by tri-junctions.

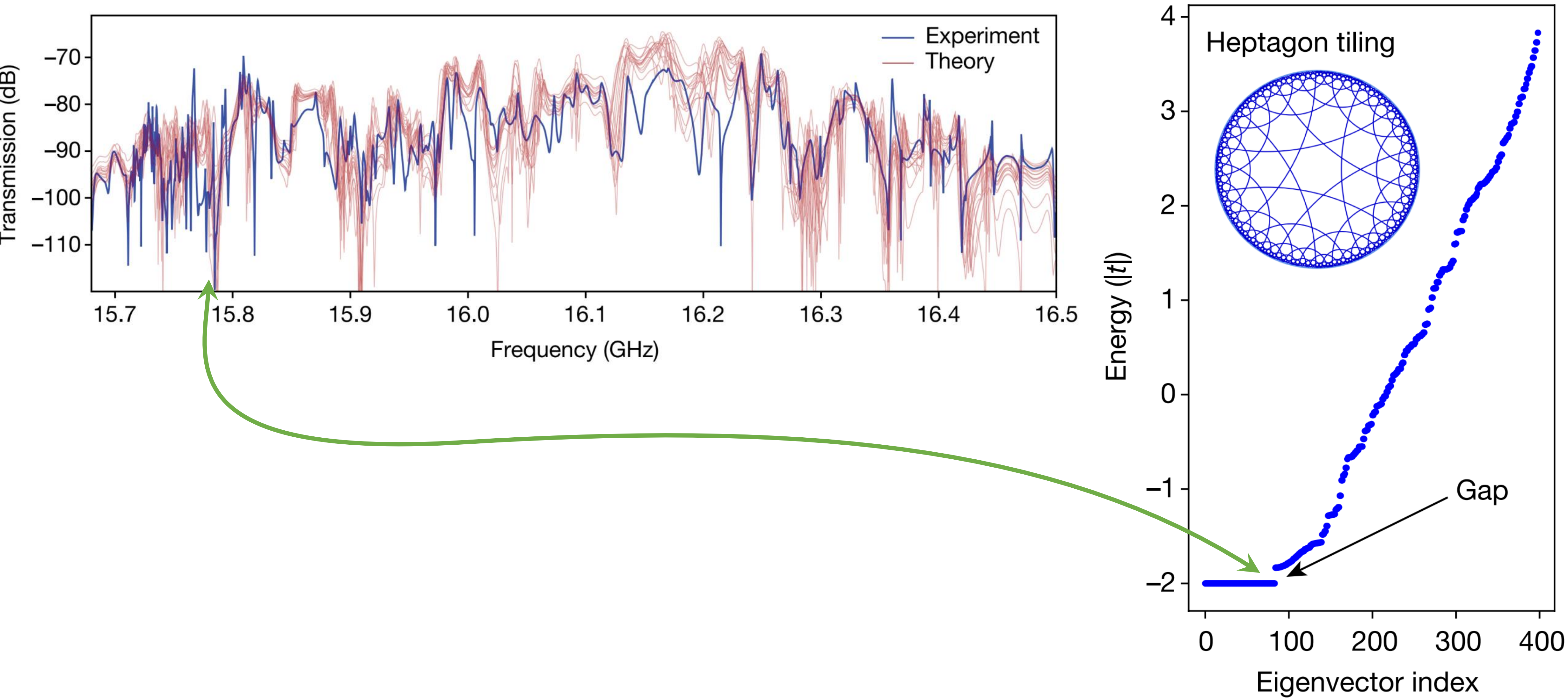
$$\mathcal{H}_{\mathrm{TB}} = \omega_0 \sum_i a_i^\dagger a_i - t \sum_{\langle i,j \rangle} \left( a_i^\dagger a_j + a_j^\dagger a_i \right)$$






A. J. Kollár, M. Fitzpatrick, and A. A. Houck, *Nature* **571**, 45—50 (2019)

# Hyperbolic lattice in circuit QED




A. J. Kollár, M. Fitzpatrick, and A. A. Houck, *Nature* **571**, 45—50 (2019)

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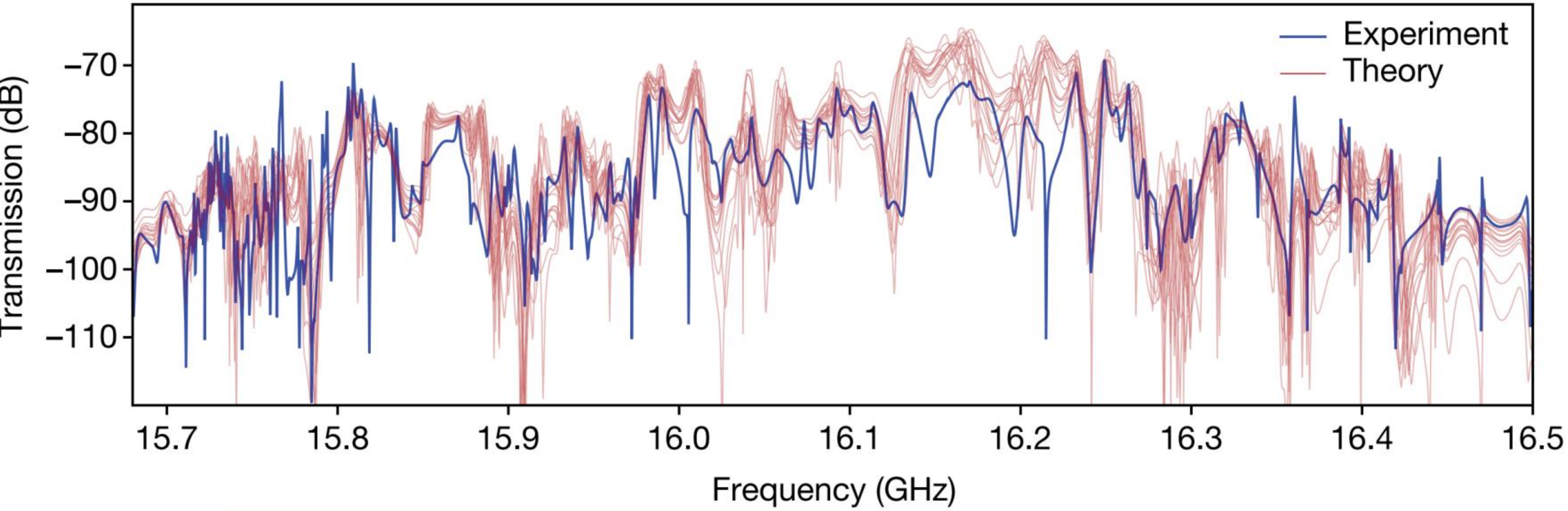
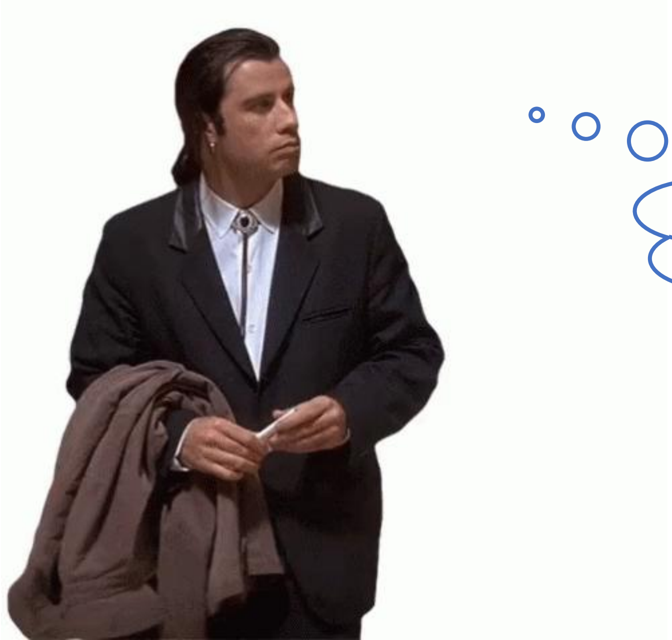
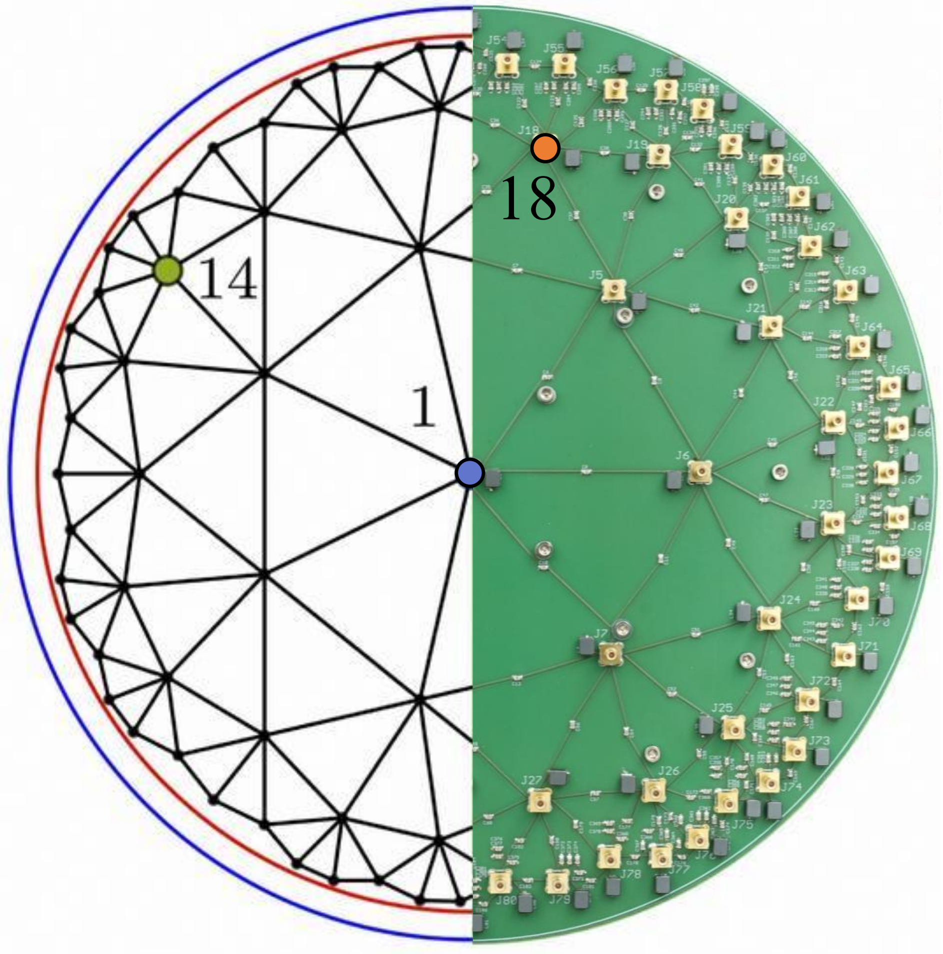


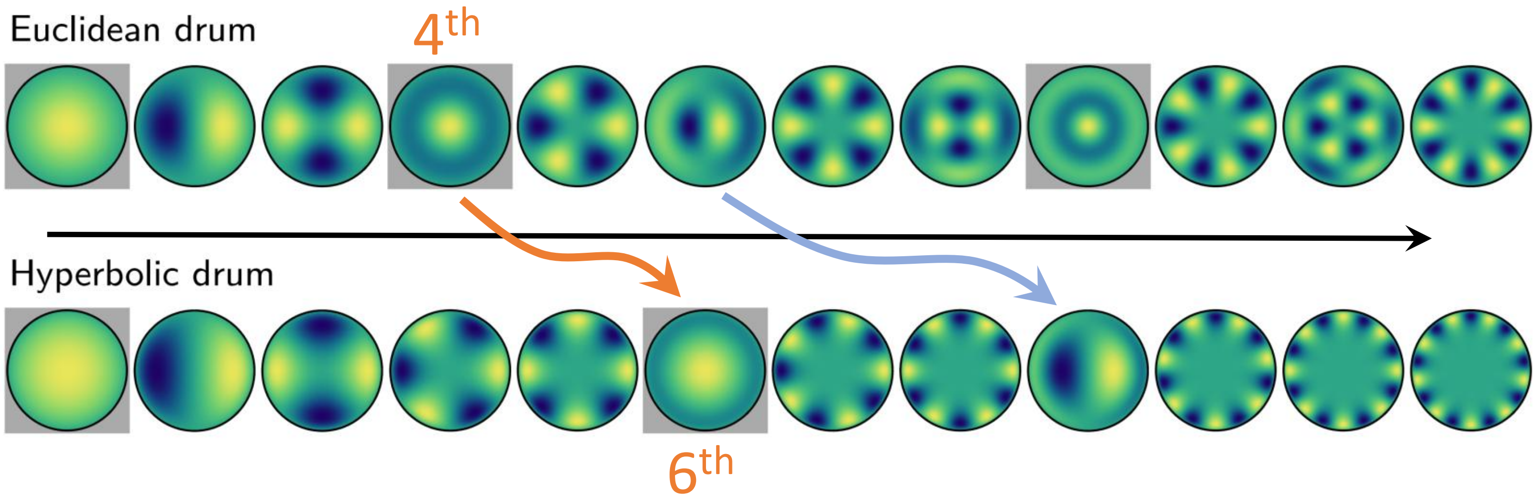
curvature where?

# Hyperbolic lattices in electric circuits

Experimental setup:



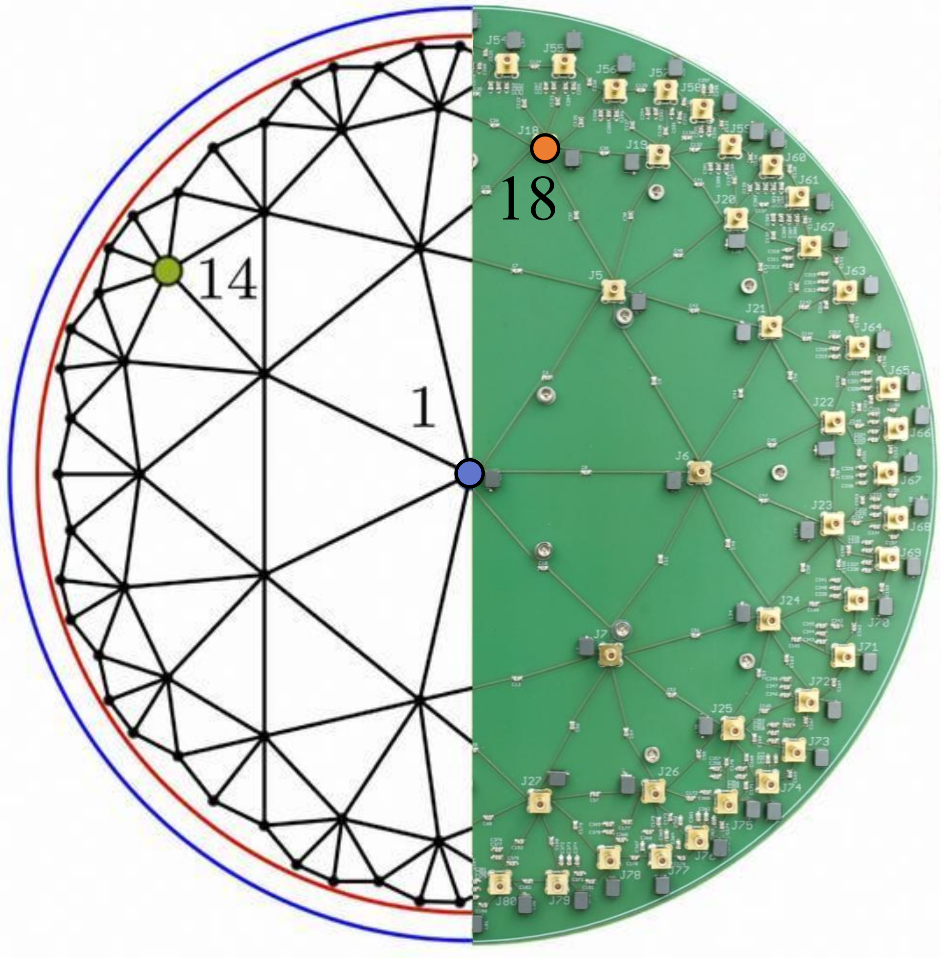

Predictions in flat vs. curved *continuum*:



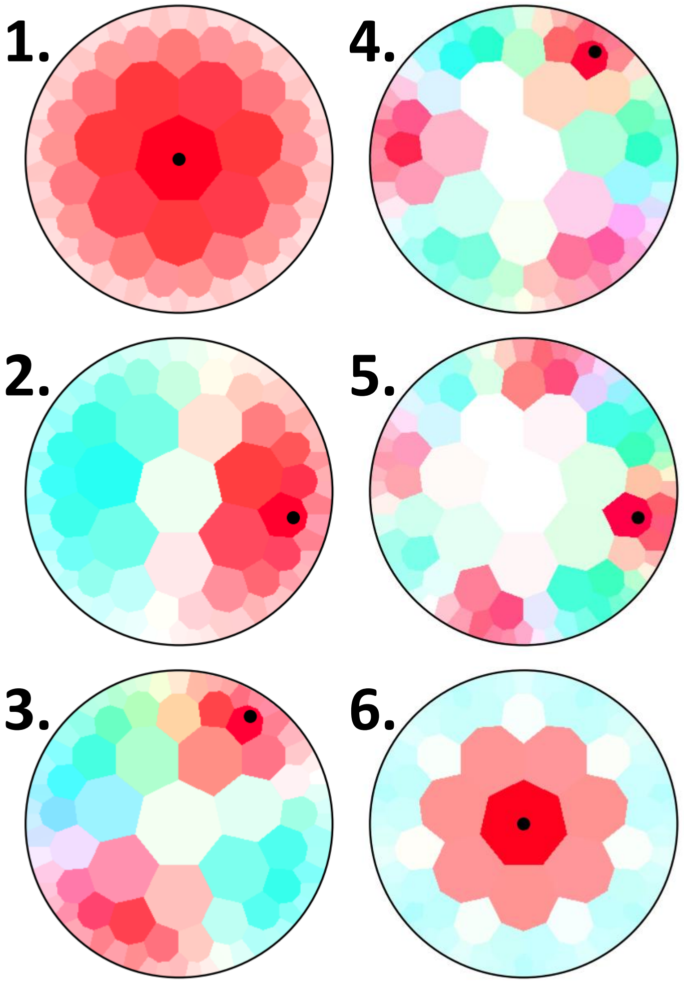

P. M. Lenggenhager, A. Stegmaier, TB, *et al.*, Nat. Commun. **13**, 4373 (2022)

# Hyperbolic lattices in electric circuits

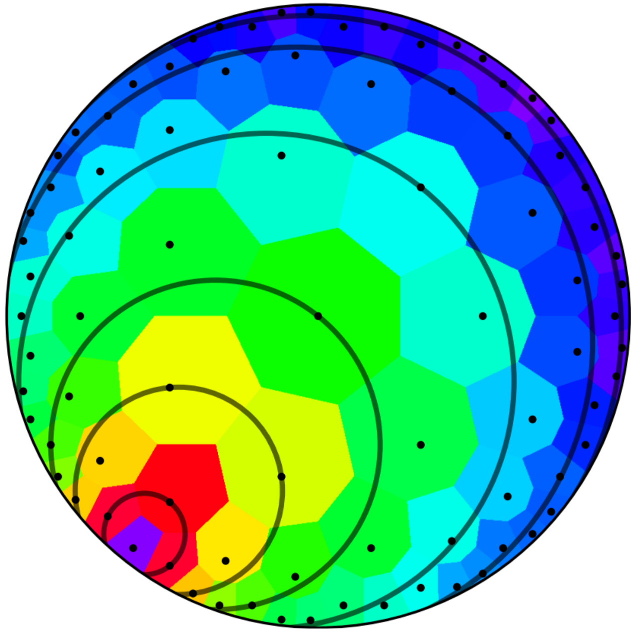
## Experimental setup:

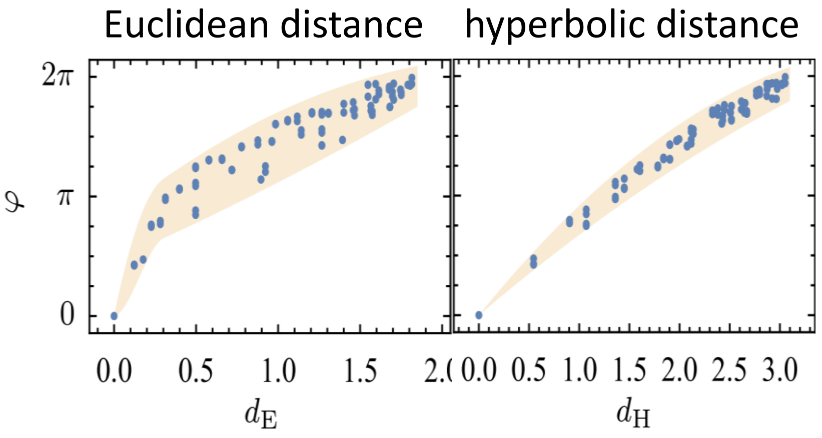



## Spectral reordering:




## Wave propagation:

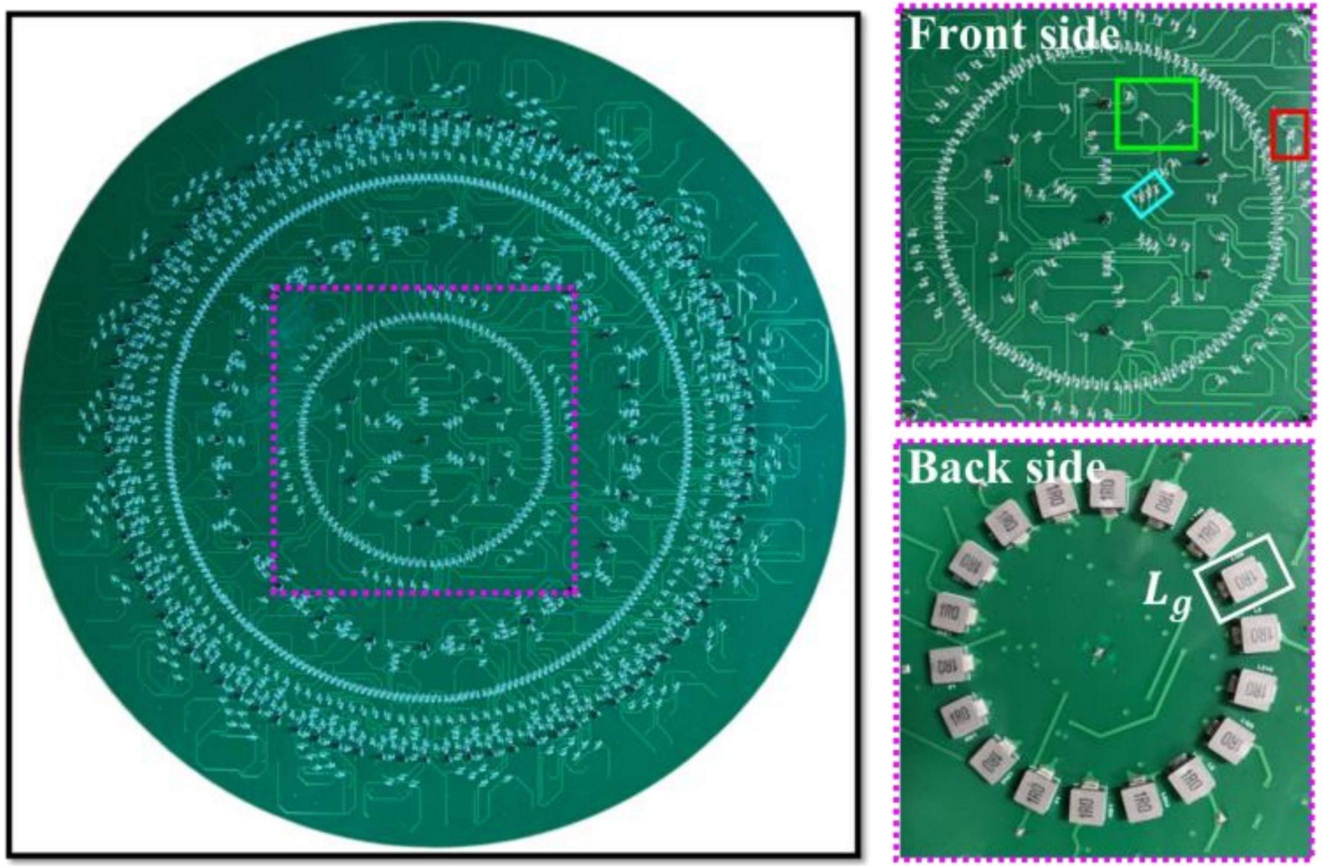





P. M. Lenggenhager, A. Stegmaier, TB, *et al.*, Nat. Commun. **13**, 4373 (2022)

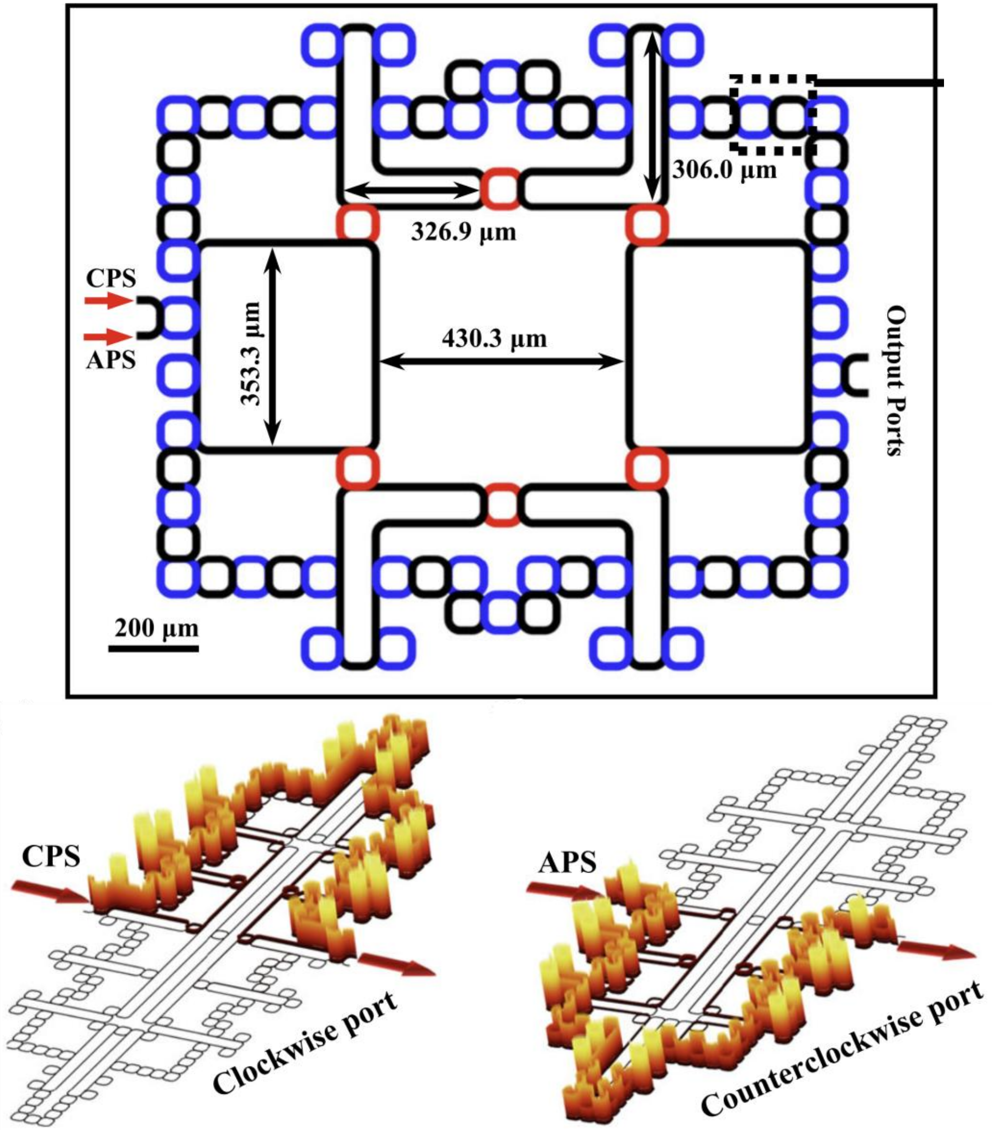
# Realizations of hyperbolic Chern insulators

## With electric circuits:



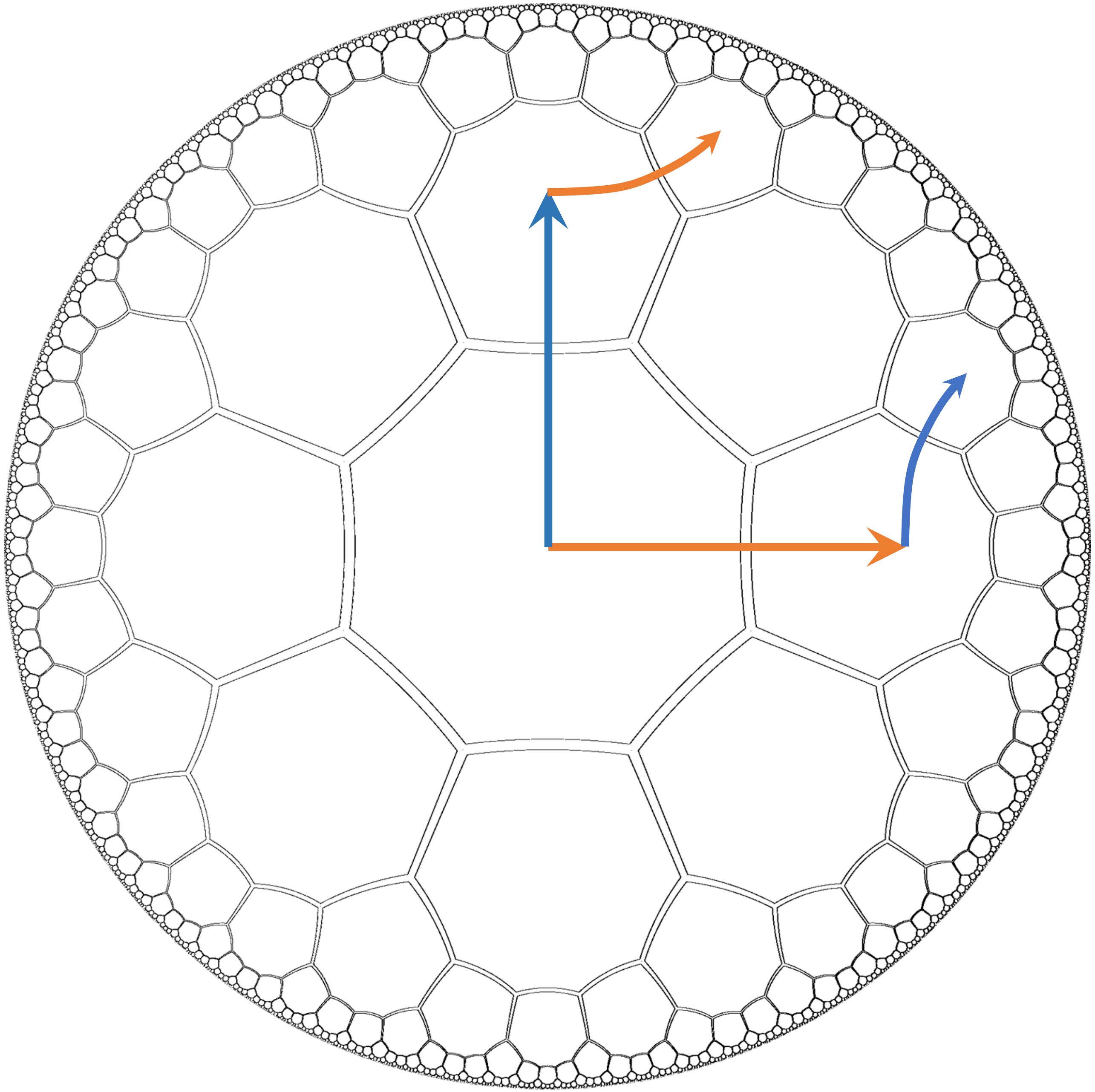

W. Zhang, *et al.*, Nat. Commun. **13**, 2937 (2022)

## In silicon photonics:




L. Huang, *et al.*, Nat. Commun. **15**, 1647 (2024)

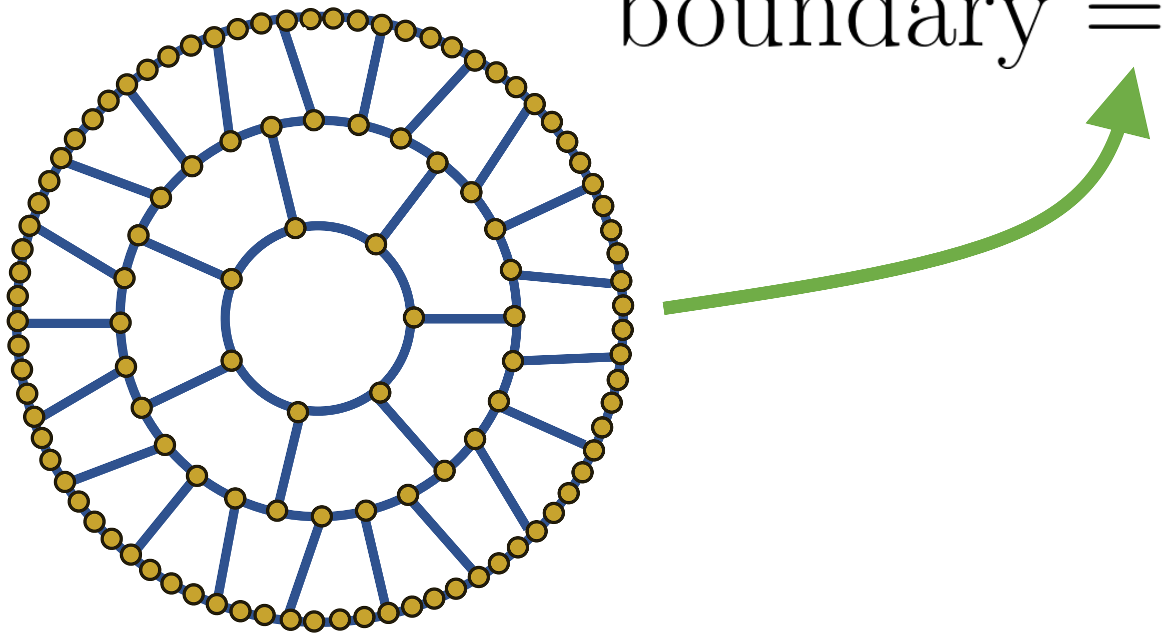
# (1.) Infinite lattice: non-commuting translations!



# (2.) Disk geometry (OBC): macroscopic boundary

$$0 < \text{fraction} = \frac{1}{2}\left(s + \sqrt{s^2 - 4s}\right) < 1 \qquad s = 2p + 2q - pq$$

$$\text{boundary} = \text{fraction} \times \text{sites}$$

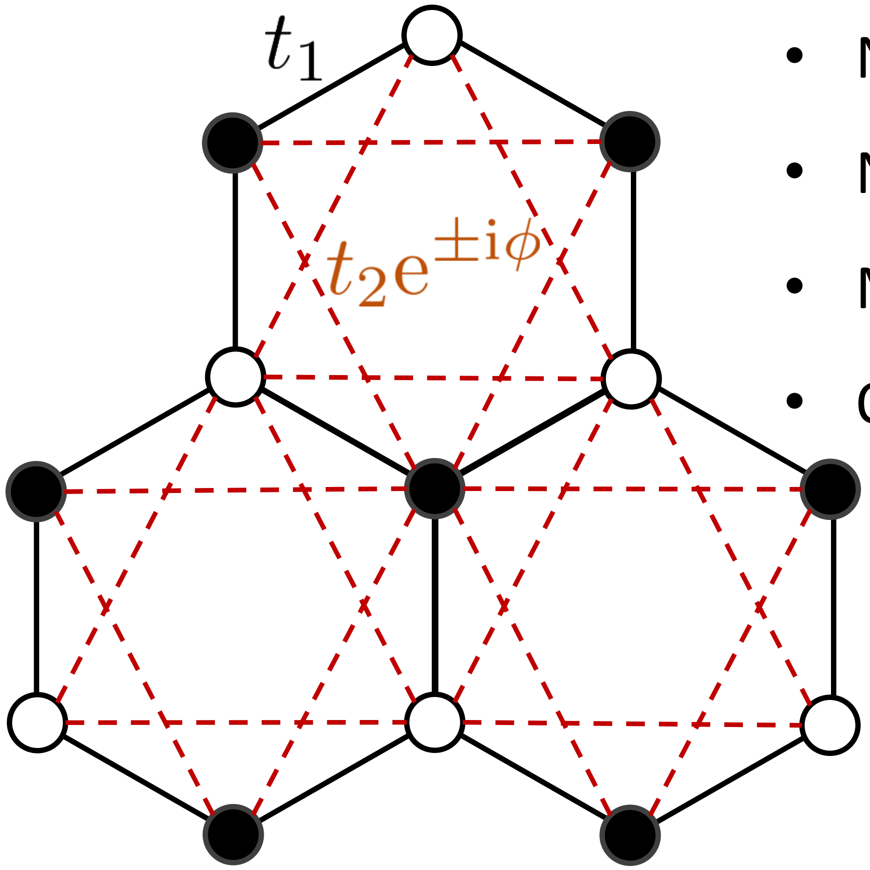


**Hyperbolic crocheting…**

P. Basteiro, *et al.*, SciPost Phys. **13**, 103 (2022)

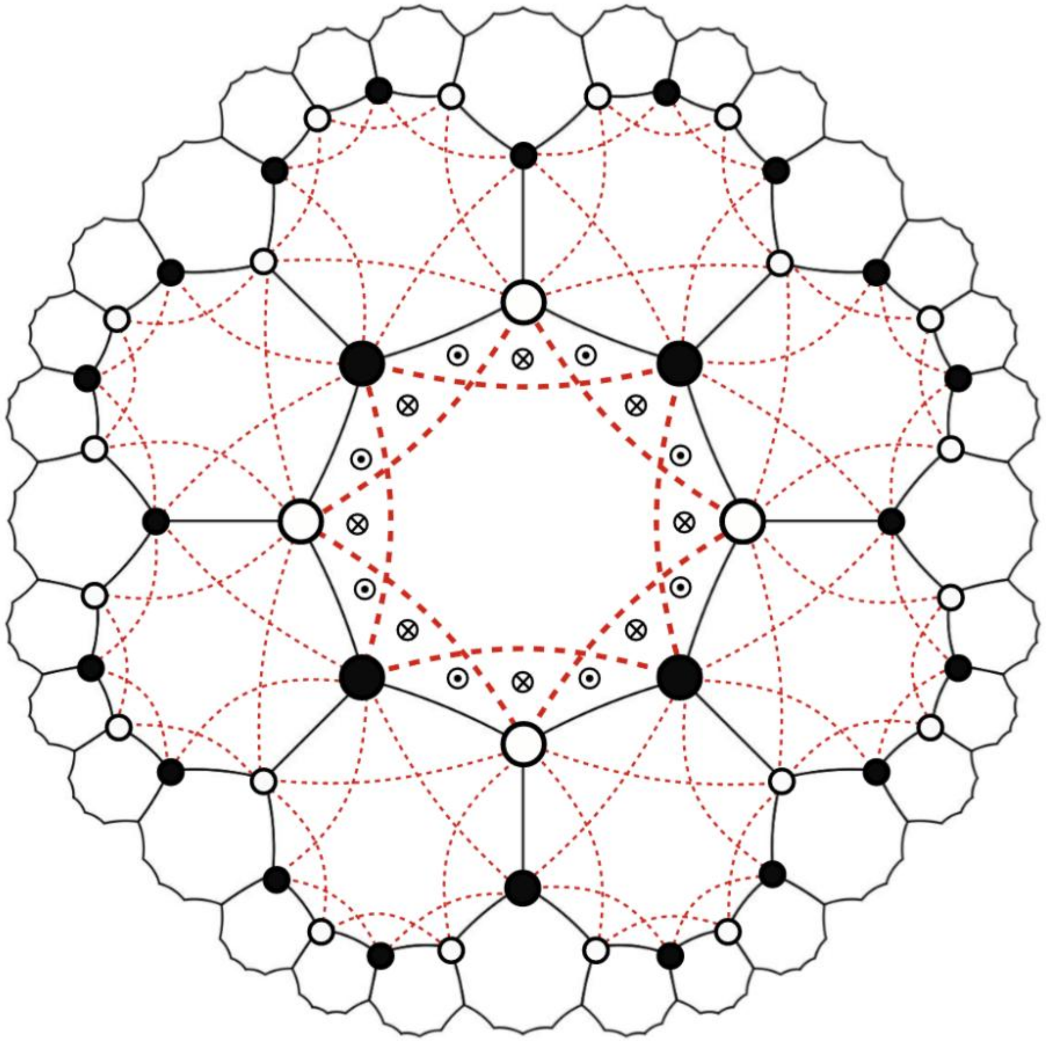
Image source: D. Taimiņa, *Crocheting Adventures with Hyperbolic Planes* (book, 2009)

# Hyperbolic Haldane model




**4 ingredients:**

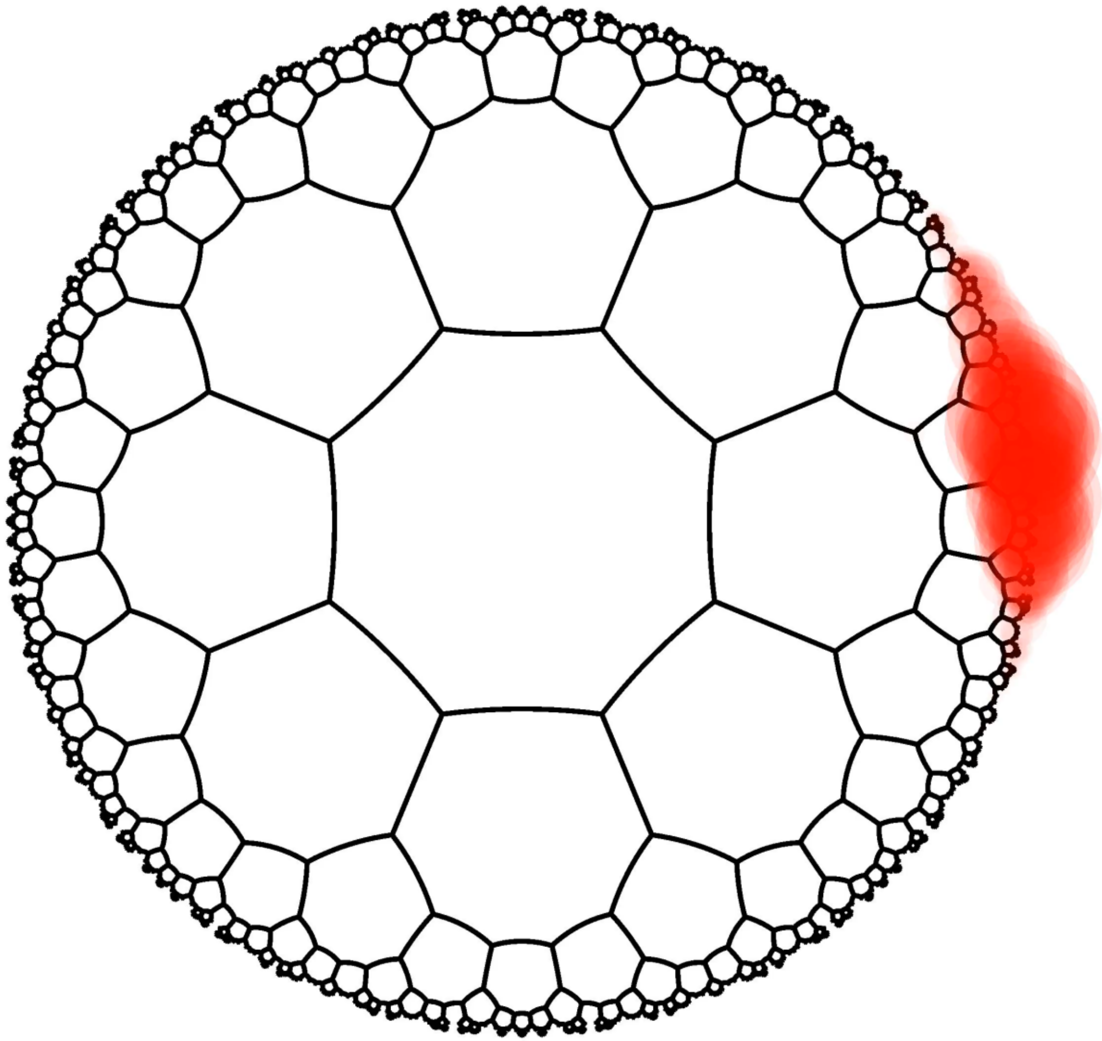
- NN hopping ($t_1$ = 1)
- NNN hopping ($t_2$)
- Magnetic flux (ϕ)
- On-site potential ($\pm M$)

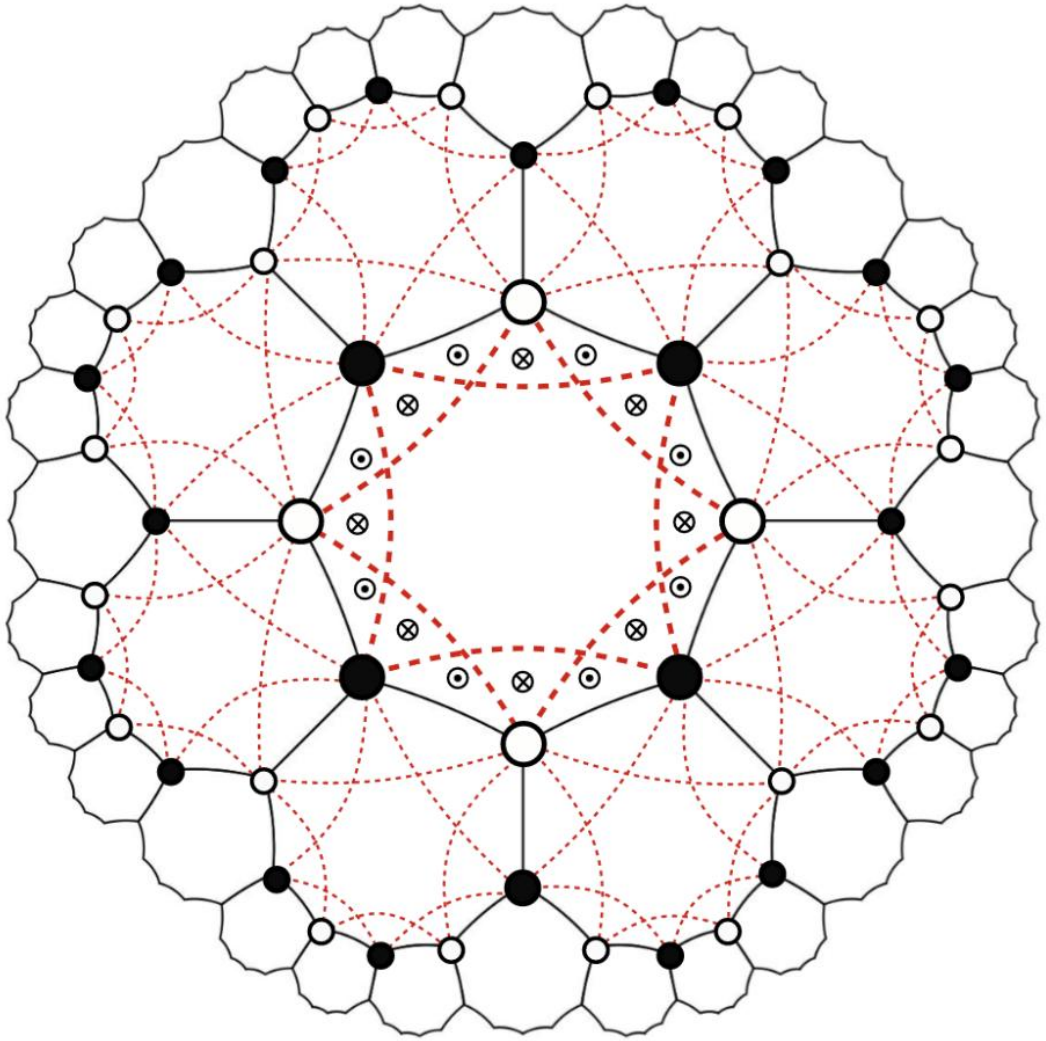


F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)

Phys. Rev. Lett. **129**, 246402 (2022)

# Hyperbolic Haldane model





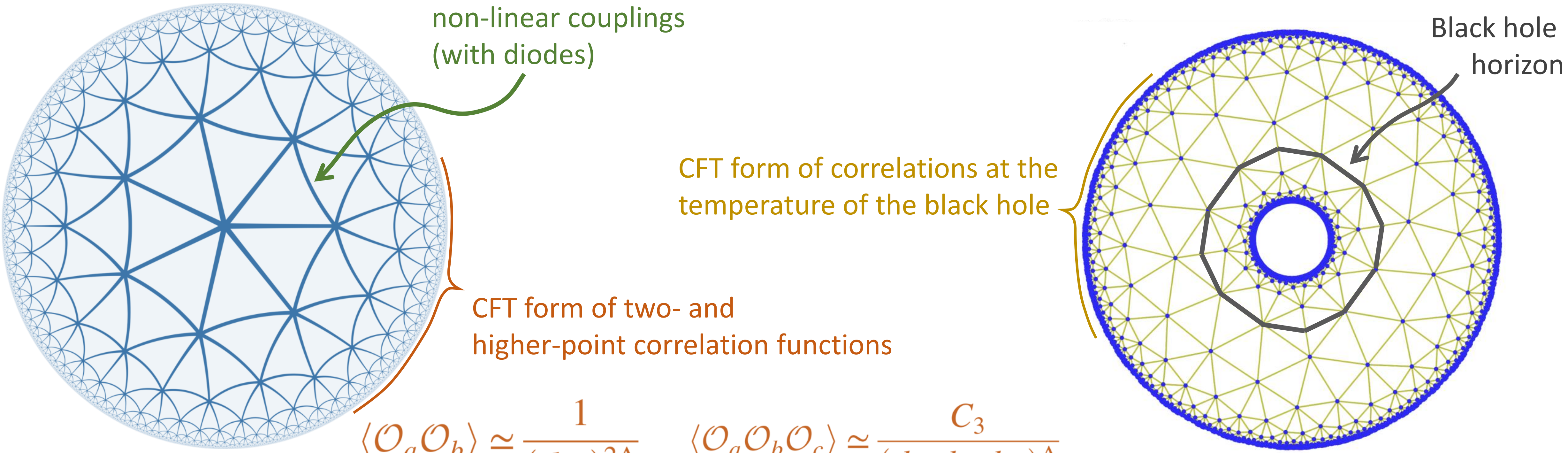
Phys. Rev. Lett. **129**, 246402 (2022)

PHYSICAL REVIEW LETTERS **133,** 061603 (2024)

Editors' Suggestion

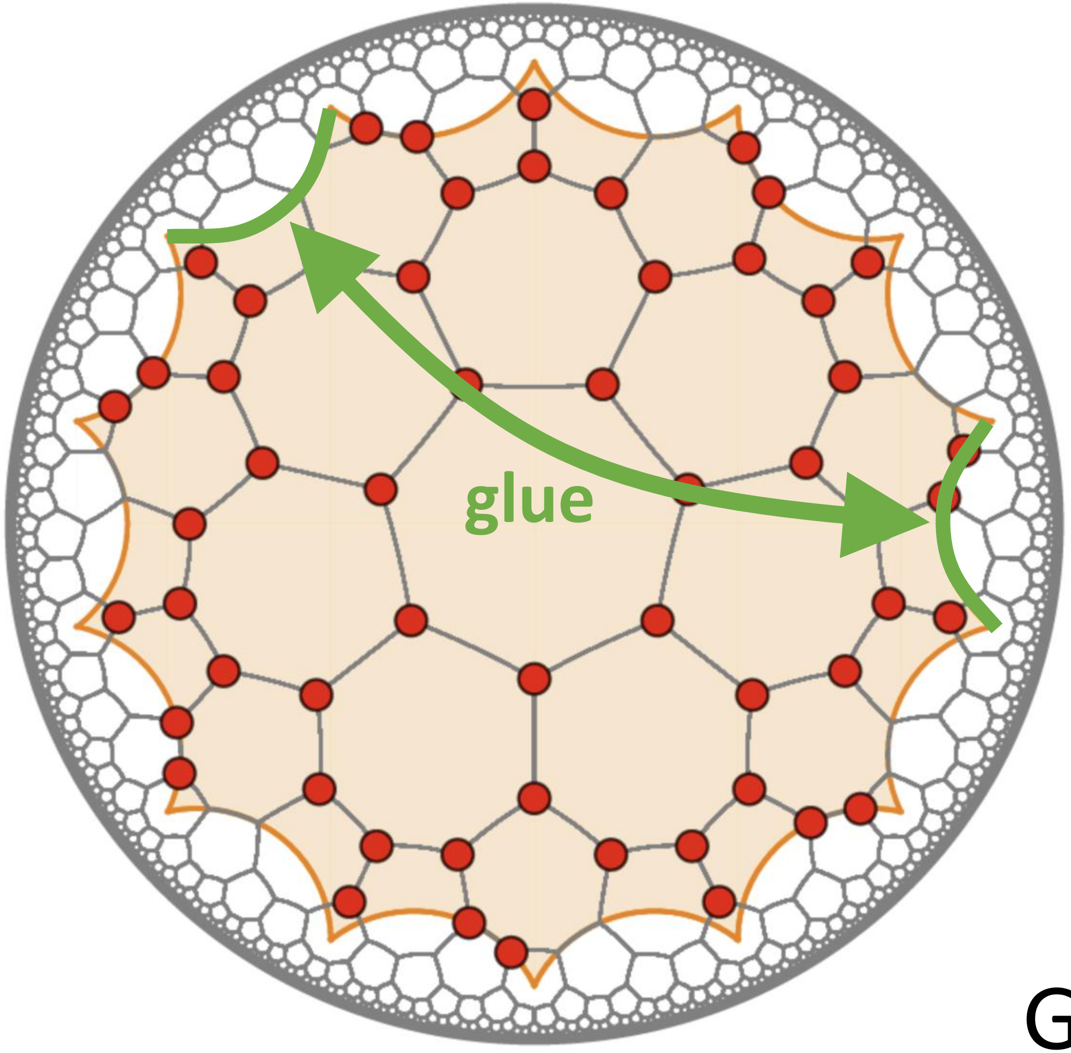
# Simulating Holographic Conformal Field Theories on Hyperbolic Lattices

Santanu Dey,[1,2,*] Anffany Chen,[1,2,†] Pablo Basteiro,[3,4] Alexander Fritzsche,[3,4,5] Martin Greiter,[3,4] Matthias Kaminski,[6] Patrick M. Lenggenhager,[4,7,8,9,10] René Meyer,[3,4] Riccardo Sorbello,[3,4] Alexander Stegmaier,[3,4] Ronny Thomale,[3,4] Johanna Erdmenger,[3,4] and Igor Boettcher[1,2]




$$\langle \mathcal{O}_a \mathcal{O}_b \rangle \simeq \frac{1}{(d_{ab})^{2\Delta}} \qquad \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle \simeq \frac{C_3}{(d_{ab} d_{ac} d_{bc})^{\Delta}}$$

# (3.) Finite hyperbolic lattices: "hyperbolic PBC clusters"




Euler characteristic:

$$\chi = V - E + F = 2 \times (1 - g)$$

$$E = Vq/2 \qquad F = Vq/p$$

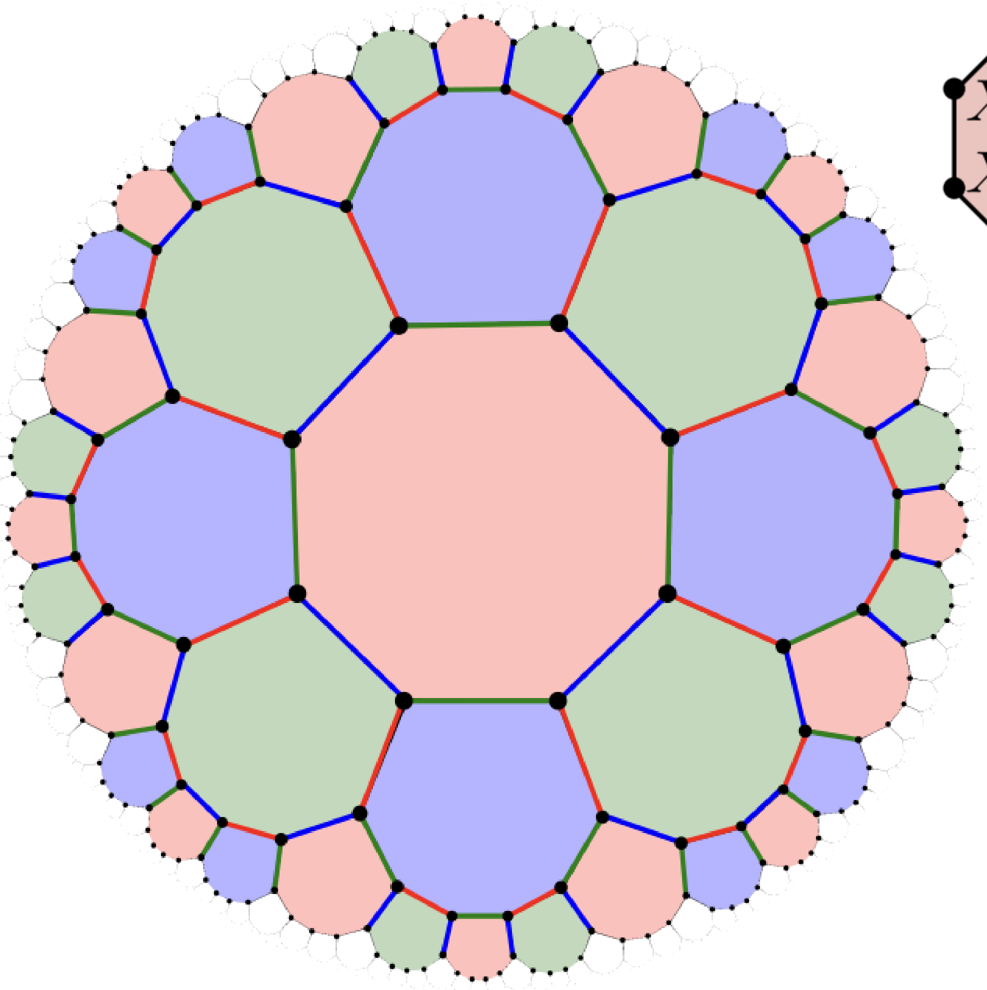
Genus of regular {*p*,*q*} lattices:

$$g = 1 + \frac{V}{2}\left(\frac{q}{2} - \frac{q}{p} - 1\right)$$

Genus grows linearly with # vertices.

I. Boettcher *et al.*, *Crystallography of Hyperbolic Lattices*, Phys. Rev. B **105**, 125118 (2022)

# Hyperbolic surface codes



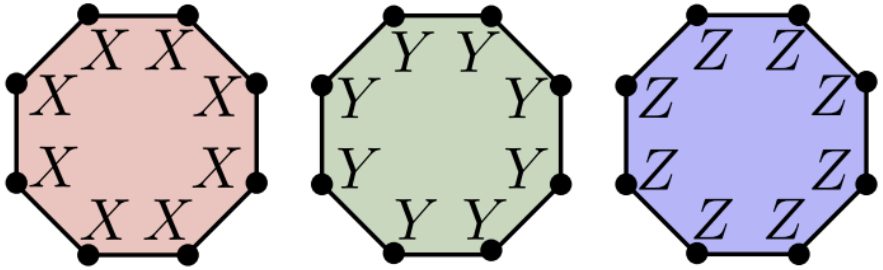


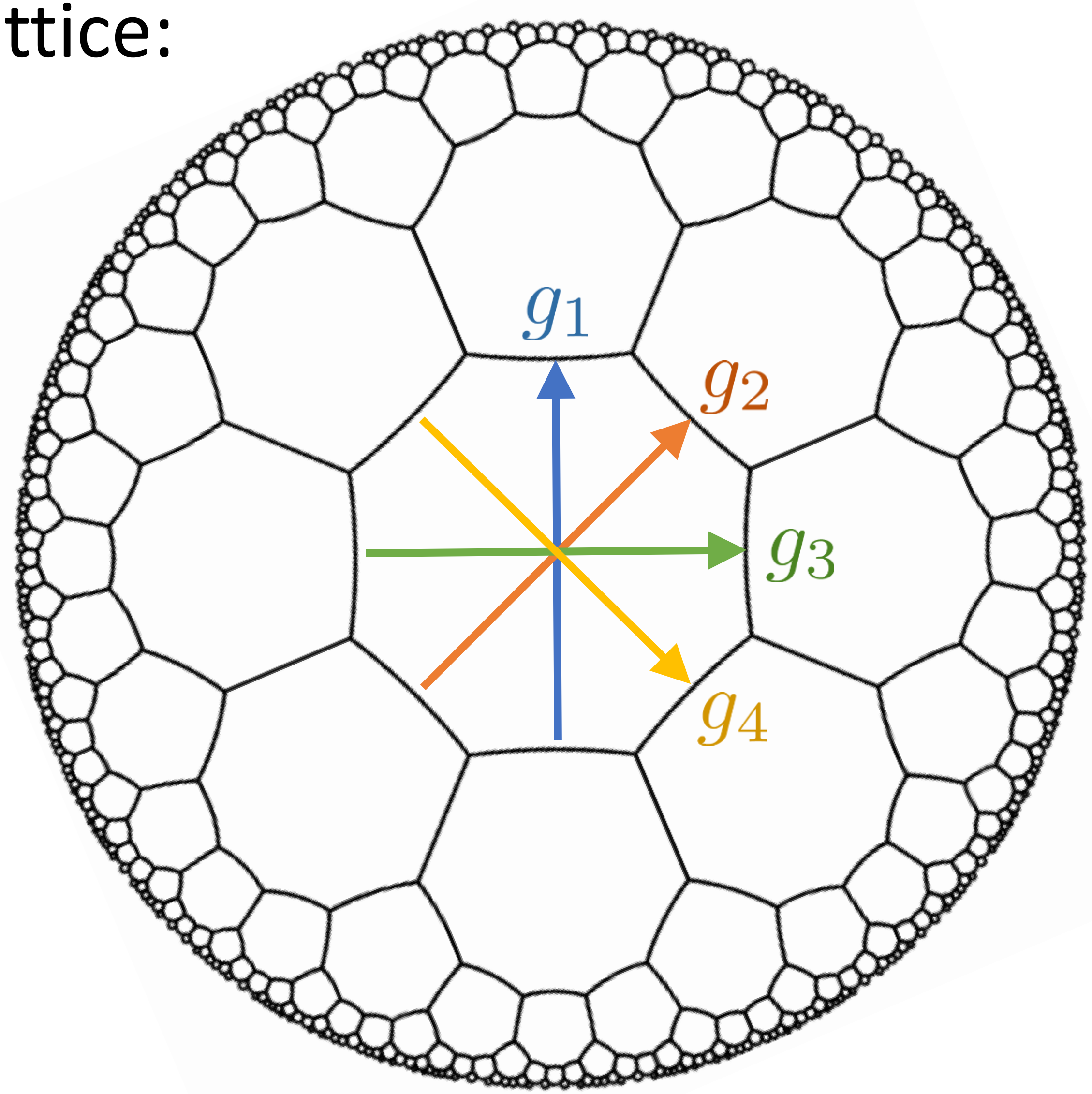

FIG. 7. A symplectic basis for the logical operators of the hyperbolic Floquet code derived from the 8.8.8 tiling of the Bolza surface, which has genus 2 and encodes four logical qubits into 16 physical data qubits. Opposite sides of the tiling are identified. The logical $\bar{X}$ and $\bar{Z}$ operators of logical qubit $i$ are denoted by $\bar{X}_i$ and $\bar{Z}_i$, respectively. For each logical, the gray highlighted path is its associated homologically nontrivial logical path, which defines how the logical is updated in each subround (see Fig. 8).

O. Higgott and N. Breuckmann, PRX Quantum **5**, 040327 (2024)

# Symmetry of hyperbolic {8,3} lattice

Four translation generators on {8,3} lattice:

$$g_1 g_2^{-1} g_3 = \text{rotation by } \frac{2\pi}{8}$$



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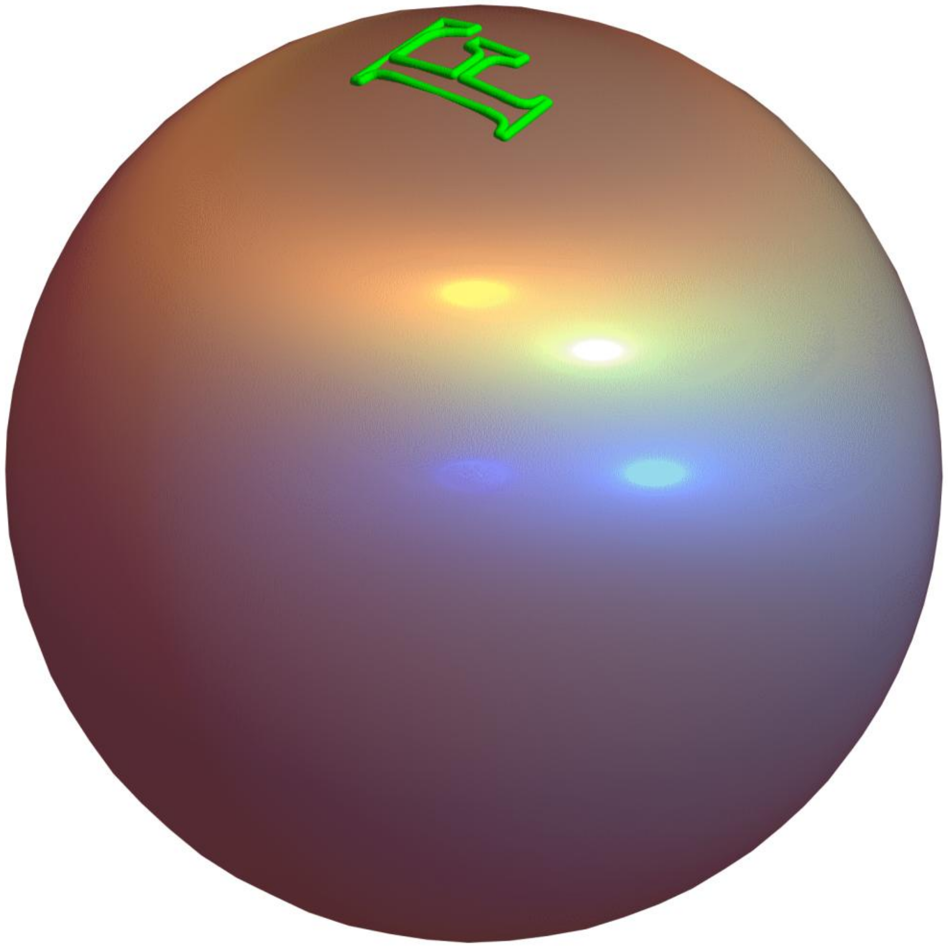


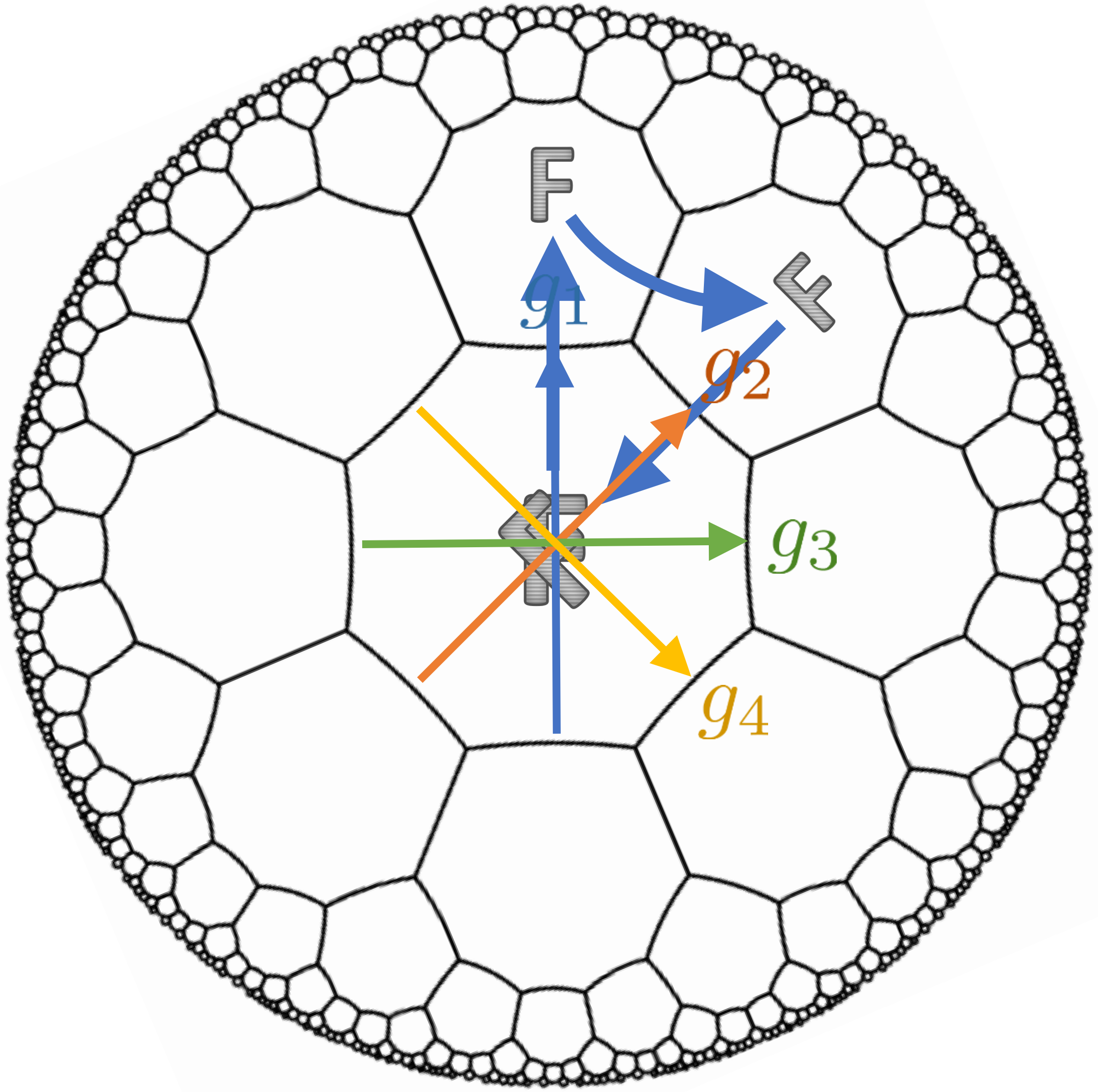

# Symmetry of hyperbolic {8,3} lattice

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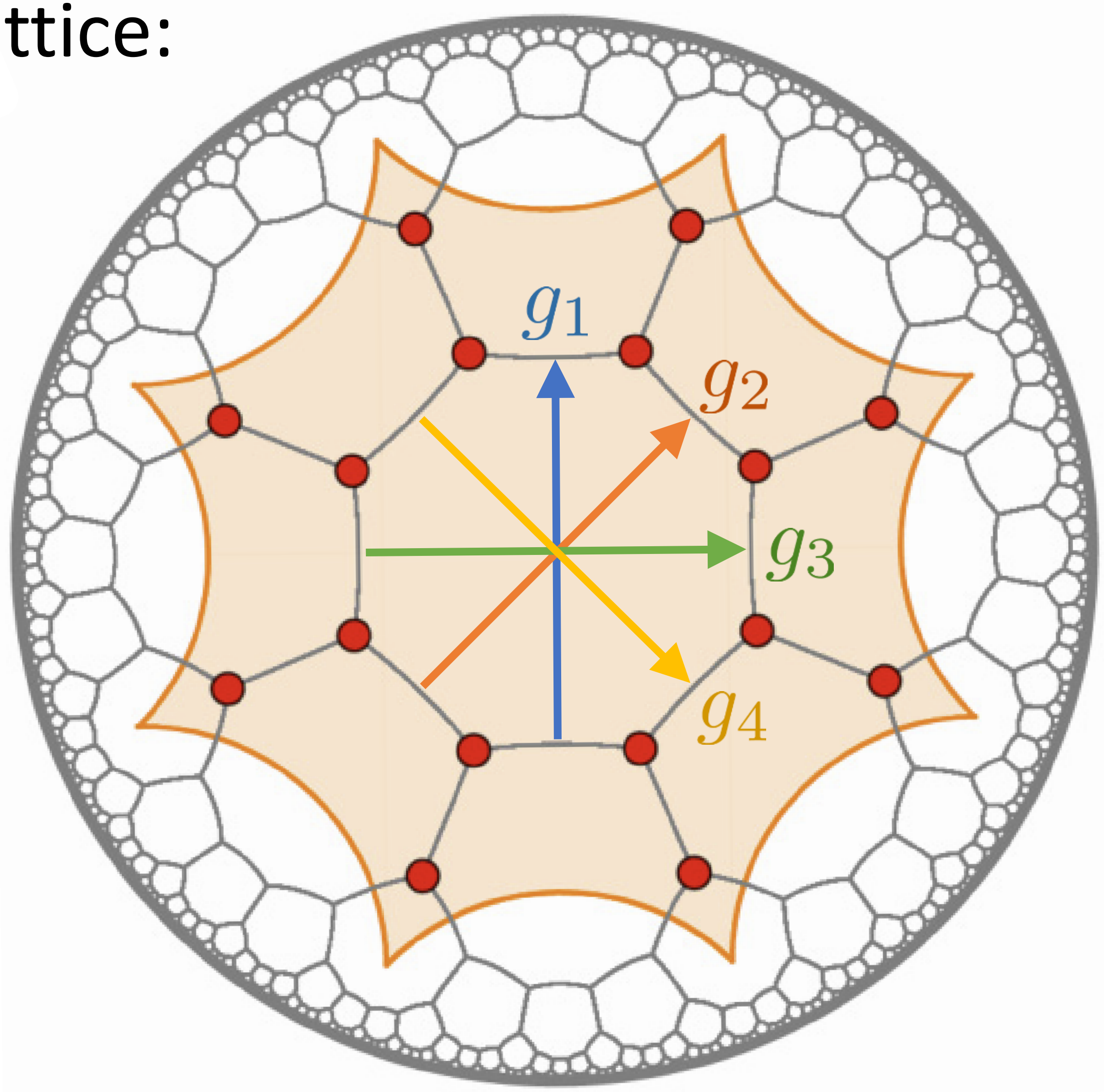
$$g_1 g_2^{-1} g_3 = \text{rotation by } \frac{2\pi}{8}$$

**Translation group should**
**not contain rotations!**

**Gauss-Bonnet theorem:**
unit cell area must be 4πn.



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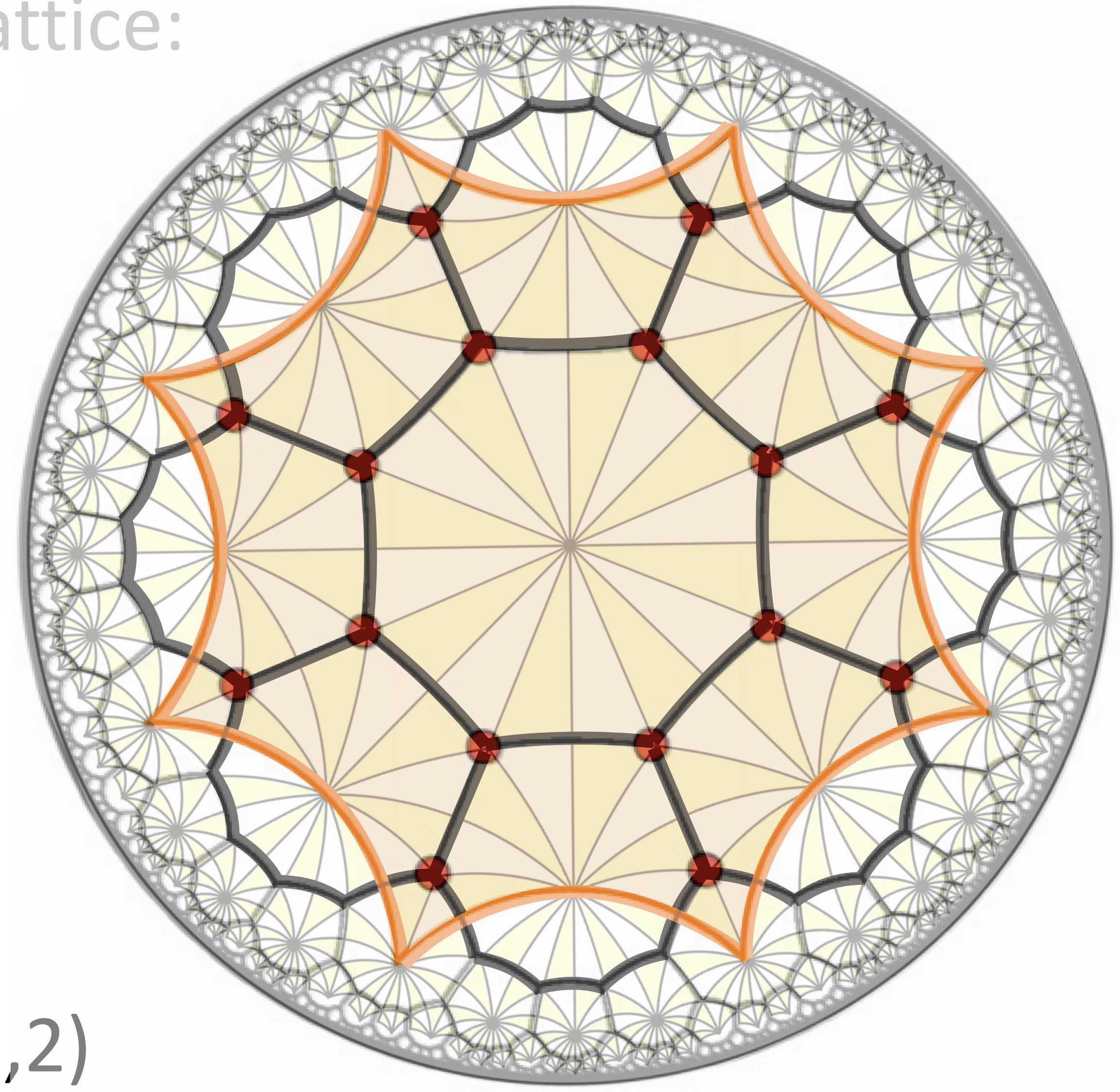
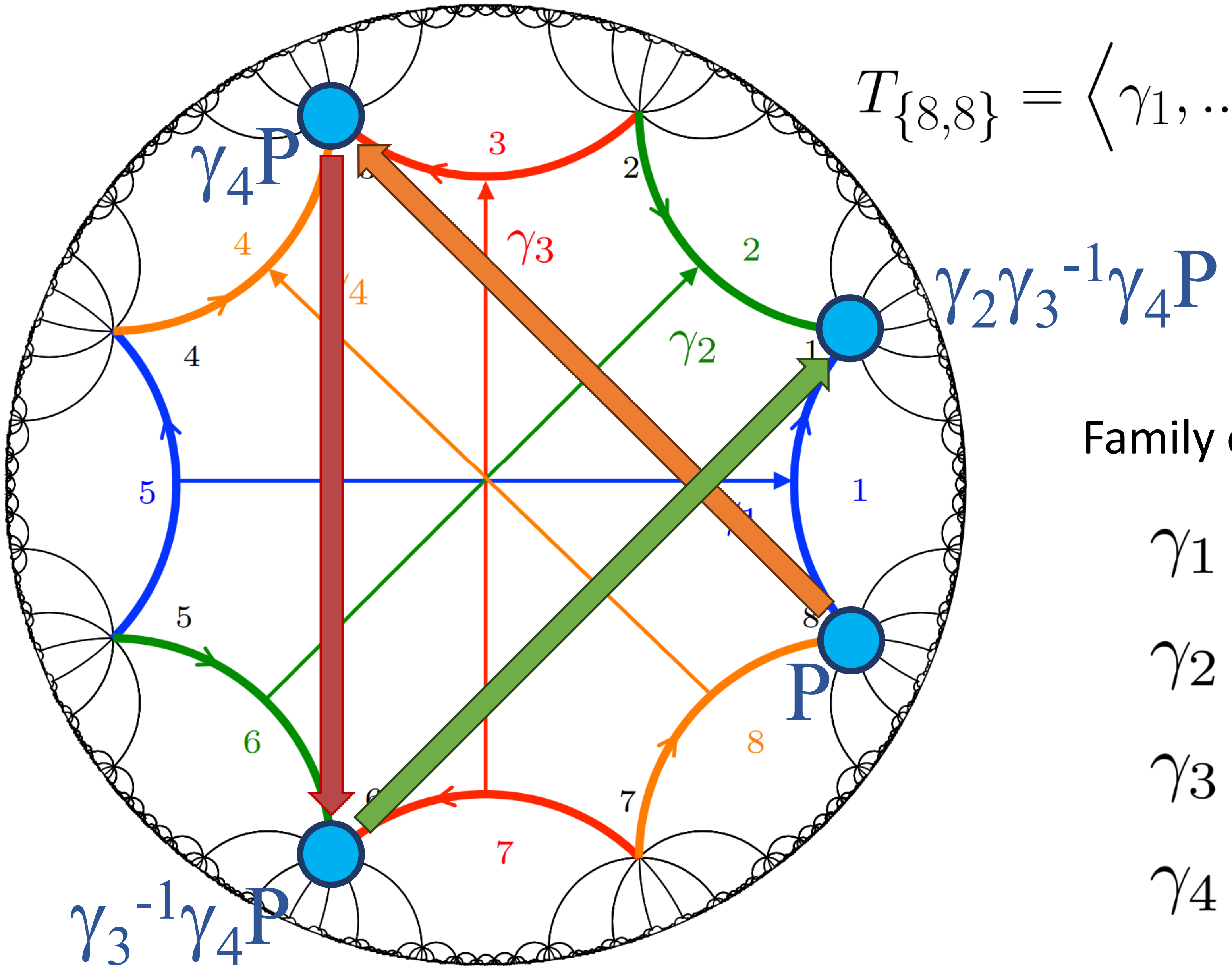


*Mathematically:*
Torsionfree normal subgroup of Δ(p,q,2)

# # Vertices in the primitive unit cell

| p,q | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | X | X | X | 1 | 24 | 6 | 36 | 12 | 60 | 4 | 84 | 21 |
| 4 | X | 1 | 24 | 4 | 24 | 2 | 8 | 4 | 120 | 2 | 168 | 4 |
| 5 | X | 30 | 12 | 20 | 360 | 30 | 380 | 1 | 60 | 20 | 600 | 360 |
| 6 | 2 | 6 | 24 | 2 | 48 | 6 | 6 | 6 | 60 | 2 | 84 | 6 |
| 7 | 56 | 42 | 504 | 56 | 24 | 42 | 56 | 504 | ? | 56 | 84 | 1 |
| 8 | 16 | 4 | 48 | 8 | 48 | 1 | 16 | 8 | 240 | 4 | 336 | 8 |
| 9 | 108 | 18 | 684 | 9 | 72 | 18 | 4 | 45 | ? | 18 | ? | 63 |
| 10 | 40 | 10 | 2 | 10 | 720 | 10 | 50 | 2 | 120 | 10 | ? | 10 |
| 11 | 220 | 330 | 132 | 110 | ? | 330 | ? | 132 | 60 | 110 | ? | ? |
| 12 | 16 | 6 | 48 | 4 | 96 | 6 | 24 | 12 | 120 | 1 | 168 | 12 |
| 13 | 364 | 546 | 1560 | 182 | 156 | 546 | ? | ? | ? | 182 | 84 | 156 |
| 14 | 98 | 14 | 1008 | 14 | 2 | 14 | 98 | 14 | ? | 14 | 168 | 2 |

# Hyperbolic band theory




$$T_{\{8,8\}} = \left\langle \gamma_1, \ldots, \gamma_4 \,\middle|\, \ldots \right.$$

Trivially fulfilled if U(1) numbers!

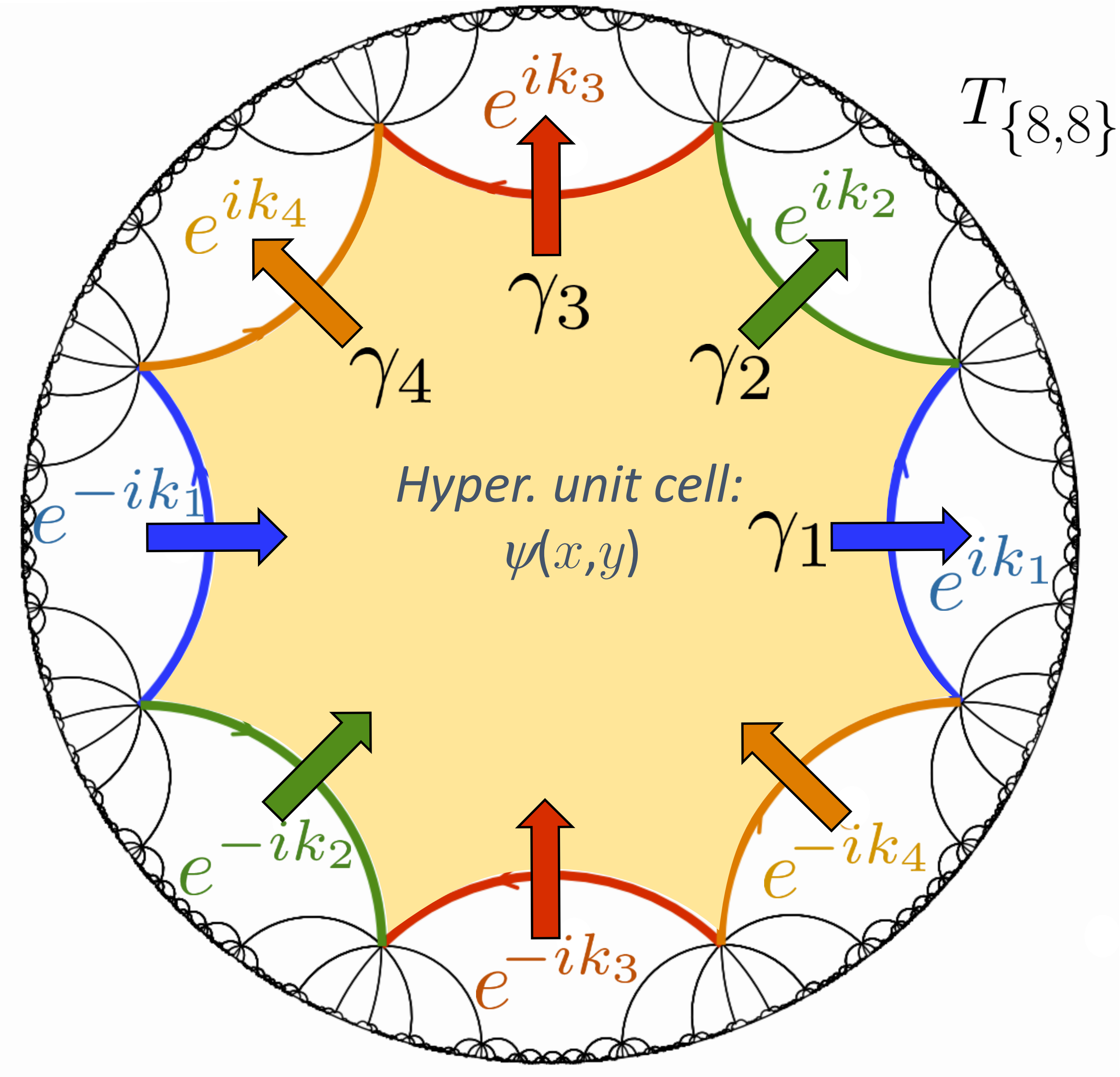
Family of 1D (Abelian) representations:

$$\gamma_1 \mapsto e^{ik_1}$$
$$\gamma_2 \mapsto e^{ik_2}$$
$$\gamma_3 \mapsto e^{ik_3}$$
$$\gamma_4 \mapsto e^{ik_4}$$

Four-dimensional Brillouin zone!

J. Maciejko and S. Rayan, Sci. Adv. **7**, abe9170 (2021)

# Hyperbolic band theory



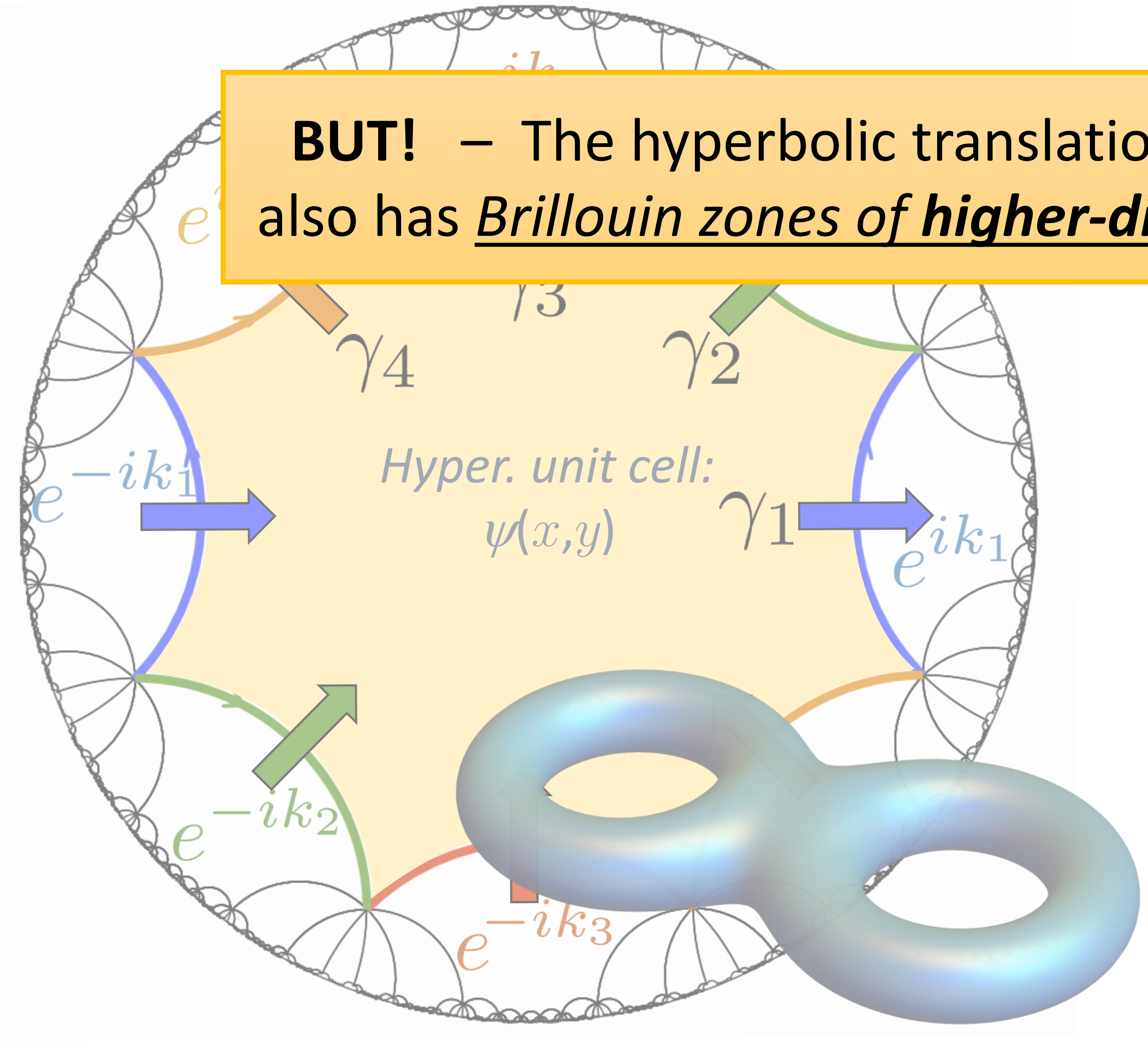

$$T_{\{8,8\}} = \left\langle \gamma_1, \ldots, \gamma_4 \middle| \underbrace{\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_1^{-1}\gamma_2\gamma_3^{-1}\gamma_4}_{\text{Trivially fulfilled if Abelianized.}} \right\rangle$$

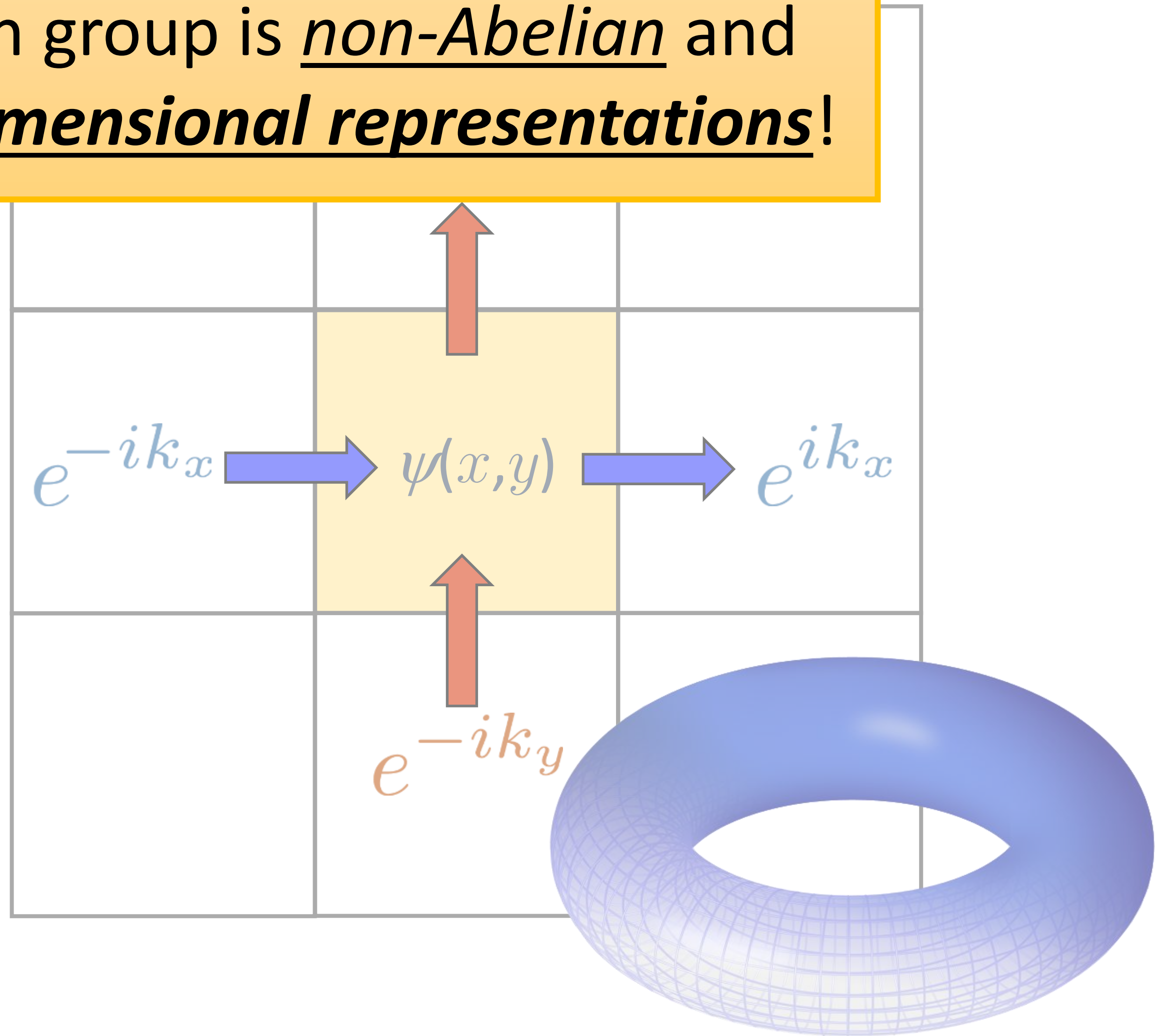
Family of 1D (Abelian) representations:

$$\left.\begin{array}{l} \gamma_1 \mapsto e^{ik_1} \\ \gamma_2 \mapsto e^{ik_2} \\ \gamma_3 \mapsto e^{ik_3} \\ \gamma_4 \mapsto e^{ik_4} \end{array}\right\} \text{Four-dimensional Brillouin zone!}$$

J. Maciejko and S. Rayan, Sci. Adv. **7**, abe9170 (2021)

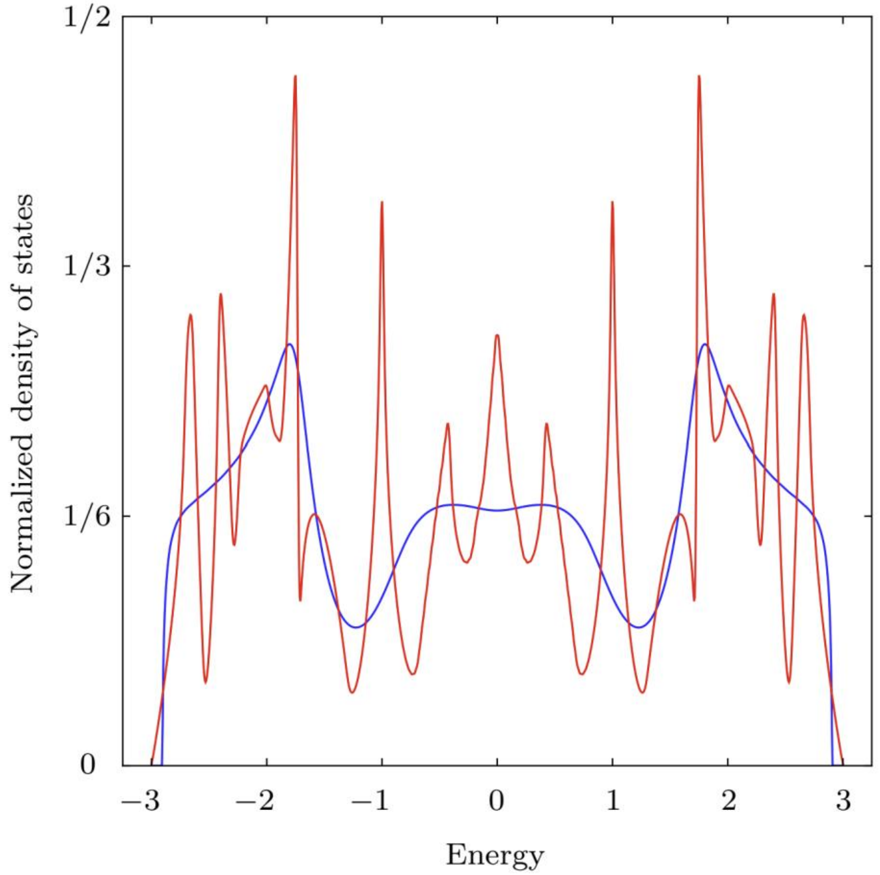
**BUT!** – The hyperbolic translation group is ***non-Abelian*** and also has ***Brillouin zones of higher-dimensional representations***!



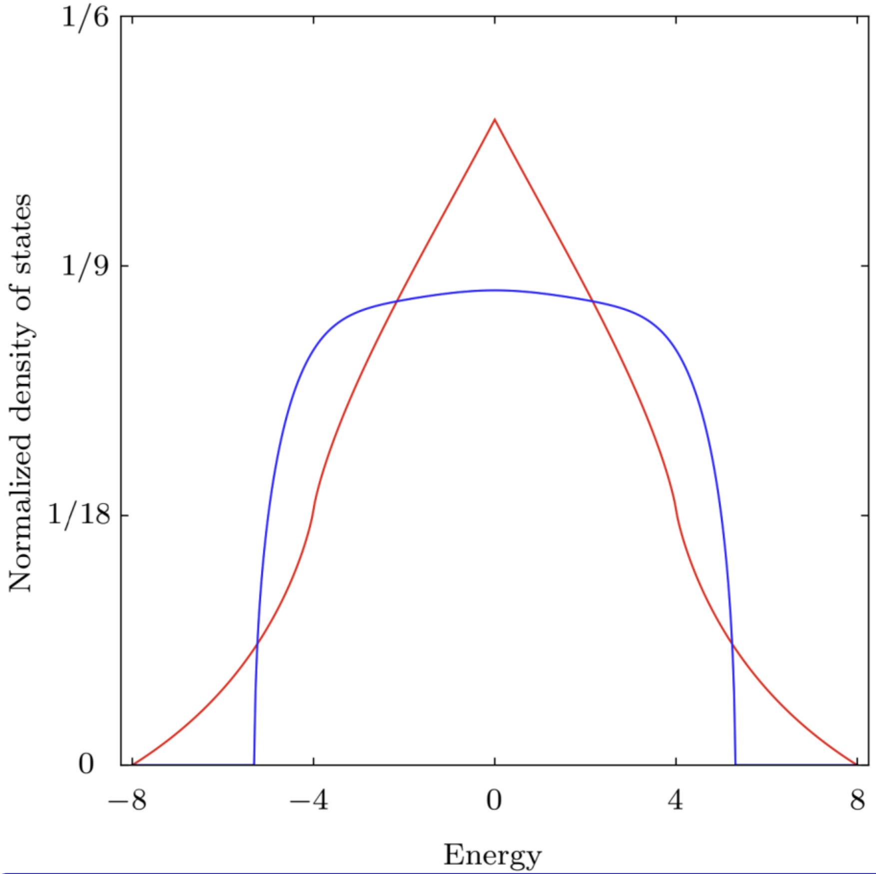



# How faithful is the Abelian approximation?

NN model on {8,3}




NN model on {8,8}



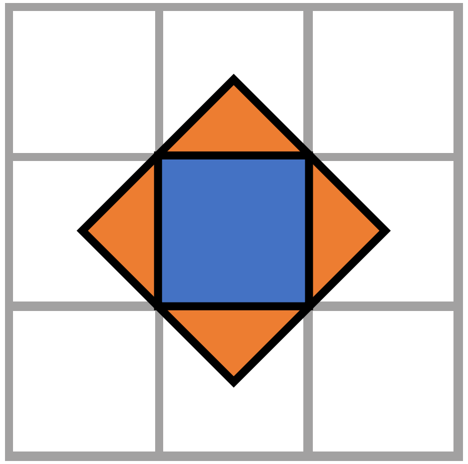

Abelian HBT vs. Exact

Continued fraction expansion:
suited for isotropic NN models.

R. Mosseri and J. Vidal, Phys. Rev. B **108**, 035154 (2023)

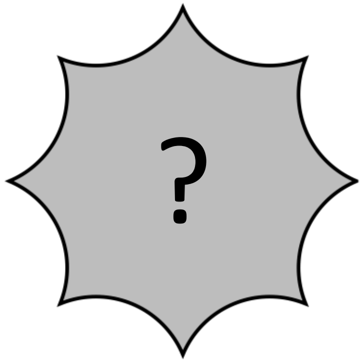
# Non-Abelian Bloch states from supercells?

**Euclidean**



Brillouin zone reduction
and band folding

**hyperbolic**




Band folding ***and also***
***enhanced BZ dimension!***

Phys. Rev. Lett. **131**, 226401 (2023)

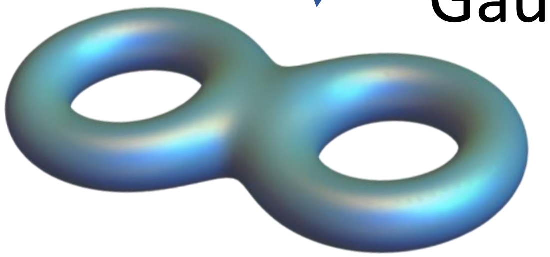
# Hyperbolic supercells

$\Delta_{\{8,8\}} \triangleright T_{\{8,8\}}$

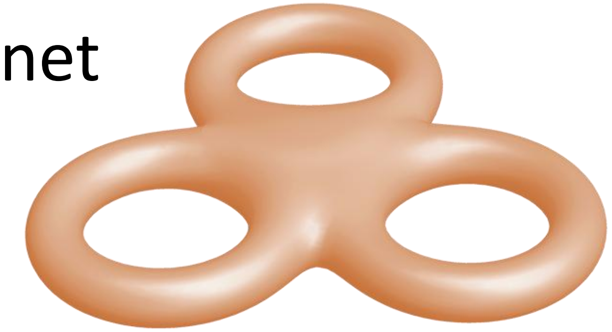
largest
transl. group

translation
*subgroup*

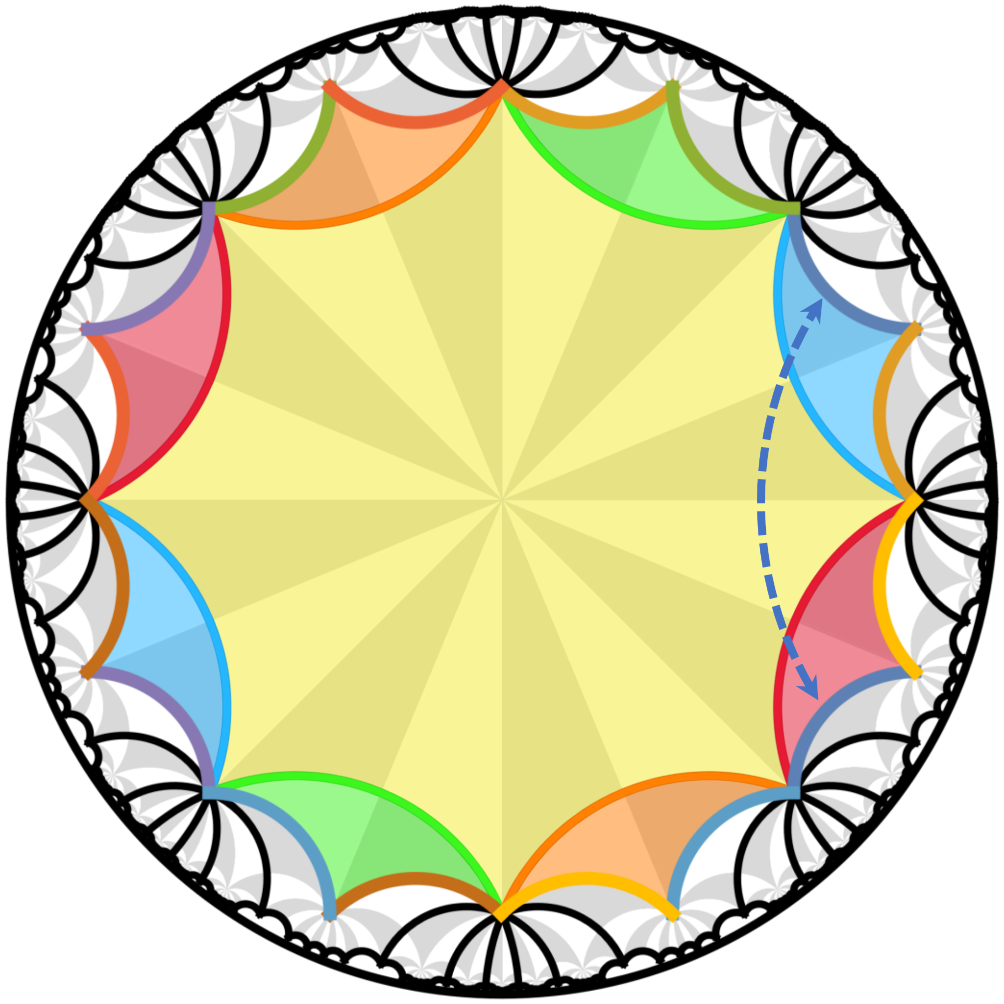
Gauss-Bonnet



(4 momenta)



(6 momenta)



**Mathematically: on a subgroup, some higher-dim. irreps reduce to new 1D irreps.**

Phys. Rev. Lett. **131**, 226401 (2023)

# Hyperbolic supercells

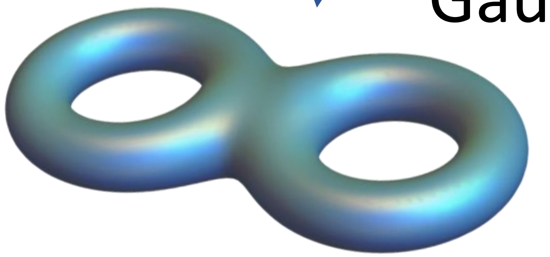
... repeat → repeat → repeat → ...

$\Delta_{\{8,8\}} \triangleright T_{\{8,8\}} \triangleright \widetilde{T}_{\{8,8\}}$

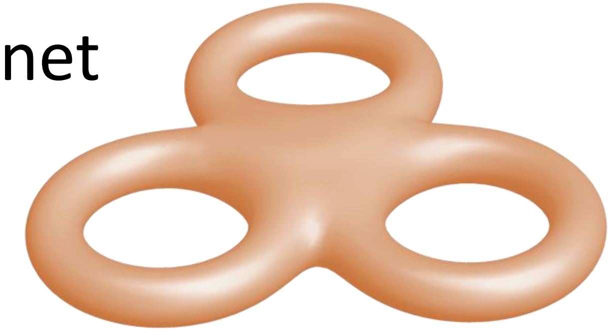
largest transl. group

translation *subgroup*

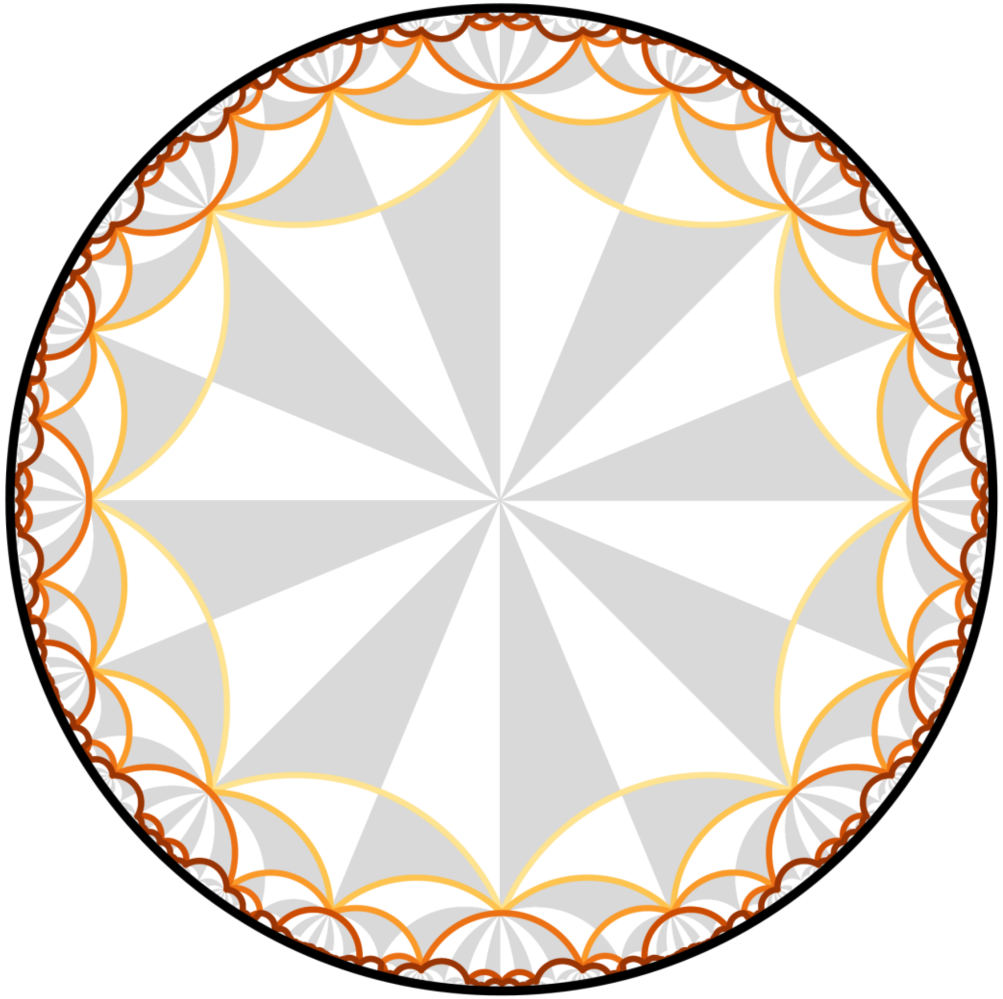
Gauss-Bonnet



(4 momenta)



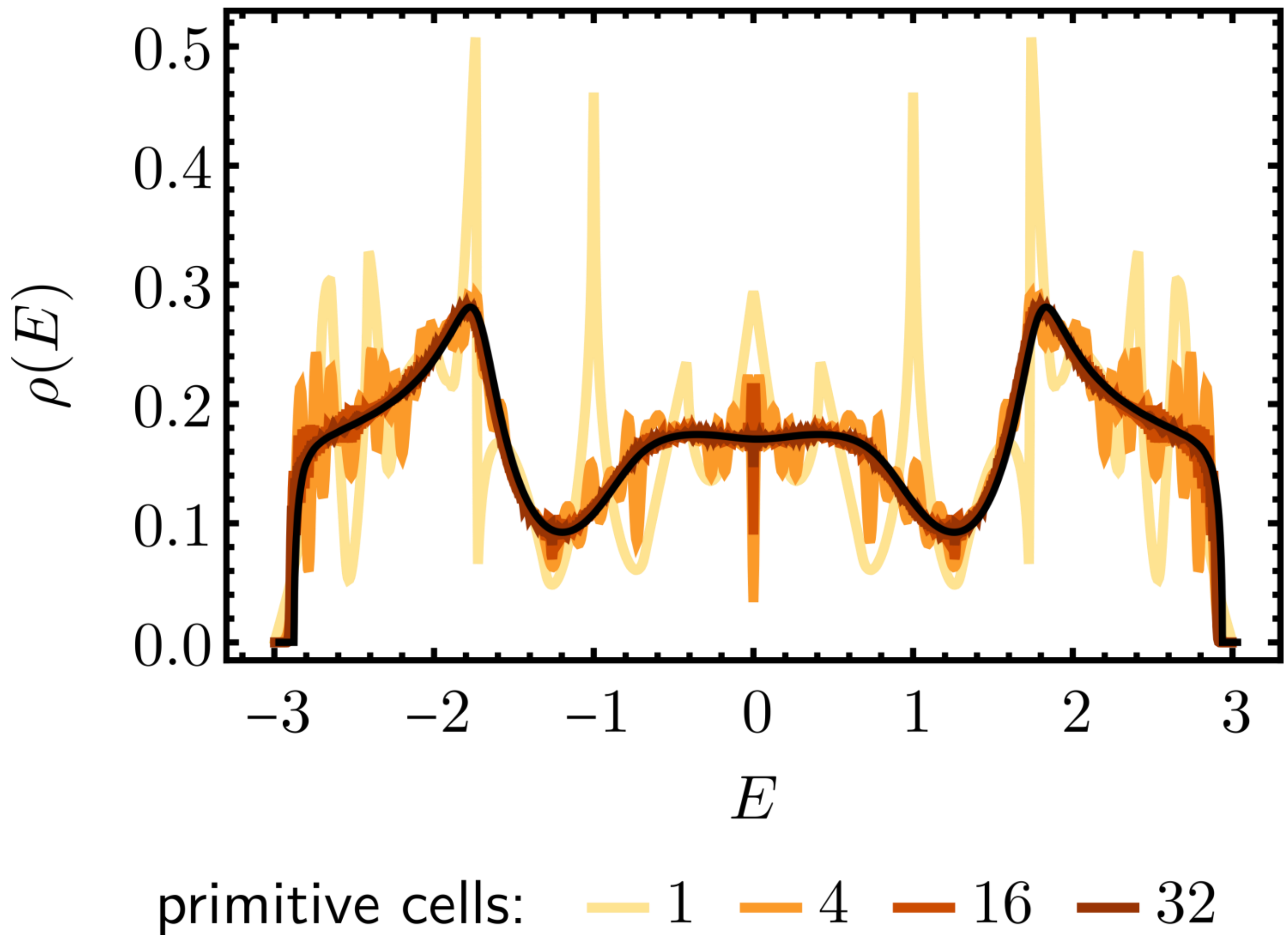
(6 momenta)

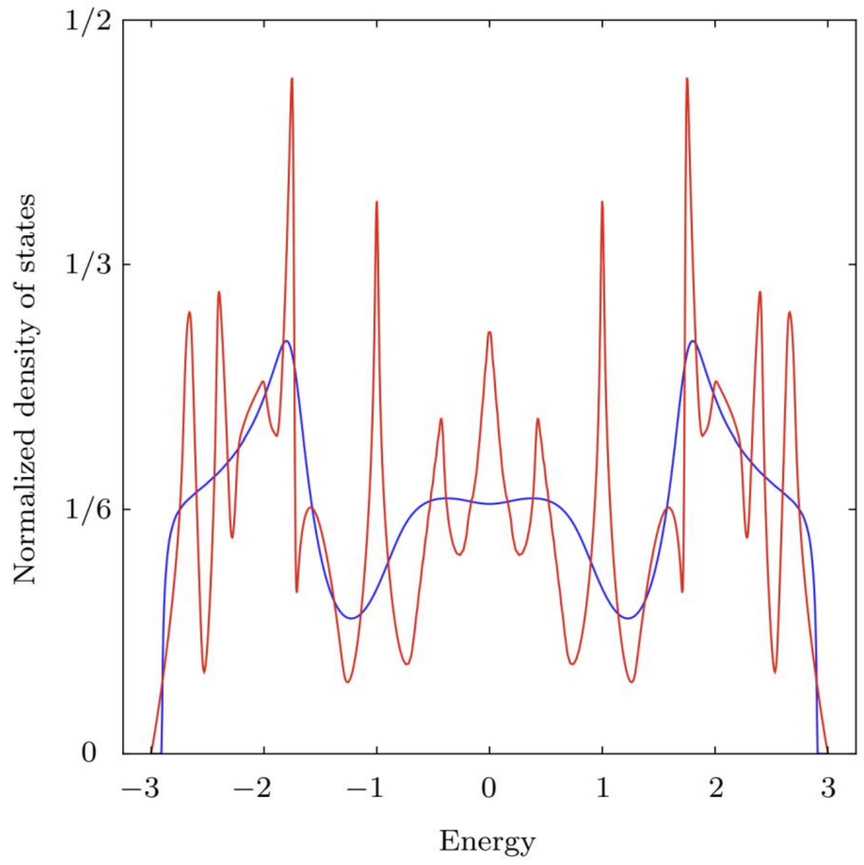


**Mathematically: on a subgroup, some higher-dim. irreps reduce to new 1D irreps.**

Phys. Rev. Lett. **131**, 226401 (2023)

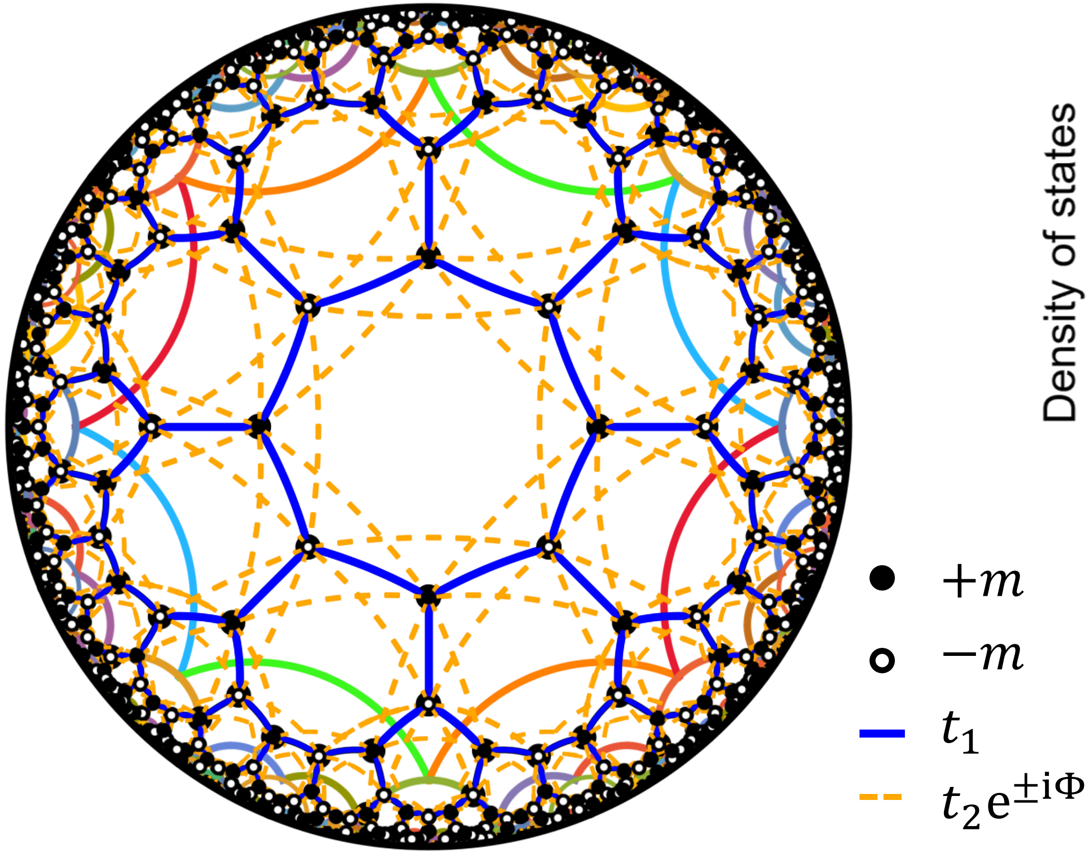
# NN model on {8,3} revisited

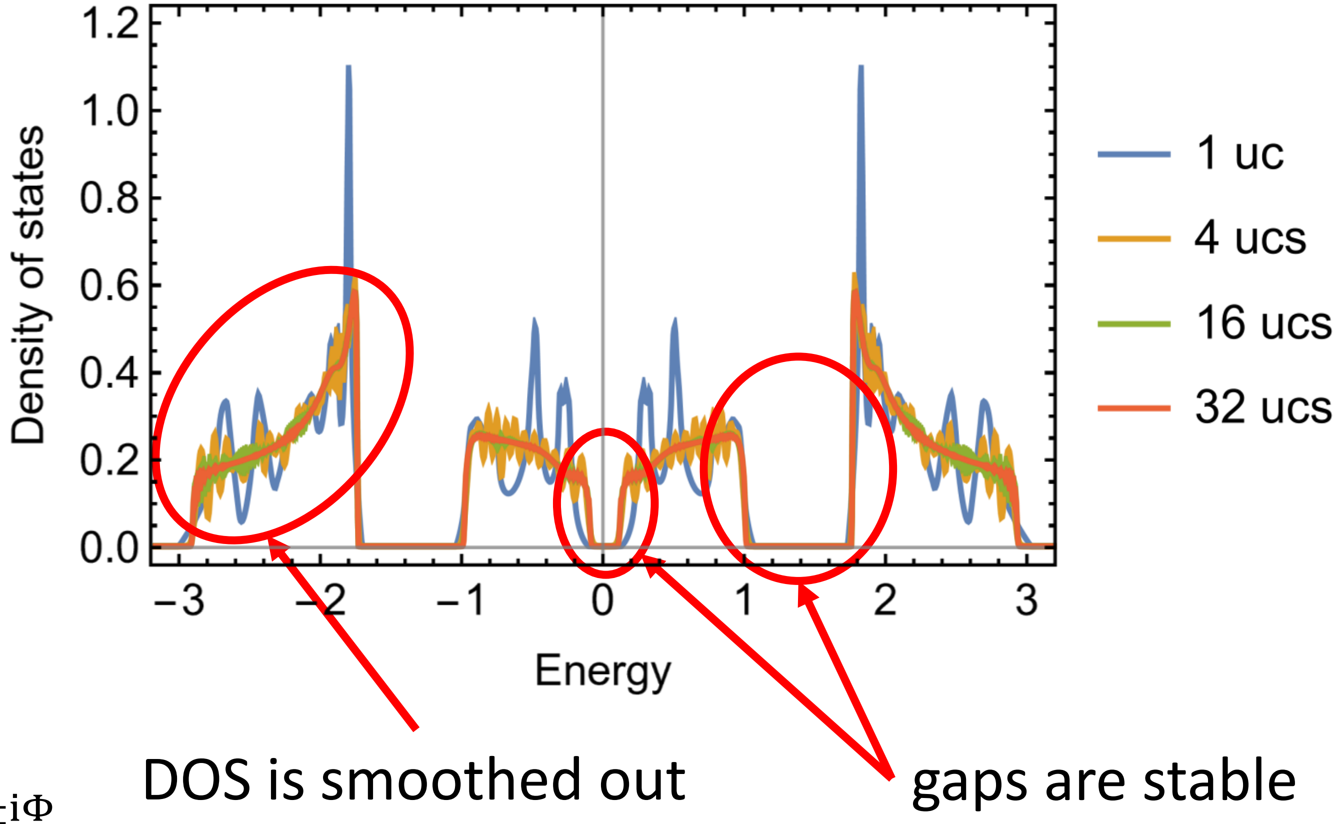


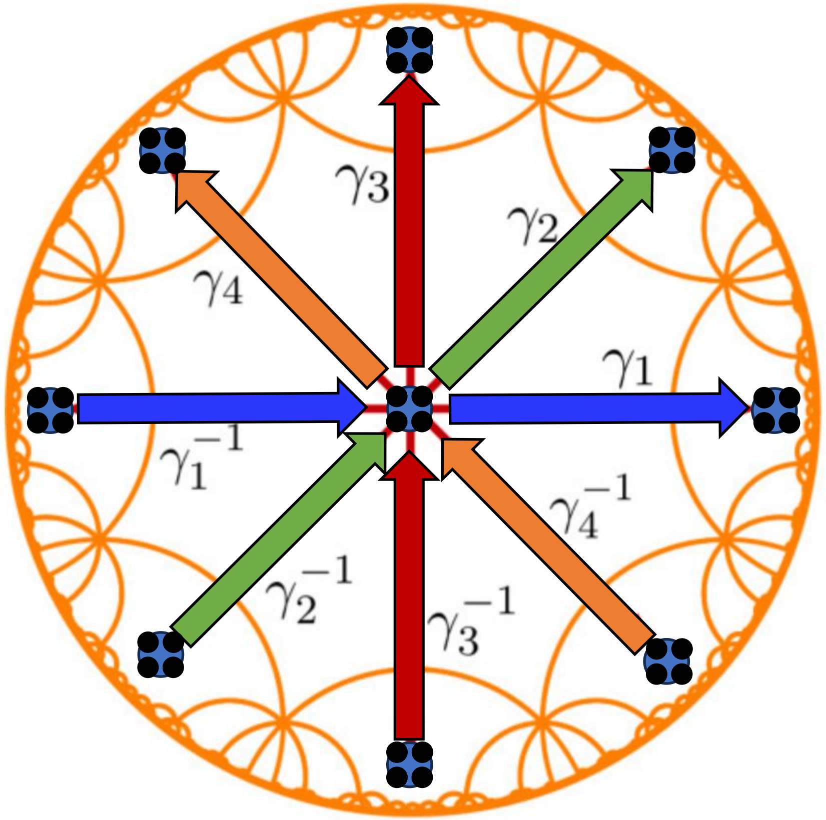

Abelian HBT vs. Exact

# Hyperbolic Haldane model



# Hyperbolic {8,8} lattice with second Chern number



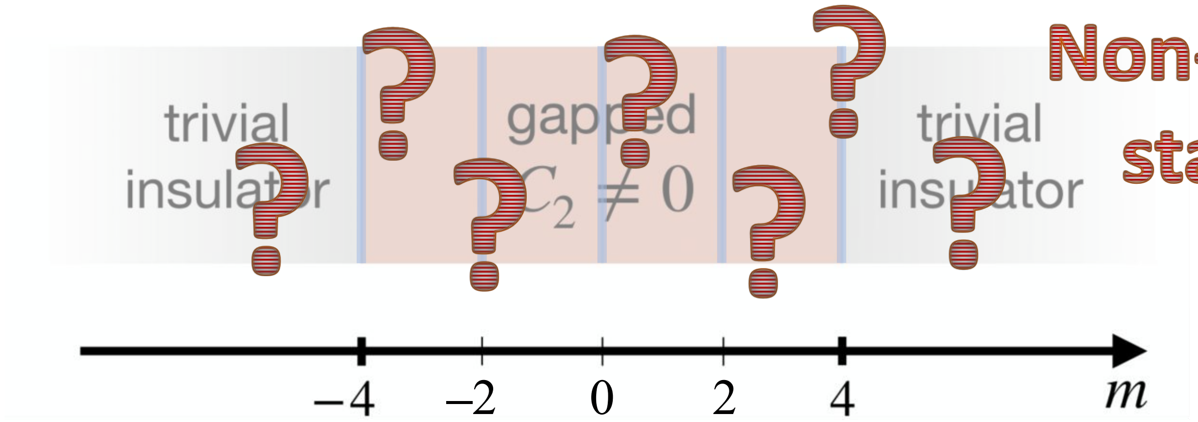

Eight nearest neighbors on hyperbolic {8,8} lattice

4 orbitals per vertex, with hopping matrices:

along $\gamma_j$: $t_j = \frac{1}{2i}\Gamma_j - \frac{1}{2}\Gamma_5$

on-site term: $t_0 = m\Gamma_5$

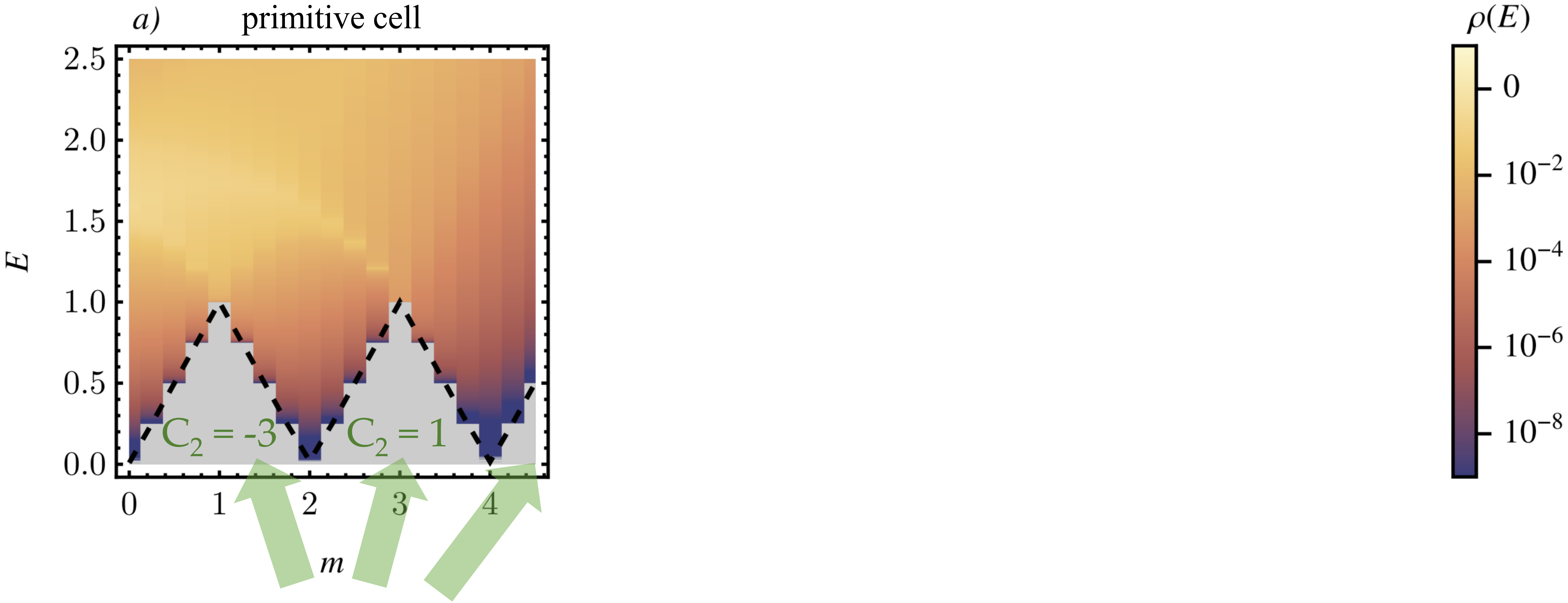
$$\mathcal{H}(\boldsymbol{k}^{4D}; m) = \sum_{j=1}^{4} \sin k_j \Gamma_j + \left( m - \sum_{j=1}^{4} \cos k_j \right) \Gamma_5$$




**Non-Abelian states!!!**

Phys. Rev. Lett. **132**, 206601 (2024)

# Model with second Chern number revisited




Spectral gaps

semimetal

metal

insulator

Phys. Rev. Lett. **132**, 206601 (2024)

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mpipks

MAX-PLANCK-GESELLSCHAFT

Joseph Maciejko

UNIVERSITY OF ALBERTA

**Marcelo Looser**

University of Zurich[UZH]

- Software packages:

  HyperCells (GAP)

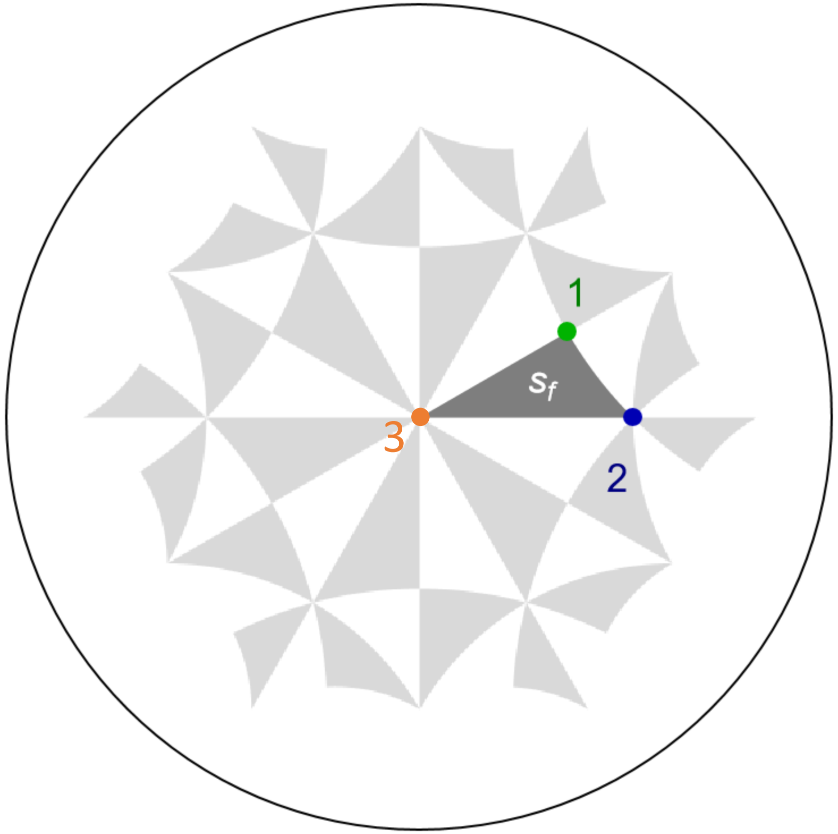
  & HyperBloch (Mathematica)

- More functions & detailed tutorials


visit [www.hypercells.net](http://www.hypercells.net)

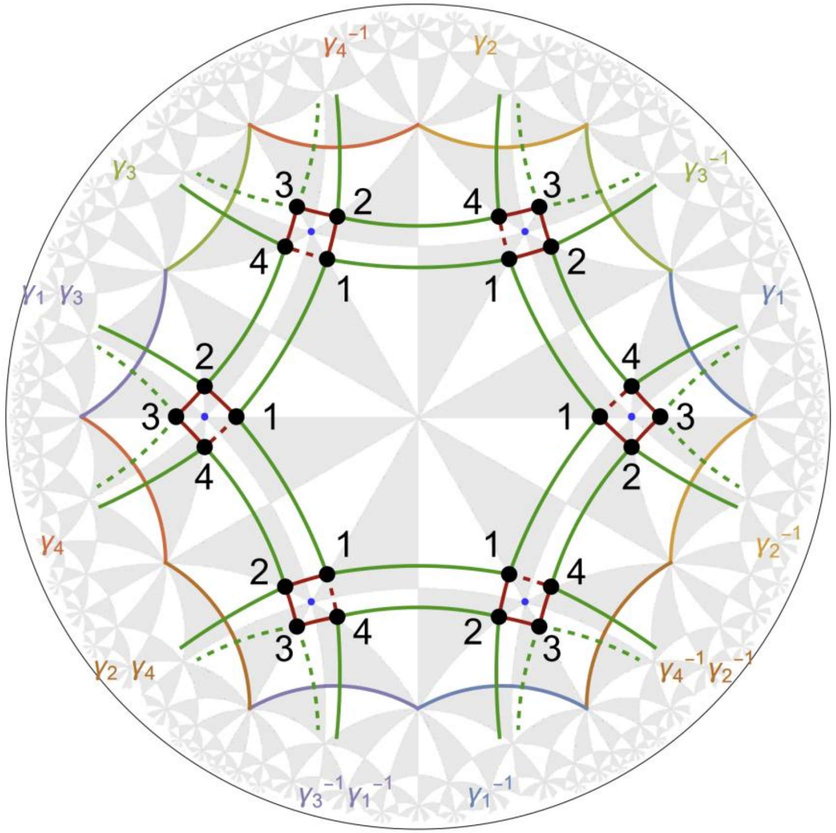
# Some applications of HyperCells & HyperBloch

**Sites at maximally symmetric Wyckoff positions**



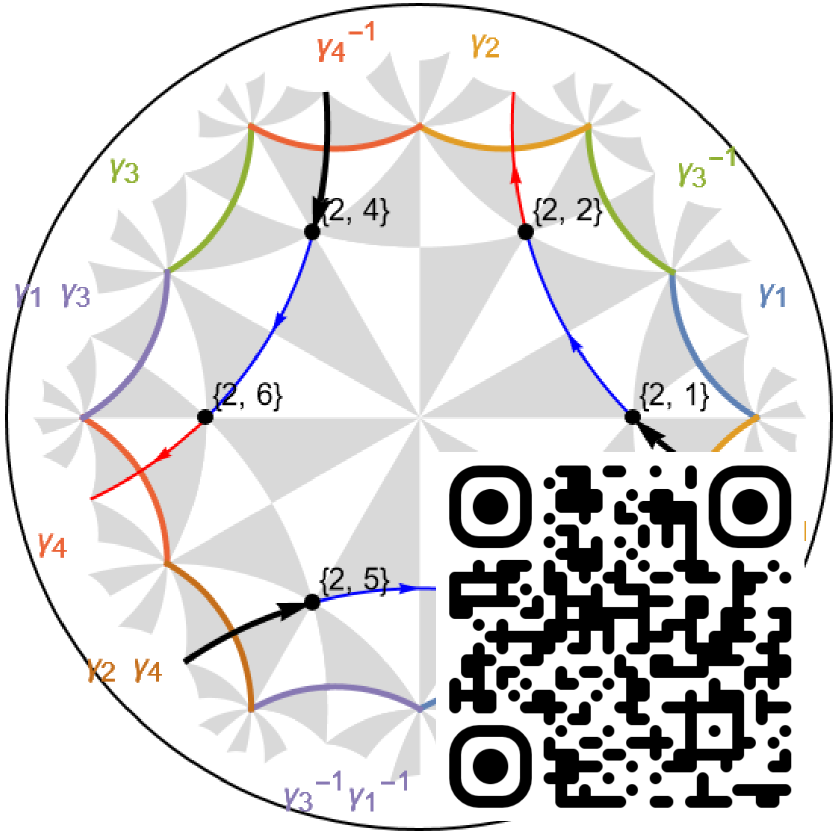

kagome, Lieb, dice and other decorations of $\{p,q\}$ lattices

**Multiple orbitals per sites, coupled by hopping *matrices***




Model with second Chern #, hyperbolic version of BBH, etc.

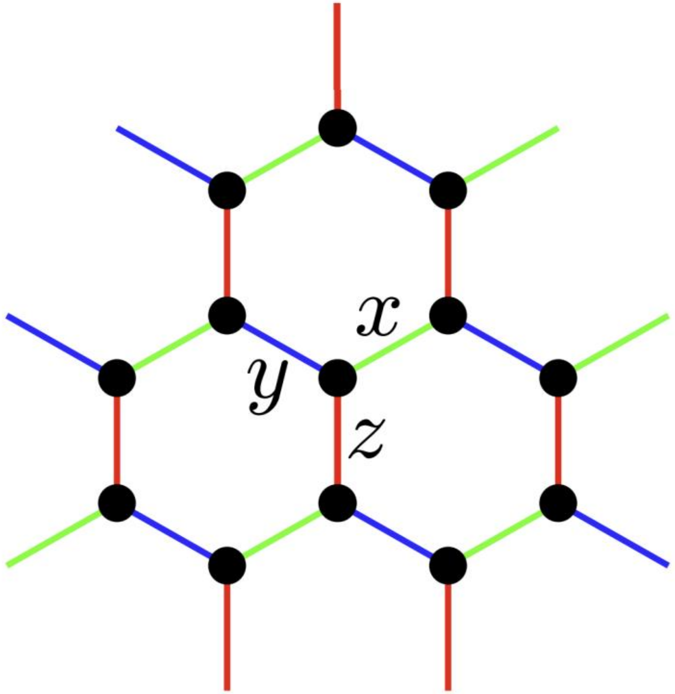
**Anisotropic or non-reciprocal hopping, on-site gain and loss**




www.hypercells.net

Hyperbolic SSH and skin effect studies

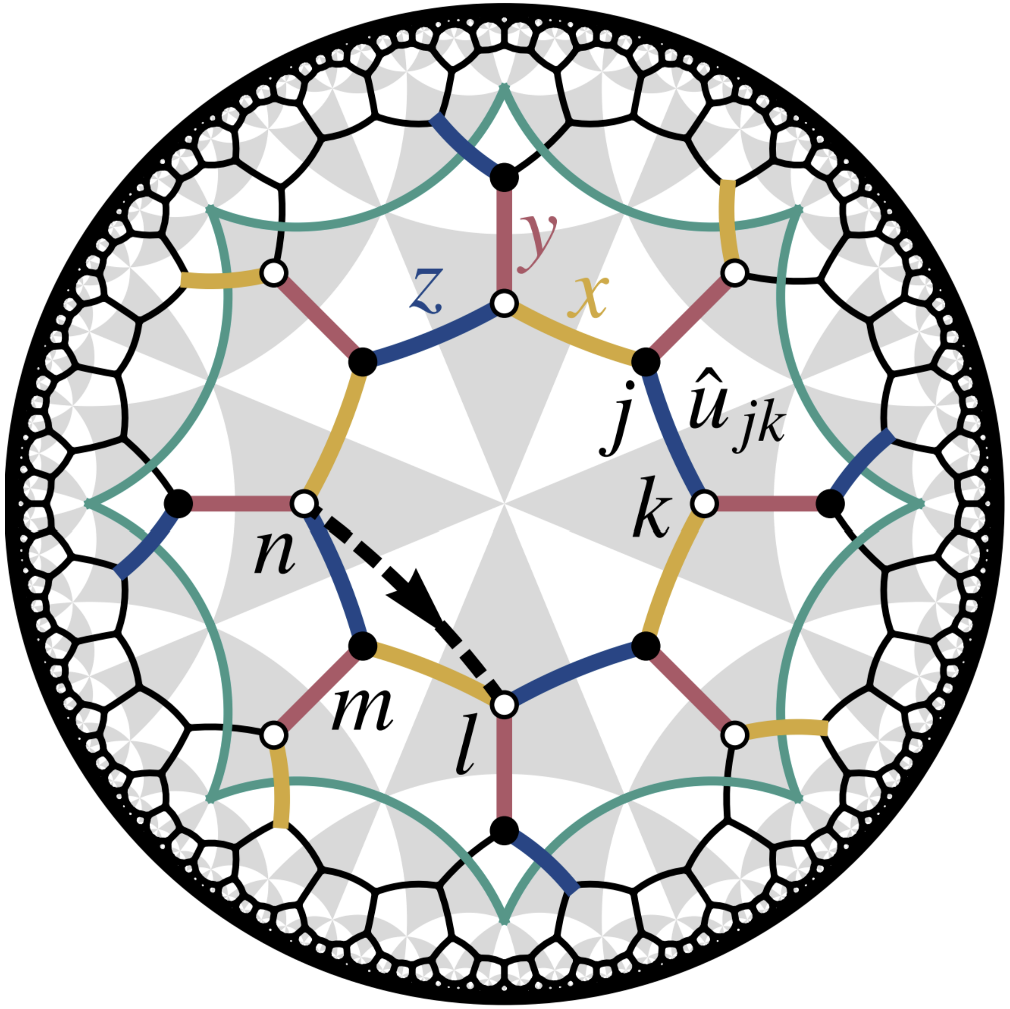
# Hyperbolic spin liquids




$S^x S^x$

$S^z S^z$

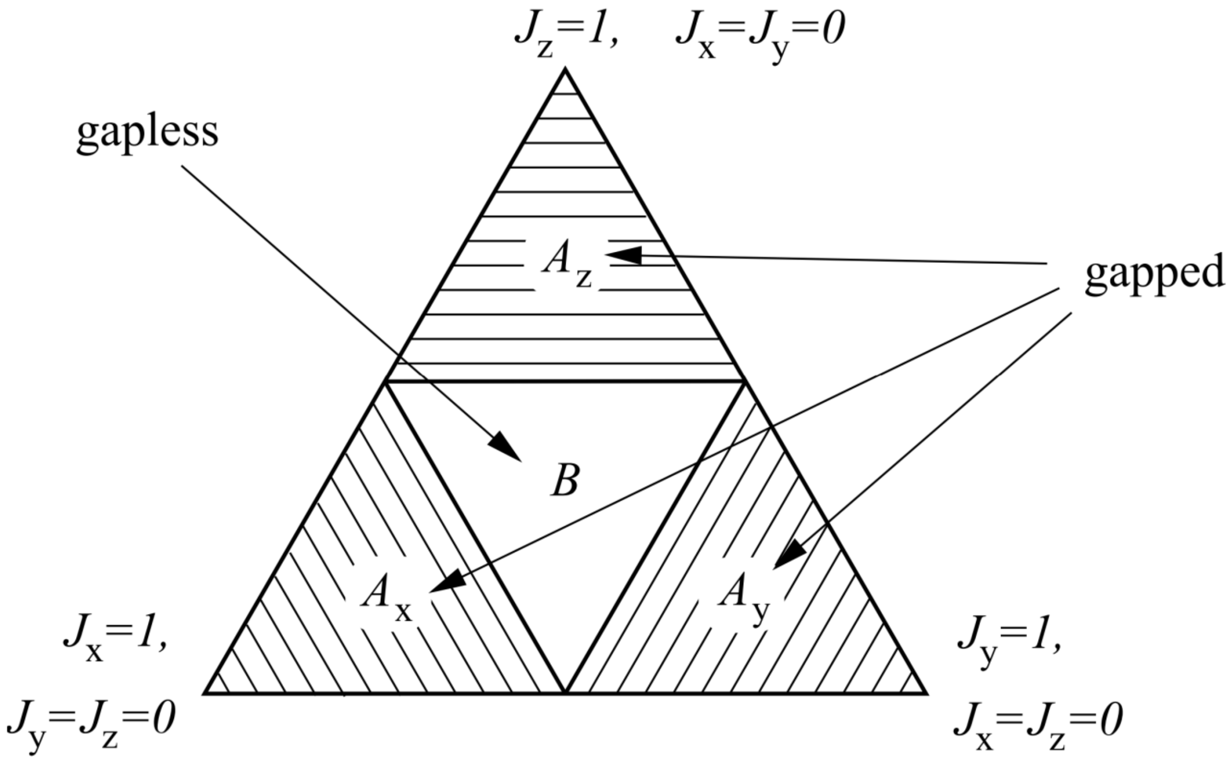
$S^y S^y$



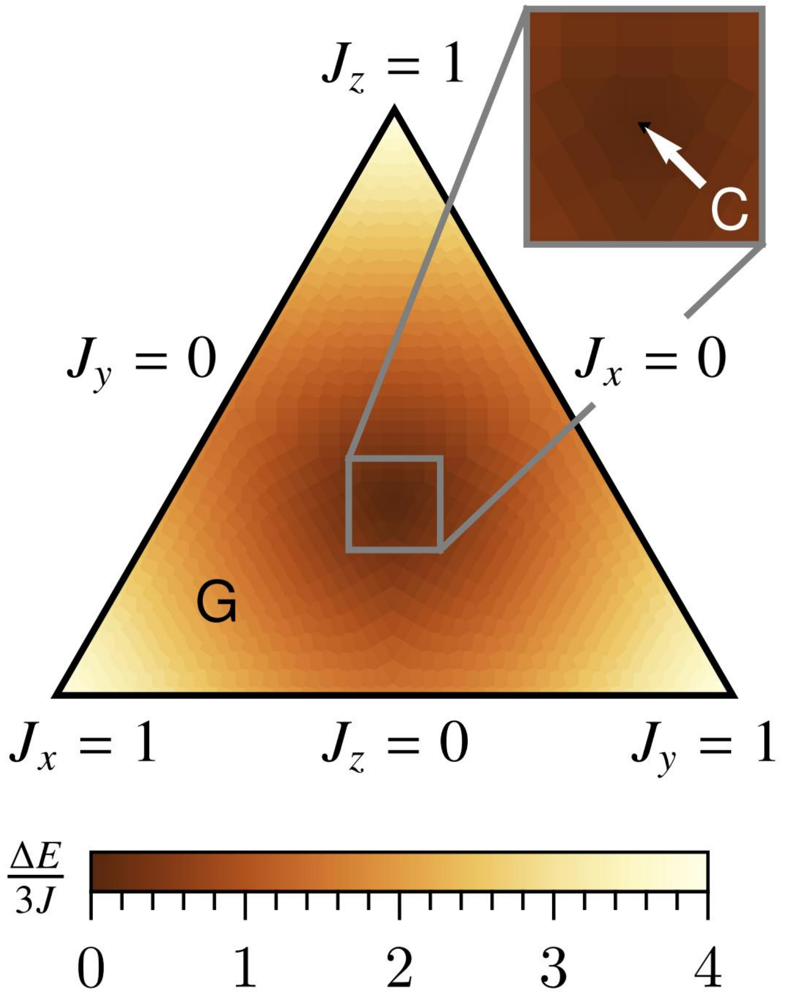

$$\hat{\mathcal{H}} = -\sum_{\langle j,k\rangle_\alpha} J_\alpha \hat{\sigma}_j^\alpha \hat{\sigma}_k^\alpha$$

P. M. Lenggenhager, S. Dey, TB, and J. Maciejko, arXiv:2407.09601

# Phase diagram of Kitaev QSL model




**Euclidean, {6,3}**

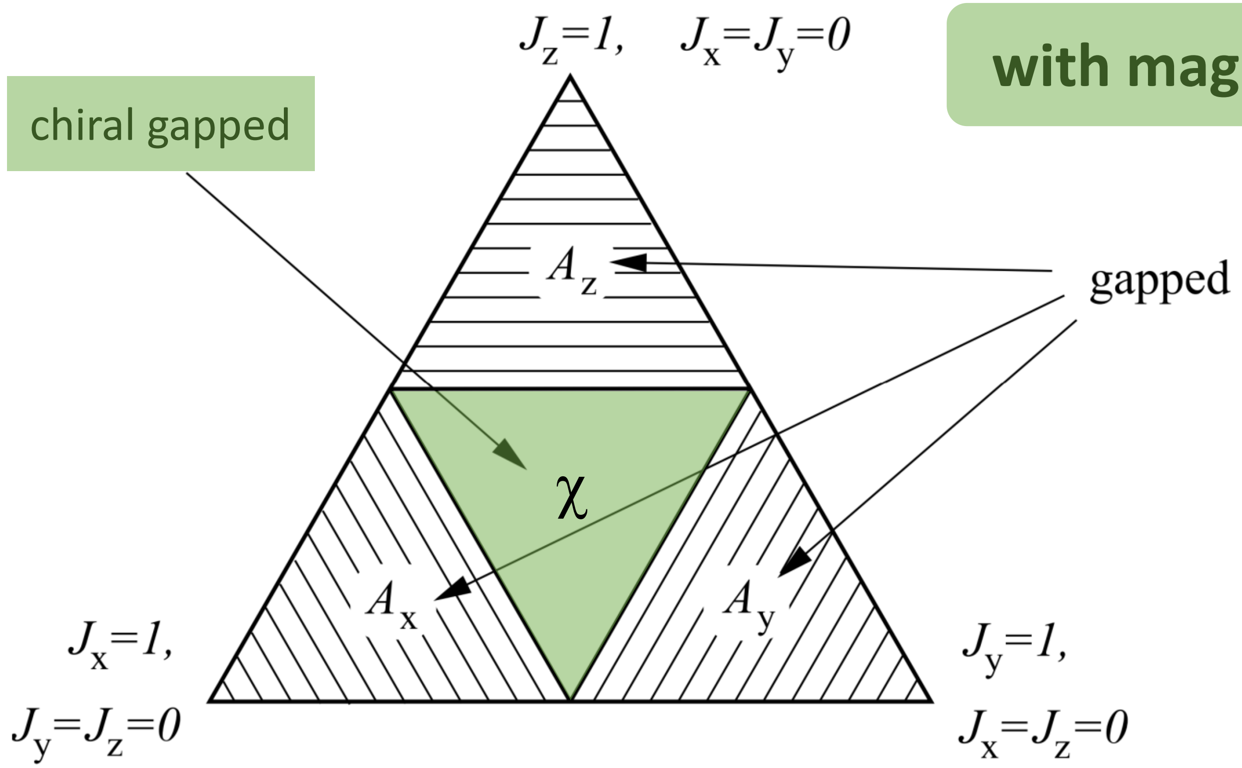



**hyperbolic, {8,3}**

A. Kitaev, Ann. Phys. **321**, 2—111 (2006)

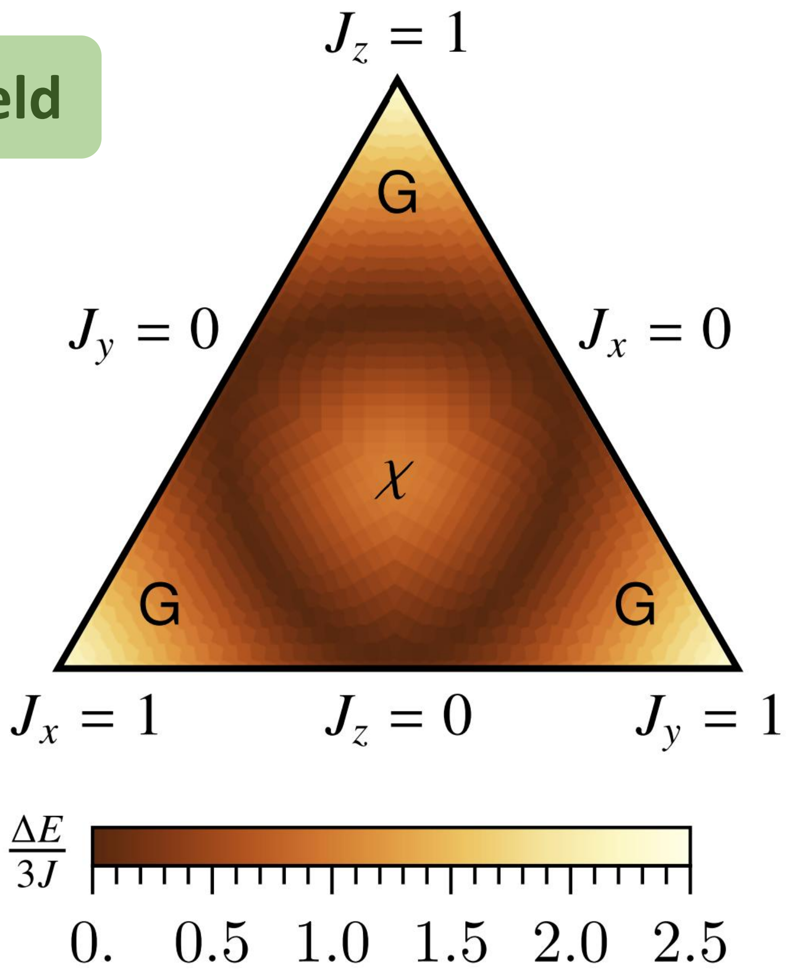
P. M. Lenggenhager, S. Dey, TB, and J. Maciejko, arXiv:2407.09601

# Phase diagram of Kitaev QSL model




**Euclidean, {6,3}**

**with magnetic field**




**hyperbolic, {8,3}**

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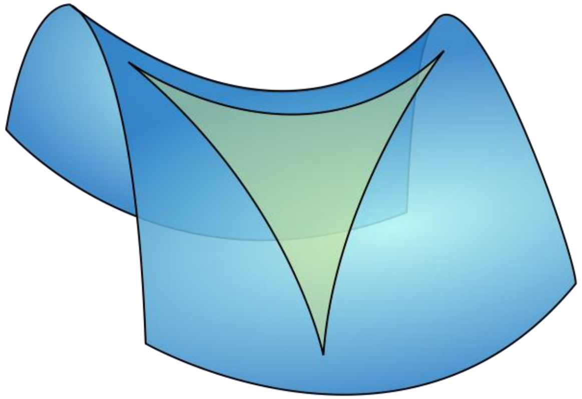
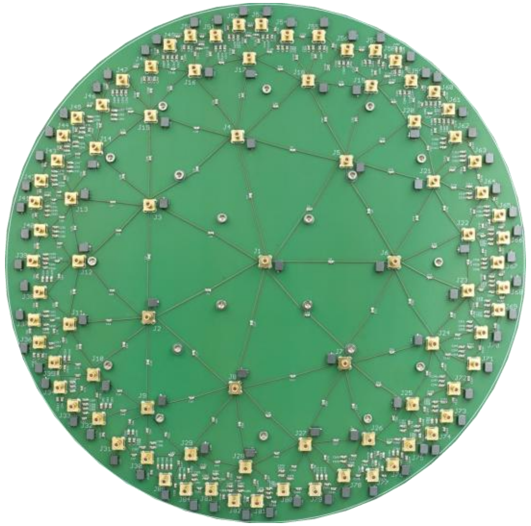
P. M. Lenggenhager, S. Dey, TB, and J. Maciejko, arXiv:2407.09601

# Summary



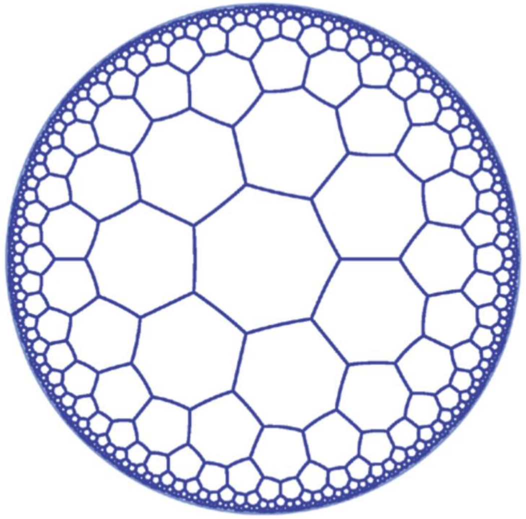
Negative curvature



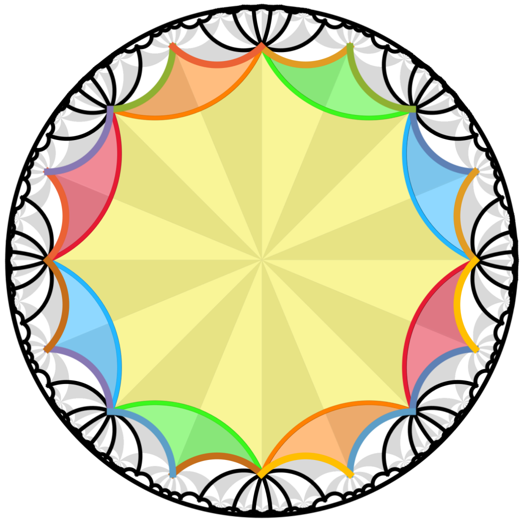
Experimental realizations

HyperCells/Bloch

{*p*,*q*} lattices



Supercellls



Kitaev QSLs