Scaling of many-body localization transitions: Fock-space and real-space quantum dynamics and spectral observables

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Our recent research that is NOT covered by this talk

- Charge and heat transport and noise in fractional quantum Hall edges (and edge junctions) with counterpropagating modes

  $\longrightarrow$ don't miss the talk by Christian Spånslätt on Friday
- Quantum dynamics and phase transitions induced by quantum measurements
- Generalized multifractality at Anderson localization transitions between metallic and insulating phases and between distinct topological phases.

Outline

- Introduction: Many-body localization.
- Spin models and their Fock-space representation.
- Analytical predictions for critical disorder of MBL transition $W_c(n)$.
  Role of Fock-space correlations in Hamiltonian.
- Numerical results for $W_c(n)$ via spectral observables.
- Quantum dynamics: Generalized imbalance and its fluctuations.
  Phase diagrams of MBL transitions in $n - W$ plane.
- Transition width $\Delta W(n)/W_c(n)$.
  Estimates for system sizes needed to study asymptotic scaling.
- Outlook

Scoquart, Gornyi, ADM, Phys. Rev. B **109**, 214203 (2024)
Scoquart, Gornyi, ADM, arXiv:2502.16219

Many-body localization (MBL)

Fundamental theme: Ergodicity and its violation in disordered interacting many-body systems at finite energy density

Gornyi, ADM, Polyakov, Phys. Rev. Lett. 2005
Basko, Aleiner, Altshuler, Annals Phys. 2006

MBL transitions: transitions between ergodic and MBL phases.

$\longrightarrow$ Extension of Anderson-localization transitions to a (much more complex) setting of highly excited states of a many-body problem

Key questions:

• Asymptotics of critical disorder $W_c(n)$ of the MBL transition and of the transition width $\Delta W(n)/W_c(n)$ for large system size (e.g., number of spins) n?

• Scaling behavior of various observables around the transition?

• Phase diagram. Physics of a broad intermediate regime seen in numerical simulations? Does its width shrink asymptotically to zero? Or are there two transitions asymptotically, with an intermediate phase in the $n \to \infty$ limit?

Comment on MBL and topology: MBL may protect topological order by promoting it to any (also infinite) temperature.

MBL transition in models with short-range interaction

Analytical predictions:

- $d = 1$: $W_c(n \to \infty) = \text{const}$ — Gornyi, ADM, Polyakov, 2005; Basko, Aleiner, Altshuler, 2006; Ros, Müller, Scardicchio, 2015; Imbrie 2016

- $d > 1$: avalanches due to exponentially rare ergodic spots
$\longrightarrow$ slow increase of $W_c(n)$ (slower than any power law) — De Roeck, Huveneers 2017; Thiery, Huveneers, Müller, De Roeck, 2018; Gopalakrishnan, Huse, 2019; Doggen, Gornyi, ADM, Polyakov, 2020

Numerics for 1D models: (mainly random-field Heisenberg model)

- finite-size drift of $W_c(n)$ — Pal, Huse, 2010; Luitz, Laflorencie, Alet, 2015; Doggen, Schindler, Tikhonov, ADM, Neupert, Polyakov, Gornyi, 2018

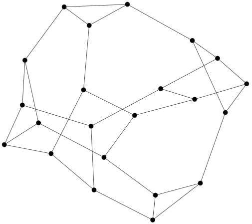
- several works: hypothesis of $W_c(n) \sim n$ — Šuntajs, Bonča, Prosen, Vidmar, 2020; Sels, Polkovnikov 2021 - 2023

- several works: hypothesis of intermediate phase — Weiner, Evers, Bera, 2019; Biroli, Hartmann, Tarzia, 2024; Colbois, Alet, Laflorencie, 2024

Localization transition on random regular graphs

Random regular graph – random graph with constant connectivity $m+1$

Locally tree-like (as Bethe lattice) but without boundary



Anderson localization on RRG $(\varepsilon_i \longrightarrow$ disorder $W)$

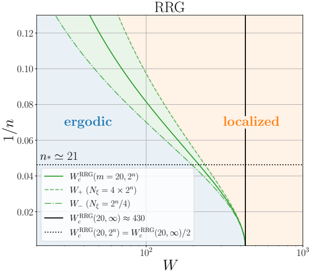
$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(c_i^+ c_j + c_j^+ c_i \right) + \sum_{i=1} \varepsilon_i c_i^+ c_i$$

Relation to the MBL problem:

Hilbert space size $N \sim m^L$ where L is "linear size"

Sites $\longleftrightarrow$ many-body basis states, links $\longleftrightarrow$ interaction matrix elements

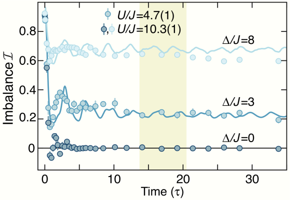
- For $N \to \infty$, transition at $W_c \sim m \ln m$
- Substantial finite-size drift of W_c; becomes stronger with increasing m
- Relation to RRG used to derive $W_c(n)$ for MBL models with long-range interaction (including several models in this talk)



Tikhonov, ADM, 2016-2021

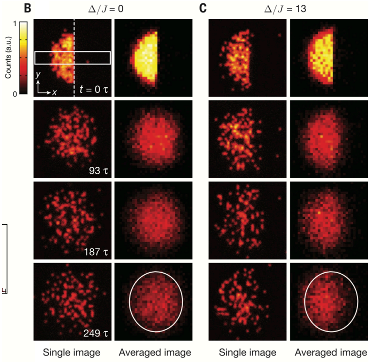
Herre, Karcher, Tikhonov, ADM, 2023

MBL transition in quantum dynamics of cold atoms in optical lattices:



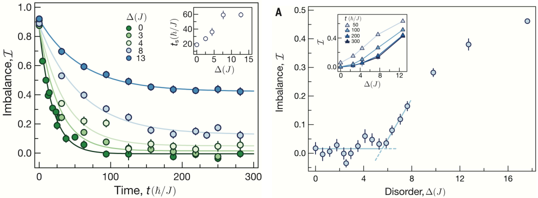
1D, even-odd imbalance as indicator of the transition

Schreiber et al, Science 2015



2D, left-right imbalance as indicator of the transition

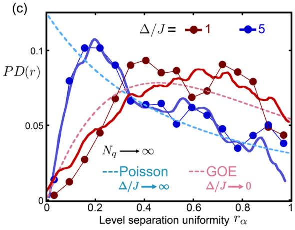
Choi et al, Science 2016



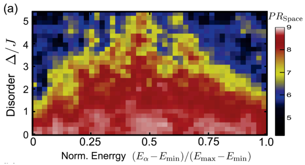
MBL transition in experiment: Coupled superconducting qubits

Spectroscopy of a chain of 9 superconducting qubits

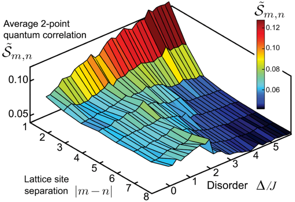
Roushan et al, Science 2017



level statistics

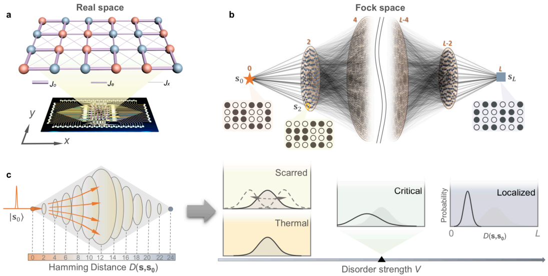


spatial IPR of eigenstates



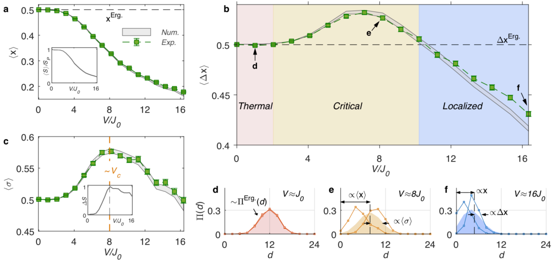
spatial correlations

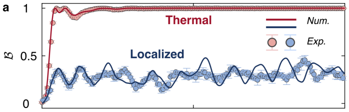
MBL transition in experiment:
2D superconducting quantum processor



Fock-space dynamics in 6×4 qubit array: From ergodicity to MBL

Yao et al, Nature Phys. 2023





Fock-space representation of spin-$\frac{1}{2}$ models

- Generic model of n interacting spins $\frac{1}{2}$

Many-body Hilbert space ≡ Fock space: vertices of n-dimensional hypercube

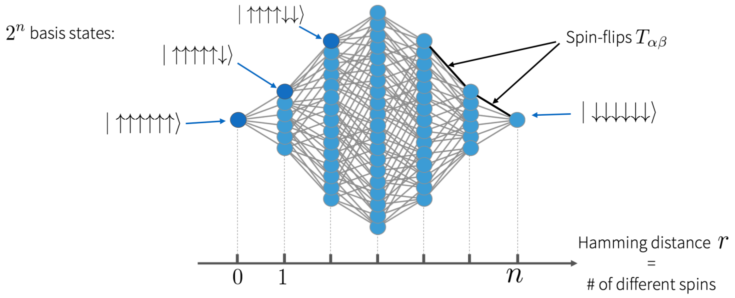
Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{H}_1 = \sum_\alpha E_\alpha |\alpha\rangle\langle\alpha| + \sum_{\alpha\neq\beta} T_{\alpha\beta} |\alpha\rangle\langle\beta|$

$\longrightarrow$ tight-binding model on a graph in Fock space

- Only no-spin-flip and single-spin-flip terms in $\hat{H}$

$\longrightarrow$ graph is formed by hypercube edges; coordination number is n

All models that we consider here are of this type



1D model

$$\hat{H}^{1\mathrm{D}} = \sum_{i=1}^{n} \epsilon_i \hat{S}_i^z + 2\sum_{i=1}^{n} V_{i,i+1}^z \hat{S}_i^z \hat{S}_{i+1}^z + 2\sum_{i=1}^{n} \sum_{a\in\{x,y\}} \left[V_{i,i+1}^a \hat{S}_i^z \hat{S}_{i+1}^a + V_{i,i-1}^a \hat{S}_i^z \hat{S}_{i-1}^a \right]$$

ϵ_i – random field, box distribution on $[-W,W]$

$V_{i,i+1}^z,\ V_{i,i+1}^x,\ V_{i,i-1}^x,\ V_{i,i+1}^y,\ V_{i,i-1}^y$ – random interactions $\sim \mathcal{N}(0,1)$

Fock-space representation:

$$E_\alpha = \langle\alpha|\,\hat{H}^{1\mathrm{D}}\,|\alpha\rangle = \frac{1}{2}\sum_{i=1}^{n} s_i^{(\alpha)}\epsilon_i + \frac{1}{2}\sum_{i=1}^{n} s_{i,i+1}^{(\alpha)} V_{i,i+1}^z \sim \mathcal{N}\left(0, \frac{nW^2+3n}{12}\right)$$

$$T_{\alpha\beta} = \langle\alpha|\,\hat{H}^{1\mathrm{D}}\,|\beta\rangle = \frac{1}{2}\left(V_{k-1,k}^x s_{k-1}^{(\alpha)} + \mathrm{i}\,V_{k-1,k}^y s_{k-1,k}^{(\alpha)} + V_{k+1,k}^x s_{k+1}^{(\alpha)} + \mathrm{i}\,V_{k+1,k}^y s_{k+1,k}^{(\alpha)}\right)$$
$$\sim \mathcal{N}\left(0,\frac{1}{2}\right) + \mathrm{i}\,\mathcal{N}\left(0,\frac{1}{2}\right)$$

$s_i^{(\alpha)} \equiv \langle\alpha|\,\sigma_i^z\,|\alpha\rangle = \pm 1 \qquad s_{i,i+1}^{(\alpha)} \equiv s_i^{(\alpha)} s_{i+1}^{(\alpha)} = \pm 1$

Correlations $(C_E)_{\alpha\beta} = \langle E_\alpha E_\beta\rangle$ and $(C_T)_{\alpha\beta\mu\nu} = \langle T_{\alpha\beta}^* T_{\mu\nu}\rangle$ crucially important!

$$(C_E^{1\mathrm{D}})_{\alpha\beta} = \frac{W^2}{12}(n-2r_{\alpha\beta}) + \frac{1}{4}(n-2q_{\alpha\beta})$$

$r_{\alpha\beta}$ – Hamming distance

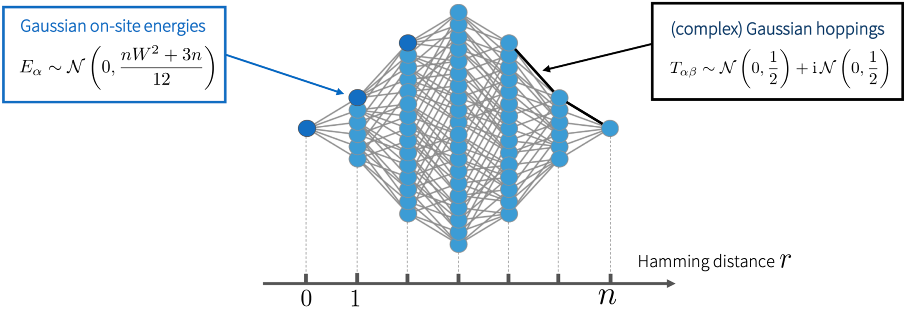
$q_{\alpha\beta}$ – number of sites i with $s_{i,i+1}^{(\alpha)} = -s_{i,i+1}^{(\beta)}$

$$(C_T^{1\mathrm{D}})_{\alpha\beta\mu\nu} = \frac{1}{2}\left(s_{k-1}^{(\alpha)} s_{k-1}^{(\mu)} + s_{k+1}^{(\alpha)} s_{k+1}^{(\mu)}\right)$$

for parallel links $(\alpha\to\beta)$ and $(\mu\to\nu)$

Covariances $(C_E^{1\mathrm{D}})_{\alpha\beta}$ and $(C_T^{1\mathrm{D}})_{\alpha\beta\mu\nu}$ are **not** functions of Hamming distance.

1D model (cont'd)



Correlations $(C_E)_{\alpha\beta} = \langle E_\alpha E_\beta \rangle$ and $(C_T)_{\alpha\beta\mu\nu} = \langle T^*_{\alpha\beta} T_{\mu\nu} \rangle$ crucially important!

For 1D model, they depend on the corresponding Fock-space vertices and links **not** solely via Hamming distance. This reflects the 1D real-space geometry.

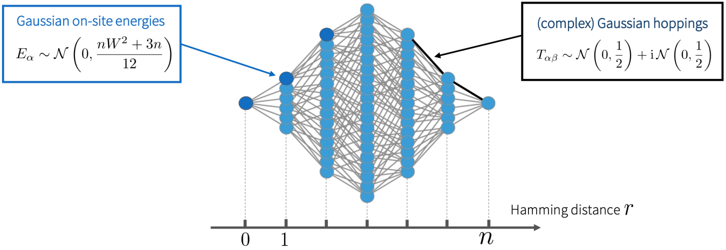
For all models considered here:

$$E_\alpha \sim \mathcal{N}\left(0, \tfrac{nW^2+3n}{12}\right) \quad \text{and} \quad T_{\alpha\beta} \sim \mathcal{N}\left(0, \tfrac{1}{2}\right) + \mathrm{i}\mathcal{N}\left(0, \tfrac{1}{2}\right)$$

The models differ in correlations C_E and/or C_T

Quantum dot (QD) model

$$\hat{H}^{\mathrm{QD}} = \sum_{i=1}^{n} \epsilon_i \hat{S}_i^z + \frac{2}{\sqrt{n}} \sum_{i,j=1}^{n} V_{ij}^z \hat{S}_i^z \hat{S}_j^z + \frac{1}{\sqrt{n}} \sum_{i,j=1}^{n} \sum_{a\in\{x,y\}} V_{ij}^a \left(\hat{S}_i^z \hat{S}_j^a + \mathrm{H.c.} \right)$$



$E_\alpha \sim \mathcal{N}\left(0, \frac{nW^2+3n}{12}\right)$ and $T_{\alpha\beta} \sim \mathcal{N}\left(0, \frac{1}{2}\right) + \mathrm{i}\mathcal{N}\left(0, \frac{1}{2}\right)$ – same as for 1D model

Difference – in correlations C_E and C_T:

$$(C_E^{\mathrm{QD}})_{\alpha\beta} = n\left[\frac{W^2}{12}\left(1 - \frac{2r_{\alpha\beta}}{n}\right) + \frac{1}{4}\left(1 - \frac{2r_{\alpha\beta}}{n}\right)^2 \right] \qquad (C_T^{\mathrm{QD}})_{\alpha\beta\mu\nu} = 1 - \frac{2r_{\alpha\mu}}{n}$$

Depend on Hamming distance only. Reflect real-space structure of QD model.

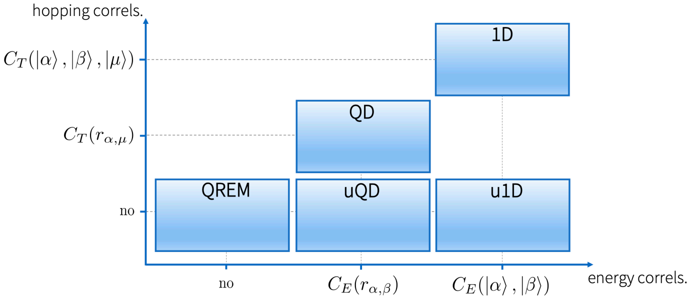
Overview of models

$E_\alpha \sim \mathcal{N}\left(0, \frac{nW^2+3n}{12}\right)$ and $T_{\alpha\beta} \sim \mathcal{N}\left(0, \frac{1}{2}\right) + \mathrm{i}\mathcal{N}\left(0, \frac{1}{2}\right)$ for all models

u1D – obtained from 1D by removing hopping correlations

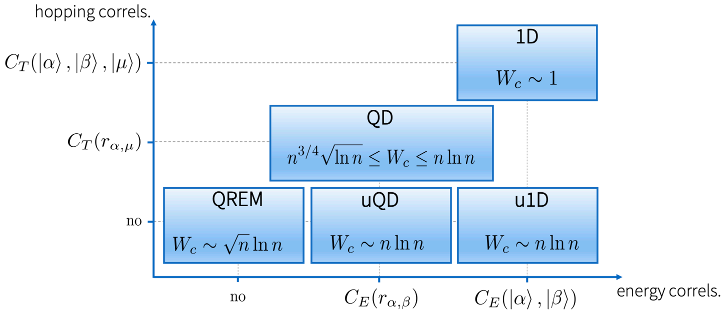
uQD – obtained from QD by removing hopping correlations

QREM (quantum random energy model) – obtained by removing both, energy and hopping correlations



Role of correlations C_E and C_T ? $\longrightarrow$ compare $W_c(n)$ for all models

Overview of models: Analytical results for $W_c(n)$



- QREM, u1D, uQD: no hopping correlations
  - $\longrightarrow$ suppressed interference between paths on a graph
  - $\longrightarrow$ RRG results can be applied, with coordination number $m + 1 \mapsto n$
- QD: more careful analysis needed; yields the behavior akin to RRG
- 1D: strong cancellations of $k!$ paths in k-th order of perturbation theory
  - $\longrightarrow$ totally different result: $W_c(n) \sim 1$.

$W \gg 1 \longrightarrow \sim n/W$ resonant spins, separated by distances $\sim W \gg 1$

$\longrightarrow$ do not interact $\longrightarrow$ do not thermalize the system $\longrightarrow$ MBL phase

MBL transition via level statistics

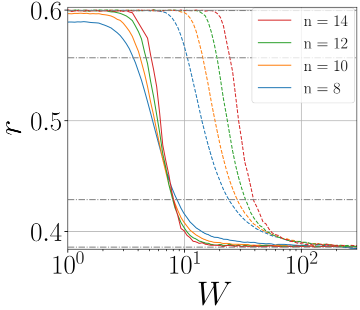
Analytical predictions: Mean adjacent gap ratio r of energy-level spectrum

MBL phase: Poisson level statistics, $r_{\mathrm{P}} \simeq 0.3863$

Ergodic phase: GUE level statistics, $r_{\mathrm{GUE}} \simeq 0.5996$

Mean gap ratio r has a jump from r_{GUE} to r_{P} at $W_c(n)$ for $n \to \infty$

Numerics: 1D (solid) and u1D (dashed) as an example:



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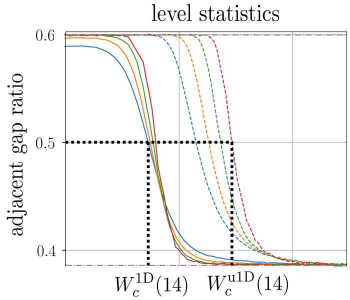
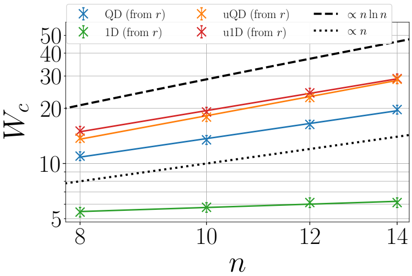
Mean gap ratio r has a jump from r_{GUE} to r_{P} at $W_c(n)$ for $n \to \infty$

Numerics: $W_c(n)$ from level statistics, defined via $r(W) = \dfrac{r_{\mathrm{GUE}} + r_{\mathrm{P}}}{2} \simeq 0.493$





MBL transition via eigenstate IPR

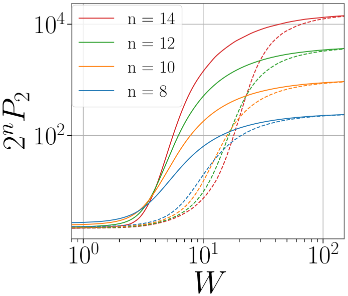
Analytical predictions: Fock-space inverse participation ratio (IPR)

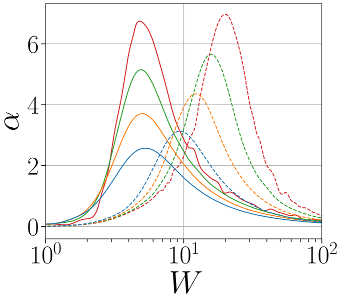
$P_2 = \sum_\beta |\langle\beta|J\rangle|^4$ $\quad|J\rangle$ - eigenstate $\quad\beta$ - basis states (graph vertices)

"Fractal dimension" $\dfrac{-\ln P_2}{\ln N} = \dfrac{-\ln P_2}{n\ln 2}$ has a jump at $W_c(n)$ for $n \to \infty$

$\longrightarrow \quad \alpha = \dfrac{\partial \ln P_2}{\partial \ln W}$ has a maximum at $W_c(n)$

Numerics: 1D (solid) and u1D (dashed) as an example:





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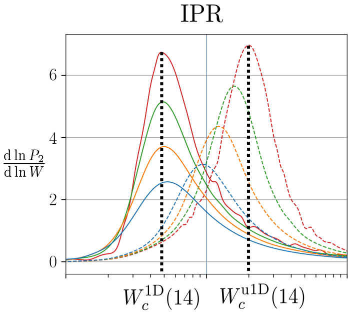
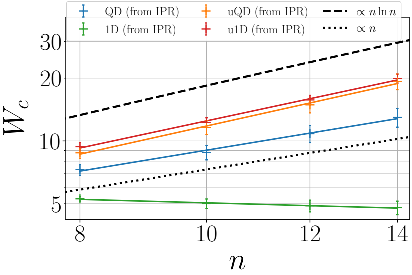
$\longrightarrow \quad \alpha = \dfrac{\partial \ln P_2}{\partial \ln W}$ has a maximum at $W_c(n)$

Numerics: $W_c(n)$ from IPR (maximum of $\partial \ln P_2 / \partial W$)





Quantum dynamics: Generalized imbalance

Consider quantum dynamics starting from a basis state $|\alpha\rangle$

Spin autocorrelation function $\equiv$ Generalized imbalance

$\mathcal{I}^{(\alpha)}(t) = \langle\alpha|\,\frac{1}{n}\sum_{j=1}^{n}\sigma_j^z(t)\sigma_j^z\,|\alpha\rangle = \langle\psi(t)|\,\hat{\mathcal{I}}^{(\alpha)}\,|\psi(t)\rangle \qquad |\psi(t)\rangle = \hat{U}(t)\,|\alpha\rangle$

$$\hat{\mathcal{I}}^{(\alpha)} = \frac{1}{n}\sum_{j=1}^{n} s_j^{(\alpha)}\sigma_j^z = 1 - \frac{2\hat{x}}{n} \qquad\qquad \hat{x} = \frac{1}{2}\sum_{i=1}^{n}\left(1 - s_i^{(\alpha)}\sigma_i^z\right)$$

$\hat{x}$ – operator of Hamming distance with respect to the basis state α

- MBL phase: $\mathcal{I}^{(\alpha)}(t) \xrightarrow{t\to\infty} \mathcal{I}_\infty, \qquad 0 < \mathcal{I}_\infty \le 1$
- ergodic phase: $\mathcal{I}^{(\alpha)}(t) \xrightarrow{t\to\infty} 0$, up to small finite-size corrections

$\longrightarrow \quad \overline{\mathcal{I}^{(\alpha)}}(t\to\infty)$ – indicator of the MBL transition $W_c(n)$

Quantum and mesoscopic variances: additional indicators, maximum at $W_c(n)$

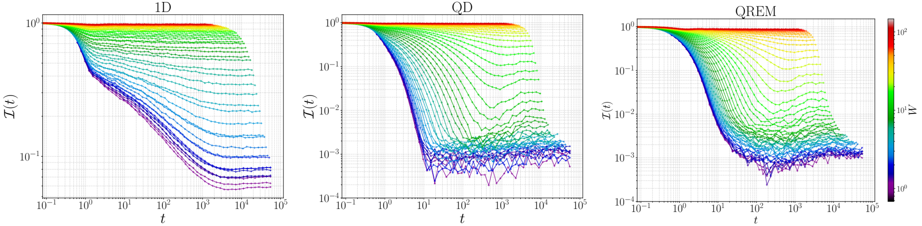
$$v_q(t) = \overline{\langle[\hat{\mathcal{I}}^{(\alpha)}]^2(t)\rangle - \langle\hat{\mathcal{I}}^{(\alpha)}(t)\rangle^2} = \left(\tfrac{2}{n}\right)^2\left[\overline{\langle x^2(t)\rangle - \langle x(t)\rangle^2}\right]$$

$$v_m(t) = \overline{\langle\hat{\mathcal{I}}^{(\alpha)}(t)\rangle^2} - \overline{\langle\hat{\mathcal{I}}^{(\alpha)}(t)\rangle}^{\,2} = \left(\tfrac{2}{n}\right)^2\left[\overline{\langle x(t)\rangle^2} - \overline{\langle x(t)\rangle}^{\,2}\right]$$

Time evolution of average imbalance

Dynamics of average generalized imbalance $\overline{\mathcal{I}(t)}$ for 1D, QD, QREM models

System size $n = 14$ for all the models; disorder range from $W \simeq 1$ to $W \simeq 100$



- sharp drop of $\overline{\mathcal{I}_\infty}$ from $\overline{\mathcal{I}_\infty} \sim 1$ to $\overline{\mathcal{I}_\infty} \approx 0$ at some W
  $\longrightarrow$ manifestation of MBL transition at $W \approx W_c(n)$
- ergodic phase, $W < W_c(n)$:
  power-law decay of $\overline{\mathcal{I}(t)}$ for 1D model (manifestation of Griffiths effects) vs fast (exponential) decay for QD and QREM
- transition region – much stronger disorder $W \approx W_c(n)$ for QD, QREM as compared to 1D, in consistency with analytical predictions

Time evolution of average imbalance (cont'd)

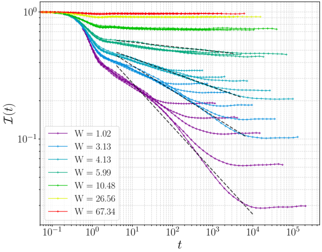
Analytical predictions for the ergodic phase and numerics ($n = 8 - 16$):

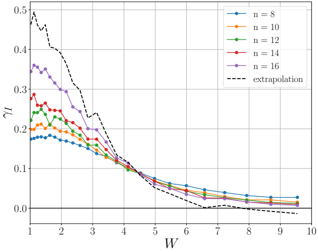
1D model:

$\overline{\mathcal{I}(t)} \sim t^{-\gamma_I}$

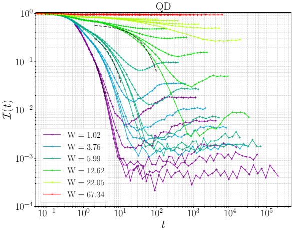
Griffiths effects

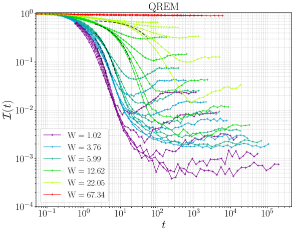
$\gamma_I \to 0$ for $W \to W_c$

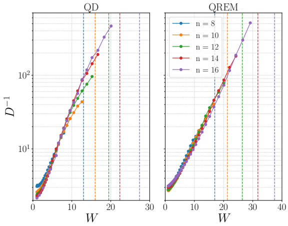




QREM and QD models: $\overline{\mathcal{I}(t)} \sim e^{-2Dt}$ $\ln D^{-1} \propto W$ (pre-critical regime)



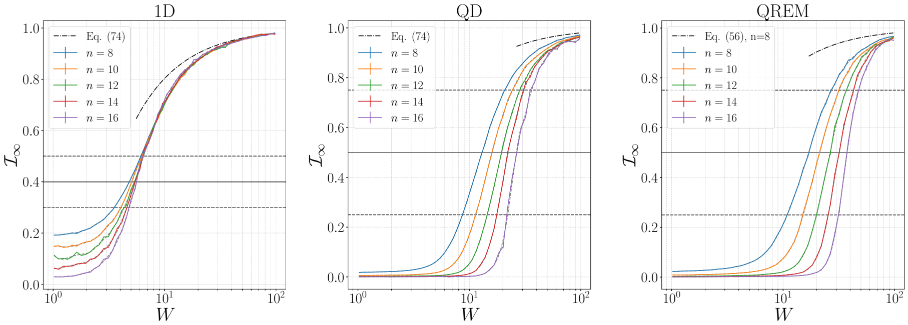




Average imbalance at $t \to \infty$

Analytical predictions and numerics ($n = 8 - 16$):

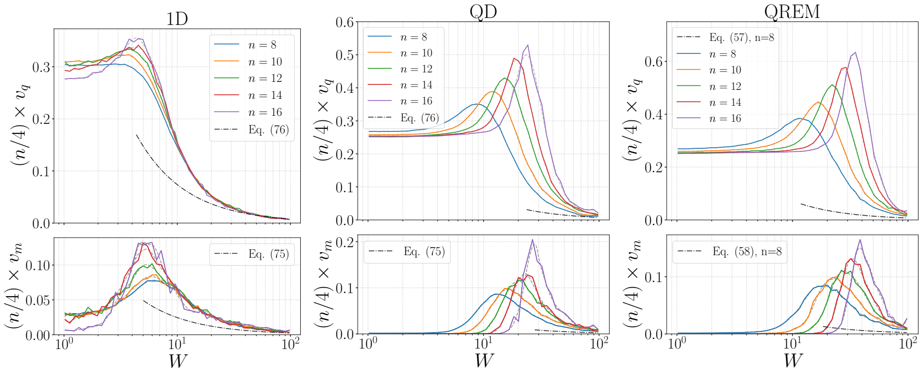
- In the limit $n \to \infty$, a jump of $\overline{\mathcal{I}}_\infty$ at $W_c(n)$ from 0 in the ergodic phase to
  - 1 in the QD model and QREM
  - $1 - p_c$ with $0 < p_c < 1$ in the 1D model
- MBL phase: 1D, QD models: $1 - \overline{\mathcal{I}}_\infty \simeq \dfrac{\pi^{3/2}}{2\sqrt{2}\,W}$ QREM: $1 - \overline{\mathcal{I}}_\infty \simeq \dfrac{\pi\sqrt{3}}{\sqrt{n}\,W}$
- $W_c^{1D}(n) \simeq \text{const}$ $W_c^{QD}(n) \gtrsim n^{3/4} \ln^{1/2} n$ $W_c^{QREM}(n) \sim n^{1/2} \ln n$



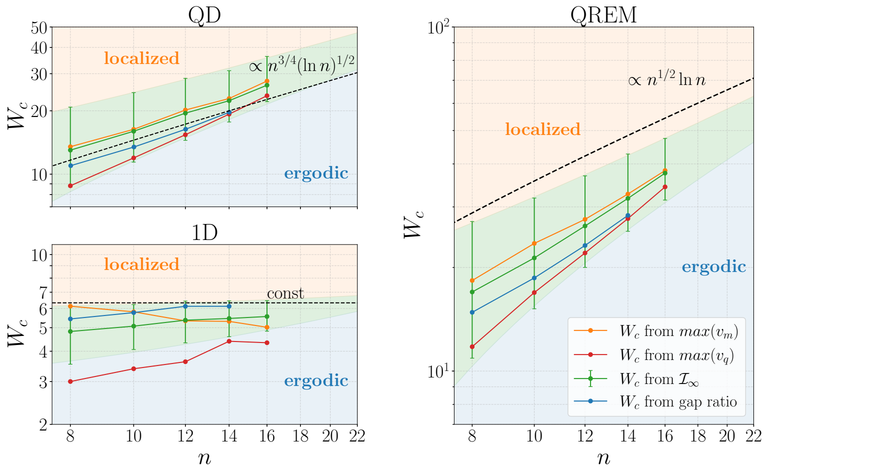
Quantum & mesoscopic imbalance fluctuations at $t \to \infty$

Analytical predictions & numerics for imbalance variances $v_q, v_m(t \to \infty)$

- MBL phase: 1D, QD: $v_q \simeq 3v_m \simeq \dfrac{3\pi^{3/2}}{4\sqrt{2}\,nW}$ QREM: $v_q \simeq 3v_m \simeq \dfrac{3\pi\sqrt{3}}{2\,n^{3/2}W}$
- ergodic phase, all models: $v_q \simeq 1/n$ v_m – exponentially small
- $W_c^{\rm 1D}(n) \simeq \text{const}$ $W_c^{\rm QD}(n) \gtrsim n^{3/4}\ln^{1/2} n$ $W_c^{\rm QREM}(n) \sim n^{1/2}\ln n$



Phase diagrams of the MBL transitions



Indicators of $W_c(n)$: average imbalance; level statistics (gap ratio); mesoscopic and quantum fluctuations of imbalance.

Transition width $\Delta W(n)$ – from average imbalance

MBL transition width

- Variable $\overline{X}(W)$ (e.g. average imbalance or level statistics gap ratio), with a jump $(\Delta X)_{\mathrm{t}}$ at the transition (at $n \to \infty$)
- Select window $(\Delta\overline{X})_{\mathrm{w}}$ of width $(\Delta X)_{\mathrm{t}}/2R$ with $R \sim 1$

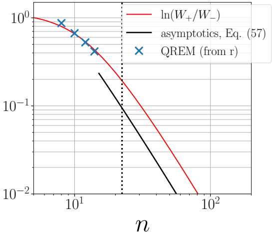
$\longrightarrow$ disorder interval associated with the transition: $[W_-(n); W_+(n)]$

$\longrightarrow$ transition width $\qquad \delta(n) = R \ln \dfrac{W_+(n)}{W_-(n)} \simeq R \dfrac{\Delta W(n)}{W_c(n)}$

Analytical predictions:

- QREM: asymptotically $\delta(n) \sim n^{-3} \ln^2 n$

However, this behavior is reached only for $n > n_*$, where $n_* \approx 22$ *(see figure)*

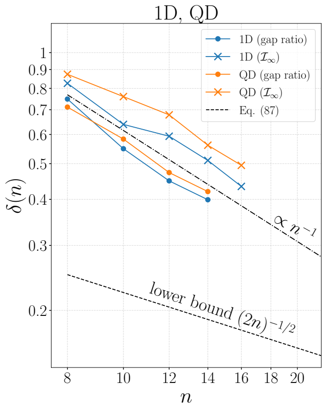


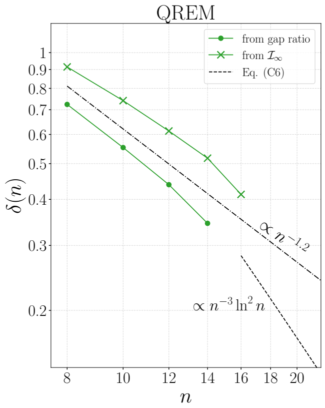
- 1D, QD models: lower bound $\quad \delta(n) \geq \dfrac{1}{\sqrt{2n}}$

related to Harris bound,
cf. Chandran, Laumann, Oganesyan, arXiv:1509.04285

MBL transition width (cont'd)

Numerical results, fits $\delta(n) \sim n^{-\mu}$ in the range $n = 8 \ldots 16$





Comparison to lower bound
$\longrightarrow$ asymptotic regime to be reached for $n > n_*$, where $n_* \lesssim 80$

Pre-critical regime as expected analytically.
Asymptotic regime to be reached for $n > n_*$, where $n_* \approx 22$

Outlook

- Extension to other models within the same class of spin models
  - power-law $1/r^\alpha$ interaction: from 1D ($\alpha = \infty$) to QD ($\alpha = 0$).
  - 2D model
- MBL phase: mechanism of transition to ergodicity
  - proliferation of system-wide resonances
  - phenomenological RG schemes for MBL transition:
    connection to numerics / experiments?
    manifestation in observables for realistic system sizes?

Summary

- Family of spin-$\frac{1}{2}$ models with single-spin-flip random interaction
  - pair interactions: 1D, QD
  - models with (partly) removed Fock-space correlations: QREM, u1D, uQD
- Scaling of MBL transition point $W_c(n)$: $W_c^{1D}(n \to \infty) = \text{const}$,
  in stark contrast to $W_c^{u1D}(n) \approx W_c^{uQD}(n) \sim n \ln n$,
  $W_c^{QREM}(n) \sim n^{1/2} \ln n$, and $W_c^{QD}(n) \gtrsim n^{3/4} \ln^{1/2} n$
  Crucial role of Fock-space Hamiltonian correlations.
- Observables: spectral (level and eigenstate statistics) and
  quantum-dynamics (imbalance and its fluctuations)
  $\longrightarrow$ indicators of MBL transition $W_c(n)$; all in mutual agreement
- Direct MBL-to-ergodicity transition, no evidence of intermediate phase
- Transition width $\Delta W(n)/W_c(n) \sim n^{-\mu}$ with $\mu \approx 1$ for all models for $n \leq 16$.
  "Pre-critical regime", not the asymptotic behavior! Asymptotic scaling of
  width: $n > n_*$. Estimate $n_* \approx 22$ for QREM and $n_* \lesssim 80$ for 1D and QD.
- Outlook: Extension to power-law interaction and to 2D.
  Transition mechanism: development of resonances in MBL phase, RG, ...