stochastic simulation of reheating and/or warm inflation

mikko laine

aec, itp, university of bern

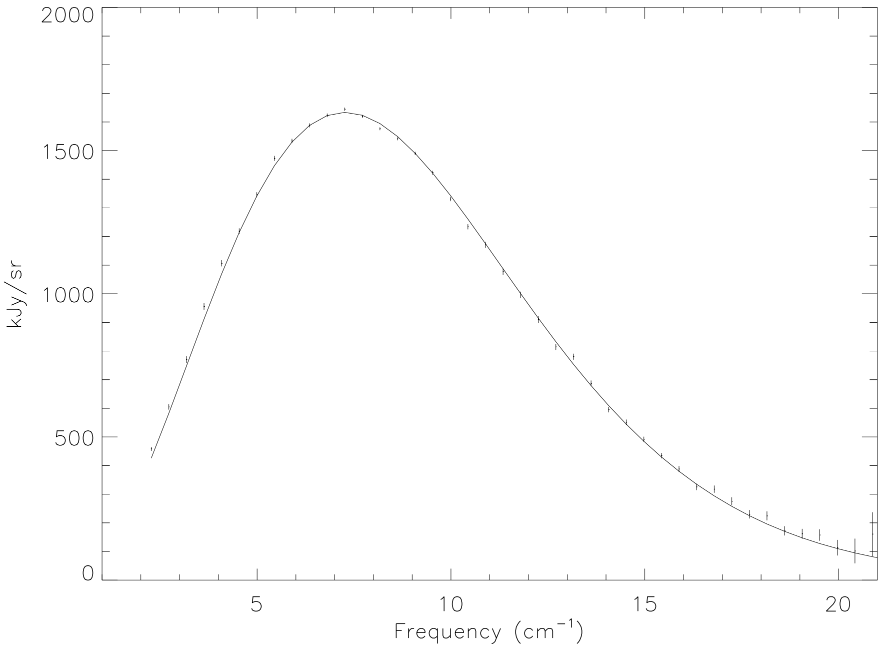
in collaboration with simona procacci and alica rogelj

(i) why is reheating difficult to understand?

(ii) an effective-theory framework

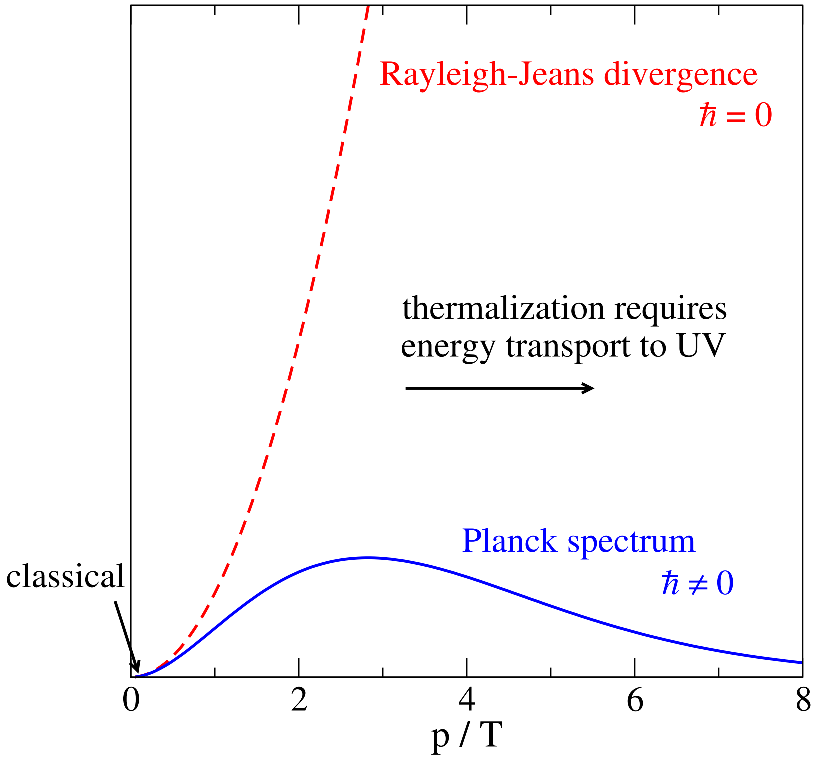
(iii) application: embedding axion-like warm inflation in the standard model

goal: how to go from inflation to a thermal universe?[1]



[1] D.J. Fixsen, E.S. Cheng, J.M. Gales, J.C. Mather, R.A. Shafer and E.L. Wright, *The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data Set*, astro-ph/9605054

challenge: blackbody ≡ Planck spectrum is not classical!



work-around: elementary fields $\to$ hydrodynamic fields

at distances $\gg \lambda_{\rm mfp}$, there are many quanta, and at times $\gg \tau_{\rm coll}$, enough time for decoherence, so temperature is a classical notion

but still: the initial state is a quantum one (φ in bunch-davies vacuum)

$\Rightarrow$ need to interpolate between quantum and classical domains

(ii) an effective-theory framework

[$\sim$ complexified scalar extension of fluctuating relativistic hydrodynamics]

a two-component approach (φ + radiation plasma)[2,3]

minimal set of dynamical variables: $\overbrace{\varphi}^{\text{inflaton}}\,,\ \overbrace{T,\ u^\mu}^{\text{plasma}},\ \overbrace{g_{\mu\nu}}^{\text{metric}}$

matching coefficients: $V(\varphi, T)$, $\Upsilon(\varphi, T)$, $e_r(T)$, $p_r(T)$

dynamics of φ: langevin equation

$$\overbrace{\varphi^{;\mu}{}_{;\mu} - V_{,\varphi}}^{\text{klein}-\text{gordon}}\ \overbrace{-\Upsilon\, u^\mu \varphi_{,\mu} + \varrho}^{\text{detailed balance}} = 0\,,\ \Big\langle\, \overbrace{\varrho_k(t)\,\varrho_k(t')}^{\text{momentum space}}\,\Big\rangle = \overbrace{\Omega_k}^{\text{from }\Upsilon\text{ and }T}\ \delta\big(t-t'\big)$$

dynamics of plasma: energy-momentum conservation

$$T_{\mu\nu} \approx \varphi_{,\mu}\varphi_{,\nu} - \frac{g_{\mu\nu}\,\varphi_{,\alpha}\varphi^{,\alpha}}{2} + (e+p)\,u_\mu u_\nu + p\,g_{\mu\nu}\,, \quad T_{\mu\nu}{}^{;\mu} = 0$$

---

[2] A. Albrecht, P.J. Steinhardt, M.S. Turner and F. Wilczek, *Reheating an Inflationary Universe*, PRL 48 (1982) 1437

[3] full account of perturbations: ML, S. Procacci, A. Rogelj, *Evolution of coupled scalar perturbations through smooth reheating. Part I. Dissipative regime*, 2407.17074; *Part II. Thermal fluctuation regime*, 2507.12849

linearization (in principle not necessary)

scalar perturbations on the matter side ($\delta\varphi$, δT, δv)

$$\varphi = \bar{\varphi} + \delta\varphi \ , \quad T = \bar{T} + \delta T \ , \quad \delta u^i = a^{-1}\delta v^i = -a^{-1}\partial_i \delta v$$

scalar perturbations on the metric side (in a general gauge)

$$g_{\mu\nu} \ = \ a^2 \begin{pmatrix} -1 - 2h_0 & \partial_i h \\ \partial_i h & (1 - 2h_{\mathrm{D}})\delta_{ij} + 2\Big(\partial_i \partial_j - \delta_{ij}\frac{\nabla^2}{3}\Big)\vartheta \end{pmatrix}$$

gauge-invariant variables (curvature perturbations)

$$
\begin{aligned}
\mathcal{R}_\varphi &\equiv -\left(h_\mathrm{D} + \frac{\nabla^2 \vartheta}{3}\right) - H\,\frac{\delta\varphi}{\dot{\bar{\varphi}}}\ , \\
\mathcal{R}_v &\equiv -\left(h_\mathrm{D} + \frac{\nabla^2 \vartheta}{3}\right) + aH\,(h - \delta v)\ , \\
\mathcal{R}_T &\equiv -\left(h_\mathrm{D} + \frac{\nabla^2 \vartheta}{3}\right) - H\,\frac{\delta T}{\dot{\bar{T}}}
\end{aligned}
$$

their differences are called isocurvature perturbations

$$
\mathcal{S}_v \equiv (\bar{e} + \bar{p})(\mathcal{R}_v - \mathcal{R}_\varphi)\ , \quad \mathcal{S}_T \equiv \bar{e}_{,T}\,\dot{\bar{T}}\,(\mathcal{R}_T - \mathcal{R}_\varphi)
$$

(bardeen potentials ϕ, ψ are sourced by $\mathcal{R}_\varphi$, $\mathcal{S}_v$, $\mathcal{S}_T$)

quantum mechanics lies in initial conditions ($k/a_1 \gg T, H$)

$$\mathcal{R}_\varphi \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}} \left[w_{\mathbf{k}} \overbrace{\mathcal{R}_{\varphi k}(t)}^{\text{mode function}} e^{i\mathbf{k}\cdot\mathbf{x}} + w_{\mathbf{k}}^\dagger \mathcal{R}_{\varphi k}^*(t)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

mode functions are promoted to stochastic variables

$$\underbrace{\left\langle \mathcal{P}_{\mathcal{R}_\varphi}(t,k) \right\rangle}_{\text{power spectrum}} \equiv \frac{k^3}{2\pi^2} \underbrace{\left\langle \overbrace{\left|\mathcal{R}_{\varphi k}(t)\right|^2}^{\text{quantum } \langle 0|\ldots|0\rangle} \right\rangle}_{\text{statistical}} \equiv \underbrace{\left\langle \left|\mathcal{R}_k(t)\right|^2 \right\rangle}_{\text{rescaling}}$$

$$\Rightarrow \dot{\mathcal{R}}_k(t_1) \approx -i\frac{k}{a_1}\mathcal{R}_k(t_1) \quad \text{forward-propagating}$$

$$\Rightarrow \mathcal{R}_k(t_1) \approx \frac{H_1}{2\pi\dot{\bar{\varphi}}_1}\frac{k}{a_1} \quad \text{canonically normalized}$$

time evolution of mode functions (mukhanov-sasaki++)

$$
\begin{aligned}
\ddot{\mathcal{R}}_k &= -\frac{\varrho_k H}{\dot{\bar{\varphi}}} - \dot{\mathcal{R}}_k \left[\Upsilon + 2\mathcal{F} + 3H \right] - \mathcal{R}_k \left[\frac{k^2}{a^2} \right] \\
&\quad + \mathcal{S}_v \left[\frac{4\pi G(\Upsilon + 2\mathcal{F})}{H} \right] - \mathcal{S}_T \left[\frac{4\pi G}{H}\left(1 - \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}}}{\dot{\bar{\varphi}}\,\bar{e}_{,T}} \right], \\
\dot{\mathcal{S}}_v &= -\dot{\mathcal{R}}_k \left[\bar{e} + \bar{p}\right] - \mathcal{S}_v \left[3H + \frac{4\pi G \dot{\bar{\varphi}}^2}{H} \right] + \mathcal{S}_T \left[\frac{\bar{p}_{,T}}{\bar{e}_{,T}} \right] \\
\dot{\mathcal{S}}_T &= \varrho_k\, \dot{\bar{\varphi}} H + \dot{\mathcal{R}}_k \left[\Upsilon \dot{\bar{\varphi}}^2 + \frac{8\pi G \bar{e}(\bar{e} + \bar{p})}{H} \right] - \mathcal{R}_k \left[(\bar{e} + \bar{p}) \frac{k^2}{a^2} \right] \\
&\quad - \mathcal{S}_v \left[\frac{k^2}{a^2} + \frac{4\pi G}{H}\left(\Upsilon \dot{\bar{\varphi}}^2 + \frac{8\pi G \bar{e}(\bar{e} + \bar{p})}{H} \right) \right] \\
&\quad + \mathcal{S}_T \left[\frac{\dot{H} - 4\pi G(\bar{e} + \bar{p})}{H} - 3H\left(1 + \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{(V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}})\,\dot{\bar{\varphi}}}{\bar{e}_{,T}} \right]
\end{aligned}
$$

quantum mechanics also appears in the noise autocorrelator

$$\Omega_k \underset{\text{cl} \leftrightarrow \text{qm}}{\overset{\text{matching}}{\simeq}} 2\Upsilon\, \epsilon_k\, n_{\rm B}\big(\epsilon_k\big) \frac{(k/a)^3}{2\pi^2} \,, \quad n_{\rm B}(x) \equiv \frac{1}{e^{\hbar x/T} - 1}$$

for quantum modes ($\hbar\epsilon_k \gg T$):

$$n_{\rm B}(x) \ll 1 \quad \Rightarrow \quad \text{no thermal noise}$$

for classical modes ($\hbar\epsilon_k \ll T$):

$$n_{\rm B}(x) \underset{\hbar x \ll T}{\approx} \frac{T}{\hbar x} \gg 1 \quad \Rightarrow \quad \Omega_k \underset{\hbar\epsilon_k \ll T}{\approx} \underbrace{2\Upsilon T}_{\substack{\text{text}-\text{book} \\ \text{fluctuation}- \\ \text{dissipation}- \\ \text{relation}}} \frac{(k/a)^3}{2\pi^2}$$

summary of basic ingredients

no slow-roll approximations

no assumptions about inflaton equilibration

correct quantum and classical limits

vanishing isocurvature when $k/a \ll H$ ($\mathcal{R}_\varphi = \mathcal{R}_v = \mathcal{R}_T$)

inflaton decouples dynamically when $e_\varphi \ll e_r$

acoustic oscillations in $\mathcal{R}_v, \mathcal{R}_T$ start when $k/a \gg H$

but for the moment we work on the linearized level

numerical method for stochastic evolution: itô equation

$$\mathcal{V} \equiv \left(\mathcal{R}_k \, \dot{\mathcal{R}}_k \, \mathcal{S}_v \, \mathcal{S}_T\right)^{\mathrm{T}} \in \mathbb{C}^4$$

time discretization:

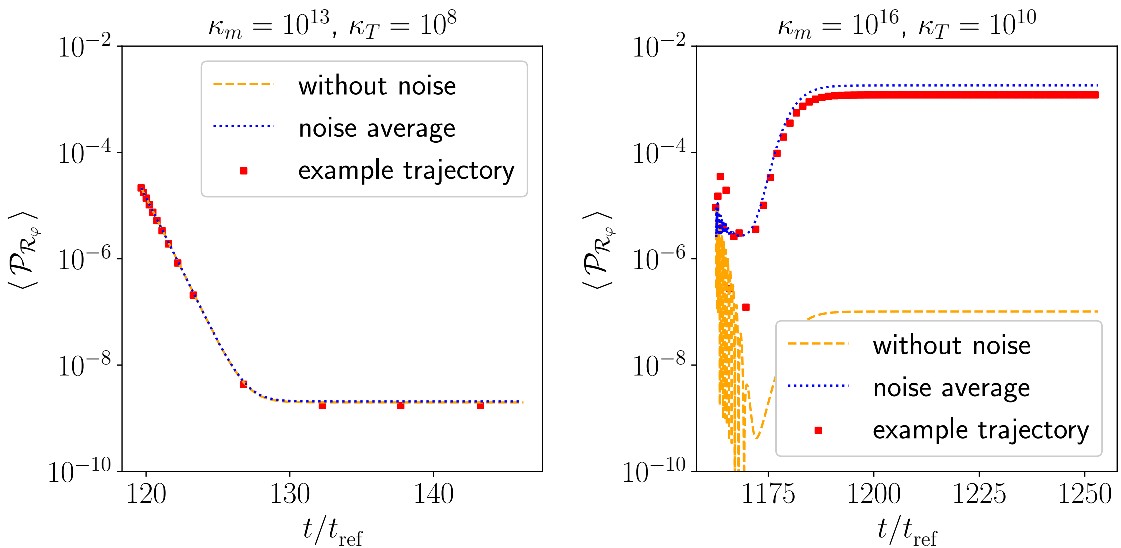
$$\mathcal{V}_{i+1} = \Big(1 + \overbrace{\epsilon_i}^{\text{time step}} \underbrace{M_i}_{\text{drift}}\Big)\mathcal{V}_i + \sqrt{\epsilon_i}\,\underbrace{\mathcal{N}_i}_{\text{noise}} \,, \quad \epsilon_i = \frac{\texttt{tol}}{\underbrace{\max\{|\lambda_i|\}}_{\text{eigenvalues of } M_i}}$$

evolution becomes deterministic if only need power spectra:

$$\left\langle \mathcal{V}_{i+1}\mathcal{V}_{i+1}^{\dagger} \right\rangle = \left(1 + \epsilon_i M_i\right)\left\langle \mathcal{V}_i \mathcal{V}_i^{\dagger} \right\rangle \left(1 + \epsilon_i M_i^{\dagger}\right) + \epsilon_i \underbrace{\left\langle \mathcal{N}_i \mathcal{N}_i^{\dagger} \right\rangle}_{\propto\, \Omega_k}$$

examples of solutions

$$\Upsilon \equiv \frac{\kappa_m\, m^3 + \kappa_T\, (\pi T)^3}{(4\pi)^3 f_a^2}\,, \quad * \equiv \begin{cases} \text{pivot scale} \\ \text{horizon exit} \end{cases}$$



"weak regime" ($\Upsilon_* < H_*$) "strong regime" ($\Upsilon_* > H_*$)

(iii) application: embedding axion-like warm inflation[4,5] in the standard model[6]

[4] K.V. Berghaus, P.W. Graham and D.E. Kaplan, *Minimal warm inflation,* 1910.07525

[5] M. Laine and S. Procacci, *Minimal warm inflation with complete medium response,* 2102.09913; ...

[6] K.V. Berghaus, M. Drewes and S. Zell, *Warm Inflation with the Standard Model,* 2503.18829

$\varphi \equiv$ **QCD axion?** $\left[\, \mathcal{L} \supset -\frac{\varphi\, \alpha_s \epsilon^{\mu\nu\rho\sigma} F^c_{\mu\nu} F^c_{\rho\sigma}}{16\pi f_a} \,\right]$

very much studied; constraints mostly from $|\varphi| \ll m_{\rm pl}$, $T \ll$ TeV

for inflation, use same $\alpha_{\rm s}$, f_a, but the domain $|\varphi| \sim m_{\rm pl}$, $T \gg$ TeV

$$V(\varphi) \overset{\text{postulated}}{\equiv} \lambda\, \varphi^4 \,, \quad \text{for } |\varphi|,\ T \ \gg\ \text{TeV}$$

$$\Upsilon(\varphi, T) \underset{\text{computed}}{\overset{!}{\approx}} \frac{T^2}{2 f_a^2} \Bigg[\underbrace{\frac{1}{\alpha_{\rm s}^5 N_{\rm c}^5 T}}_{\text{sphaleron reheating}} + \underbrace{\frac{2 N_{\rm f}}{N_{\rm c} H(\varphi, T)}}_{\text{fermion dilution}} \Bigg]^{-1},$$

$$N_{\rm c} \equiv 3\,, \quad N_{\rm f} \equiv 5$$

even if there are strong self-interactions, the (approximate) shift symmetry protects the flatness of the potential from IR corrections

background values ($pX \equiv 10^{+X}$)

	adjusted input		our output ($k_*/a_0 \equiv 0.05\,\mathrm{Mpc}^{-1}$)		
λ	f_a [10^{12} GeV]	α_s	T_*/H_*	$Q_* = \Upsilon_*/3H_*$	$\lvert\dot{\varphi}\rvert_*/H_*^2$
10^{-21}	$0.146 \to 0.153$	0.0269	7798	13.9	$1.11p8$
10^{-20}	$0.264 \to 0.278$	0.0263	3534	8.59	$2.91p7$
10^{-19}	$0.518 \to 0.516$	0.0257	1531	4.88	$7.24p6$
10^{-18}	$1.05 \to 1.025$	0.0251	618	2.30	$1.69p6$
10^{-17}	$2.30 \to 2.27$	0.0246	213	0.711	$3.68p5$
10^{-16}	$4.95 \to 4.98$	0.0243	60.9	0.0919	$8.38p4$
10^{-15}	$8.69 \to 8.56$	0.0242	20.4	0.0150	$2.33p4$

perturbations ($mX \equiv 10^{-X}$)

adjusted input			our output		
λ	f_a [10^{12} GeV]	α_s	A_s	n_s	r
10^{-21}	0.153	0.0269	$2.11m9$	0.968	$2.12m11$
10^{-20}	0.278	0.0263	$2.09m9$	0.970	$5.00m10$
10^{-19}	0.516	0.0257	$2.11m9$	0.969	$1.31m8$
10^{-18}	1.025	0.0251	$2.10m9$	0.968	$4.27m7$
10^{-17}	2.27	0.0246	$2.10m9$	0.967	$1.68m5$
10^{-16}	4.98	0.0243	$2.09m9$	0.971	$4.79m4$
10^{-15}	8.56	0.0242	$2.09m9$	0.980	$6.26m3$

(Planck + ACT: $n_s = 0.974 \pm 0.003$)

conclusions & outlook

two subsystems: inflaton (φ), strongly self-interacting plasma (T)

transfer of perturbations $\varphi \leftrightarrow T$ is tractable

framework model-independent; model dependence in V, Υ, e_r, p_r

a viable implementation with φ as QCD axion

$\Rightarrow$ gw signature from hydrodynamic fluctuations? inflaton is weakly coupled and shear viscosity is large, $\hat{\eta} \propto \lambda^{-2}$ (cf. simona's talk)