Numerical Simulations of Primordial Black Holes

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Numerical Simulations of Early Universe Sources of Gravitational Waves

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Fellini
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Introduction: a very brief overview

- **Primordial Black Holes (PBHs)** [**Zeldovich & Novikov** (1967), **Hawking** (1971)] could form from the collapse of cosmological perturbation during the radiation dominated era.

$$p = \frac{\rho}{3}$$

- **PBHs could span a large wide range of masses and if not evaporated** [BH evaporation **Hawking** (1974)]: PBHs with ***$M > 10^{15}\,g$*** are interesting candidates for dark matter, intermediate mass black holes and the seeds of supermassive black holes.

- **Numerical hydrodynamical simulations in spherical symmetry** of a cosmological perturbation, characterized by an amplitude δ, have shown:

    - $\delta > \delta_c \;\Rightarrow\;$ PBH formation
    - $\delta < \delta_c \;\Rightarrow\;$ perturbation bounce
    - $\delta_c \sim c_s^2 \equiv \dfrac{\partial p}{\partial \rho}$ **(Carr 1975)**

Equation of State of the Early Universe

The easy Universe goes through 3 main transitions before matter-radiation equality:

- Electroweak phase-transition
- **QCD phase- transition** (crossover)

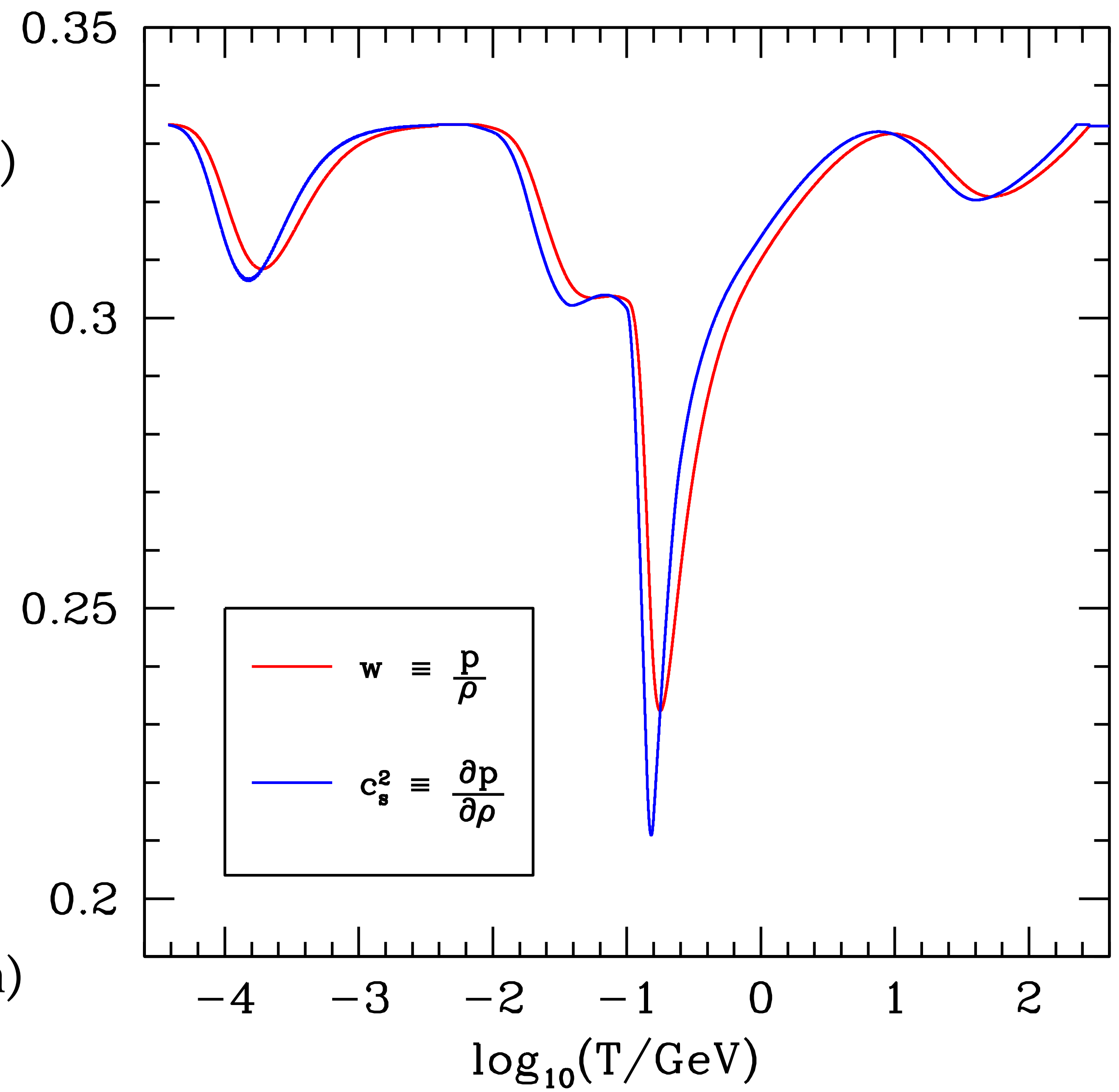
$$\rho = \frac{\pi^2}{30} g_{\text{eff}}(T) T^4 \qquad s = \frac{2\pi^2}{45} h_{\text{eff}}(T) T^3$$

$$p = sT - \rho = w(T)\rho$$

$$w(T) = \frac{4h_{\text{eff}}(T)}{3g_{\text{eff}}(T)} - 1$$

$$c_s^2(T) = \frac{\partial p}{\partial \rho} = \frac{4(4h_{\text{eff}} + T h'_{\text{eff}})}{3(4g_{\text{eff}} + T g'_{\text{eff}})} - 1$$

- Nucelosynthesis (e+e- annihilation)



Relativistic Hydrodynamics in spherical symmetry

$$\mathrm{d}s^2 = -A^2(t,r)\mathrm{d}t^2 + B^2(t,r)\mathrm{d}r^2 + R^2(t,r)\mathrm{d}\Omega^2$$ - comoving (cosmic time) slicing

$$U \equiv D_t R \equiv \frac{1}{A}\frac{\partial R}{\partial t}\bigg|_r \qquad \Gamma \equiv D_r R \equiv \frac{1}{B}\frac{\partial R}{\partial r}\bigg|_t$$

$$D_t U = -\frac{\Gamma}{\rho + p_\mathrm{r}}\left[D_r p_\mathrm{r} + \frac{2\Gamma}{R}\left(p_\mathrm{r} - p_\mathrm{t}\right)\right] - \frac{M}{R^2} - 4\pi R p_\mathrm{r}$$

$$\frac{D_t \rho_0}{\rho_0} = -\frac{1}{R^2\Gamma} D_r\left(R^2 U\right)$$

$$\frac{D_t \rho}{\rho + p_\mathrm{r}} = \frac{D_t \rho_0}{\rho_0} + \frac{2U}{R}\frac{p_\mathrm{r} - p_\mathrm{t}}{\rho + p_\mathrm{r}}$$

$$\frac{D_r A}{A} = -\frac{1}{\rho + p_\mathrm{r}}\left[D_r p_\mathrm{r} + \frac{2\Gamma}{R}\left(p_\mathrm{r} - p_\mathrm{t}\right)\right]$$

$$D_r M = 4\pi R^2 \Gamma \rho$$

$$D_t M = -4\pi R^2 U p_\mathrm{r}$$

$$D_t \Gamma = -\frac{U}{\rho + p_\mathrm{r}}\left[D_r p_\mathrm{r} + \frac{2\Gamma}{R}\left(p_\mathrm{r} - p_\mathrm{t}\right)\right]$$

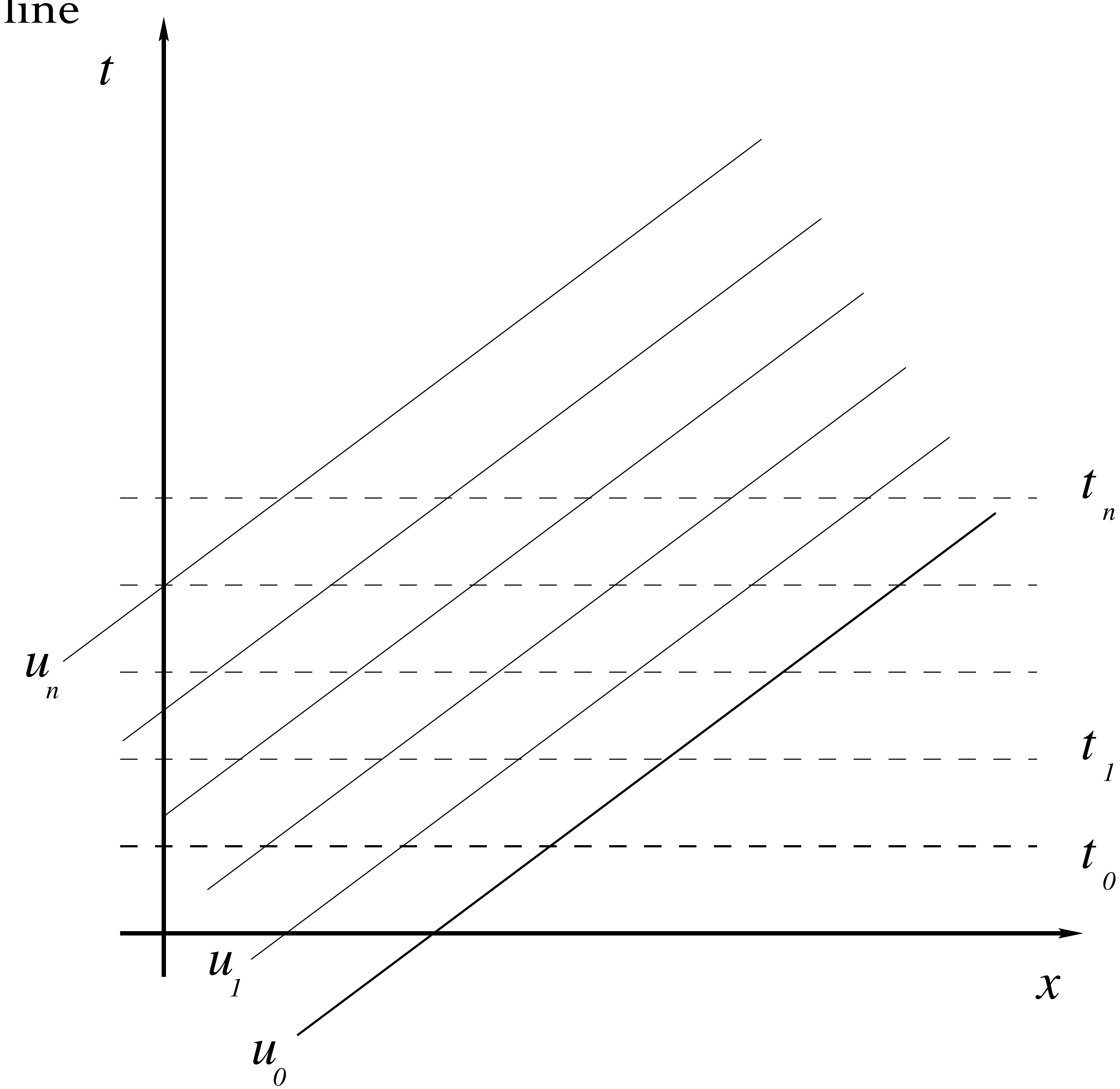
$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

Numerical technique

The diagram show two different time foliation (space-time slicing):

1. comoving (cosmic time) - solid line
2. null (observer time) - dashed line

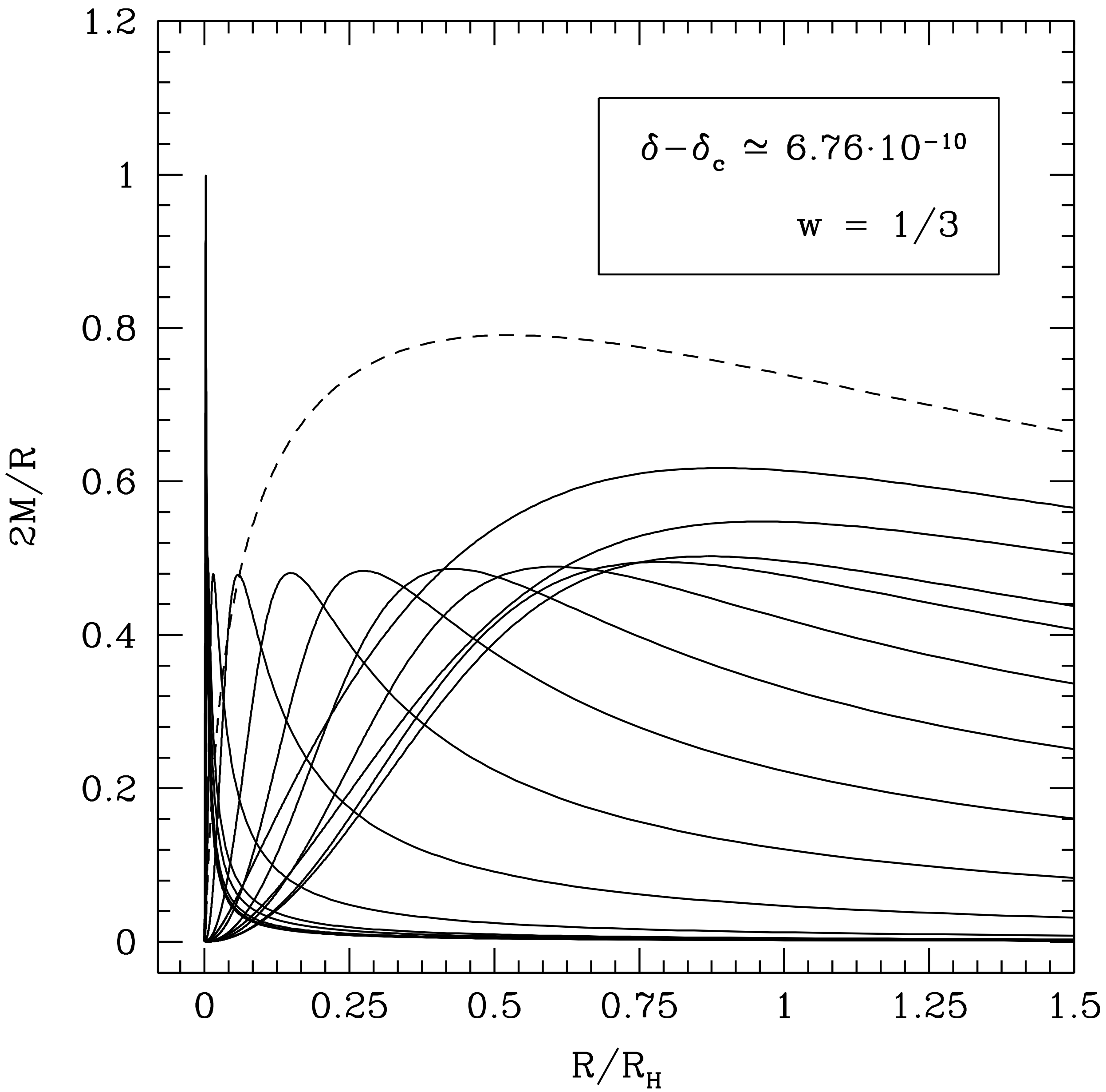
The initial conditions are specified on the cosmic time slicing. To follow the full evolution, and avoid the central singularity, one can transform the calculation into the observer time slicing, where the formation of the apparent horizon formation is infinitely redshifted. This allow to fully compute the final mass of the BH.

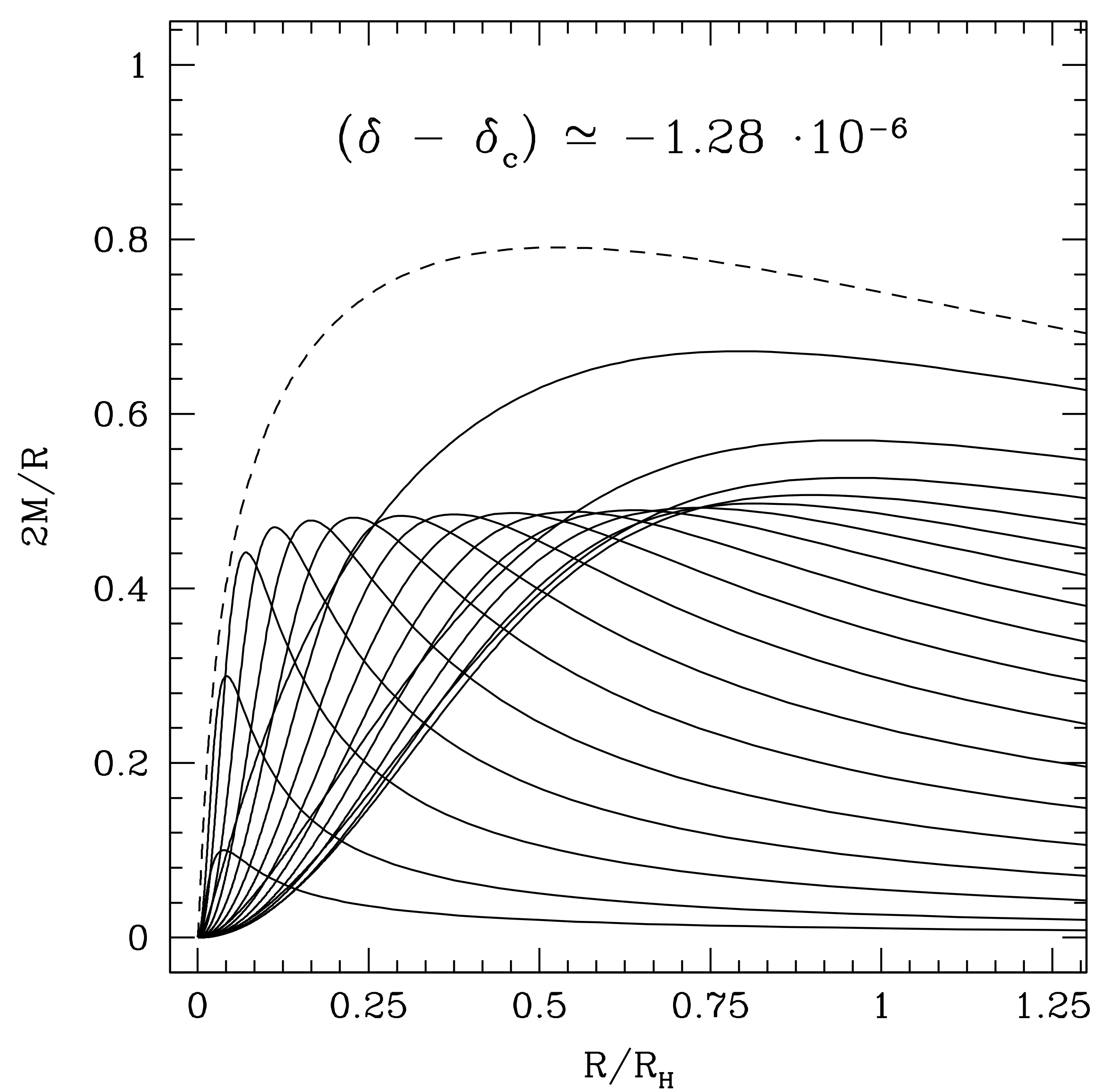


The code is using AMR.

Numerical Results: PBH formation/bounce

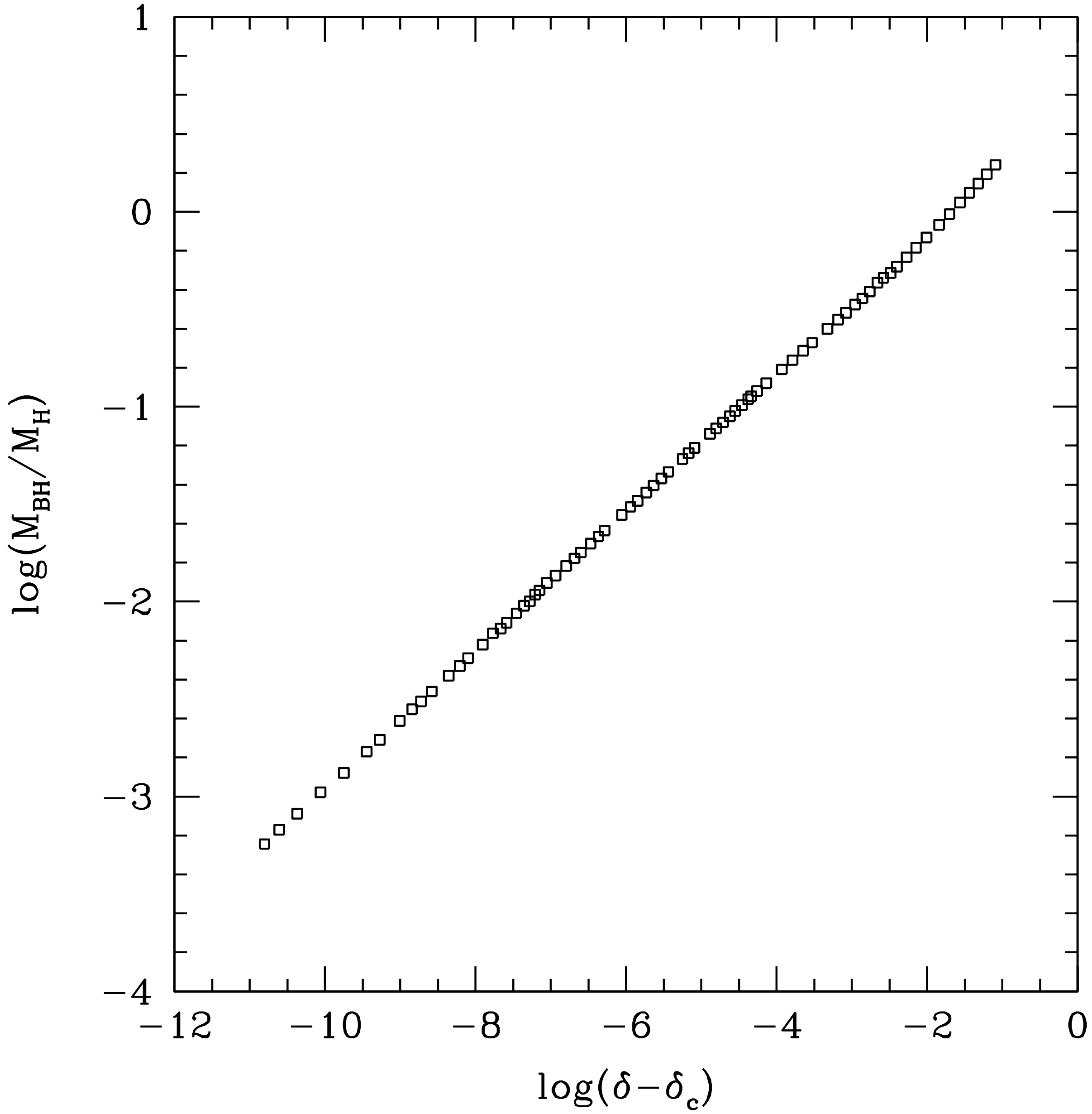
$$R(r,t) = 2M(r,t)$$





IM, J. Miller, Rezzolla - CQG (2005)

Numerical Results: Scaling Law / Critical collapse



$$M_{PBH} = \mathcal{K}(\delta - \delta_c)^\gamma M_H$$

M_H – cosmological horizon mass

$\mathcal{K}, \delta_c$ – shape dependent

$$\gamma \simeq 0.36$$

***IM, Miller, Polnarev* - CQG (2009, 2013)**

Initial conditions: curvature profile

- The asymptotic metric ($t \to 0$), describing super-horizon cosmological perturbations in the comoving synchronous gauge can be written as:

$$ds^2 \simeq -dt^2 + a^2(t)e^{2\zeta(r)}\left[dr^2 + r^2 d\Omega^2\right] = -dt^2 + a^2(t)\left[\frac{d\tilde{r}^2}{1 - K(\tilde{r})\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right]$$

- In the "linear regime" of cosmological perturbations, adiabatic perturbations on super horizon scales can be described by a time independent curvature profile using the **quasi-homogeneous / gradient expansion approach**.

$$\frac{\delta\rho}{\rho_b} = -\left(\frac{1}{aH}\right)^2 \frac{4}{9}\left[\boxed{\nabla^2\zeta(r)} + \boxed{\frac{1}{2}\left(\nabla\zeta(r)\right)^2}\right] e^{-2\zeta(r)}$$

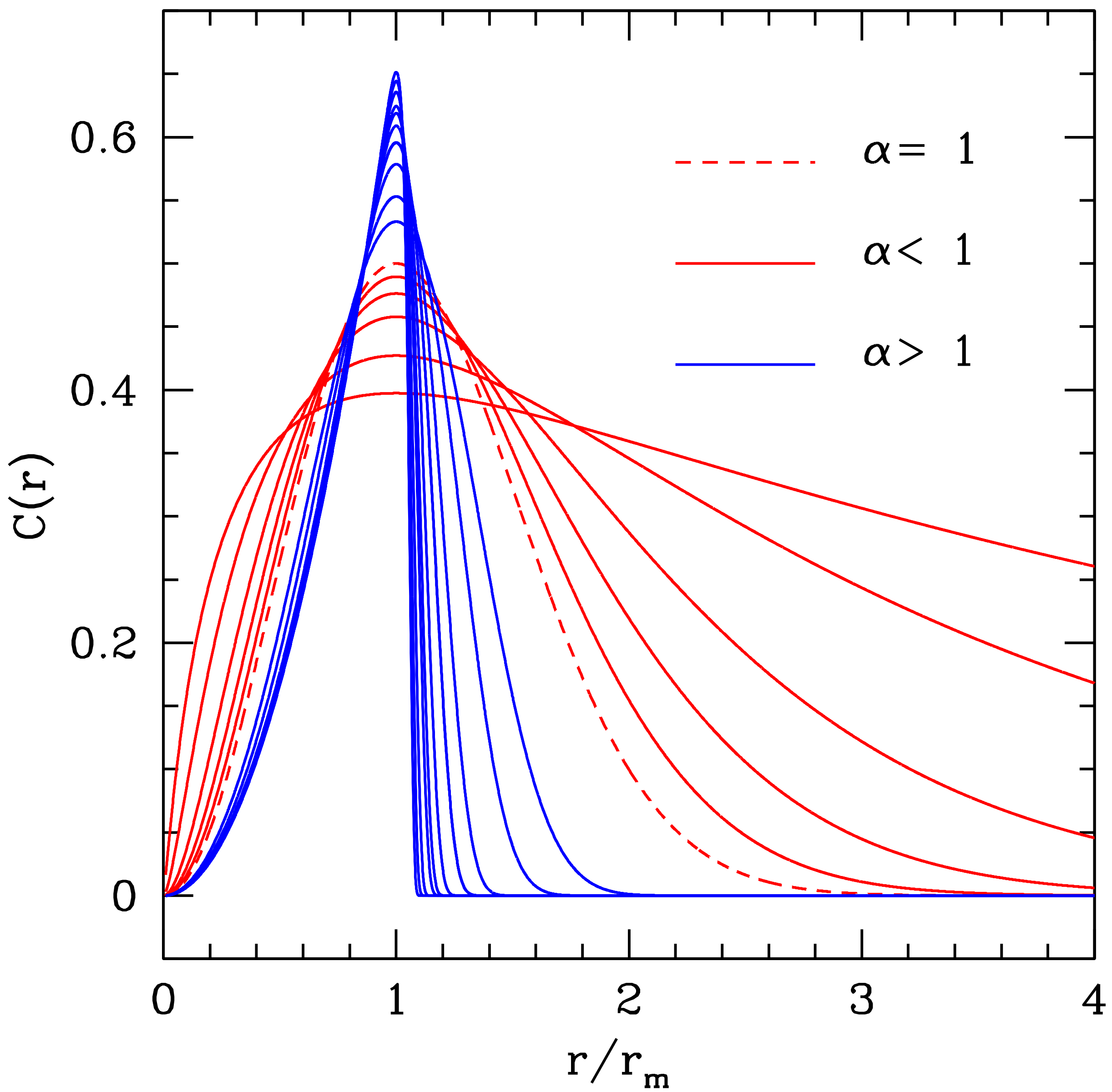
- The perturbation amplitude δ is measured by the peak of the compaction function, corresponding to the excess of mass of the over density.

$$\mathcal{C}(r) := \frac{2[M(r,t) - M_b(r,t)]}{R(r,t)} = \boxed{-\frac{4}{3}r\zeta'(r)}\left[1 + \boxed{\frac{1}{2}r\zeta'(r)}\right] \quad \Rightarrow \quad \delta = \boxed{\delta_G}\left[1 - \boxed{\frac{3}{8}\delta_G}\right]$$

Shape parameter

$$\mathcal{C}'(r_m) = 0, \quad \Phi_m \equiv -r_m \zeta'(r_m)$$

$$\boxed{\delta(r_m, t_H) = 3\frac{\delta\rho}{\rho_b}(r_m, t_H)}$$



$$\tilde{r} = r e^{\zeta(r)}$$

$$\alpha \equiv -\frac{\mathcal{C}''(\tilde{r}_m)\tilde{r}_m^2}{4\mathcal{C}(\tilde{r}_m)} = \frac{\alpha_G}{\left(1 - \frac{1}{2}\Phi_m\right)(1 - \Phi_m)}$$

$$\boxed{0.4 \leq \delta_c(\alpha) \leq \frac{2}{3}}$$

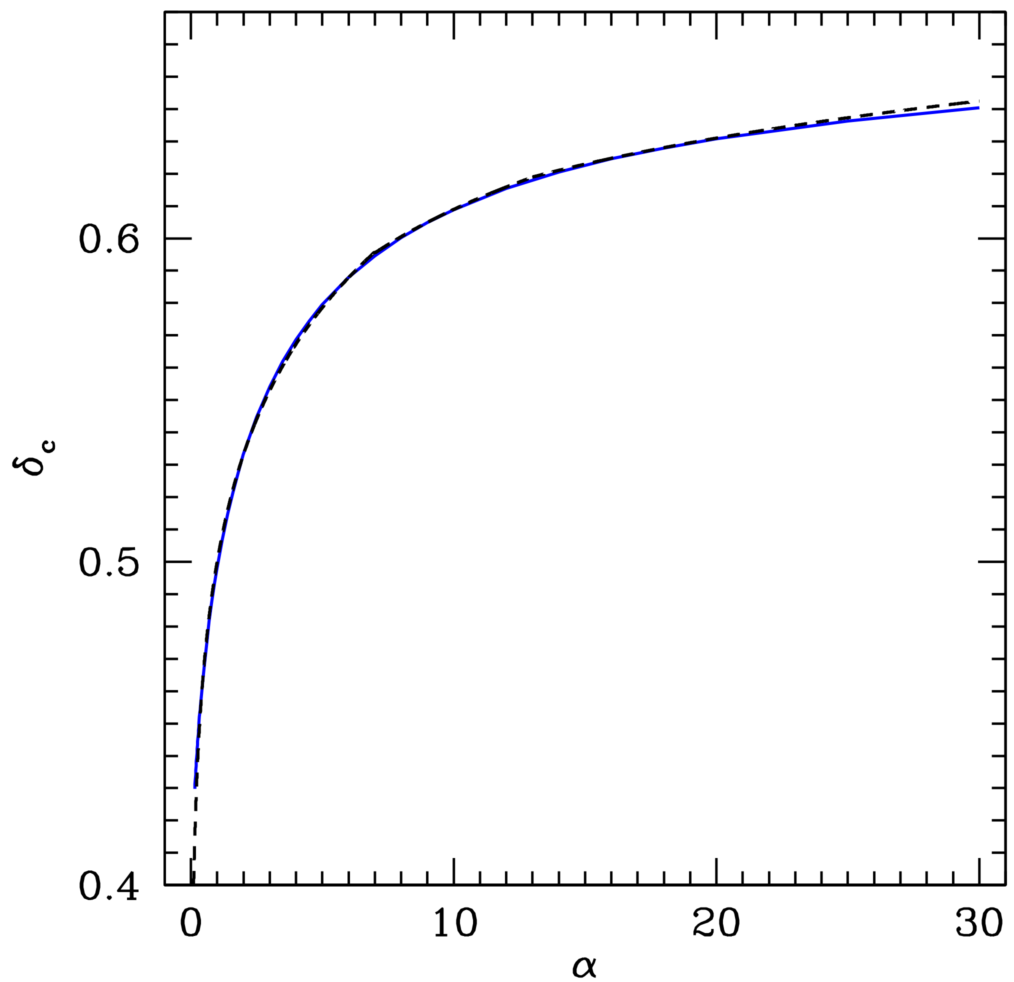
I. Musco - PRD (2019)

PBH threshold

- *Escrivá, Germani, Sheth* - PRD (2020)

$$\bar{\mathcal{C}}(r_m) = 0.4 \quad \text{(shape independent)}$$

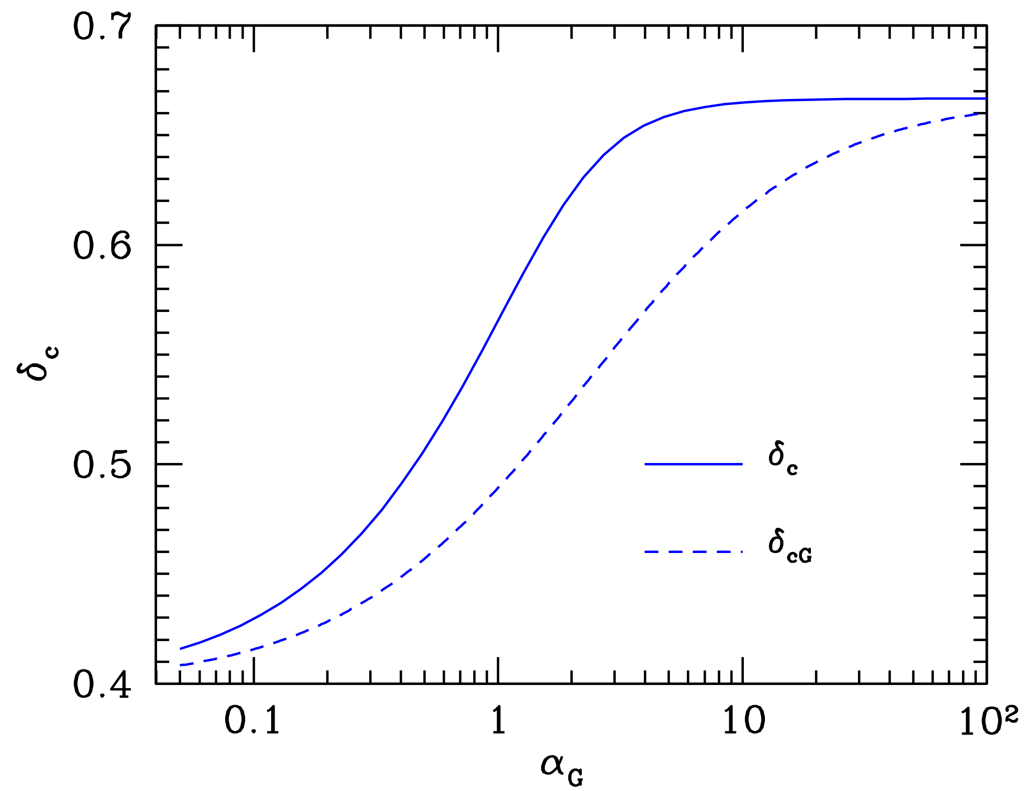
$$\delta_c \simeq \frac{4}{15} e^{-\frac{1}{\alpha}} \frac{\alpha^{1-5/2\alpha}}{\Gamma\left(\frac{5}{2\alpha}\right) - \Gamma\left(\frac{5}{2\alpha}, \frac{1}{\alpha}\right)}$$



- *IM, De Luca, Franciolini, Riotto* - PRD (2021)

$$\delta_c \simeq \begin{cases} \alpha^{0.047} - 0.50 & 0.1 \lesssim \alpha \lesssim 7 \\ \alpha^{0.035} - 0.475 & 7 \lesssim \alpha \lesssim 13 \\ \alpha^{0.026} - 0.45 & 13 \lesssim \alpha \lesssim 30 \end{cases}$$

$$\delta_m = \frac{4}{3}\Phi_m \left(1 - \frac{1}{2}\Phi_m\right) = \delta_G \left(1 - \frac{3}{8}\delta_G\right)$$



PBH Abundance (Peak Theory)

C.Germani, IM - PRL (2019)

- **PDF of δ follows a Gaussian distribution:** $P(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right)$

$$\sigma^2 = \langle \delta^2 \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\delta(k, r_m) = \left(\frac{2}{3}\right)^4 \int_0^\infty \frac{dk}{k} (kr_m)^4 T^2(k, r_m) W^2(k, r_m) \mathcal{P}_\zeta(k)$$

$$\beta_f \simeq \sqrt{\frac{2}{\pi}} \mathcal{K} \left(\frac{k_*}{a_m H_m}\right)^3 \sigma^\gamma \nu_c^{1-\gamma} \gamma^{\gamma+1/2} e^{-\frac{\nu_c^2}{2}} \qquad \nu_c \equiv \frac{\delta_c}{\sigma}$$

- **If** $M_{PBH} \sim 10^{16} g$ are Dark Matter $\Rightarrow \beta_f \simeq 10^{-8} \sqrt{\frac{M_{PBH}}{M_\odot}} \simeq 10^{-16}$

- **Narrow peak:** $\frac{k_*}{\sigma} \gg 1 \;\Rightarrow\; \nu_c \simeq 0.22 \sqrt{\frac{k_*}{\sigma \mathcal{P}_0}} \;\Rightarrow\; \mathcal{P}_0 \sim 7 \times 10^{-4} \frac{k_*}{\sigma} \gg 10^{-3}$

- **Broad peak:** $\frac{k_*}{\sigma} \ll 1 \;\Rightarrow\; \nu_c \simeq 0.46 (\mathcal{P}_0)^{-1/2} \;\Rightarrow\; \mathcal{P}_0 \sim 3 \times 10^{-3}$

- **Non linear effects:** $\delta = \delta_G \left[1 - \frac{3}{8}\delta_G\right] \quad \Rightarrow \quad 1.5 \lesssim \frac{\mathcal{P}_{0_{NL}}}{\mathcal{P}_{0_L}} = \frac{16\left(1 - \sqrt{1 - \frac{3}{2}\delta_c}\right)^2}{9\delta_c^2} \lesssim 4$

S.Young, IM, C.Byrnes JCAP (2019)

De Luca, Franciolini, Kehagias, Peloso, Riotto and Unal (2019)

PBH threshold prescription

Curvature power spectrum $\mathcal{P}_\zeta$

⇓

Characteristic overdensity scale $k_* \hat{r}_m$

⇓

Characteristic shape parameter α

⇓

Threshold δ_c

IM, De Luca, Franciolini, Riotto - PRD (2021)

1. **The power spectrum of the curvature perturbation**: take the primordial power spectrum $\mathcal{P}_\zeta$ of the Gaussian curvature perturbation and compute, on superhorizon scales, its convolution with the transfer function $T(k, \eta)$

$$P_\zeta(k,\eta) = \frac{2\pi^2}{k^3}\mathcal{P}_\zeta(k) T^2(k,\eta).$$

2. **The comoving length scale** $\hat{r}_m$ of the perturbation is related to the characteristic scale k_* of the power spectrum P_ζ. Compute the value of $k_* \hat{r}_m$ by solving the following integral equation

$$\int dk k^2 \left[(k^2 \hat{r}_m^2 - 1) \frac{\sin(k\hat{r}_m)}{k\hat{r}_m} + \cos(k\hat{r}_m) \right] P_\zeta(k,\eta) = 0\,.$$

3. **The shape parameter:** compute the corresponding shape parameter α of the collapsing perturbation, including the correction from the non linear effects, by solving the following equation

$$F(\alpha)\left[1+F(\alpha)\right]\alpha = -\frac{1}{2}\left[1 + \hat{r}_m \frac{\int dk k^4 \cos(k\hat{r}_m) P_\zeta(k,\eta)}{\int dk k^3 \sin(k\hat{r}_m) P_\zeta(k,\eta)}\right]$$

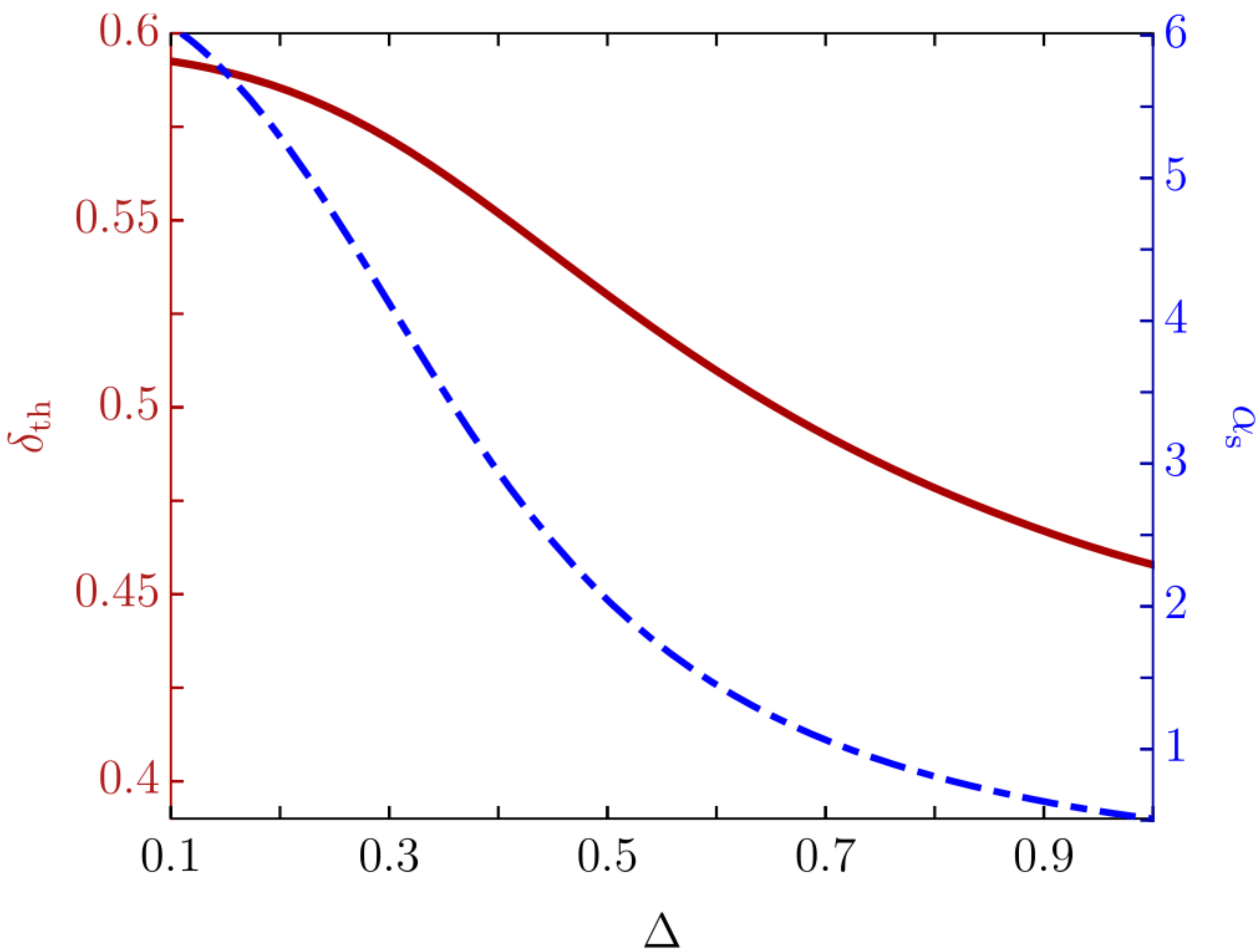
$$F(\alpha) = \sqrt{1 - \frac{2}{5} e^{-1/\alpha} \frac{\alpha^{1-5/2\alpha}}{\Gamma\left(\frac{5}{2\alpha}\right) - \Gamma\left(\frac{5}{2\alpha}, \frac{1}{\alpha}\right)}}\,.$$

4. **The threshold** δ_c**:** compute the threshold as function of α, fitting the numerical simulations, at *superhorizon scales*, making a linear extrapolation at horizon crossing ($aHr_m = 1$).

$$\delta_c \simeq \begin{cases} \alpha^{0.047} - 0.50 & 0.1 \lesssim \alpha \lesssim 7 \\ \alpha^{0.035} - 0.475 & 7 \lesssim \alpha \lesssim 13 \\ \alpha^{0.026} - 0.45 & 13 \lesssim \alpha \lesssim 30 \end{cases}$$

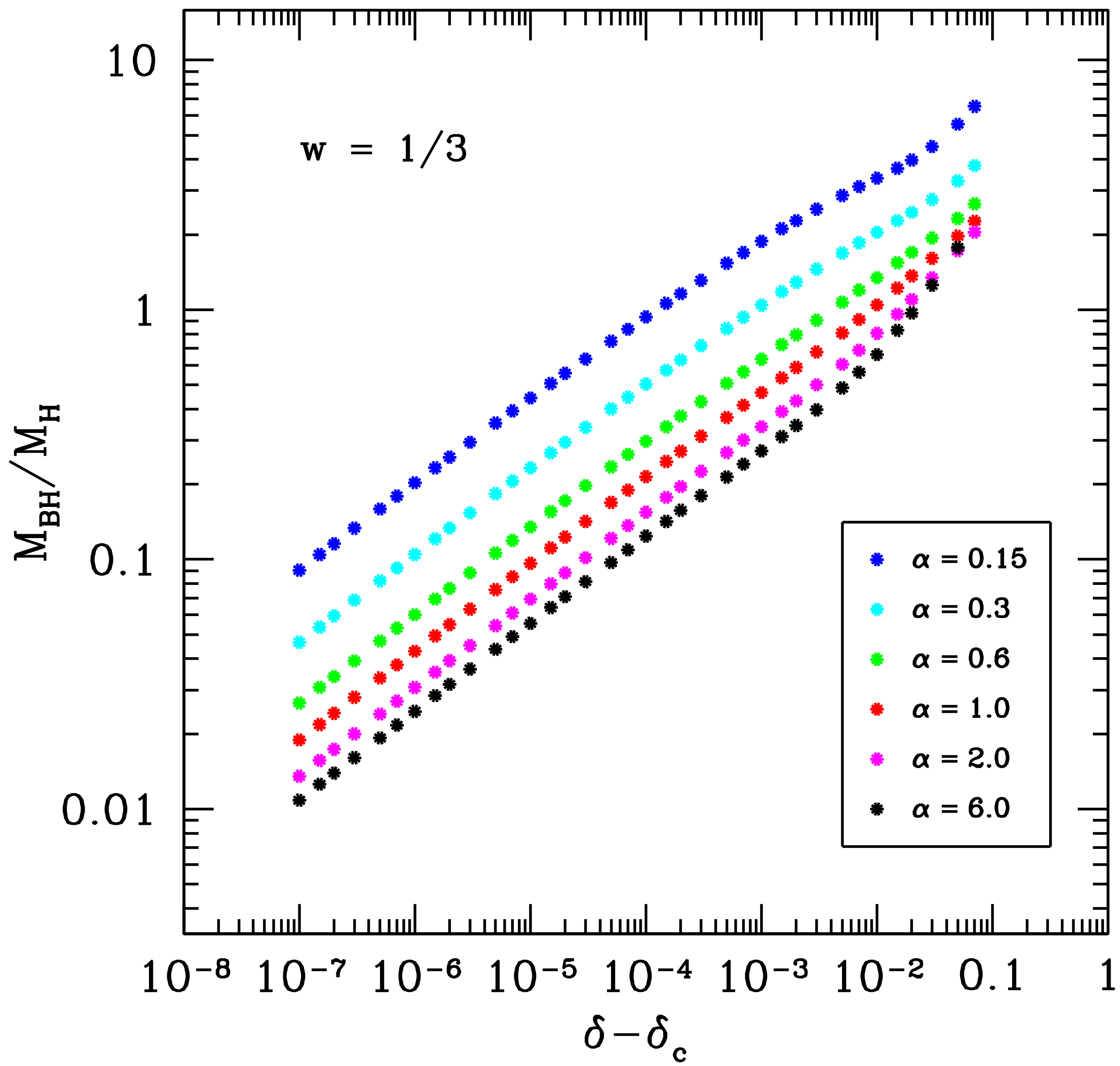
Example: Log Normal Power Spectrum

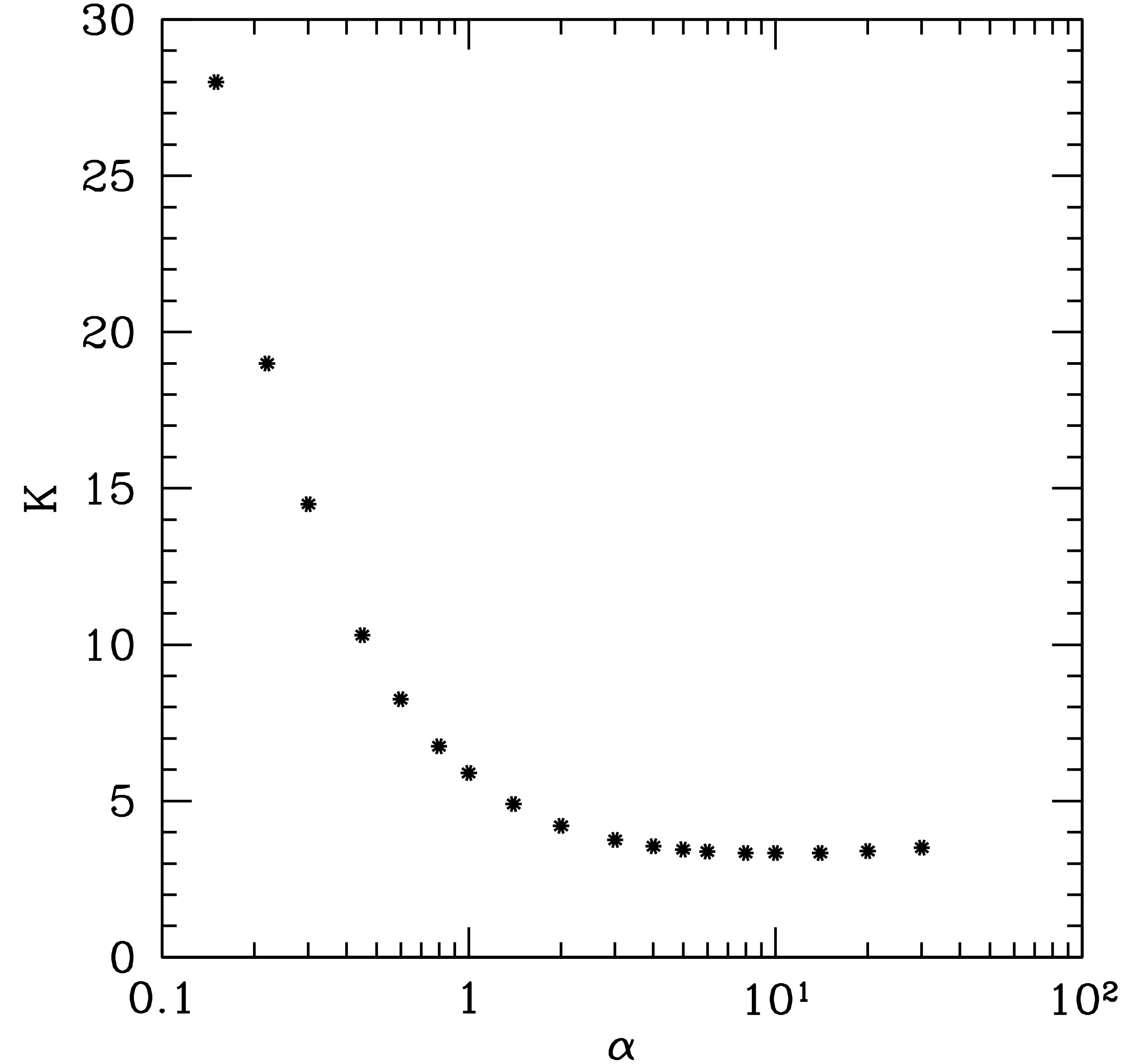
$$\mathcal{P}_\zeta(k) = \frac{A}{2\pi\Delta^2}\exp\left[-\frac{1}{2\Delta^2}\ln^2(k/k_*)\right] \;\Rightarrow\; 1 \lesssim \alpha \lesssim 6$$



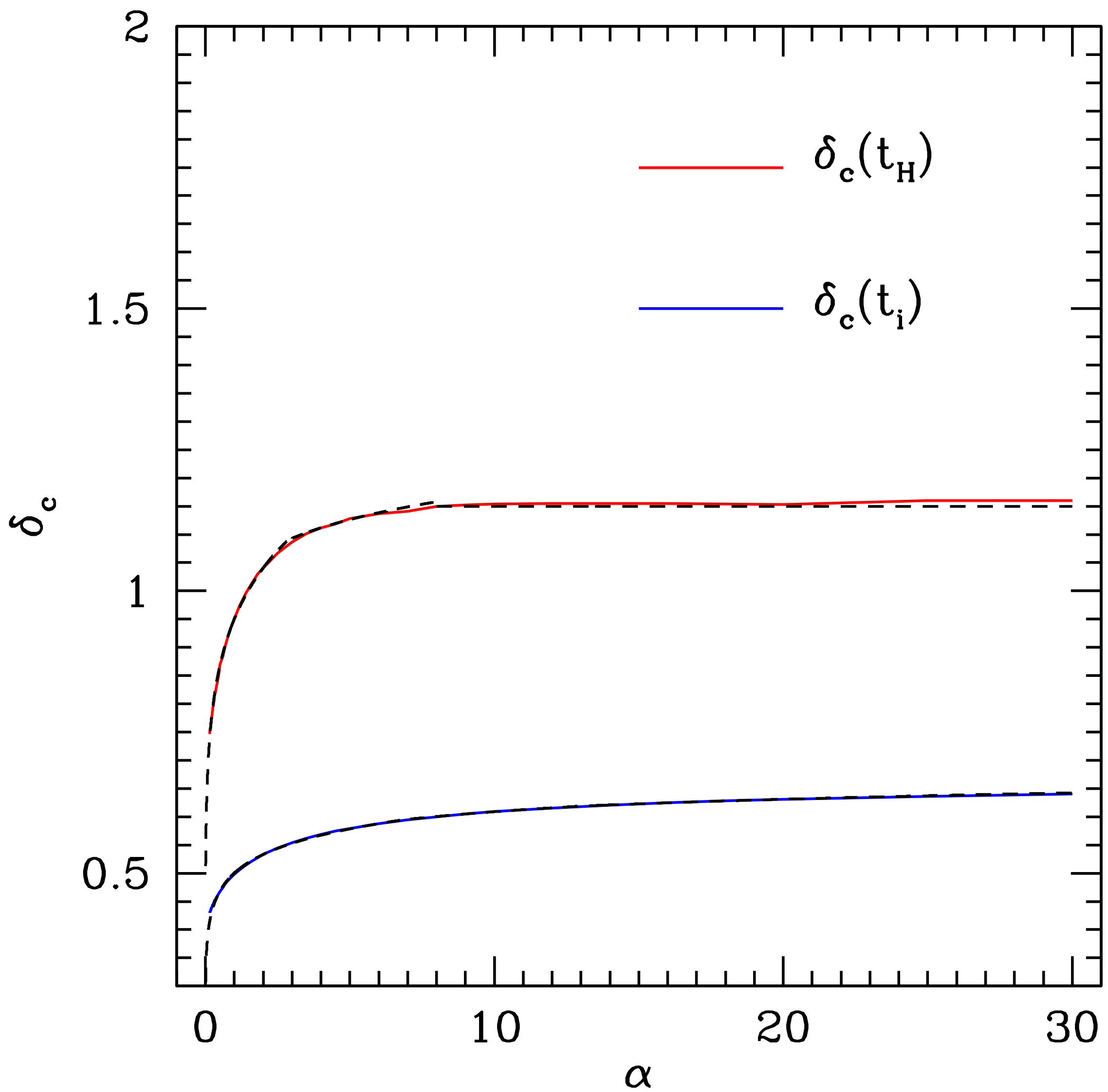
G. Franciolini, A. Iovino, V. Vaskonen, H. Veermae - PRL (2023)

Scaling law (critical collapse)





Non linear horizon crossing: a problem for PBH abundance?



$$\delta_c(t_i) \simeq \begin{cases} 0.13\alpha + 0.41 & \alpha \lesssim 0.25 \\ \alpha^{0.045} - 0.50 & 0.25 \lesssim \alpha \lesssim 7 \\ \alpha^{0.035} - 0.475 & 7 \lesssim \alpha \lesssim 13 \\ \alpha^{0.026} - 0.45 & 13 \lesssim \alpha \lesssim 30 \end{cases}$$

$$\delta_c(t_H) \simeq \begin{cases} \alpha^{0.125} - 0.05 & 0.1 \lesssim \alpha \lesssim 3 \\ \alpha^{0.06} + 0.025 & 3 \lesssim \alpha \lesssim 8 \\ 1.15 & \alpha \gtrsim 8 \end{cases}$$

$$0.4 \leq \delta_c(t_i) < 0.6 \qquad 0.7 \lesssim \delta_c(t_H) \lesssim 1.15$$

$$\text{PBH Abundance:} \quad \beta \propto e^{-\frac{\nu_c^2}{2}} \quad \nu_c \equiv \delta_c/\sigma$$

IM, De Luca, Franciolini, Riotto - PRD (2021)

Problems

- The threshold value depends on the horizon crossing (linear extrapolation of gradient expansion, non linear horizon crossing).
- If the power spectrum is broad we need to include a **window function** to cut the sub horizon mode.
- This introduces a certain level of **arbitrariness in the value of the threshold / computation of the abundance.** (peaked power spectrum De Luca, Kehagias, Riotto - arXiv:2307.13633)

Can we overcome these problems ? 🤔

A possible solution can be to compute the threshold during the critical self similar regime

Self similar equations:

$\xi \equiv -R/t$ - Self similar coordinate

$$U \equiv D_t R$$
$$\Omega \equiv 4\pi R^2 e$$
$$\Phi \equiv \frac{M}{R}$$

Self similar variables

$$\frac{d\ln U}{d\ln \xi} = \left[\frac{(\Phi + w\Omega)^2 - 2w\Gamma^2\Phi}{U^2(\Phi + w\Omega)^2 - w\Gamma^2(\Omega - \Phi)^2}\right]\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

$$\frac{d\ln \Omega}{d\ln \xi} = \frac{(1+w)(\Omega - \Phi)}{(\Phi + w\Omega)}\frac{d\ln U}{d\ln \xi} + \frac{2w}{(\Phi + w\Omega)}\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

$$\frac{d\ln \Phi}{d\ln \xi} = \frac{1}{\Phi}\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

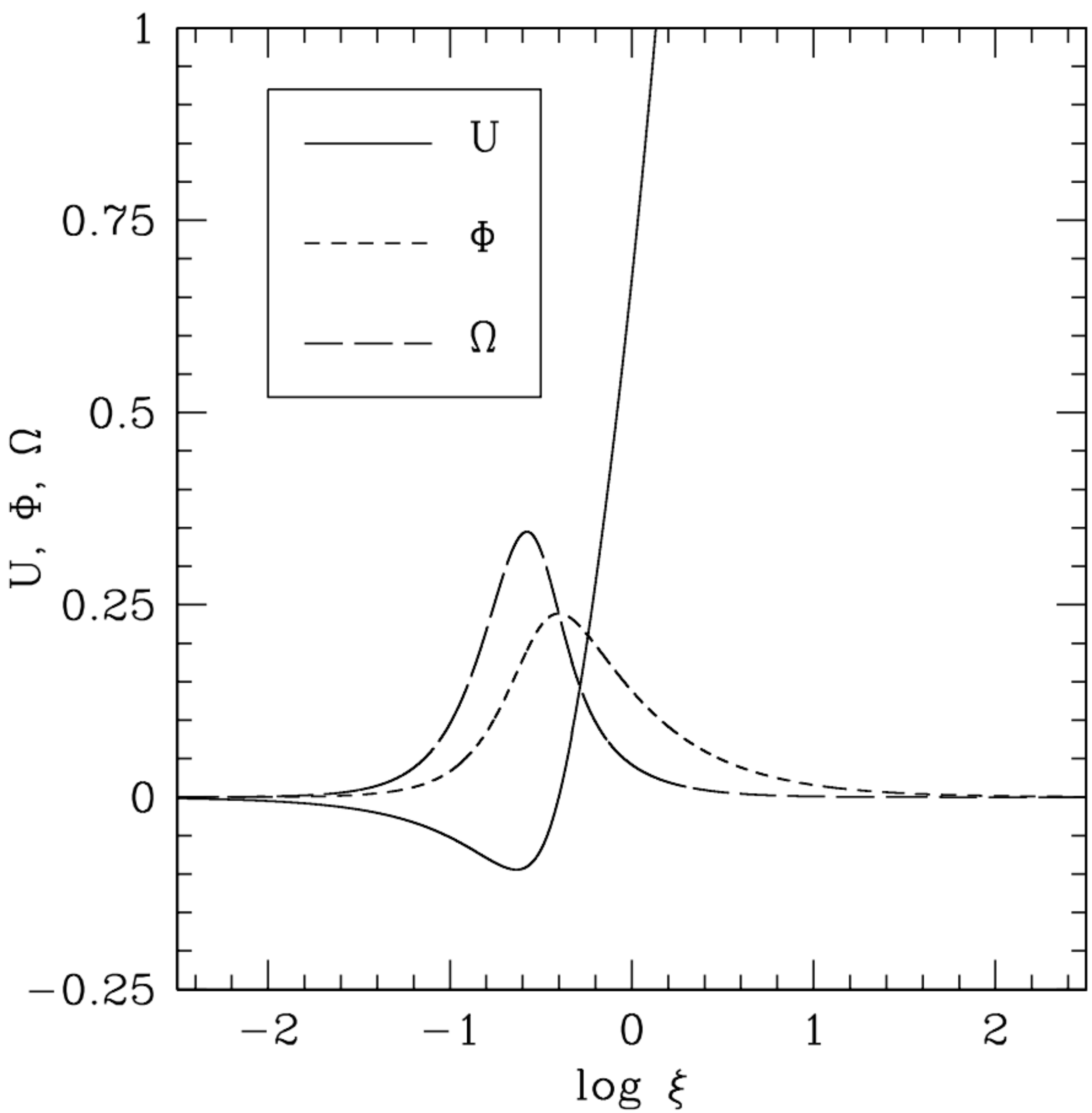
$$\Gamma^2 = 1 + U^2 - 2\Phi$$

Null foliation correction term

Self similar solution

Initial conditions : k shooting parameter

$$\begin{cases} U(\xi) = -\dfrac{2}{3(1+w)}\xi \\ \Phi(\xi) = k\xi^2 \\ \Omega(\xi) = 3k\xi^2 \end{cases}$$



- **The solution is shape independent**
- Φ_{peak} **is slicing independent**
- **Sonic point** (***U = 0***)
- Critical solution just before the formation of the apparent horizon.

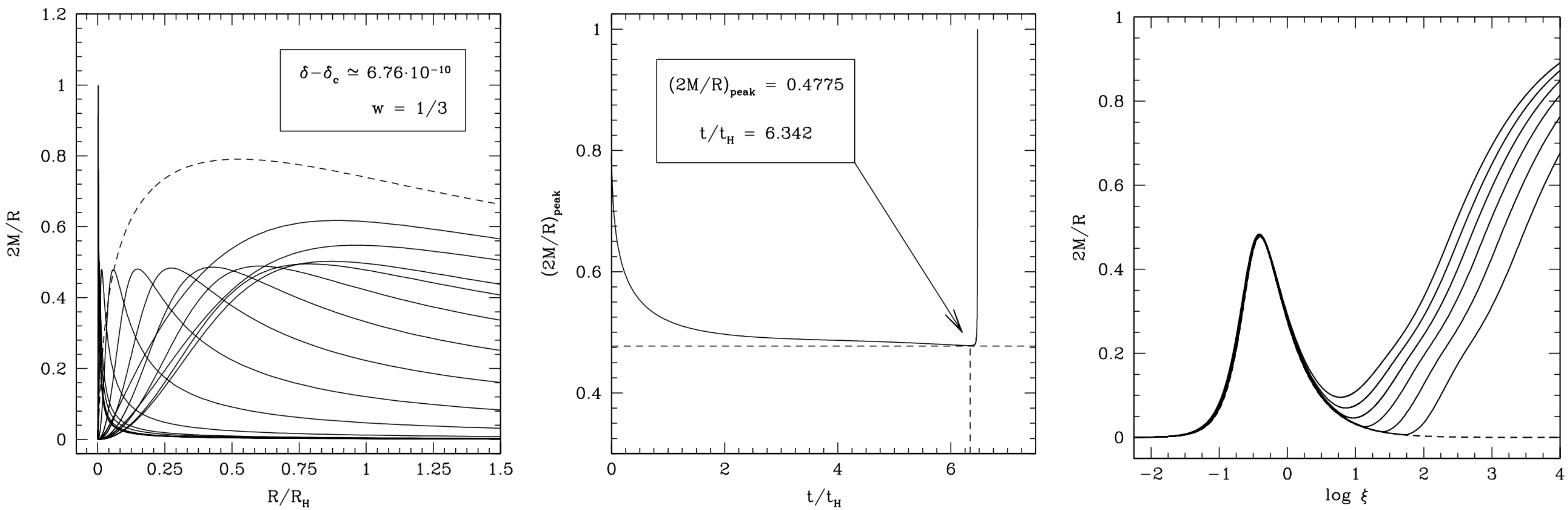
Musco & J. Miller - CQG (2013)
Harada - CQG (2001)

Perturbation amplitude: gradient expansion / self similarity

$$\delta \equiv \frac{2M}{R} - U^2 = 1 - \Gamma^2 \simeq -r\zeta'(r)\left[2 + r\zeta'(r)\right]_{r=r_m}$$

$$0.6 \leq \delta_c(\alpha) \leq 1$$

- **Self similarity** $U = 0 \quad \Rightarrow \quad \delta_c(t/t_H) = \left.\frac{2M}{R}\right|_{\text{peak}} \simeq 0.48$
- At late time all the different definitions of the compaction function converge.



Advantages:

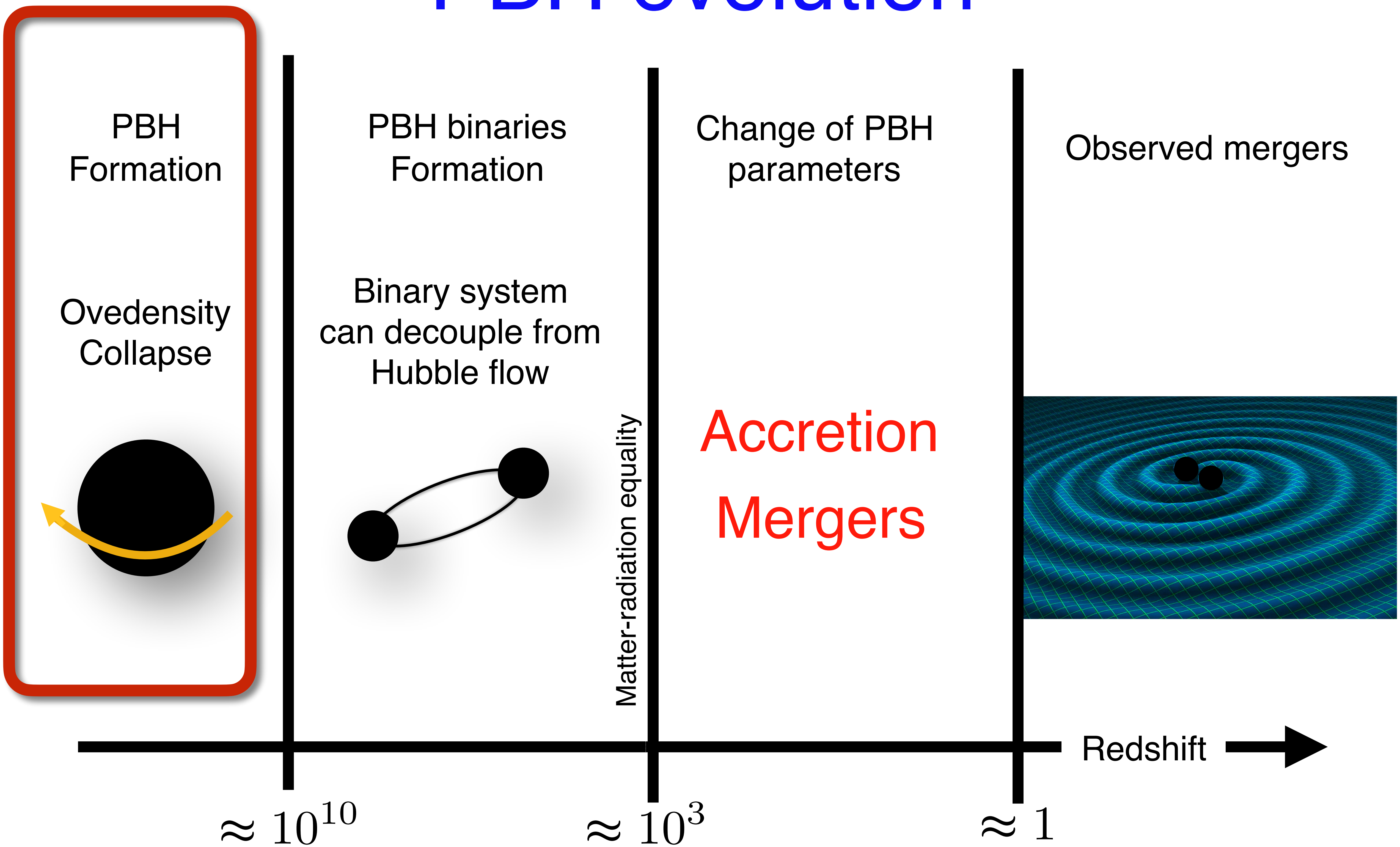
- This alternative definition of the perturbation amplitude is gauge independent, corresponding to the Shibata - Sasaki compaction function.
- In the self similar regime the peak of the compaction function is shape independent. The only dependence left is the equation of state.
- Critical collapse: $w = 1/3 \;\Rightarrow\; \delta_c \simeq 0.48;\; \gamma \simeq 0.36;\; \mathcal{K} \simeq 5\,(\alpha \gtrsim 1)$
- The threshold is computed just before the formation of the apparent horizon, no ambiguities due to the definition of the horizon crossing.

Possible drawback:

- We do not know how to compute the statistic within the horizon. We need to compute the **non linear transfer function** for a generic broad power spectrum (*IM & Young* - in progress).

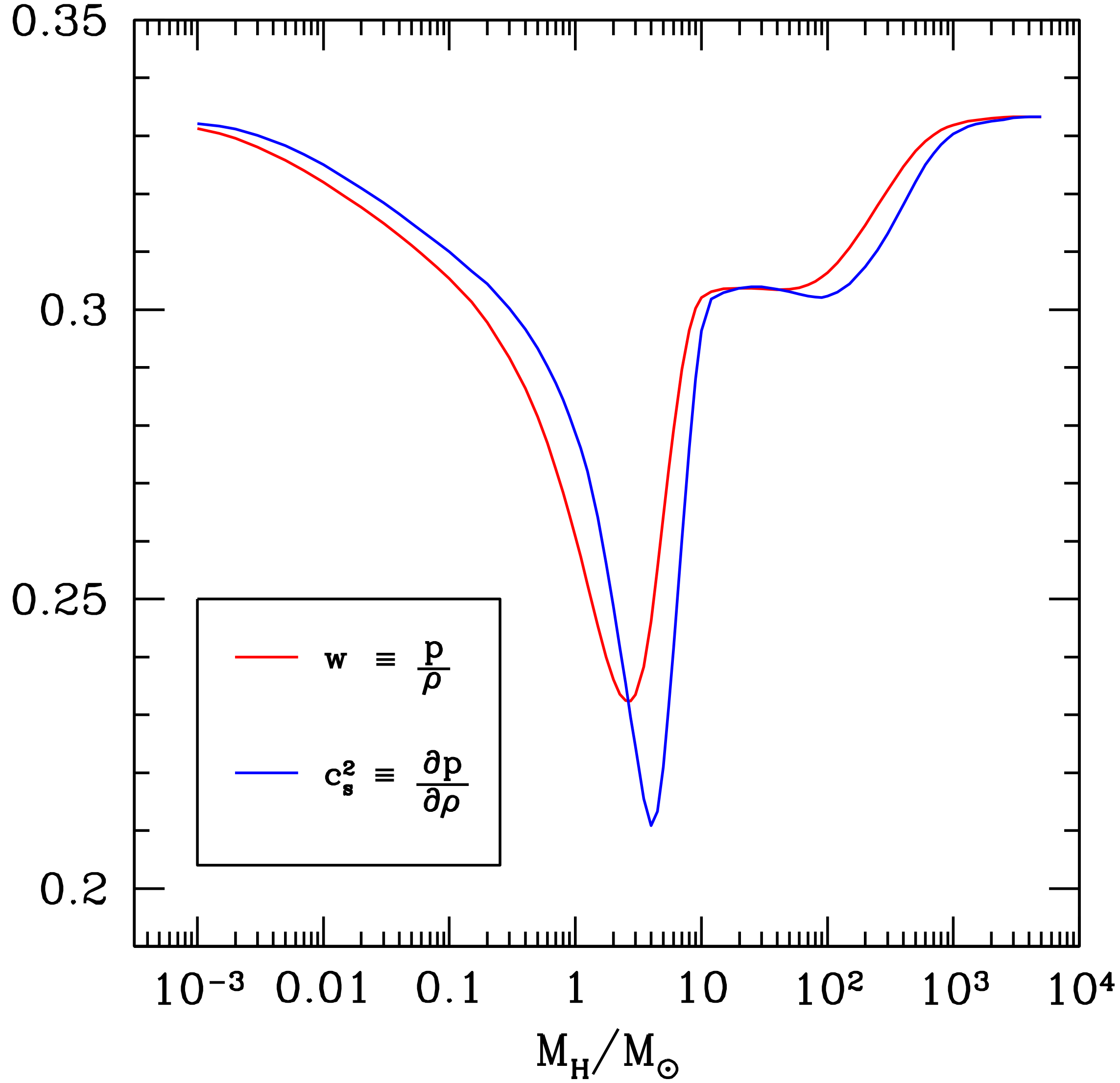
Conclusions: The calculation of the variance should take into account the effect of the shape. If we can find a consistent prescription we will have a clear prescription to compute PBH abundance.

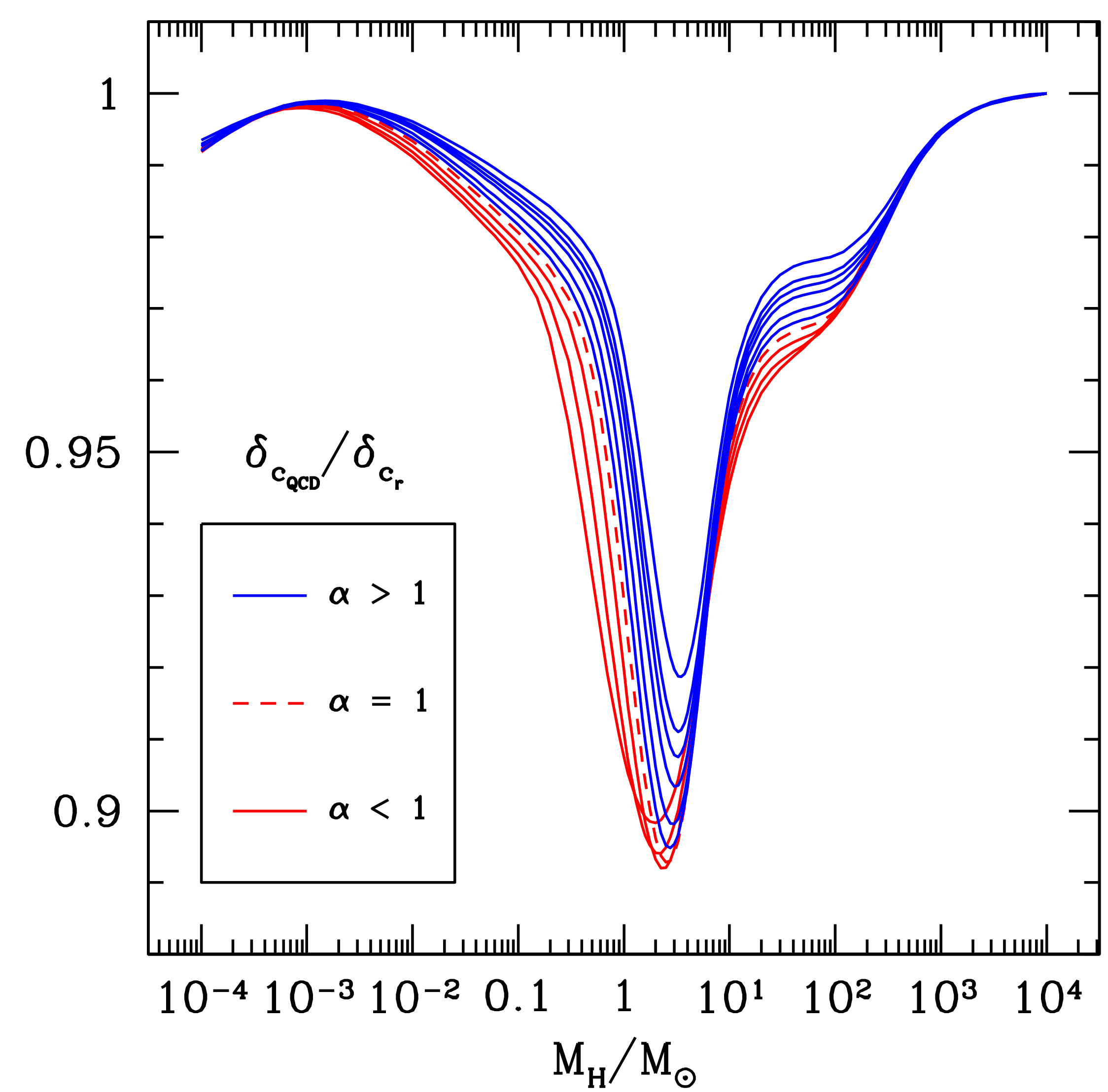
PBH evolution



PBH Threshold during the QCD

IM, K. Jedamzik, Sam Young - PRD (2024)

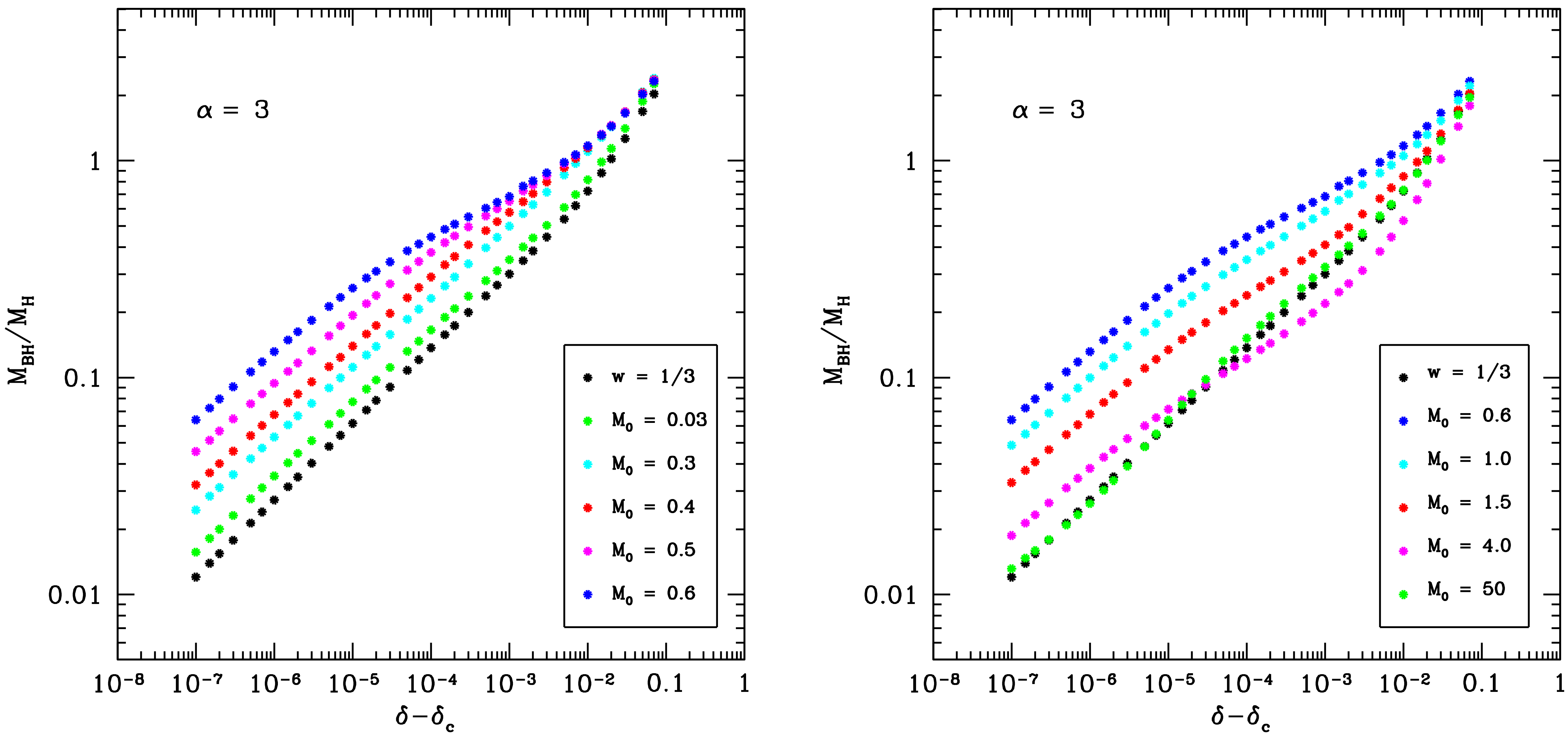




Depending on the shape, the threshold for PBH formation during the QCD phase transition is reduced about 10% around the minimum of ***w(T)*.**

Significant enhancement of PBH formation around the solar mass scale: abundance increased of about O(3) with respect radiation!

PBH scaling law during the QCD



$$M_{PBH} = \mathcal{K}(\delta - \delta_c)^\gamma M_H$$

$$\delta_c(M_H), \gamma(M_H), \mathcal{K}(M_H)$$

IM, K. Jedamzik, Sam Young - PRD (2024)

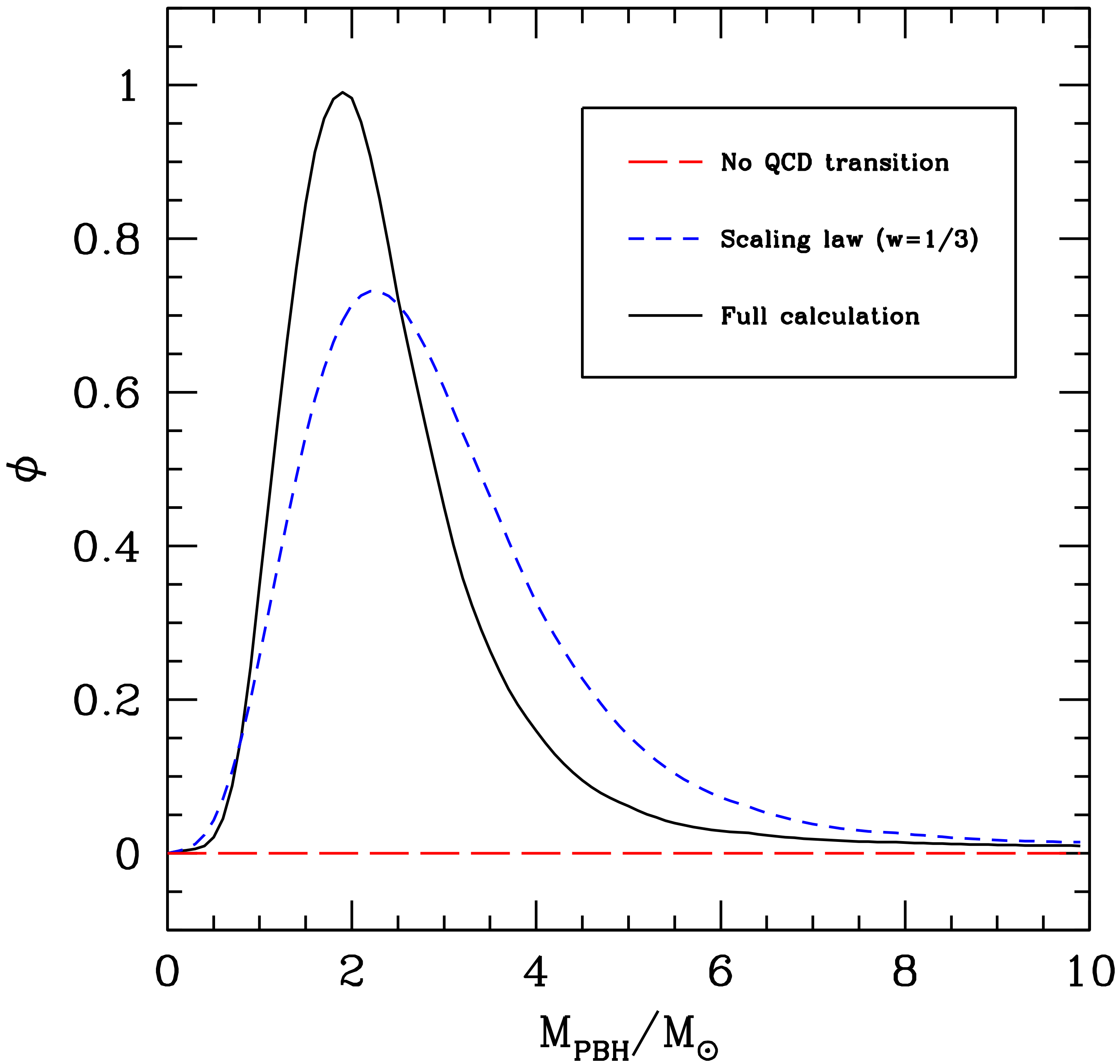
PBH mass function during the QCD

Mass Function ψ(mPBH): fraction of PBHs with mass in the infinitesimal interval of M_{PBH}

$$\phi(M_{\rm PBH}) = \frac{1}{\Omega_{\rm PBH}} \frac{d\Omega_{\rm PBH}}{dM_{\rm PBH}}$$

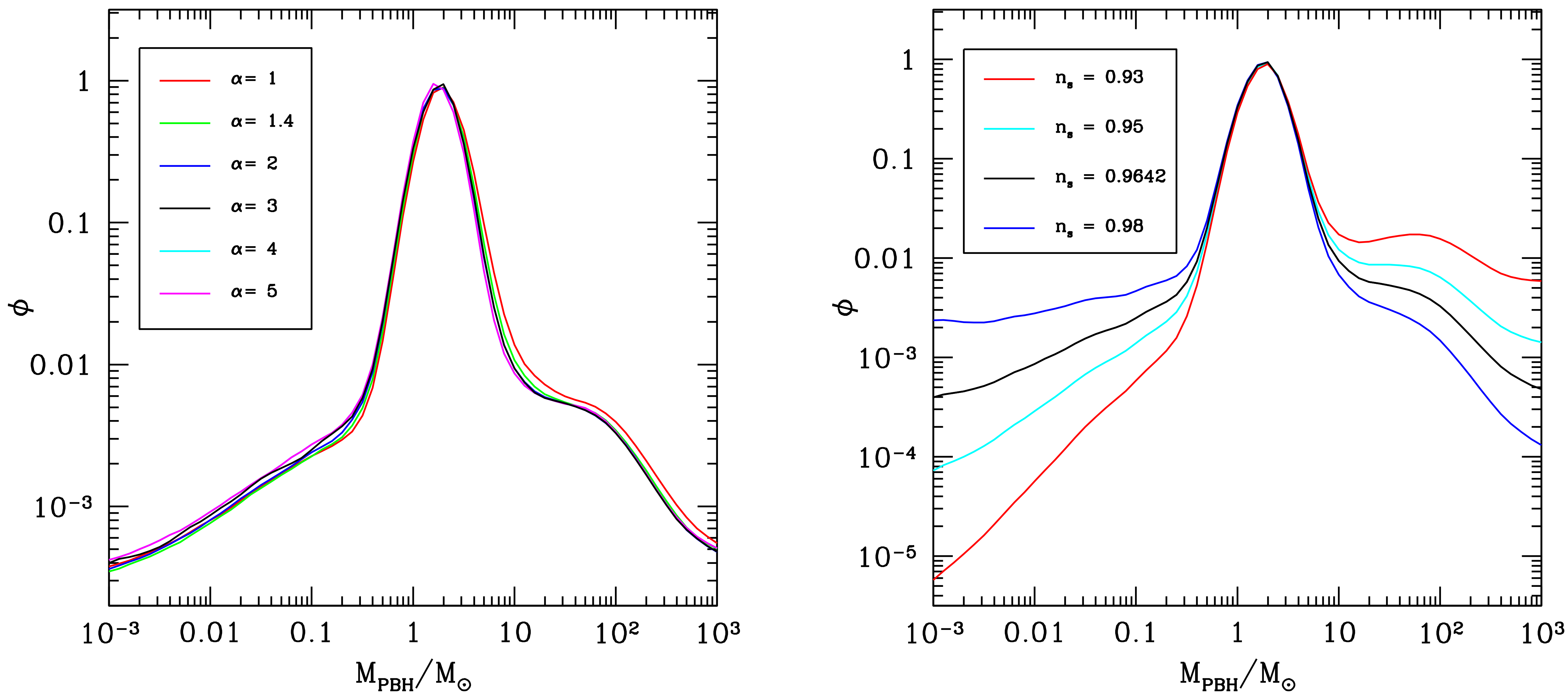
$$\int dm_{\rm PBH}\ \phi(m_{\rm PBH}) = 1$$

- The main effect is given by the modification of the threshold.
- The modified scaling law gives a pile up of PBHs on smaller masses.



IM, K. Jedamzik, Sam Young - PRD (2024)

PBH mass function during the QCD: shape/tilt dependence



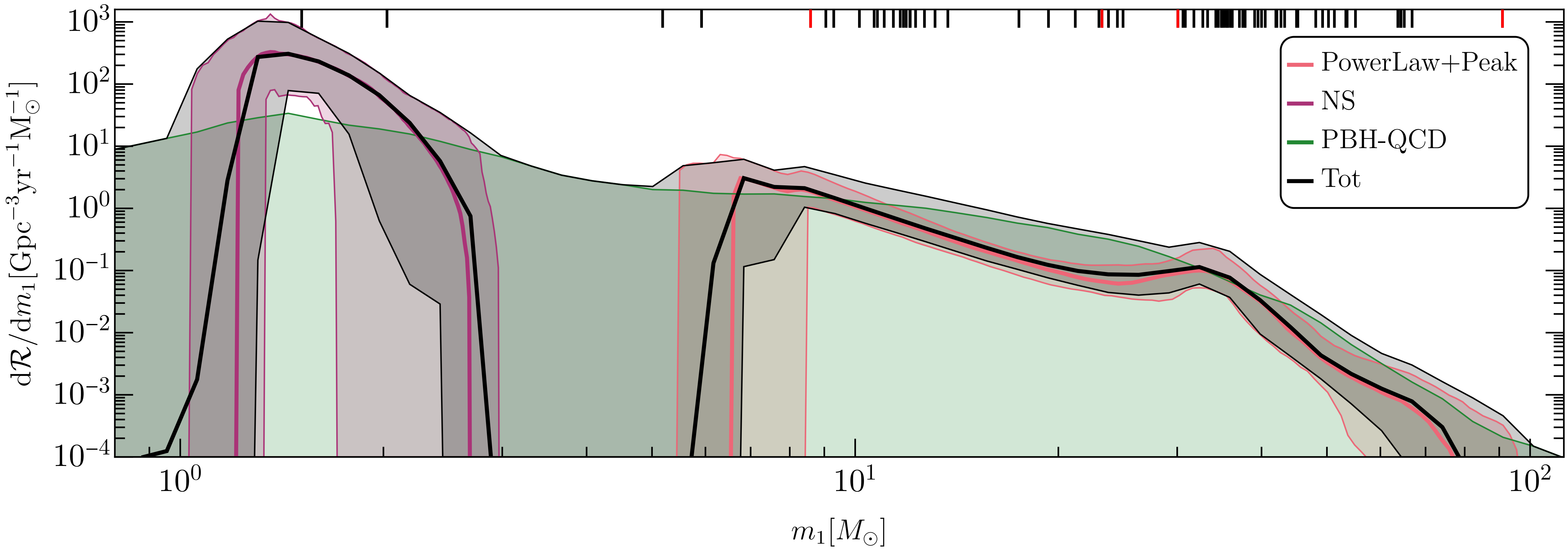
- Given the PBH abundance, the shape does not play a significant role on the mass function (attractor solution)!
- The tilt of the power spectrum does not affect the peak of the mass function.

IM, K. Jedamzik, Sam Young - PRD (2024)

GWs from PBH mergers

G. Franciolini, IM, P.Pani, A urbano - PRD (2022)

- Making **Bayesian inference analysis** we found that a sub-population of PBHs is compatible with the LVK catalog.
- PBHs give a natural explanation for the events in with BH mass gap: in particular GW190814 falling within the lower mass gap (predictions for O4 and O5).



GW event	PBH prob. [%]	$m_1[M_\odot]$	$m_2[M_\odot]$
GW151012	1.2	$23.2^{+14.9}_{-5.5}$	$13.6^{+4.1}_{-4.8}$
GW190412	25.4	$30.1^{+4.7}_{-5.1}$	$8.3^{+1.6}_{-0.9}$
GW190512_180714	1.6	$23.3^{+5.3}_{-5.8}$	$12.6^{+3.6}_{-2.5}$
GW190519_153544	1.5	$66.0^{+10.7}_{-12.0}$	$40.5^{+11.0}_{-11.1}$
GW190521	7.2	$95.3^{+28.7}_{-18.9}$	$69.0^{+22.7}_{-23.1}$
GW190602_175927	2.7	$69.1^{+15.7}_{-13.0}$	$47.8^{+14.3}_{-17.4}$
GW190701_203306	1.4	$53.9^{+11.8}_{-8.0}$	$40.8^{+8.7}_{-12.0}$
GW190706_222641	1.3	$67.0^{+14.6}_{-16.2}$	$38.2^{+14.6}_{-13.3}$
GW190828_065509	2.8	$24.1^{+7.0}_{-7.2}$	$10.2^{+3.6}_{-2.1}$
GW190924_021846	40.3	$8.9^{+7.0}_{-2.0}$	$5.0^{+1.4}_{-1.9}$
GW191109_010717	2.9	65^{+11}_{-11}	47^{+15}_{-13}
GW191129_134029	1.2	$10.7^{+4.1}_{-2.1}$	$6.7^{+1.5}_{-1.7}$
GW190425	2.8	$2.0^{+0.6}_{-0.3}$	$1.4^{+0.3}_{-0.3}$
GW190426_152155	1.2	$5.7^{+3.9}_{-2.3}$	$1.5^{+0.8}_{-0.5}$
GW190814	29.1	$23.2^{+1.1}_{-1.0}$	$2.59^{+0.08}_{-0.09}$
GW190917_114630	3.0	$9.3^{+3.4}_{-4.4}$	$2.1^{+1.5}_{-0.5}$
GW200105_162426	3.6	$8.9^{+1.2}_{-1.5}$	$1.9^{+0.3}_{-0.2}$
GW200115_042309	1.2	$5.9^{+2.0}_{-2.5}$	$1.44^{+0.85}_{-0.29}$

PBH - DM constraints

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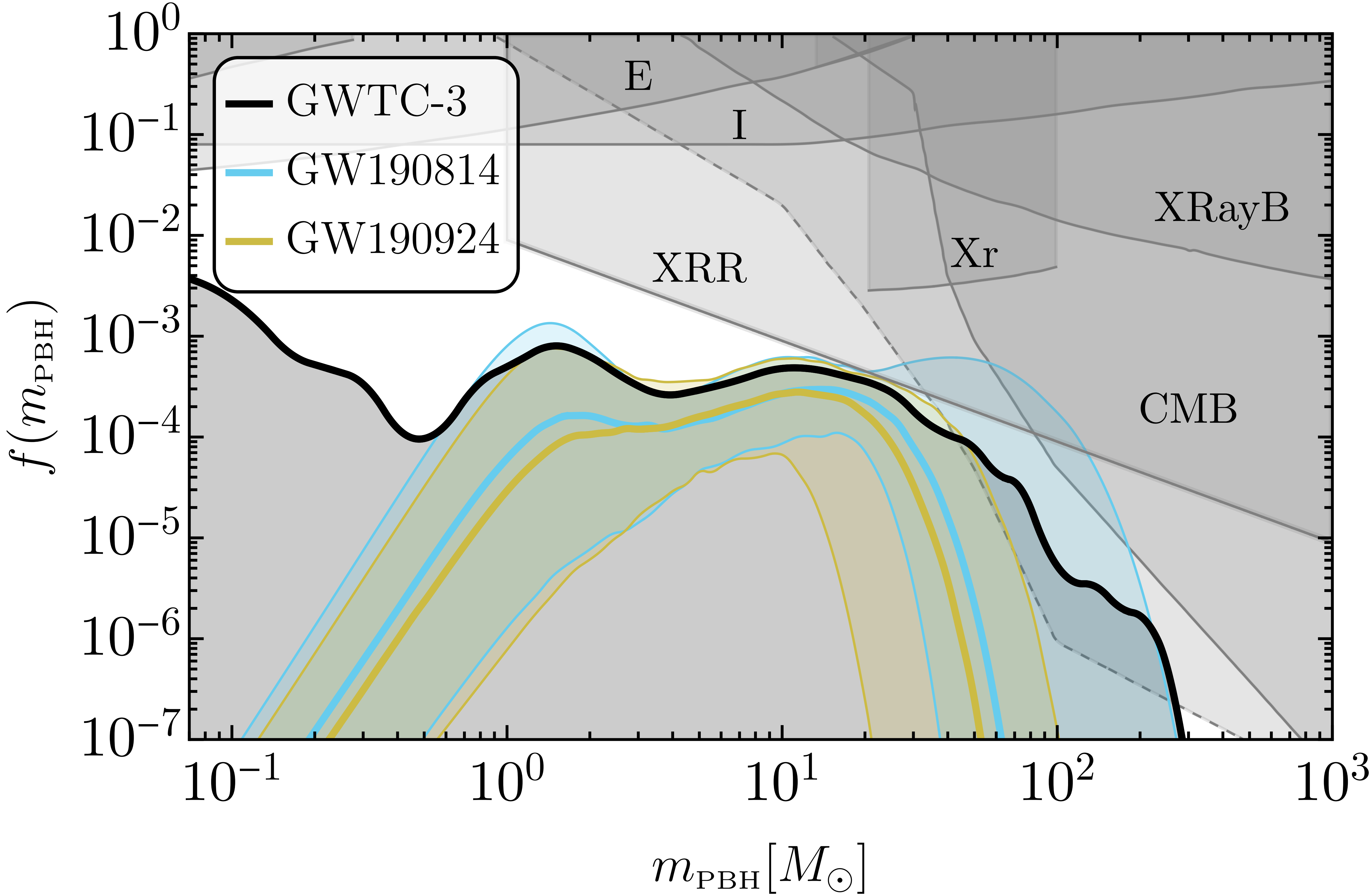
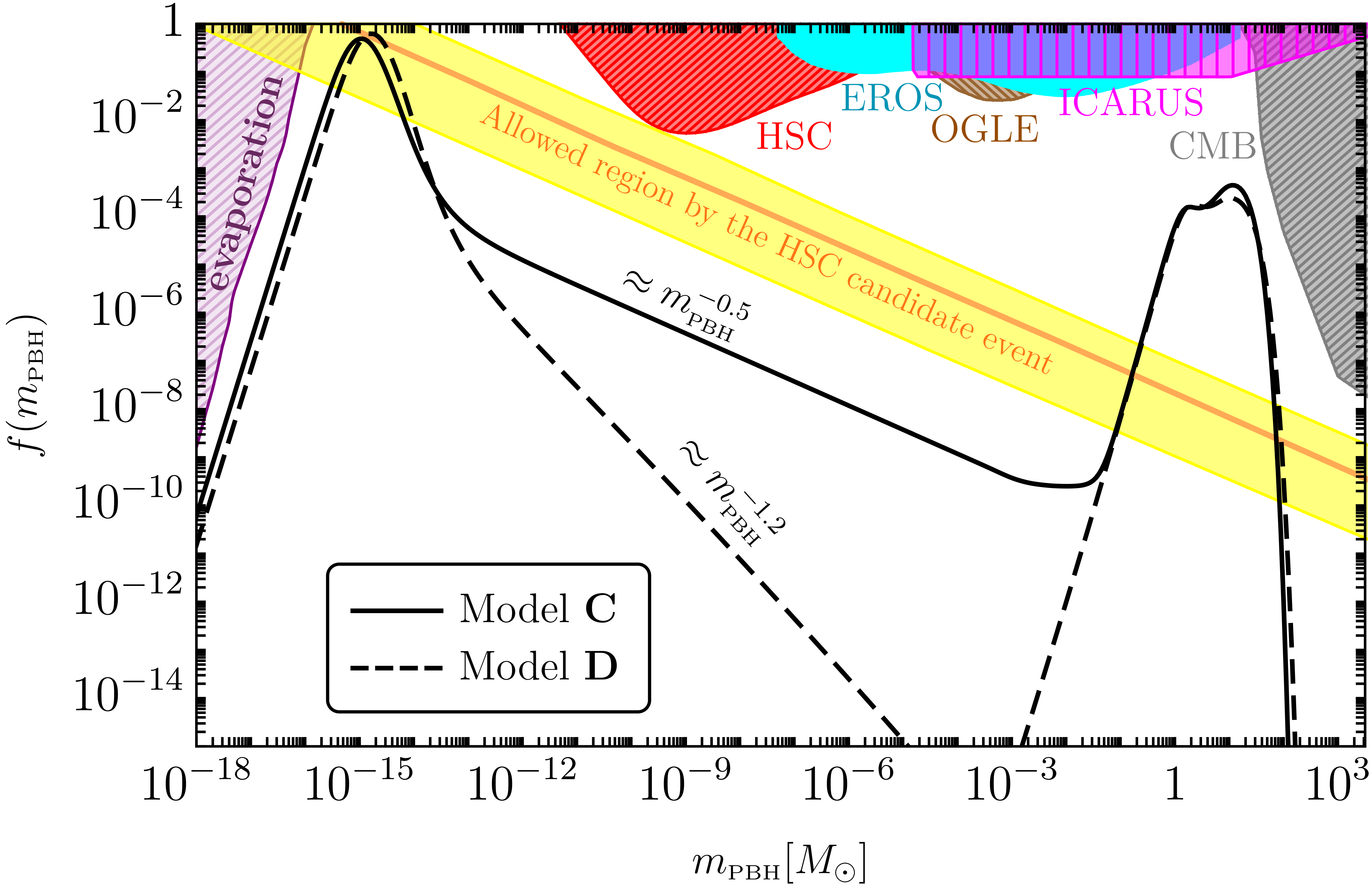


PBHs and Dark Matter (asteroidal mass)



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Conclusions

- The **non linear threshold for PBH** and **the mass function** could be fully computed from the **shape of the power spectrum of cosmological perturbations**, making relativistic numerical simulations.

- A softening of the equation of state **(QCD) significantly enhances the formation of PBHs**, with a **mass distribution peaked between 1 and 2 solar masses** (the range of heavy NSs and light BHs).

- This could gives **a sub-population of BH mergers compatible with the LVK catalog**, explaining mass gap events as **GW190814**.

- Our analysis predicts a **constraint on the abundance of DM in PBHs formed during the QCD (up to 0.1%)**, compatible with the current observational constraints.

- **A large enough feature of the power spectrum could account for all dark matter in PBHs in the asteroidal mass range** (USR inflation models).