Numerical Simulations of (Axion) Inflation

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Axions in Stockholm, Nordita, Stockholm, Sweden, Europe, Earth, Solar System, ...

Lattice l Simulations of (Axion) Inflation

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Lattice Simulations
of (Axion) Inflation

LATTICE
COSMOLOGY
- techniques -

LATTICE
COSMOLOGY

LATTICE COSMOLOGY

Techniques to Simulate Early Universe dynamics

LATTICE
COSMOLOGY

Techniques to Simulate Early Universe dynamics

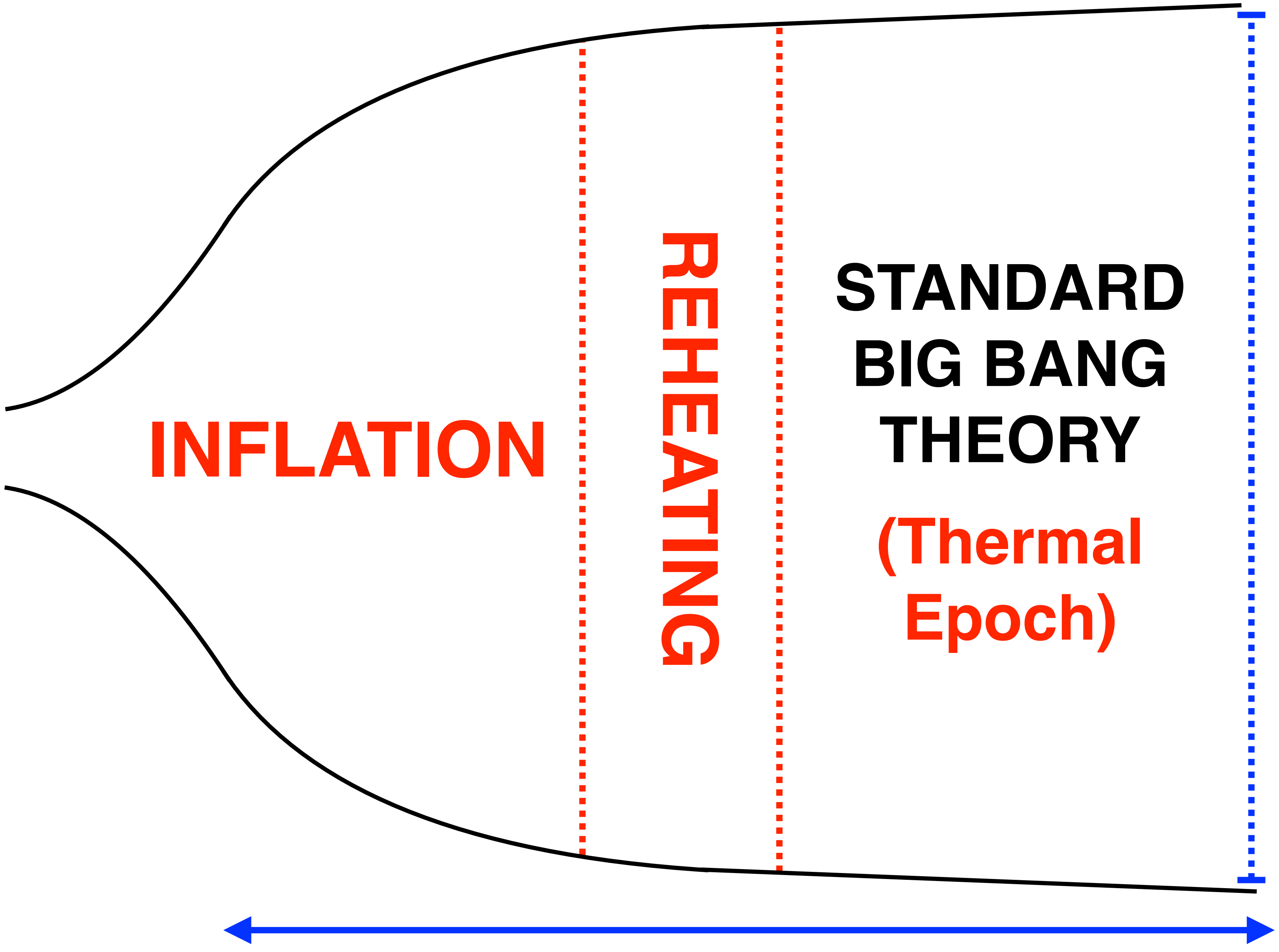
(When do we need to simulate it ?)

LATTICE
COSMOLOGY

Techniques to Simulate Early Universe dynamics

When things get complicated:
non-linear, strong coupling,
non-perturbative, etc

The Early Universe



INFLATION

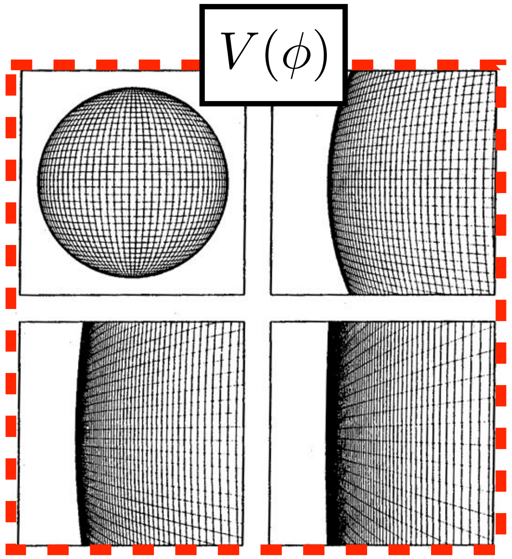
REHEATING

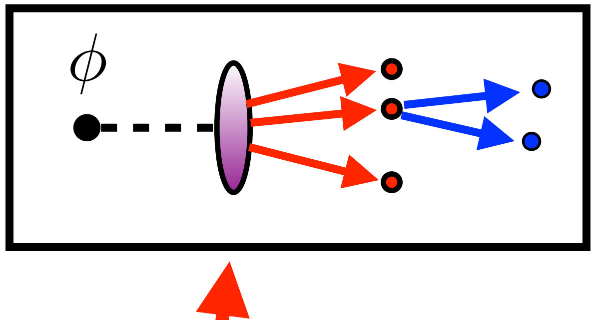
STANDARD BIG BANG THEORY

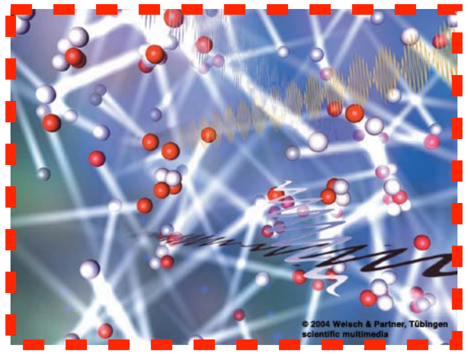
(Thermal Epoch)

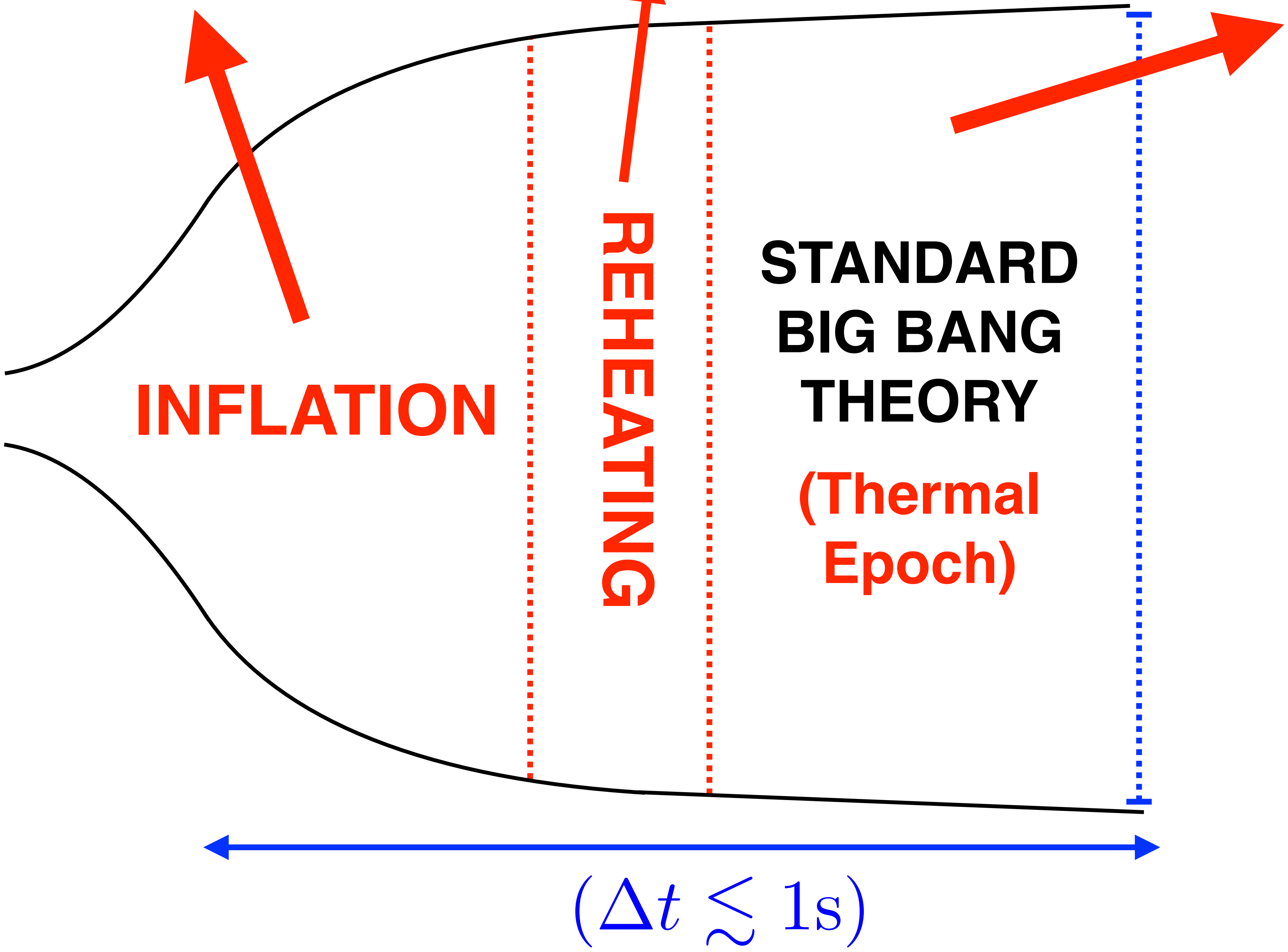
$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe









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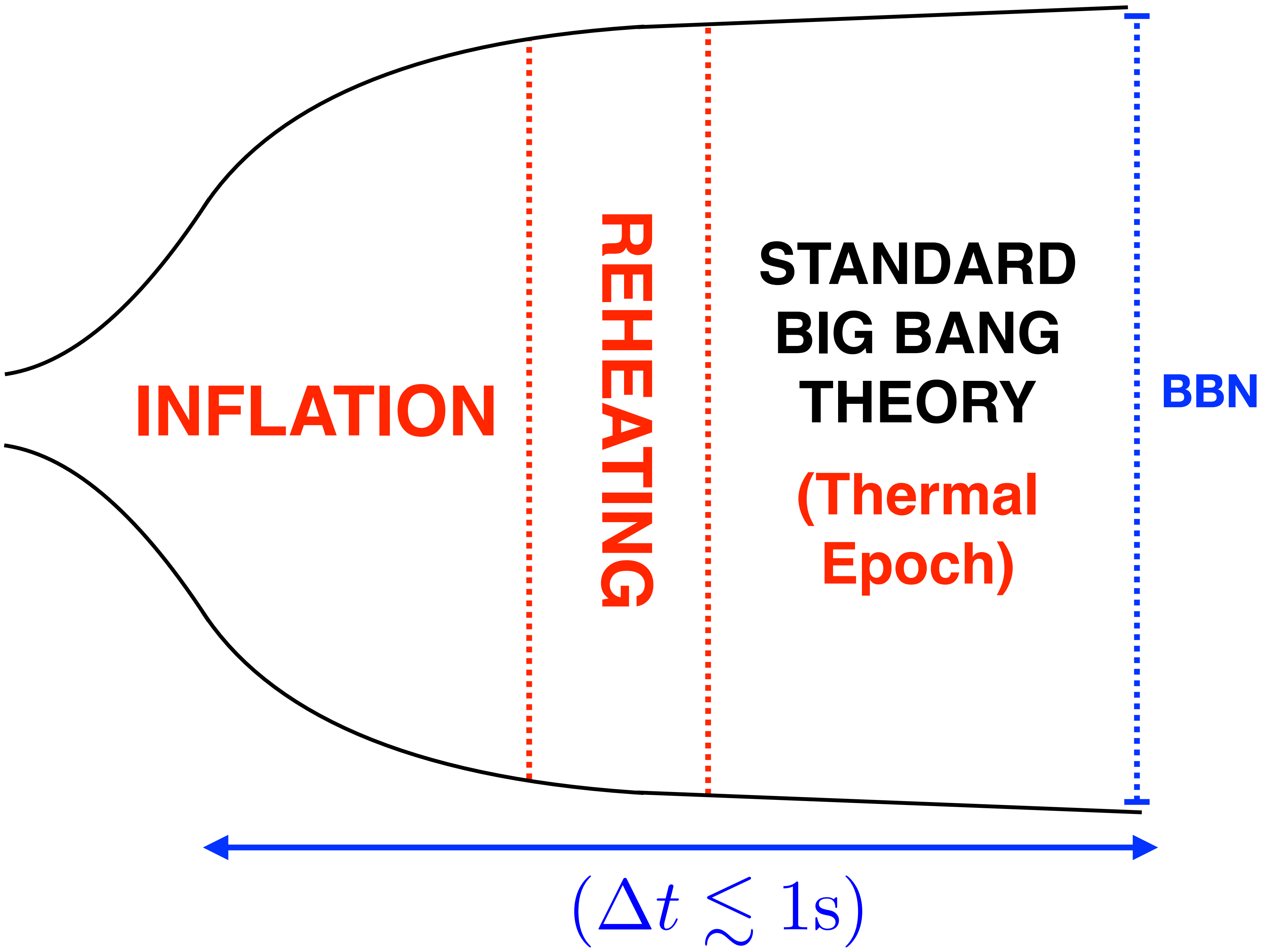
The Early Universe



INFLATION

REHEATING

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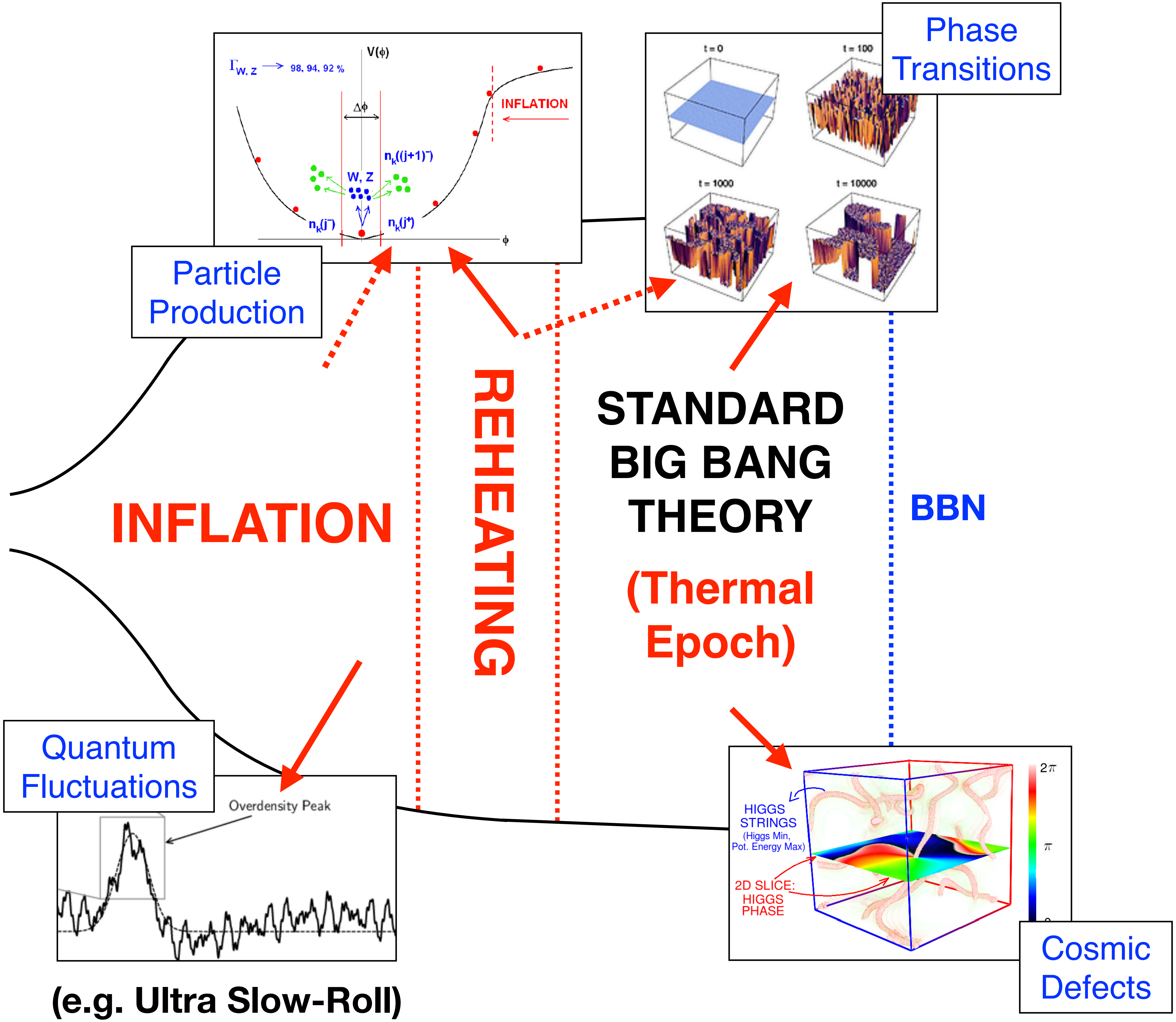
(Thermal Epoch)

BBN

$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



Particle Production

Phase Transitions

INFLATION

REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

BBN

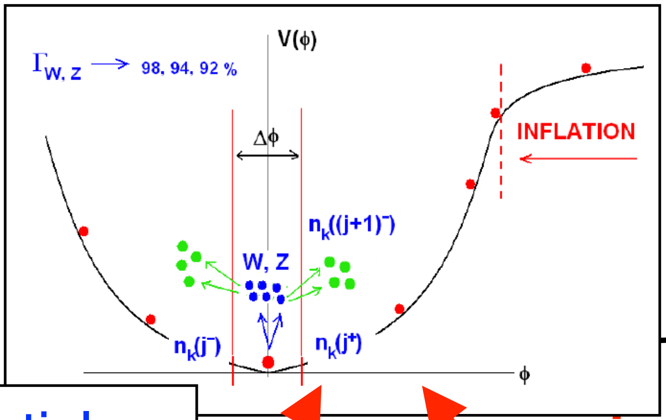
Quantum Fluctuations

(e.g. Ultra Slow-Roll)

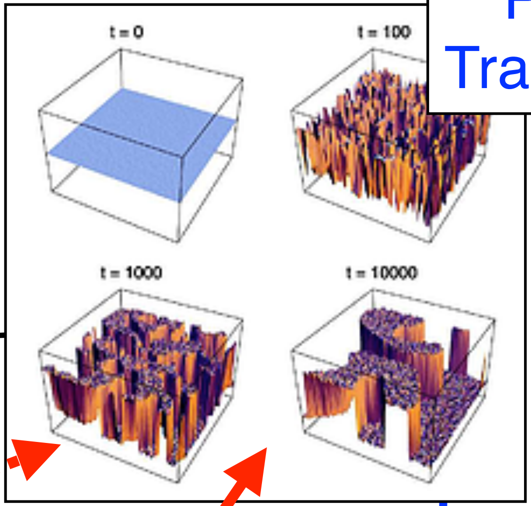
Cosmic Defects

The Early Universe

Particle Production



Phase Transitions



INFLATION

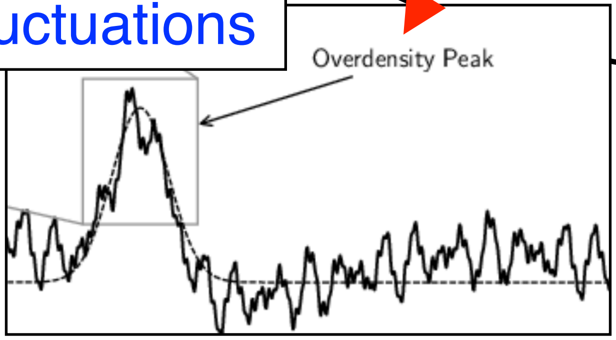
REHEATING

STANDARD BIG BANG THEORY

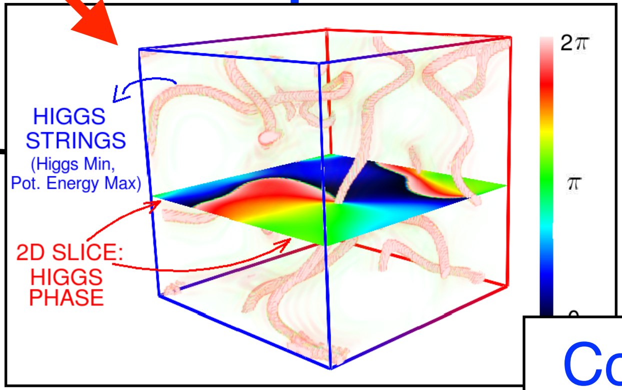
(Thermal Epoch)

Common Factor ?

Quantum Fluctuations

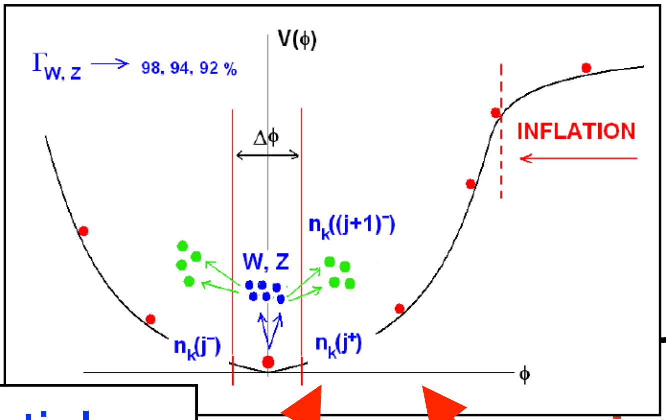


(e.g. Ultra Slow-Roll)

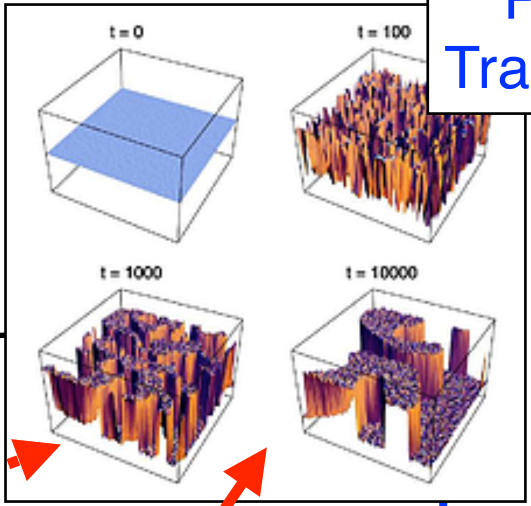


Cosmic Defects

The Early Universe



Particle Production



Phase Transitions

INFLATION

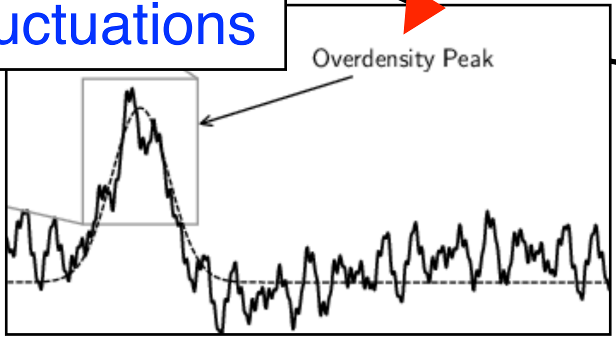
REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

Non-linear field dynamics

Quantum Fluctuations



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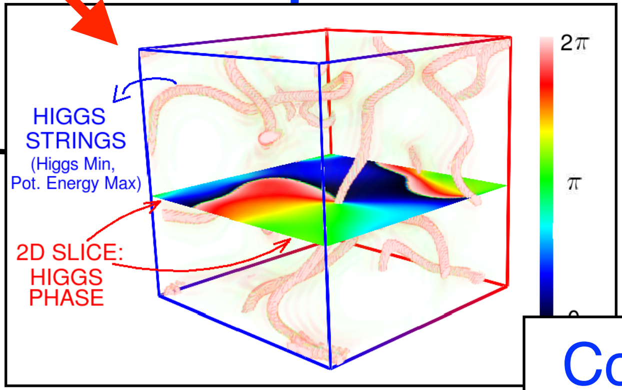

Cosmic Defects

The Early Universe

Particle Production

Phase Transitions

Non-linear field dynamics

Curvature Fluctuations

Cosmic Defects

The Early Universe

Particle Production

Phase Transitions

Non-linear field dynamics

Curvature Fluctuations

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-linear field dynamics

Curvature Fluctuations

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Baryo-genesis

Non-linear field dynamics

Curvature Fluctuations

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Baryo-genesis

Non-linear field dynamics

Magneto-genesis

Curvature Fluctuations

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Baryogenesis

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Curvature Fluctuations

Black Hole Formation

Cosmic Defects

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Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Non-linear field dynamics

Magneto-genesis

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Cosmic Defects

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Baryo-genesis

Non-linear field dynamics

Non-minimal Kinetic Theories

Magneto-genesis

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Baryogenesis

Non-linear field dynamics

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Magnetogenesis

Turbulence
Thermalisation
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Non-linear

≡ Numerical simulations

Non-minimal Kinetic Theories

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Baryo-genesis

Non-linear

≡ Lattice

simulations

Non-minimal Kinetic Theories

Magneto-genesis

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Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Latticeeasy / Clustereasy, Defrost,

CUDAEasy, HLattice, PyCOOL,

GABE, PSpectRe, Stella, GFiRe, …

+ GRChombo, GABERel, …

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation ….

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

~ 10-25 yrs

The Early Universe

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Magneto-genesis

+ GRCho

(Scalar) Field Theory Codes

Curvature Fluctuations

Bla

For

~ 10-25 yrs

The Early Universe

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Baryo-genesis

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Non-minimal Kinetic Theories

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Turbulence Thermalisation ….

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Cosmic Defects

~ 5-10 yrs

The Early Universe

Gravitational waves

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Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Gen. Rel. (GR) Codes

Non-minimal Kinetic Theories

Magneto-genesis

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Turbulence Thermalisation ….

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Magneto-genesis

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Turbulence
Thermalisation
….

Curvature Fluctuations

+ PENCIL, …

Black Hole Formation

Cosmic Defects

~ 5-10 yrs

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

MHD Codes (Fluid Dynamics)

GRChombo, GABERel, …

Non-minimal Kinetic Theories

Magneto-genesis

+ PENCIL, …

Turbulence Thermalisation
….

Curvature Fluctuations

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Cosmic Defects

~ 5-10 yrs

The Early Universe

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Non-minimal Kinetic Theories

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Cosmic Defects

~ 10-25 yrs

The Early Universe

Lattice Cosmology

Gravitational waves

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CosmoLattice

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Magneto-genesis

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Curvature Fluctuations

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Cosmic Defects

~ 4 yrs

The Early Universe

Gravitational waves

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Baryo-genesis

CosmoLattice

+ CosmoLattice-MHD

+ CosmoLattice-GR

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation
….

Curvature Fluctuations

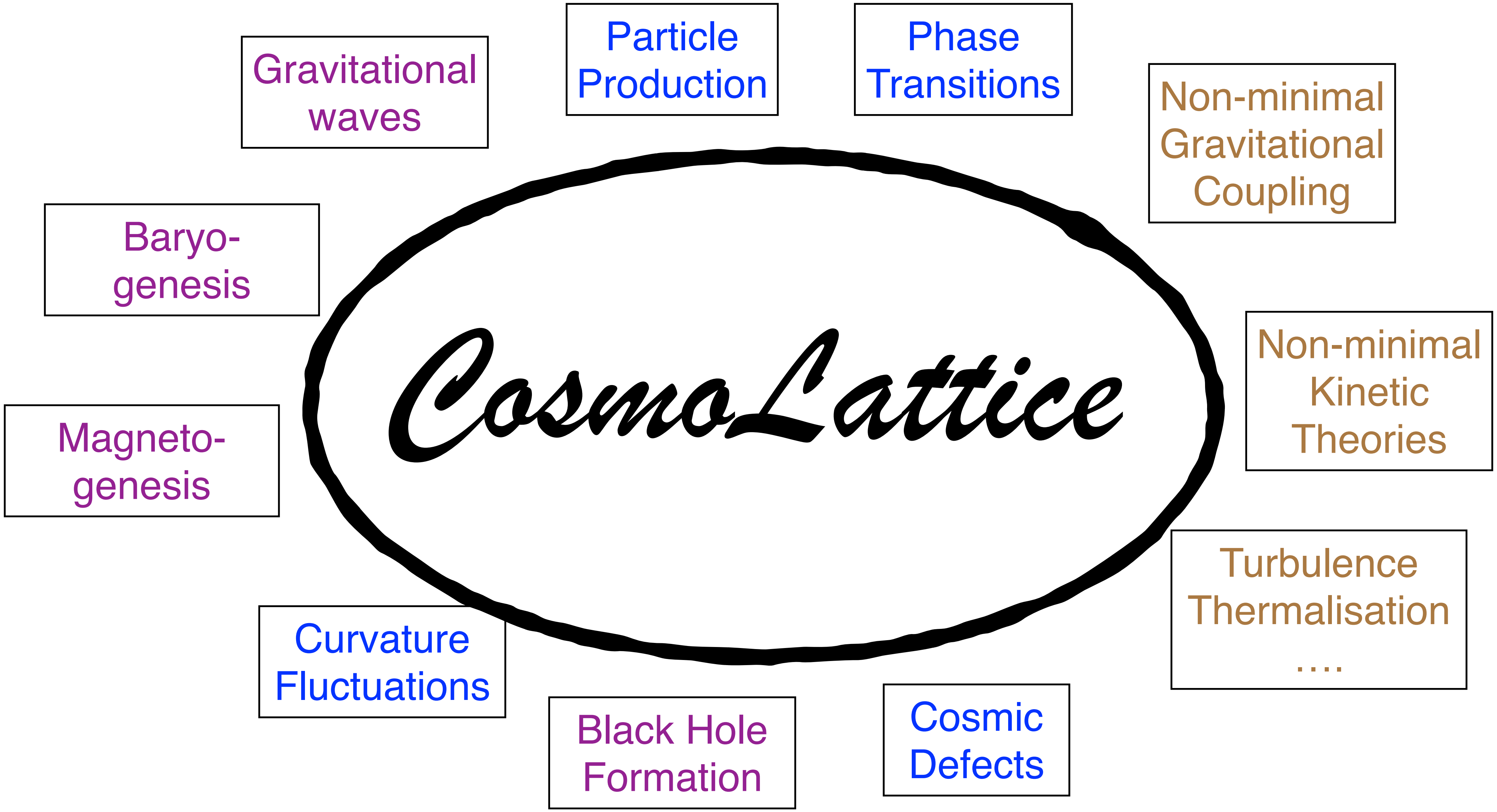
Black Hole Formation

Cosmic Defects

~ 4 yrs

[+ upcoming]

The Early Universe



CosmoLattice

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Non-minimal Kinetic Theories

Turbulence Thermalisation ….

Cosmic Defects

Black Hole Formation

Curvature Fluctuations

Magneto-genesis

Baryo-genesis

~ 4 yrs

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory: arXiv: 2006.15122 (+100 pages)

Code Manual: arXiv: 2102.01031 (+100 pages)

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

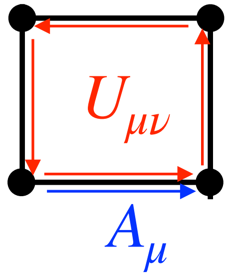
Lattice Theory: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar-gauge field dynamics** [w. **self-consistent** expanding background]

[U(1) x SU(2)]

Links & plaquettes
(~ lattice-QCD)



CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- **Parallellized** in multiple spatial dimensions (**but you write in serial !**)

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- **Family** of evolution **algorithms**, accuracy ranging from $\delta\mathcal{O}(\delta t^2) - \delta\mathcal{O}(\delta t^{10})$

  [*LeapFrog, Verlet, Runge-Kutta, Yoshida, …*]

CosmoLattice

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http://www.cosmolattice.net/

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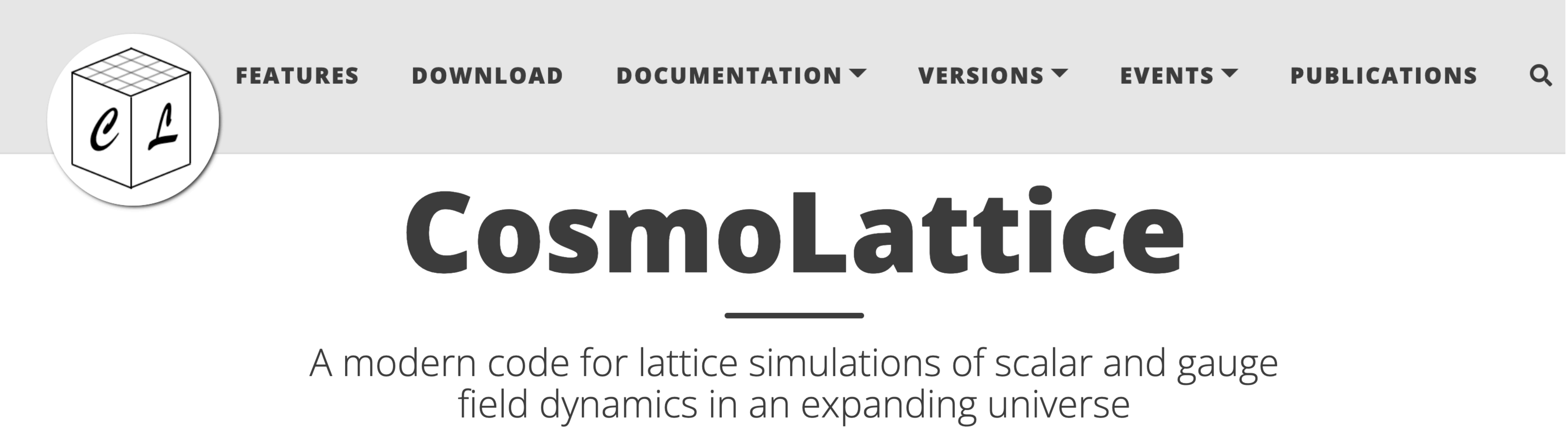

CosmoLattice

A modern code for lattice simulations of scalar and gauge field dynamics in an expanding universe

CosmoLattice – Default Field Content

- **Matter content:**

$$\boxed{S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}}$$

CosmoLattice – Default Field Content

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$$\boxed{\phi \in \mathscr{R}e}$$

Scalar sector

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$
$$D_\mu^{\mathrm{A}} \equiv \partial_\mu - iQ_A^{(\varphi)}g_{_A}A_\mu$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

U(1) gauge sector

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1 \\ \varphi_2 + i\varphi_3\end{pmatrix}$$
$$D_\mu \equiv \mathscr{I}D_\mu^{\mathrm{A}} - ig_BQ_BB_\mu^aT_a$$
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SU(2) gauge sector

Scalar potential

CosmoLattice – Default Field Content

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$$\boxed{\phi \in \mathscr{Re}}$$

Scalar sector

$$\boxed{\begin{aligned} \varphi &\equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1) \\ D_\mu^{\mathrm{A}} &\equiv \partial_\mu - iQ_A^{(\varphi)}g_A A_\mu \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}}$$

U(1) gauge sector

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SU(2) gauge sector

Scalar potential

- Background Metric:

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j \left\{\begin{array}{l} \text{➤ } \textbf{Self-consistent expansion} \text{ (Friedmann equations)} \\ \text{➤ } \textbf{Fixed power-law background} \quad a(t) \sim t^{\frac{2}{3(1+w)}} \end{array}\right.$$

CosmoLattice – Default Field Content

- **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \underbrace{(D_\mu^{A}\varphi)^*(D_A^{\mu}\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \underbrace{(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\}} + V(\phi,|\varphi|,|\Phi|)\right\}$$

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Scalar potential

- **Background Metric:**

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j \;\Bigg\{$$

- **Self-consistent expansion** (Friedmann equations)
- **Fixed power-law background** $a(t) \sim t^{\frac{2}{3(1+w)}}$

CosmoLattice – Equations of Motion

- **Hamiltonian scheme**: coupled first-order differential equations

Scalar fld example

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial \phi} \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

KICK: $(\pi_\phi)' = -a^{3+\alpha}\frac{\partial V}{\partial \phi} + a^{1+\alpha}\nabla^2\phi$

DRIFT: $\phi' \equiv \pi_\phi a^{\alpha-3}$

CosmoLattice – Equations of Motion

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- **Scalar fld example**

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial \phi} \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

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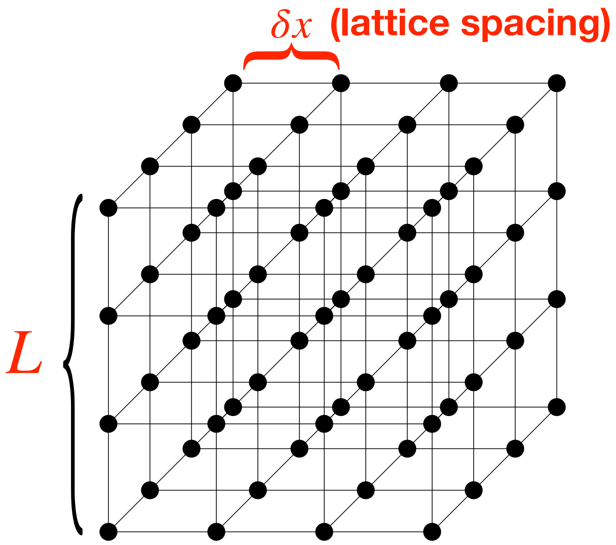
DRIFT: $\phi' \equiv \pi_\phi a^{\alpha-3}$

- **Scalar Fields and momenta** are defined in the **lattice sites**



N : number of points/dimension

$L = N \cdot \delta x$: length side

δt : time step

$\rightarrow$

Minimum and maximum momenta:

$$k_{\rm min} = \frac{2\pi}{L} \qquad k_{\rm max} = \frac{\sqrt{3}}{2}N k_{\rm min}$$

CosmoLattice – Equations of Motion

- **Hamiltonian scheme**: coupled first-order differential equations

Scalar fld example

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial \phi} \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

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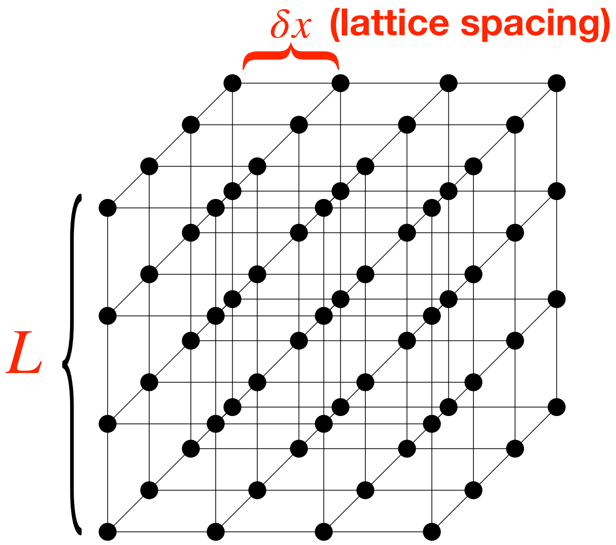

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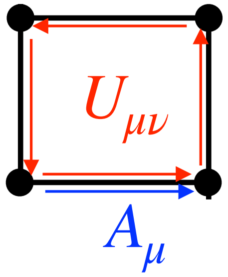
δt : time step

Minimum and maximum momenta:

$k_{\text{min}} = \frac{2\pi}{L}$ $\quad k_{\text{max}} = \frac{\sqrt{3}}{2}Nk_{\text{min}}$

- **Gauge fields** introduced via **links** and **plaquettes** (like in **lattice-QCD**)



CosmoLattice – Expansion Evolution

- Algorithms use **second Friedmann equation** to **evolve the scale factor**.
- The **first Friedmann equation** is used to check the accuracy of the simulation.

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3m_p^2}\langle(\alpha-2)(K_\phi+K_\varphi+K_\Phi)+\alpha(G_\phi+G_\varphi+G_\Phi)+(\alpha+1)V + (\alpha-1)(K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)})\rangle$$

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3m_p^2}\langle K_\phi+K_\varphi+K_\Phi+G_\phi+G_\varphi+G_\Phi+K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)}+V\rangle$$

$\langle ... \rangle$ represents volume averaging

CosmoLattice – Expansion Evolution

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$$\frac{a''}{a} = \frac{a^{2\alpha}}{3m_p^2}\Big\langle(\alpha-2)(K_\phi+K_\varphi+K_\Phi)+\alpha(G_\phi+G_\varphi+G_\Phi)+(\alpha+1)V \\ +(\alpha-1)(K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)})\Big\rangle$$

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3m_p^2}\langle K_\phi+K_\varphi+K_\Phi+G_\phi+G_\varphi+G_\Phi+K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)}+V\rangle$$

$\langle\ldots\rangle$ represents volume averaging

$$\begin{aligned} K_\phi &= \tfrac{1}{2a^{2\alpha}}\phi'^2 \\ K_\varphi &= \tfrac{1}{a^{2\alpha}}(D_0^A\varphi)^*(D_0^A\varphi) \\ K_\Phi &= \tfrac{1}{a^{2\alpha}}(D_0\Phi)^\dagger(D_0\Phi) \end{aligned} \;;\quad \begin{aligned} G_\phi &= \tfrac{1}{2a^2}\textstyle\sum_i(\partial_i\phi)^2 \\ G_\varphi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i^A\varphi)^*(D_i^A\varphi) \\ G_\Phi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i\Phi)^\dagger(D_i\Phi) \end{aligned} \;;\quad \begin{aligned} K_{U(1)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_i F_{0i}^2 \\ K_{SU(2)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_{a,i}(G_{0i}^a)^2 \\ G_{U(1)} &= \tfrac{1}{2a^4}\textstyle\sum_{i,j<i}F_{ij}^2 \\ G_{SU(2)} &= \tfrac{1}{2a^4}\textstyle\sum_{a,i,j<i}(G_{ij}^a)^2 \end{aligned}$$

(Kinetic-Scalar) (Gradient-Scalar) (Electric & Magnetic)

CosmoLattice – Expansion Evolution

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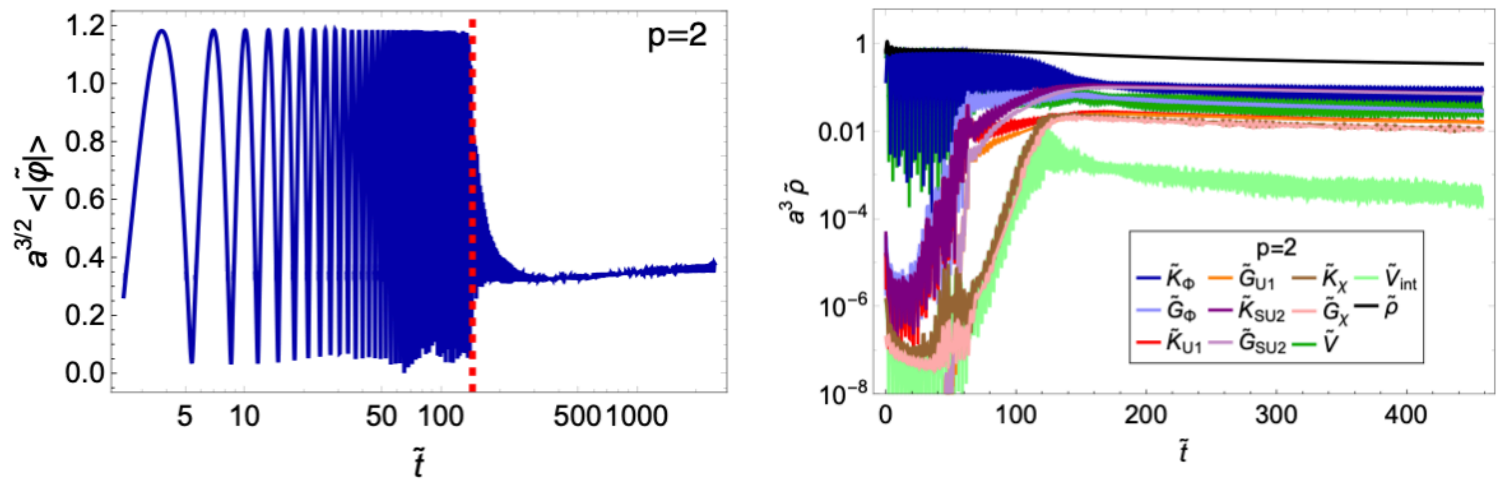
(Kinetic-Scalar) (Gradient-Scalar) (Electric & Magnetic)

CosmoLattice – Output / Observables

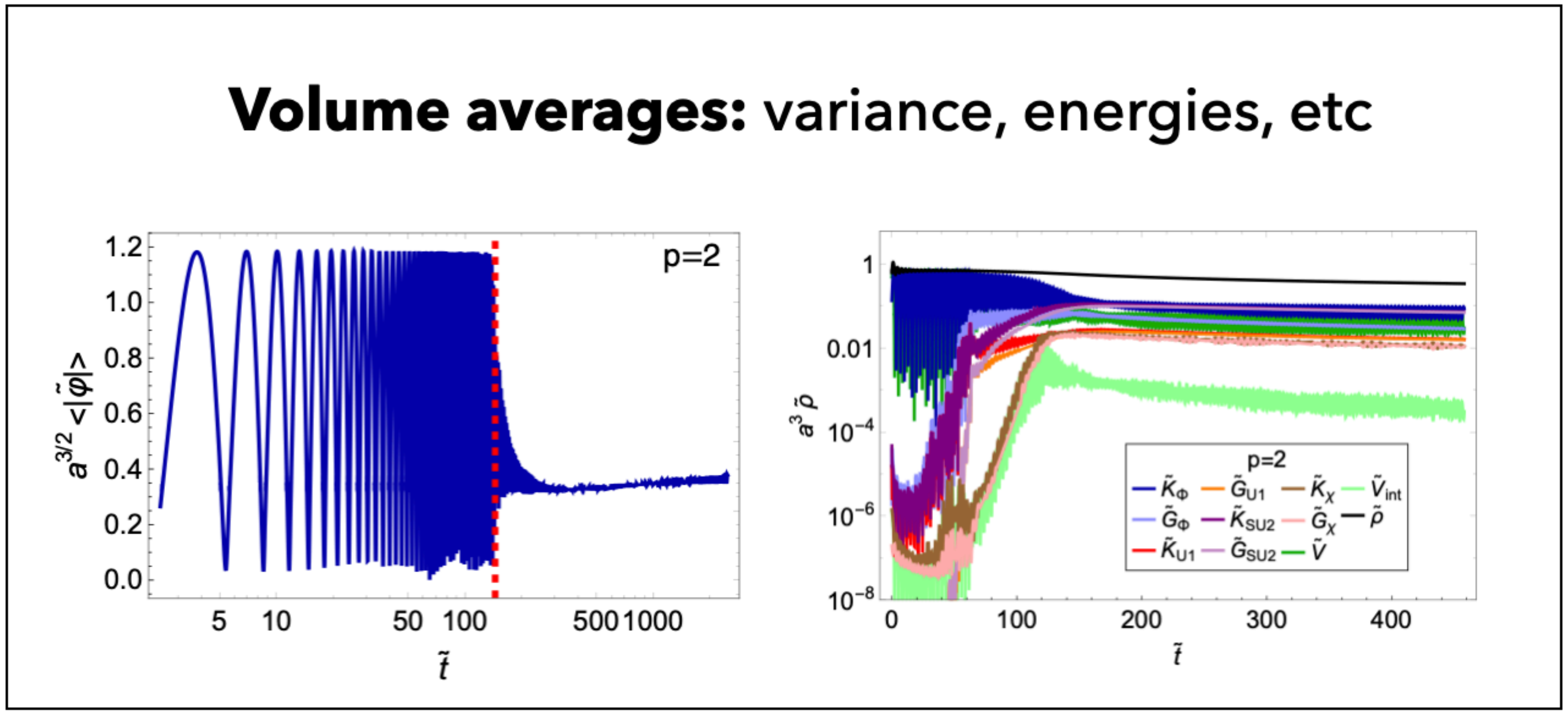
Output Types →

Volume averages: variance, energies, etc

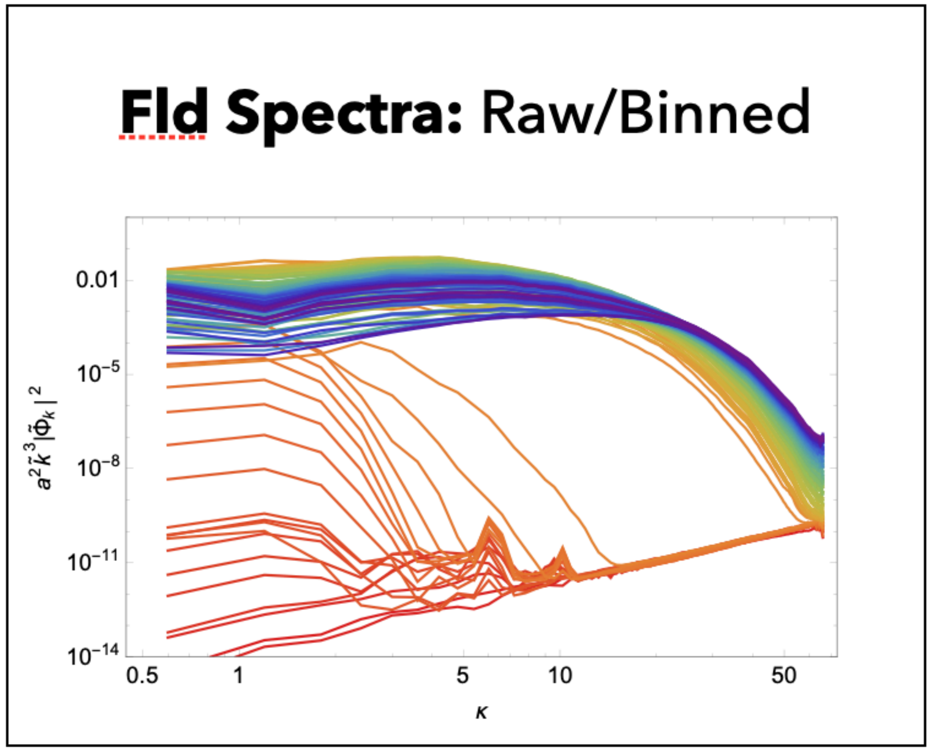


CosmoLattice – Output / Observables

Output Types



Volume averages: variance, energies, etc

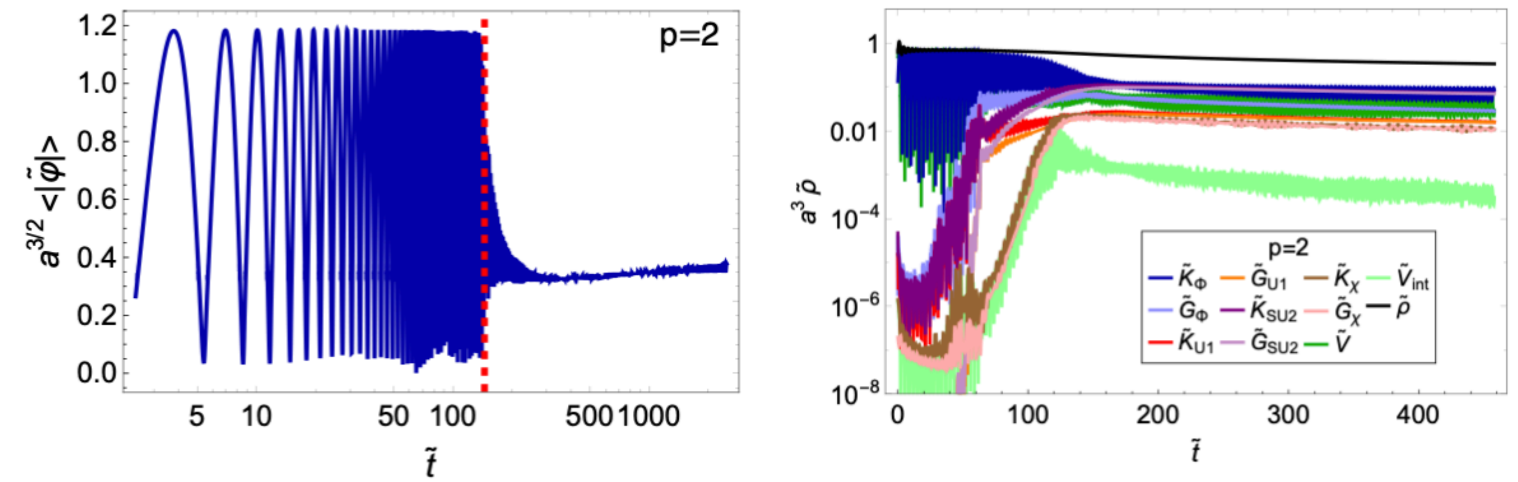


Fld Spectra: Raw/Binned

CosmoLattice – Output / Observables

Output Types

Volume averages: variance, energies, etc



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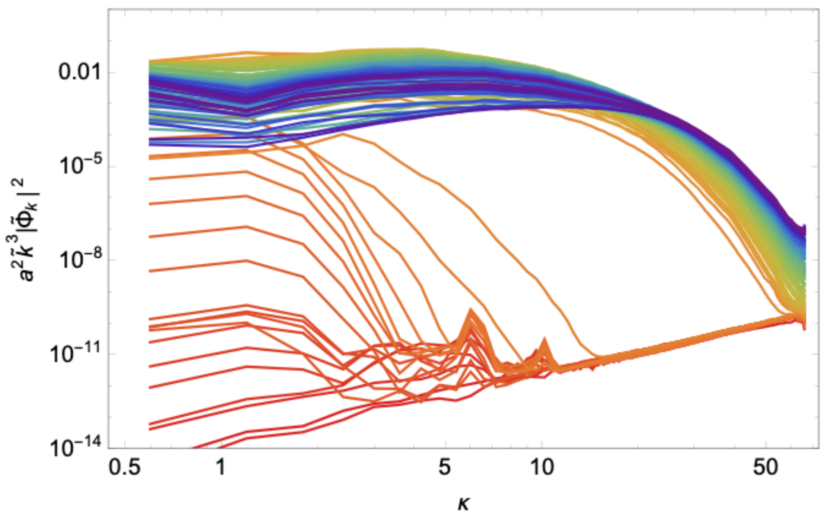
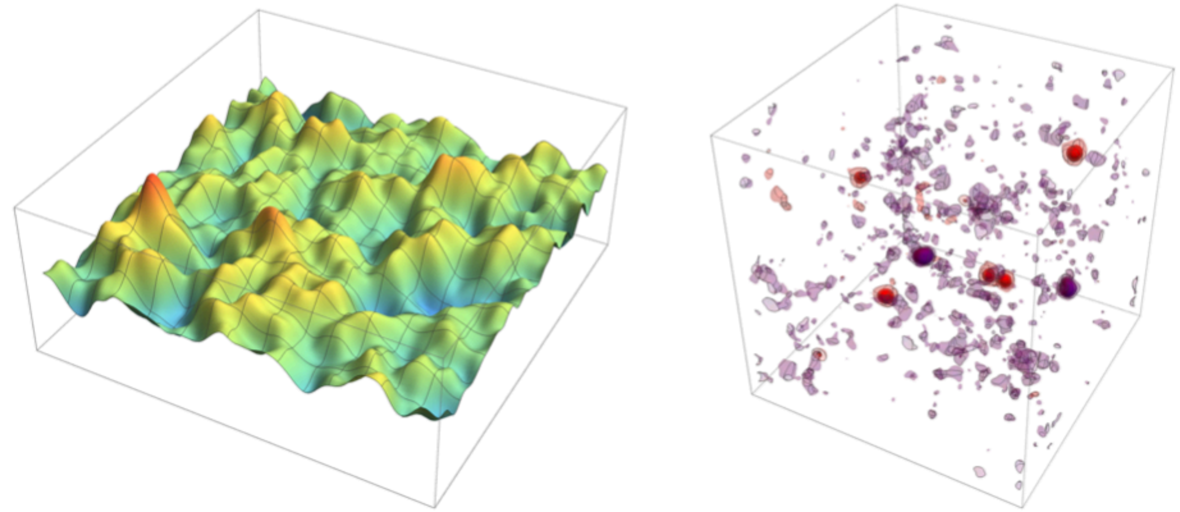


Snapshots: 2D/3D distribution



In summary ...

CosmoLattice

Theory Review
arXiv: 2006.15122

http://www.cosmolattice.net/

Code Manual
arXiv: 2102.01031

Field Th. Problem

- * Init Conditions
- * Eqs. of Motion

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CosmoLattice

- * Choose Lattice: dt, N, dx
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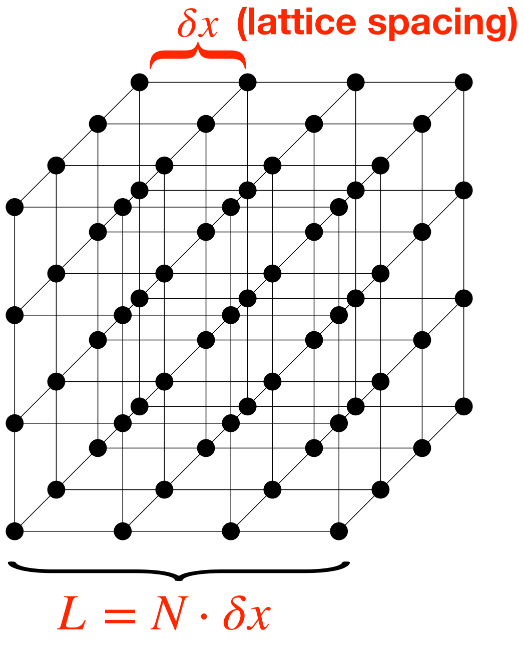

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*** Choose Observables**

Algorithms

- *Staggered LeapFrog (LF)*
- *Position-Verlet (PV2)*
- *Velocity-Verlet (VV2)*
- *Runge-Kutta (RK2, RK3, RK4)*
- *Yoshida (VV4, VV6, VV8, VV10)*

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- **Init Conditions**
- **Eqs. of Motion**

→

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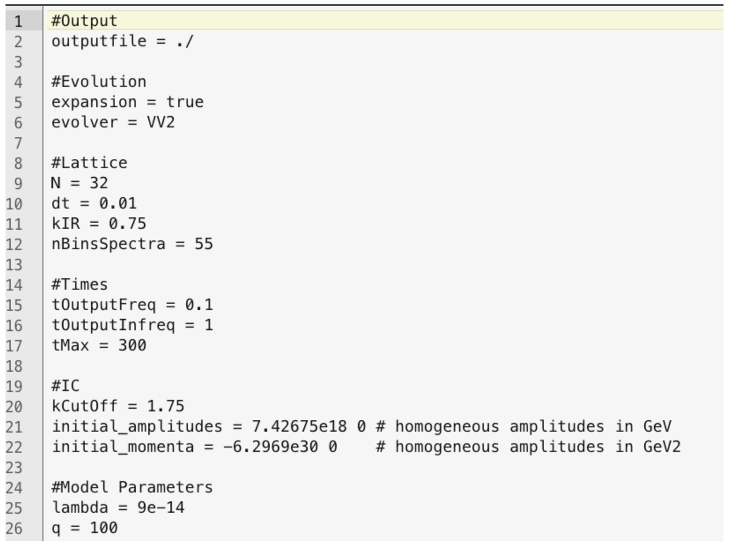
- **Choose Observables**

$$\lambda_1\,,\, \lambda_2\,,\, \ldots ,\, g_1^2\,,\, g_2^2\,,\, \ldots$$

$$m_\phi^2\,,\, m_\psi^2\,,\, \ldots , v^2,\, \Phi_*,\, \ldots$$



```
#Output
outputfile = ./

#Evolution
expansion = true
evolver = VV2

#Lattice
N = 32
dt = 0.01
kIR = 0.75
nBinsSpectra = 55

#Times
tOutputFreq = 0.1
tOutputInfreq = 1
tMax = 300

#IC
kCutOff = 1.75
initial_amplitudes = 7.42675e18 0 # homogeneous amplitudes in GeV
initial_momenta = -6.2969e30 0    # homogeneous amplitudes in GeV2

#Model Parameters
lambda = 9e-14
q = 100
```

➤ **Parameters** via **input file**

(no need to re-compile !)

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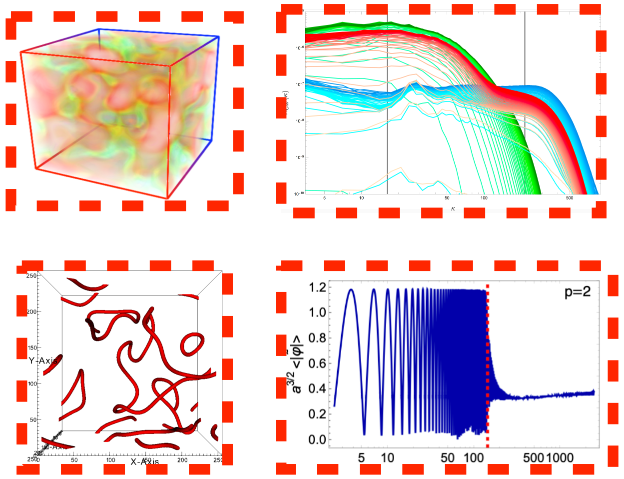

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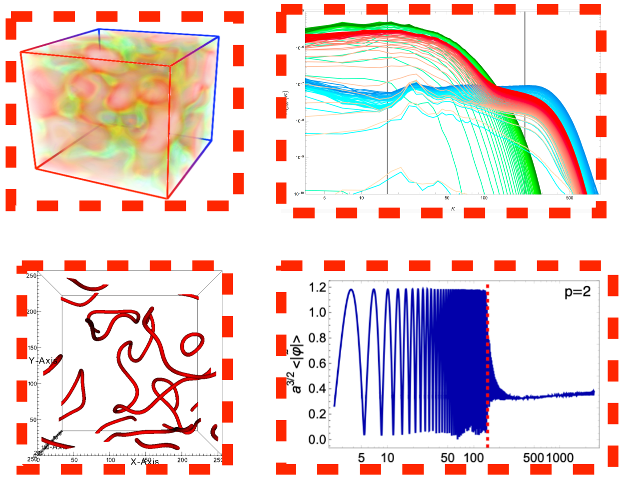

CL is a platform for field theories
You choose the problem to solve !

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Field Th. Problem

* Init Co

New Problem

Field Objects
Field Algebra

CosmoLattice

- **Choose Lattice: dt, N, dx**
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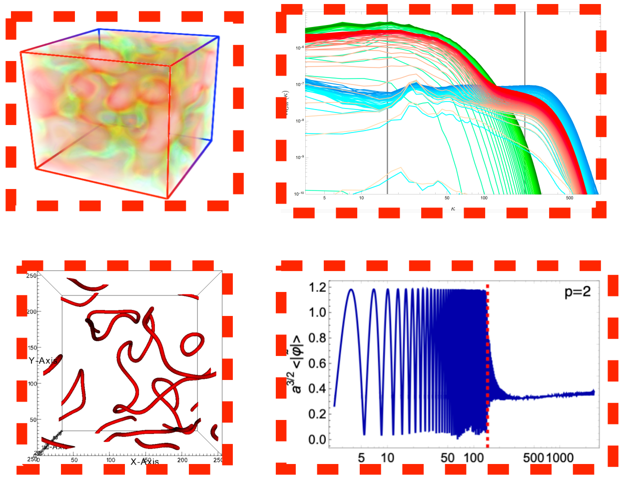
- **Choose Observables**

Output



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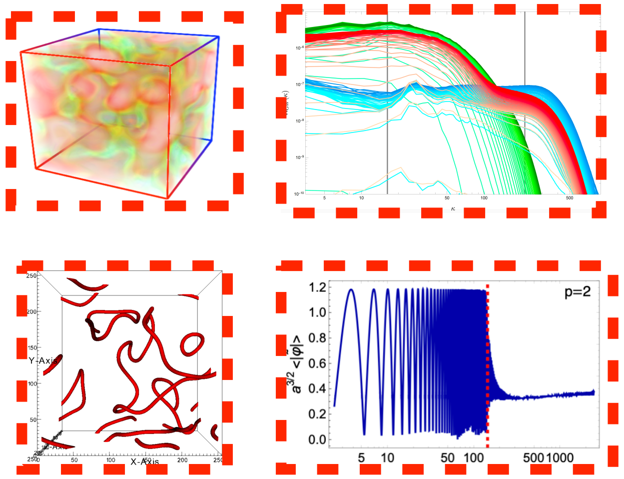

- **CL so far (v1.0, Public):**
  - Global scalar field dynamics
  - U(1) scalar-gauge dynamics
  - SU(2) scalar-gauge dynamics

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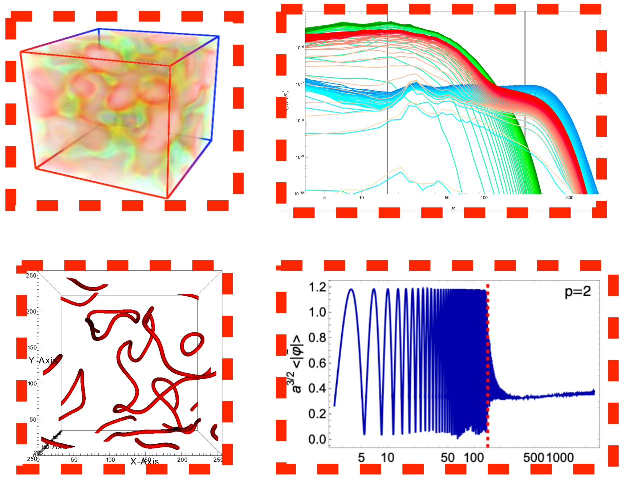

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  - GWs from Global/Abelian sectors

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New Problem

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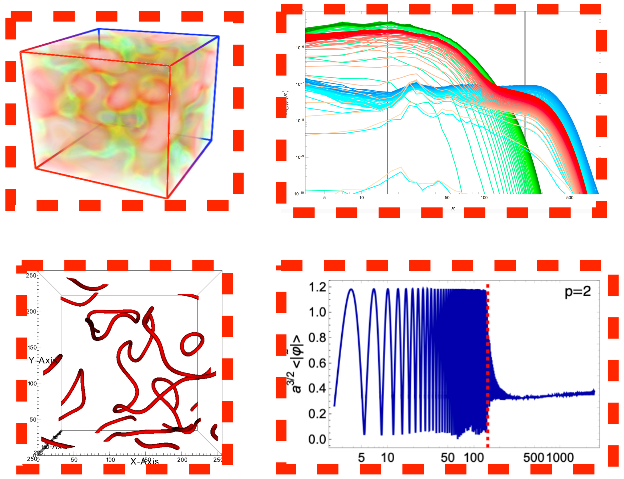

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  - SU(2) scalar-gauge dynamics
  - **GWs from Global/Abelian sectors**

One can add New Modules !

(Interactions, fields, EOM, observables, features, ...)

New Modules

* Magneto Hydro-dynamics (MHD)
* Non-minimal Grav. coupling
* Cosmic string networks
* Axion-gauge interactions

New
Modules

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* Non-minimal Grav. coupling
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Magneto Hydro-dynamics (MHD)

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu - p g^{\mu\nu} + \ldots$$

$$D_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\sigma\nu} T^{\sigma\nu} + \Gamma^\nu_{\nu\sigma} T^{\mu\sigma} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\, T^{\mu\nu}) = \ldots$$

$$\boxed{\begin{aligned} \partial_\eta \tilde{T}^{00} + \partial_i \tilde{T}^{0i} &= \tilde{S}^0[\phi, A_k, \{\tilde{T}_{lk}\}]\,, \\ \partial_\eta \tilde{T}^{0i} + \partial_j \tilde{T}^{ij} &= \tilde{S}^i[\phi, A_k, \{\tilde{T}_{lk}\}]\,, \end{aligned}}$$

Work in progress … key to GWs from PhT's !

(w/ K. Marschall, A. Midiri and A. Roper Pol)

New
Modules

* Magneto Hydro-dynamics (MHD)
* Non-minimal Grav. coupling
* Cosmic string networks
* Axion-gauge interactions

Non-minimal Grav. coupling

$$\mathcal{S}_{\rm NMC} = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\xi R\phi^2+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi,\{\varphi_{\rm m}\})\right)$$

$$[\texttt{Non}-\texttt{minimally coupled}]\left\{\begin{array}{l}\phi' = a^{\alpha-3}\pi_\phi\,, \\ \pi_\phi' = a^{1+\alpha}\,\nabla^2\phi - a^{3+\alpha}\,(\xi R\phi + V_{,\phi})\,,\end{array}\right.$$

$$[\texttt{Expanding background}]\left\{\begin{array}{l}a' = a^{\alpha-1}\pi_a\,, \\ \pi_a' = \frac{a^{2+\alpha}}{6}R\,,\end{array}\right.$$

with

$$R = \frac{1}{m_p^2}\left[\frac{2\,(1-6\xi)\,(E_G^\phi - E_K^\phi) + 4\langle V\rangle - 6\xi\langle\phi\,V_{,\phi}\rangle + (\rho_{\rm m}-3p_{\rm m})}{1+(6\xi-1)\,\xi\langle\phi^2\rangle/m_p^2}\right],$$

(w/ B. Stefanek, T. Opferkuch, and A. Florio)

Used in ***arXiv:2112.08388; arXiv:2406.02689; arXiv:2404.17654***

New
Modules

* Magneto Hydro-dynamics (MHD)
* Non-minimal Grav. coupling
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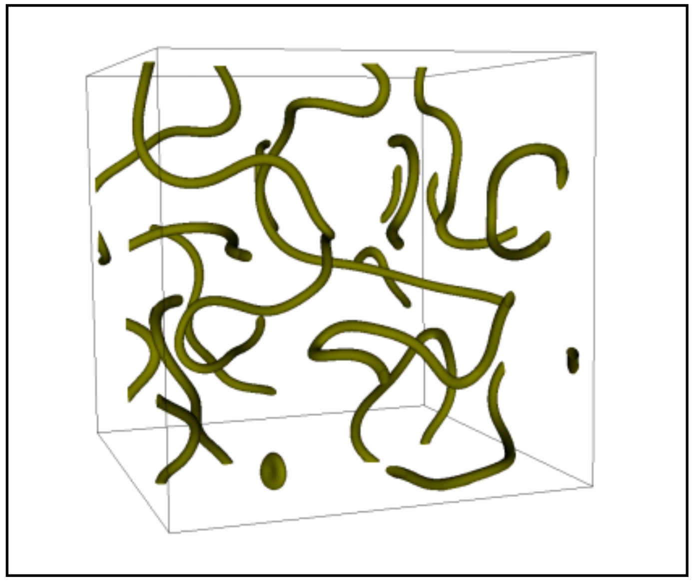
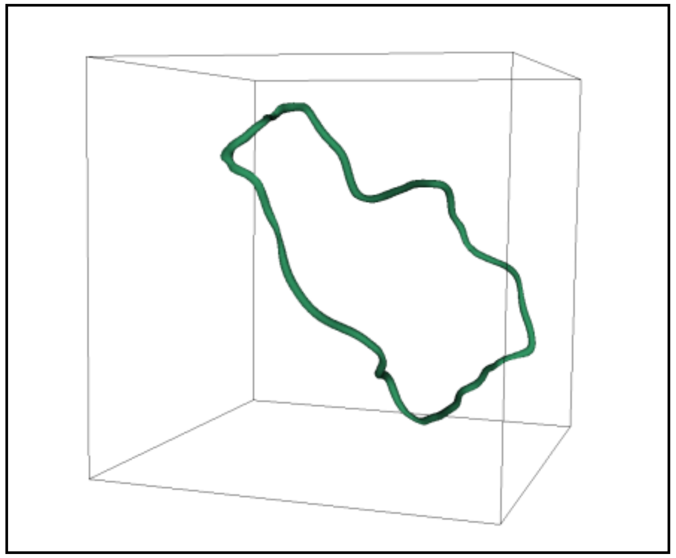
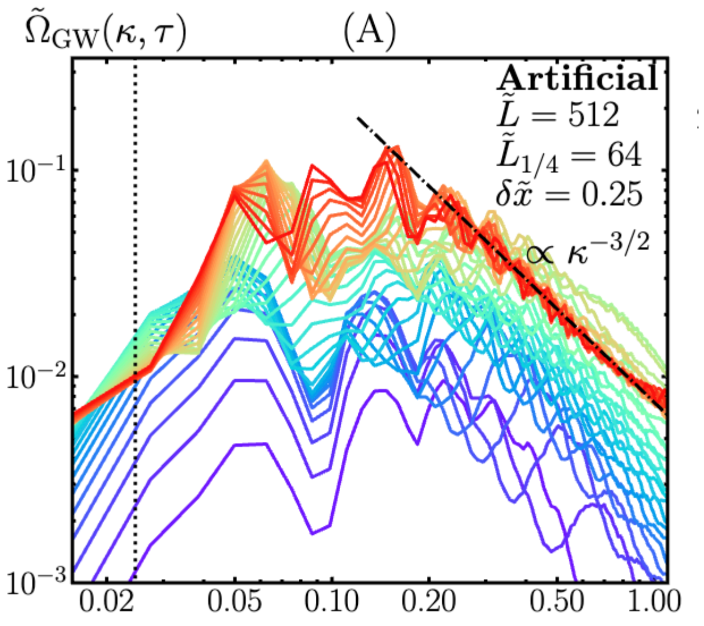
* Axion-gauge interactions

Cosmic Defect Networks

Particle & GW Emission


(String Networks)


(Isolated loops)



(GW emission)

(w/ J. Baeza-Ballesteros, E. Copeland, J. Lizarraga)

Used in ***arXiv:2308.08456*** *(Global)* & ***arXiv:2408.02364*** *(Local)*

New
Modules

* Magneto Hydro-dynamics (MHD)
* Non-minimal Grav. coupling
* Cosmic string networks
* Axion-gauge interactions

Axion-gauge interactions

$$\mathcal{S}_{\rm ax} = -\int \mathrm{d}x^4\sqrt{-g}\,\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha_\Lambda}{4}\frac{\phi}{m_p}F_{\mu\nu}\tilde{F}^{\mu\nu}\}$$

$$\left.\begin{array}{rcll}
\ddot{\phi} & = & -3H\dot{\phi} + \frac{1}{a^2}\vec{\nabla}^2\phi - V_{,\phi} + \frac{\alpha_\Lambda}{a^3 m_p}\vec{E}\cdot\vec{B}\,, & \\
\dot{\vec{E}} & = & -H\vec{E} - \frac{1}{a^2}\vec{\nabla}\times\vec{B} - \frac{\alpha_\Lambda}{a m_p}\left(\dot{\phi}\vec{B} + \vec{\nabla}\phi\cdot\vec{E}\right), & \\
\ddot{a} & = & -\frac{a}{3m_p^2}\left(2\rho_{\rm K} - \rho_{\rm V} + \rho_{\rm EM}\right)\,, & \\
\vec{\nabla}\cdot\vec{E} & = & -\frac{\alpha_\Lambda}{a m_p}\vec{\nabla}\phi\cdot\vec{B}\,, & [\texttt{Gauss law}] \\
H^2 & = & \frac{1}{3m_p^2}\left(\rho_{\rm K} + \rho_{\rm G} + \rho_{\rm V} + \rho_{\rm EM}\right)\,, & [\texttt{Hubble law}]
\end{array}\right\}$$

(w/ J. Lizarraga, N. Loayza, and A. Urio)

Used in *Phys. Rev. Lett. 131 (2023) 15, 151003*
and *Phys.Rev.D* 111 (2025) 6, 063545

New Physics

* Magneto Hydro-dynamics (MHD)
* Axion-gauge interactions
* Cosmic string networks
* Non-minimal Grav. coupling

To be released in 2025/26 !

New
Physics

* Magneto Hydro-dynamics (MHD)
* Axion-gauge interactions
* Cosmic string networks
* Non-minimal Grav. coupling
* Grav. Pert. Th / Full GR
  (w/ N. Loayza & R. Flauger)

To be released in 202X?

Applications

1) Non-linear inflation dynamics

2) GW from non-linear dynamics

3) Preheating & Equation of State after inflation

4) Cosmic defect networks (axions, AH, domain walls)

5) Single string loop dynamics

6) Non-minimal gravitational Interactions

7) Phase transitions

....

X) Your project !

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.....

X) Your project !

Applications

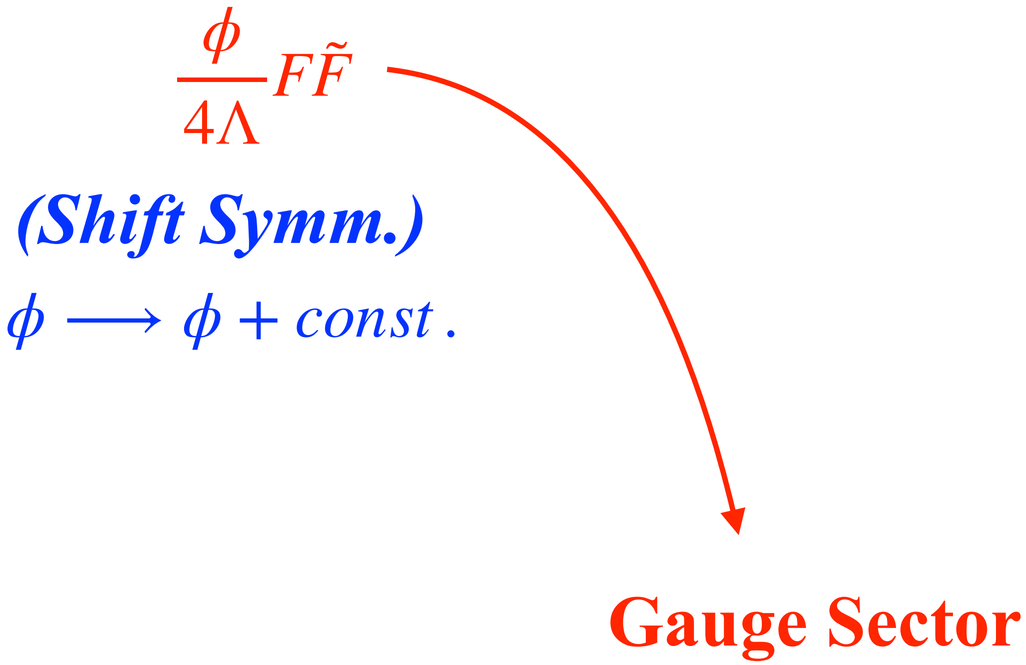
- **Non-linear inflation dynamics** (Axion-inflation)

(Non-Linear)
Field Dynamics of
Axion Inflation

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$



$$\frac{\phi}{4\Lambda} F\tilde{F}$$

(Shift Symm.)

$$\phi \longrightarrow \phi + const\,.$$

Gauge Sector

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

Potential

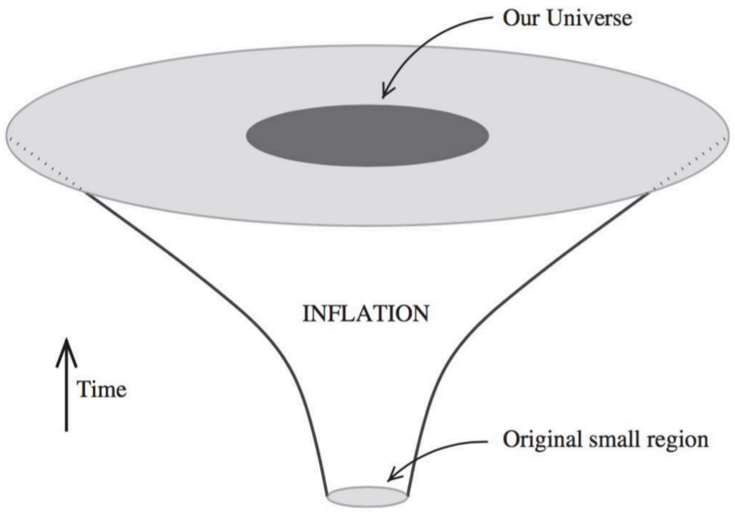
$$V(\phi) + \frac{\phi}{4\Lambda} F\tilde{F}$$

(Shift Symm.)

$$\phi \longrightarrow \phi + const\,.$$

Due to Non-Perturbative effects

Gauge Sector



Inflation

Axion-Inflation

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$$V(\phi) + \frac{\phi}{4\Lambda} F\tilde{F} \quad \Rightarrow \quad A''_{\pm} + \left(k^2 \pm \frac{k\dot{\phi}}{\Lambda H \tau}\right) A_{\pm} = 0$$

e.g. **Sorbo et al 2006-2012**

(Shift Symm.)

$$\phi \longrightarrow \phi + const\,.$$

Gauge field dynamics during inflaton

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

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Chiral instability

$$A_+ \propto e^{\pi\xi}\,, \quad |A_-| \ll |A_+|$$

$(\xi \propto \dot{\phi})$

A_+ exponentially amplified

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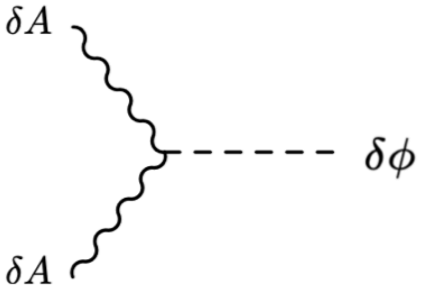
Inflaton perturbations $\delta\phi$ through inverse decay (highly non-Gaussian)



Barnaby, Peloso '10
Planck '15

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Freese, Frieman, Olinto '90; . . .

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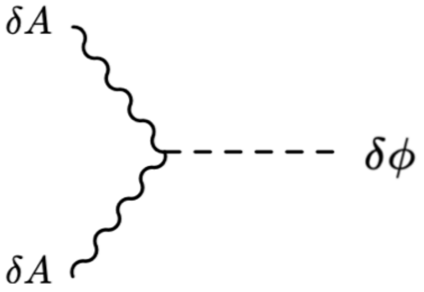
A+ exponentially amplified

Inflaton perturbations $\delta\phi$ through inverse decay (highly non-Gaussian)

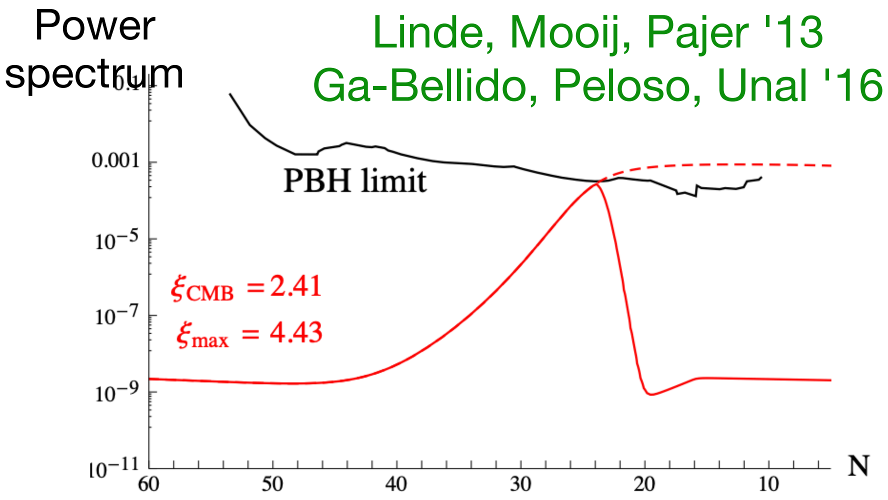


Barnaby, Peloso '10
Planck '15

Amplitude $\delta\phi$ must be bounded

Otherwise too many Primordial Black Holes (PBH) !

Linde, Mooij, Pajer '13
Ga-Bellido, Peloso, Unal '16



Axion-Inflation

Freese, Frieman, Olinto ’90; . . .

Shift symmetry $\phi \to \phi + const.$

$$V(\phi) + \frac{\phi}{4\Lambda} F\tilde{F} \quad \Rightarrow \quad A''_{\pm} + \left(k^2 \pm \frac{k\dot{\phi}}{\Lambda H \tau}\right) A_{\pm} = 0$$

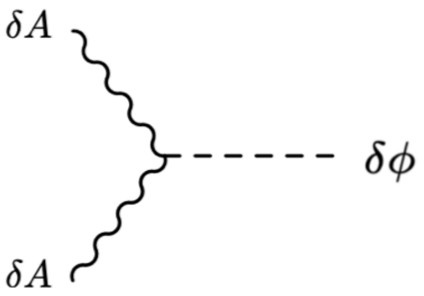
Chiral instability

$$A_+ \propto e^{\pi\xi}\,, \quad |A_-| \ll |A_+|$$

A+ exponentially amplified

$(\xi \propto \dot{\phi})$

Inflaton perturbations $\delta\phi$ through inverse decay (highly non-Gaussian)



Barnaby, Peloso ’10
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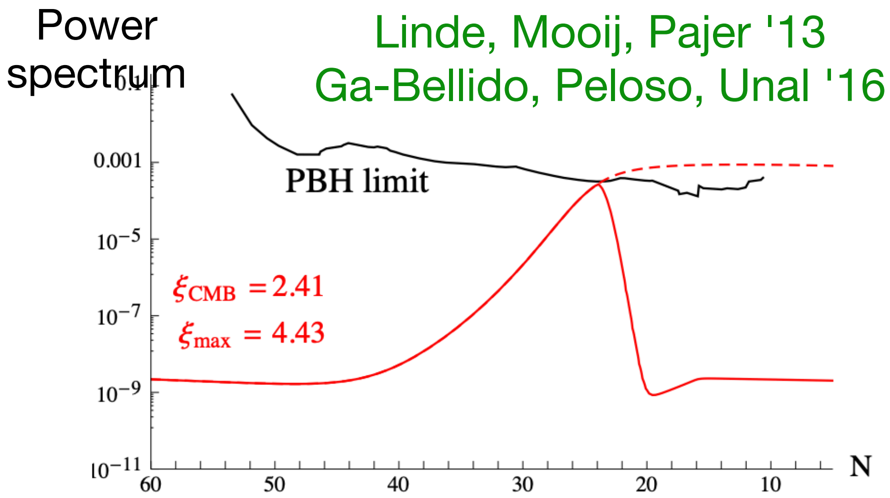

Linde, Mooij, Pajer '13
Ga-Bellido, Peloso, Unal '16

Only one chirality of gauge field then… chiral GWs !

$$\{E_iE_j + B_iB_j\}^{\rm TT}$$

$$h_L \;,\; \cancel{h_R}$$

Cook & Sorbo '11
Amber & Sorbo ‘12

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

$$V(\phi) + \frac{\phi}{4\Lambda}F\tilde{F} \quad \Rightarrow \quad A''_{\pm} + \left(k^2 \pm \frac{k\dot{\phi}}{\Lambda H \tau}\right)A_{\pm} = 0$$

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Inflaton perturbations $\delta\phi$ through inverse decay (highly non-Gaussian)

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Otherwise too many Primordial Black Holes (PBH) !

Only one chirality of gauge field then… chiral GWs !

Can we trust current pheno calculations ?

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

$$V(\phi) + \frac{\phi}{4\Lambda} F\tilde{F} \quad \Rightarrow \quad A''_{\pm} + \left(k^2 \pm \frac{k\dot{\phi}}{\Lambda H \tau}\right) A_{\pm} = 0$$

Chiral instability

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Inflaton perturbations $\delta\phi$
through inverse decay
(highly non-Gaussian)

Amplitude $\delta\phi$ must be bounded

Otherwise too many
Primordial Black Holes (PBH) !

Only
one chirality
of gauge field
then… chiral GWs !

It really depends how good non-linearities are captured !

Axion-Inflation

PROBLEM: **PNG, GW and PBH** ⟶ **Approximations (e.g. Analytical)**

Axion-Inflation

PROBLEM: **PNG, GW and PBH** → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot{\phi}\ , \quad E_i \equiv \dot{A}_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde{\pi}_\phi = a^3\pi_\phi\ , \quad \tilde{\vec{E}} = a\vec{E}\ , \ \pi_a \equiv \dot{a} \end{array}\right)$$

Let's have a look to the full problem !

$$\left(V(\phi) = \frac{1}{2}m^2\phi^2\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot{\phi}\;, \quad E_i \equiv \dot{A}_i\;, \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde{\pi}_\phi = a^3 \pi_\phi\;, \quad \tilde{\vec{E}} = a\vec{E}\;,\; \pi_a \equiv \dot{a} \end{array}\right)$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2 \phi + \dfrac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\;, \\ \\ \dot{\tilde{\vec{E}}} = -\dfrac{1}{a}\vec{\nabla}\times\vec{B} - \dfrac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}\;. \end{cases}$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot{\phi}\ , \quad E_i \equiv \dot{A}_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde{\pi}_\phi = a^3\pi_\phi\ , \quad \tilde{\vec{E}} = a\vec{E}\ ,\ \pi_a \equiv \dot{a} \end{array}\right)$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3m^2\phi + \dfrac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\ , \\[2ex] \dot{\tilde{\vec{E}}} = -\dfrac{1}{a}\vec{\nabla}\times\vec{B} - \dfrac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}\ . \end{cases}$$

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\left\langle -2\underset{\text{(Kin)}}{K_\phi} + \underset{\text{(Pot)}}{V} - \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A}\right\rangle$$

$$\left(\begin{array}{c} K_\phi \equiv \dfrac{1}{2}\pi_\phi^2 = \dfrac{1}{2a^6}\tilde{\pi}_\phi^2\ , \quad G_\phi \equiv \dfrac{1}{2a^2}\sum(\partial_i\phi)^2\ , \quad V \equiv \dfrac{1}{2}m^2\phi^2\ , \\ K_A \equiv \dfrac{1}{2}\sum_i\dfrac{E_i^2}{a^2} = \dfrac{1}{2}\sum_i\dfrac{\tilde{E}_i^2}{a^4}\ , \quad G_A \equiv \dfrac{1}{2}\sum_i\dfrac{B_i^2}{a^4} \end{array}\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot{\phi}\ , \quad E_i \equiv \dot{A}_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde{\pi}_\phi = a^3\pi_\phi\ , \quad \tilde{\vec{E}} = a\vec{E}\ ,\ \pi_a \equiv \dot{a} \end{array}\right)$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \dfrac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\ , \\ \dot{\tilde{\vec{E}}} = -\dfrac{1}{a}\vec{\nabla}\times\vec{B} - \dfrac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}\ . \end{cases}$$

→ **Non-Linear dynamics**

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\left\langle -2\underset{\text{(Kin)}}{K_\phi} + \underset{\text{(Pot)}}{V} - \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A}\right\rangle$$

$$\left(\begin{array}{c} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde{\pi}_\phi^2\ , \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\ , \quad V \equiv \frac{1}{2}m^2\phi^2\ , \\ K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i \frac{\tilde{E}_i^2}{a^4}\ , \quad G_A \equiv \frac{1}{2}\sum_i \frac{B_i^2}{a^4} \end{array}\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot{\phi}\ , \quad E_i \equiv \dot{A}_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde{\pi}_\phi = a^3\pi_\phi\ , \quad \tilde{\vec{E}} = a\vec{E}\ , \ \pi_a \equiv \dot{a} \end{array}\right)$$

EoM

Linear Regime

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \xcancel{a\vec{\nabla}^2\phi} - a^3 m^2\phi \xcancel{+ \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\,,} \\ \dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \underbrace{\frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B}}_{\text{Interaction}} + \xcancel{\frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}\,.} \end{cases}$$

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\Big\langle \underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V} \xcancel{- \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A}} \Big\rangle$$

$$\left(\begin{array}{c} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde{\pi}_\phi^2\ , \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\ , \quad V \equiv \frac{1}{2}m^2\phi^2\ , \\ K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i \frac{\tilde{E}_i^2}{a^4}\ , \quad G_A \equiv \frac{1}{2}\sum_i \frac{B_i^2}{a^4} \end{array}\right)$$

Axion-Inflation

PROBLEM: **PNG, GW and PBH** → Approximations (e.g. Analytical)

$$\left(\begin{array}{c} \pi_\phi \equiv \dot\phi\ , \quad E_i \equiv \dot A_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde\pi_\phi = a^3\pi_\phi\ , \quad \tilde{\vec E} = a\vec E\ , \ \pi_a \equiv \dot a \end{array}\right)$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde\pi}_\phi = \cancel{a\vec\nabla^2\phi} - a^3m^2\phi + \dfrac{1}{a\Lambda}\tilde{\vec E}\cdot\vec B\ , \\ \dot{\tilde{\vec E}} = -\dfrac{1}{a}\vec\nabla\times\vec B - \dfrac{1}{a^3\Lambda}\tilde\pi_\phi\vec B + \cancel{\dfrac{1}{a\Lambda}\vec\nabla\phi\times\tilde{\vec E}}\ . \end{cases}$$

Back-Reaction

Hom. EoM $(\mathbf{t})$

$$\dot\pi_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\langle -2\underset{\text{(Kin)}}{K_\phi} + \underset{\text{(Pot)}}{V} - \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A}\rangle$$

$$\left(\begin{array}{c} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde\pi_\phi^2\ , \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\ , \quad V \equiv \frac{1}{2}m^2\phi^2\ , \\ K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i\frac{\tilde E_i^2}{a^4}\ , \quad G_A \equiv \frac{1}{2}\sum_i\frac{B_i^2}{a^4} \end{array}\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left(\begin{array}{c} \pi_\phi \equiv \dot\phi\ , \quad E_i \equiv \dot A_i\ , \quad B_i \equiv \epsilon_{ijk}\partial_j A_k \\ \tilde\pi_\phi = a^3\pi_\phi\ , \quad \tilde{\vec E} = a\vec E\ , \ \pi_a \equiv \dot a \end{array}\right)$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde\pi}_\phi = \cancel{a\vec\nabla^2\phi} - a^3m^2\phi + \dfrac{1}{a\Lambda}\langle\tilde{\vec E}\cdot\vec B\,\rangle \\ \dot{\tilde{\vec E}} = -\dfrac{1}{a}\vec\nabla\times\vec B - \dfrac{1}{a^3\Lambda}\tilde\pi_\phi\vec B + \cancel{\dfrac{1}{a\Lambda}\vec\nabla\phi\times\tilde{\vec E}}\ . \end{cases}$$

Hom. (t) Approx.

Back-Reaction (Homog. Approx.)

Hom. EoM $(\mathbf{t})$

$$\dot\pi_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\Big(\big\langle -2\underset{\text{(Kin)}}{K_\phi} + \underset{\text{(Pot)}}{V}\big\rangle - \big\langle \underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\big\rangle\Big)$$

$$\left(\begin{array}{c} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde\pi_\phi^2\ , \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\ , \quad V \equiv \frac{1}{2}m^2\phi^2\ , \\ K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i\frac{\tilde E_i^2}{a^4}\ , \quad G_A \equiv \frac{1}{2}\sum_i\frac{B_i^2}{a^4} \end{array}\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\left\langle \vec{E}\vec{B} \right\rangle = -\frac{\lambda}{a^4}\int \frac{dk}{4\pi}k^3\frac{d}{d\tau}\left|A_{-\lambda}(\tau,\vec{k})\right|^2 \qquad ;$$

$$\left\langle K_A + G_A \right\rangle \equiv \left\langle \frac{E^2+B^2}{2} \right\rangle = \frac{1}{a^4}\int \frac{dk}{4\pi^2}k^2\left(\left|\frac{dA_{-\lambda}(\tau,\vec{k})}{d\tau}\right|^2 + k^2\left|A_{-\lambda}(\tau,\vec{k})\right|^2\right) \quad ;$$

$$\begin{tikzcd} & \lambda = + \,, \text{if} \phi > 0 \\ \lambda = \pm \arrow[ur] \arrow[dr] & \\ & \lambda = - \,, \text{if} \phi < 0 \end{tikzcd}$$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \cancel{a\vec{\nabla}^2\phi} - a^3m^2\phi + \dfrac{1}{a\Lambda}\left\langle \tilde{\vec{E}}\cdot\vec{B}\right\rangle \\[2ex] \dot{\tilde{\vec{E}}} = -\dfrac{1}{a}\vec{\nabla}\times\vec{B} - \dfrac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \cancel{\dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}} \,. \end{cases}$$

Hom. (t) Approx.

Back-Reaction (Homog. Approx.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\left(\left\langle -\underset{\text{(Kin)}}{2K_\phi} + \underset{\text{(Pot)}}{V}\right\rangle - \left\langle \underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\right\rangle\right)$$

$$\left(\begin{gathered} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde{\pi}_\phi^2\,, \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\,, \quad V \equiv \frac{1}{2}m^2\phi^2\,, \\ K_A \equiv \frac{1}{2}\sum_i\frac{E_i^2}{a^2} = \frac{1}{2}\sum_i\frac{\tilde{E}_i^2}{a^4}\,, \quad G_A \equiv \frac{1}{2}\sum_i\frac{B_i^2}{a^4} \end{gathered}\right)$$

Axion-Inflation

PROBLEM: PNG, GW and PBH ⟶ **Approximations (e.g. Analytical)**

$$\left\langle\vec{E}\vec{B}\right\rangle = -\frac{\lambda}{a^4}\int\frac{dk}{4\pi}k^3\frac{d}{d\tau}\left|A_{-\lambda}(\tau,\vec{k})\right|^2 \quad ;$$

$$\left\langle K_A + G_A\right\rangle \equiv \left\langle\frac{E^2+B^2}{2}\right\rangle = \frac{1}{a^4}\int\frac{dk}{4\pi^2}k^2\left(\left|\frac{dA_{-\lambda}(\tau,\vec{k})}{d\tau}\right|^2 + k^2\left|A_{-\lambda}(\tau,\vec{k})\right|^2\right) \quad ;$$

$$\lambda = \pm \quad \begin{cases} \lambda = +\,,\,\mathrm{if}\,\dot\phi > 0 \\ \lambda = -\,,\,\mathrm{if}\,\dot\phi < 0 \end{cases}$$

EoM

Local EoM $(\vec{\mathbf{x}},\mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \cancel{a\vec{\nabla}^2\phi} - a^3m^2\phi + \frac{1}{a\Lambda}\left\langle\tilde{\vec{E}}\cdot\vec{B}\right\rangle \\ \frac{d^2A_\pm(\tau,\vec{k})}{d\tau^2} + \left[k^2 \pm 2\lambda\xi kaH\right]A_\pm(\tau,\vec{k}) = 0\ \cancel{\frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}}\,. \end{cases}$$

Hom. (t) Approx.

Back-Reaction (Homog. Approx.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\left(\left\langle\underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V}\right\rangle - \left\langle\underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\right\rangle\right)$$

DallAgata et al 2019, Domcke et 2020 ⟶ **Elaborated Iterative scheme !**

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

$$\langle \vec{E}\vec{B}\rangle = -\frac{\lambda}{a^4}\int \frac{dk}{4\pi}k^3\frac{d}{d\tau}\left|A_{-\lambda}(\tau,\vec{k})\right|^2 \quad ;$$

$$\langle K_A + G_A\rangle \equiv \left\langle \frac{E^2+B^2}{2}\right\rangle = \frac{1}{a^4}\int\frac{dk}{4\pi^2}k^2\left(\left|\frac{dA_{-\lambda}(\tau,\vec{k})}{d\tau}\right|^2 + k^2\left|A_{-\lambda}(\tau,\vec{k})\right|^2\right) \quad ;$$

$\lambda = \pm$ → $\lambda = +\,,\text{if}\phi > 0$

$\lambda = \pm$ → $\lambda = -\,,\text{if}\phi < 0$

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\left\{\begin{array}{l} \dot{\tilde{\pi}}_\phi = \cancel{a\vec{\nabla}^2\phi} - a^3m^2\phi + \dfrac{1}{a\Lambda}\langle\tilde{\vec{E}}\cdot\vec{B}\,\rangle \\[2em] \dfrac{d^2A_\pm(\tau,\vec{k})}{d\tau^2} + \left[k^2 \pm 2\lambda\xi k a H\right]A_\pm(\tau,\vec{k}) = 0\ \cancel{\dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}}\ . \end{array}\right.$$

Hom. (t) Approx.

Back-Reaction (Homog. Approx.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\left(\left\langle -2\underset{\text{(Kin)}}{K_\phi} + \underset{\text{(Pot)}}{V}\right\rangle - \left\langle \underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\right\rangle\right)$$

Gorbar et al 2021 → **Gradient Expansion Formalism (GEF)**

Axion-Inflation

PROBLEM: PNG, GW and PBH → Approximations (e.g. Analytical)

Can we do better than homogeneous backreaction ?

EoM

Local EoM $(\vec{\mathbf{x}},\mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \cancel{a\vec{\nabla}^2\phi} - a^3m^2\phi + \dfrac{1}{a\Lambda}\langle\tilde{\vec{E}}\cdot\vec{B}\,\rangle \\ \dfrac{d^2A_\pm(\tau,\vec{k})}{d\tau^2} + \left[k^2 \pm 2\lambda\xi k a H\right] A_\pm(\tau,\vec{k}) = 0 \; \cancel{\dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}}\,. \end{cases}$$

Hom. (t) Approx.

Back-Reaction (Homog. Approx.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho+3p) = \frac{a}{3m_{pl}^2}\Big(\langle -2K_\phi + V\rangle - \langle K_A + G_A\rangle\Big)$$

(Kin) (Pot) (Elec) (Mag)

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

Yes, we need a full lattice approach

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \cancel{a\vec\nabla^2\phi} - a^3 m^2 \phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B} \\ \frac{d^2A_\pm(\tau,\vec{k})}{d\tau^2} + \left[k^2 \pm 2\lambda\xi k a H\right] A_\pm(\tau,\vec{k}) = 0 \; \cancel{\frac{1}{a\Lambda}\vec\nabla\phi\times\tilde{\vec{E}}} \, . \end{cases}$$

Back-Reaction (Source InHom.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\Big\langle \underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V} - \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A} \Big\rangle$$

Axion-Inflation

PROBLEM: PNG, GW and PBH → **Approximations (e.g. Analytical)**

Yes, we need a full lattice approach

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = \cancel{a\vec{\nabla}^2\phi} - a^3 m^2 \phi + \frac{1}{a\Lambda}\vec{\tilde{E}}\cdot\vec{B} \\ \dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \cancel{\frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{\tilde{E}}}\,. \end{cases}$$

Back-Reaction (Source InHom.)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\left\langle \underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V} - \underset{\text{(Elec)}}{K_A} - \underset{\text{(Mag)}}{G_A} \right\rangle$$

Axion-Inflation

PROBLEM: PNG, GW and PBH ⟶ **Approximations (e.g. Analytical)**

Yes, we need a full lattice approach

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \frac{1}{a\Lambda}\vec{\tilde{E}}\cdot\vec{B} \\ \dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{\tilde{E}}\ . \end{cases}$$

Back-Reaction (Fully Local)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\left\langle \underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V} - \left(\underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\right)\right\rangle$$

Axion-Inflation

PROBLEM: PNG, GW and PBH ⟶ Approximations (e.g. Analytical)

Need to "latticize" the EOM !

EoM

Local EoM $(\vec{\mathbf{x}}, \mathbf{t})$

$$\begin{cases} \dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \dfrac{1}{a\Lambda}\vec{\tilde{E}}\cdot\vec{B} \\ \dot{\vec{\tilde{E}}} = -\dfrac{1}{a}\vec{\nabla}\times\vec{B} - \dfrac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \dfrac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{\tilde{E}}\,. \end{cases}$$

Back-Reaction (Fully Local)

Hom. EoM $(\mathbf{t})$

$$\dot{\pi}_a = -\frac{a}{6m_{pl}^2}(\rho + 3p) = \frac{a}{3m_{pl}^2}\Big\langle \underset{\text{(Kin)}}{-2K_\phi} + \underset{\text{(Pot)}}{V} - \big(\underset{\text{(Elec)}}{K_A} + \underset{\text{(Mag)}}{G_A}\big)\Big\rangle$$

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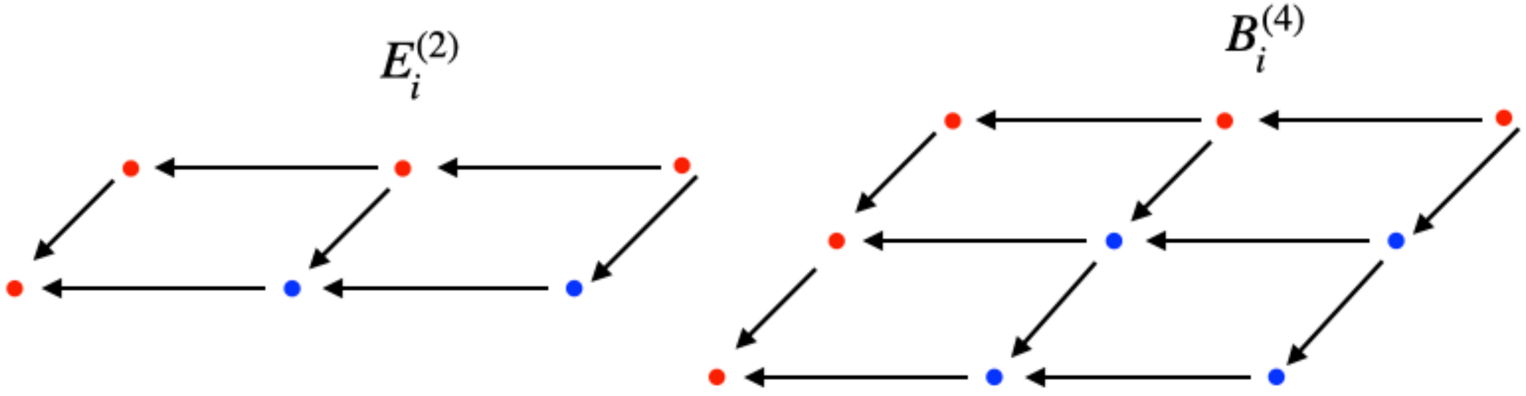
Lattice version of EOM

DGF, Shaposhnikov 2017
Canivete, DGF 2018



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Canivete, DGF 2018

1. Lattice Gauge Inv: $A_\mu \longrightarrow A_\mu + \Delta_\mu^+ \alpha$
2. Cont. Limit to $\mathcal{O}(dx^2)$
3. Lattice Bianchi Identities: $\Delta_i^-(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}) = 0\,, \ldots$
4. Topological Term: $(F_{\mu\nu}\tilde{F}^{\mu\nu})_L = \Delta_\mu^+ K^\mu$ (**CS current**)

   $[\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu\,]$

Exact Shift Sym. on the lattice !

Axion-Inflation

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Now we show our recent work on inflation:

DGF, Lizarraga, Urio, Urrestilla (+Loayza)

Phys.Rev.Lett. 131 (2023) 15, 151003 **ArXiv:2303.17436**
Phys.Rev.D 111 (2025) 6, 063545 **ArXiv: 2411.16368**

Axion-Inflation

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[preheating: Giblin et al 1502.06506, 1606.08474, 1805.04550, 2311.01504 **]**
[inflation: Caravano et al 2204.12874, Brandenburg et al 2411.04854, …**]**
[spectator axion: Caravano & Peloso 2207.13405, …**]**

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \ ; \ \frac{\phi}{4\Lambda}F\tilde{F} \ ; \ \alpha_\Lambda = 18\right)$

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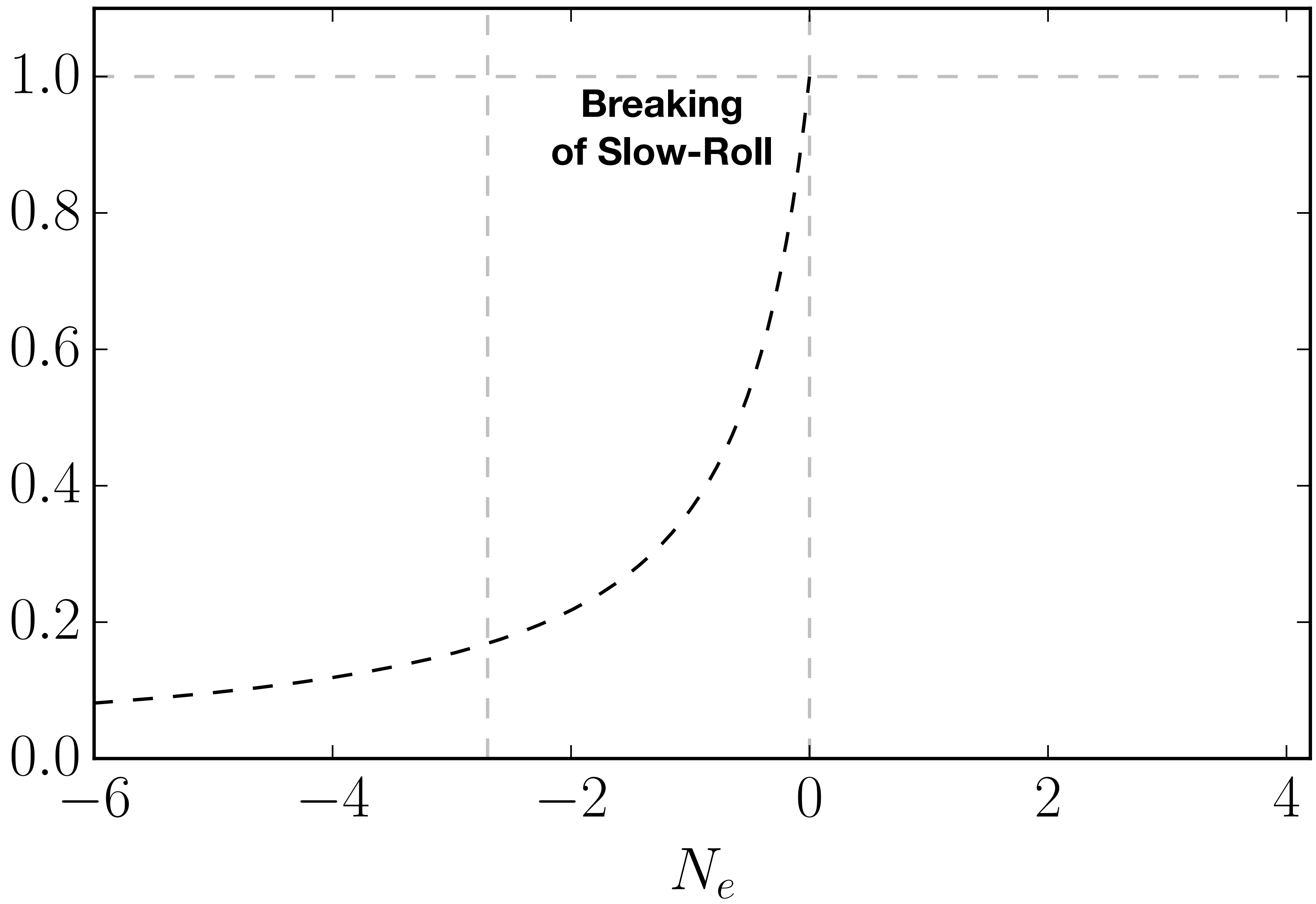
Linear regime (- - -)

Epsilon Parameter

$\epsilon_H \equiv -\frac{\dot{H}}{H^2}$



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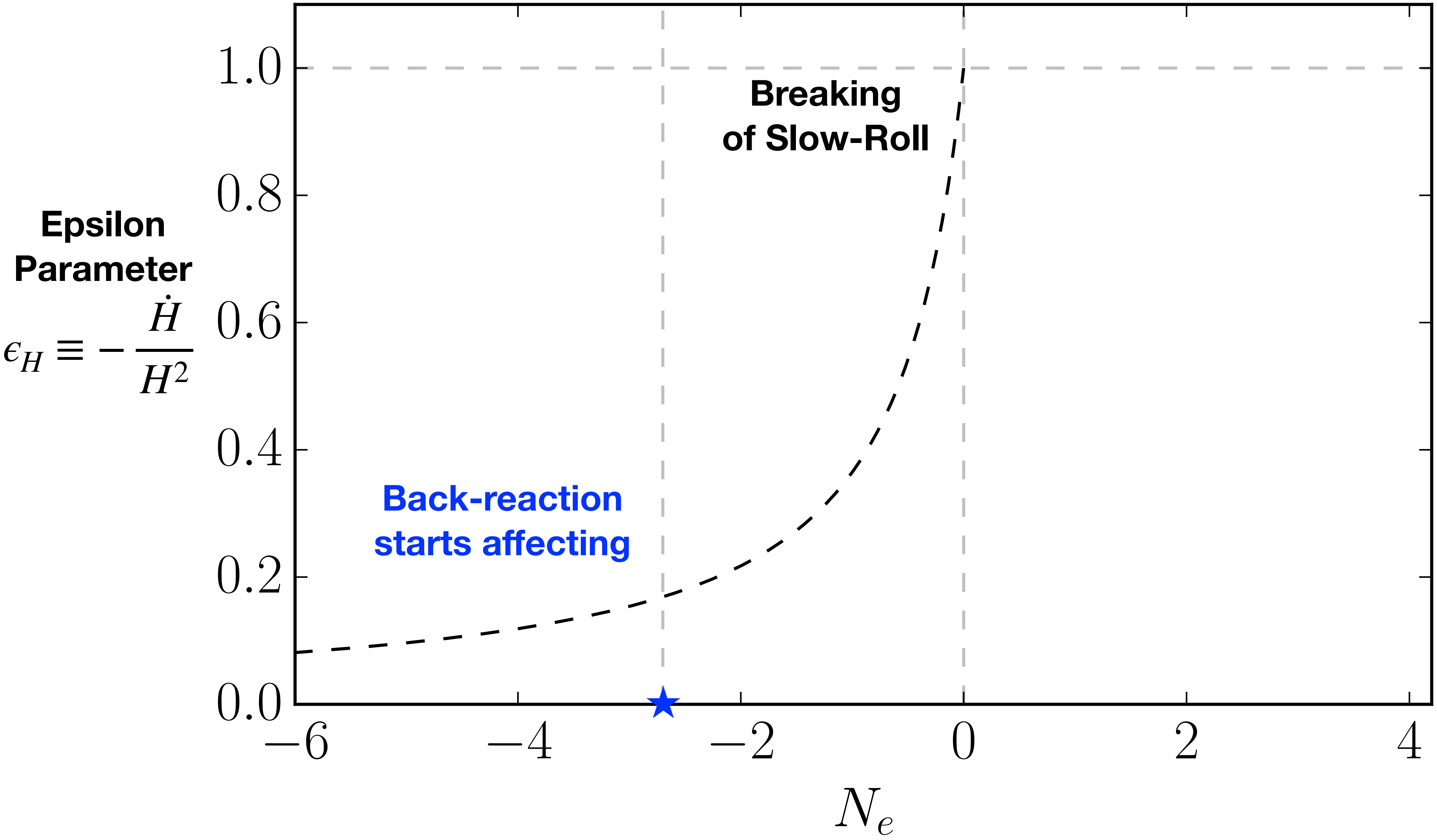
Linear regime (– – –)



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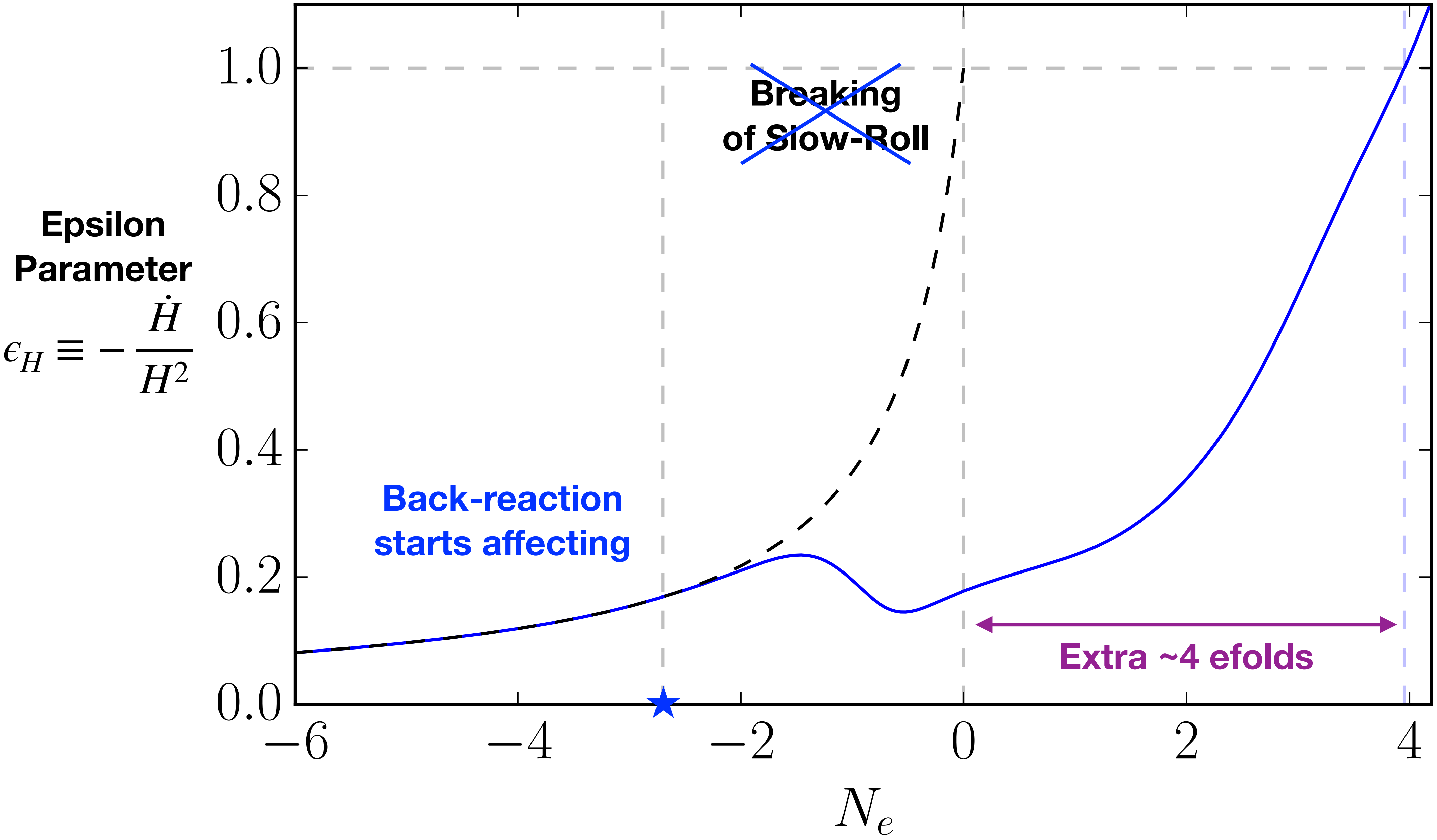
Linear regime (– – –) **Non-Linear regime (——)**



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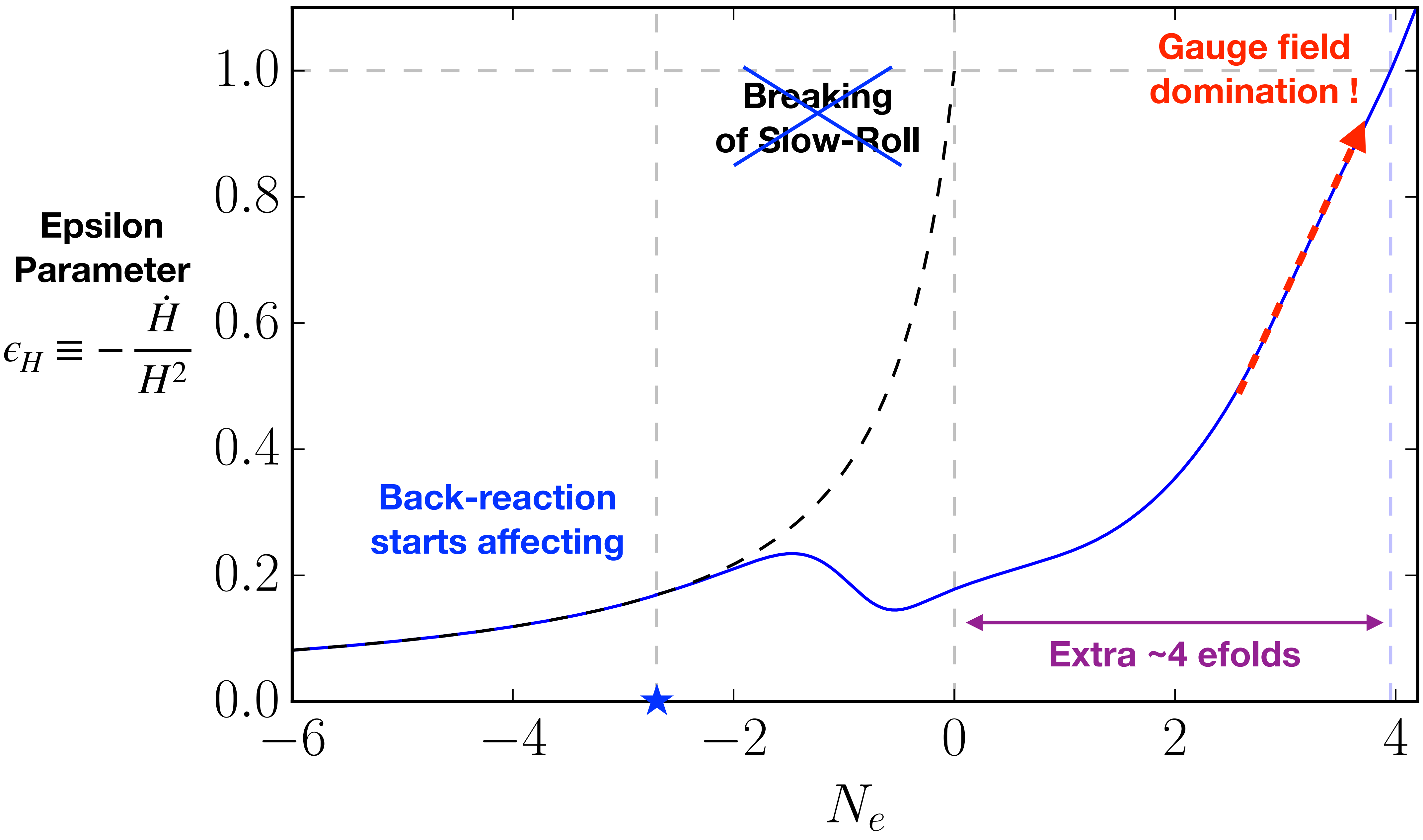
Linear regime (- - -) **Non-Linear regime (——)**



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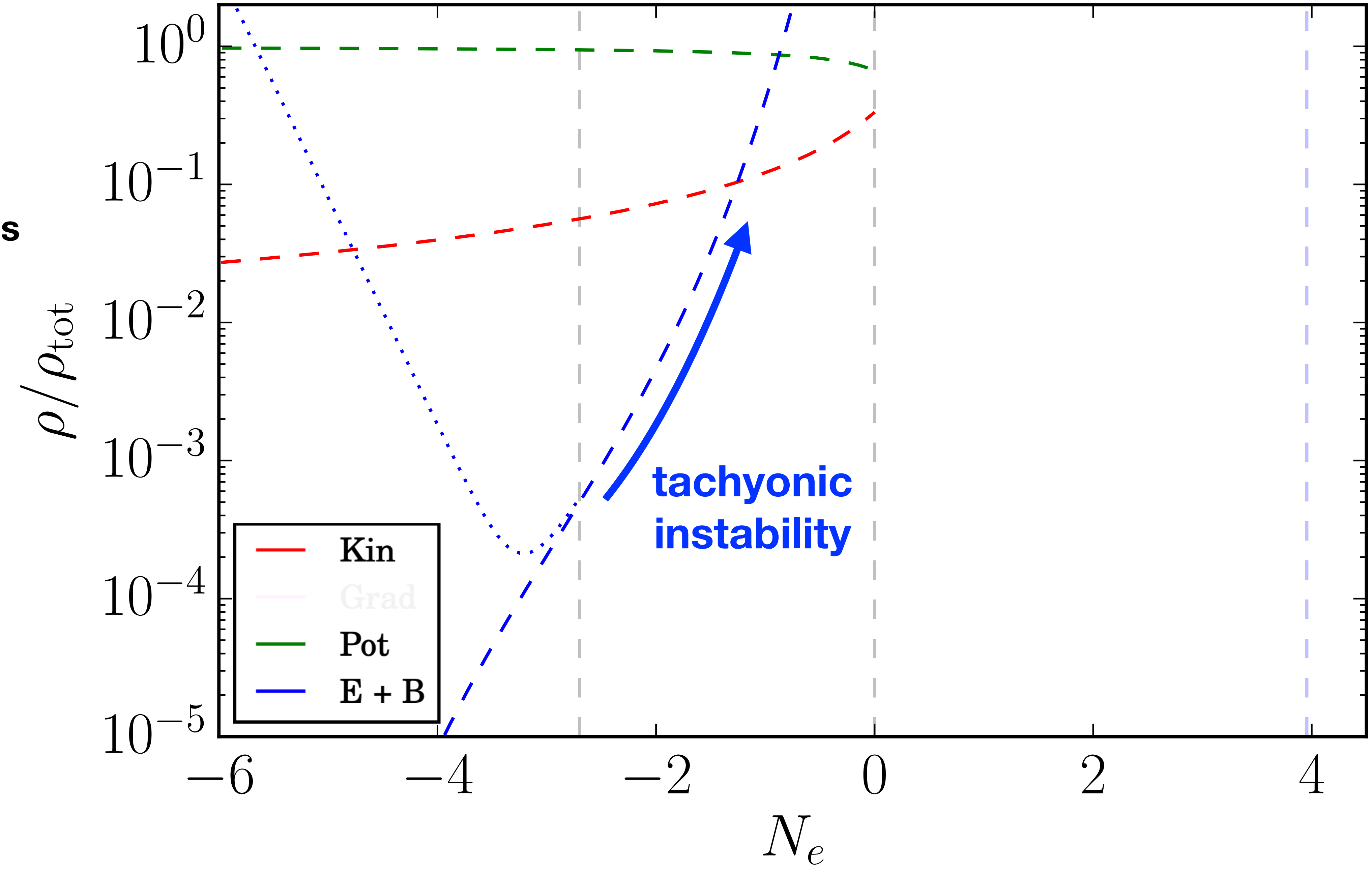
Linear (– – –)

Energy Densities



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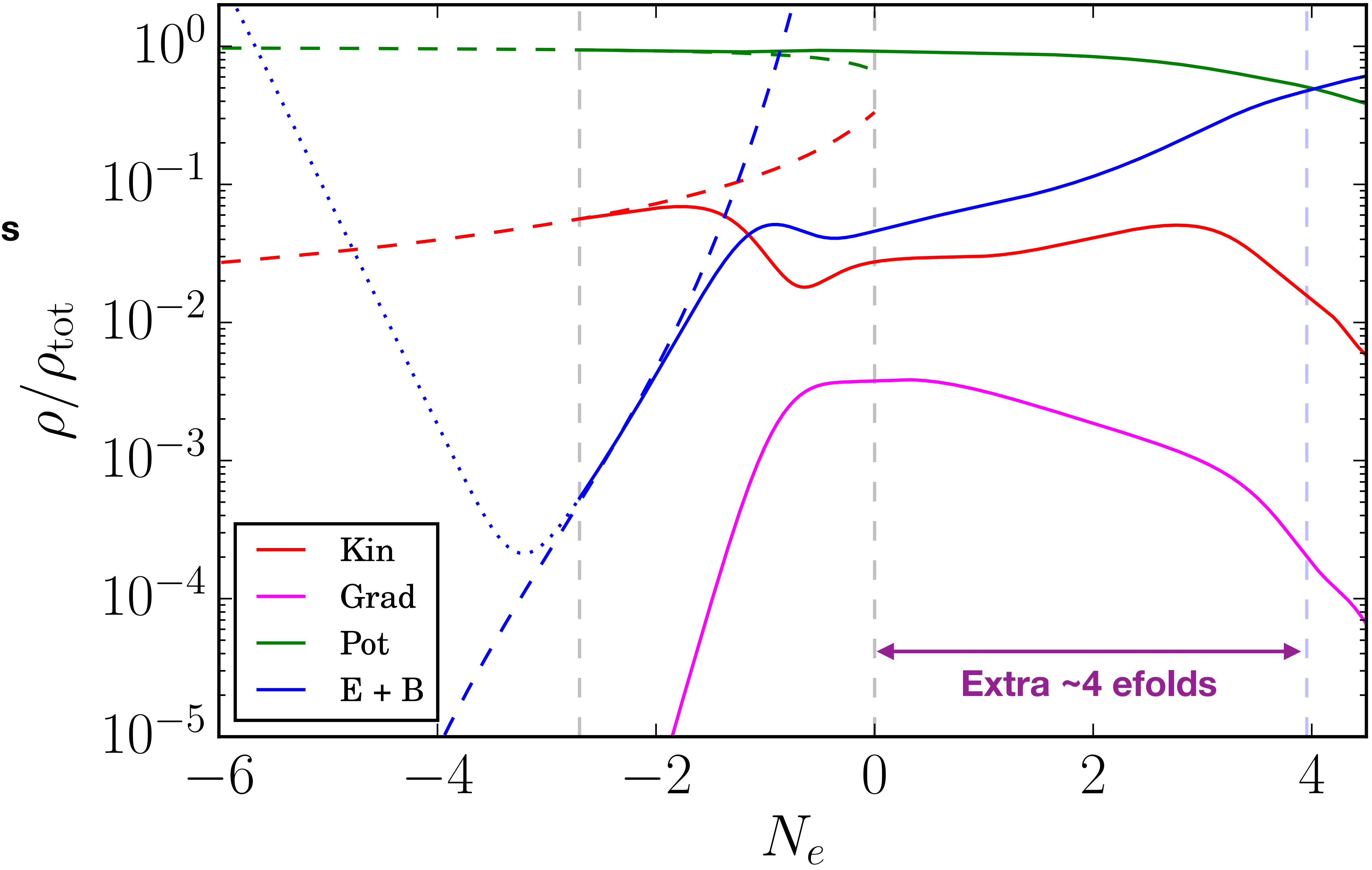
Linear (- - -) **Non-Linear (——)**

Energy Densities



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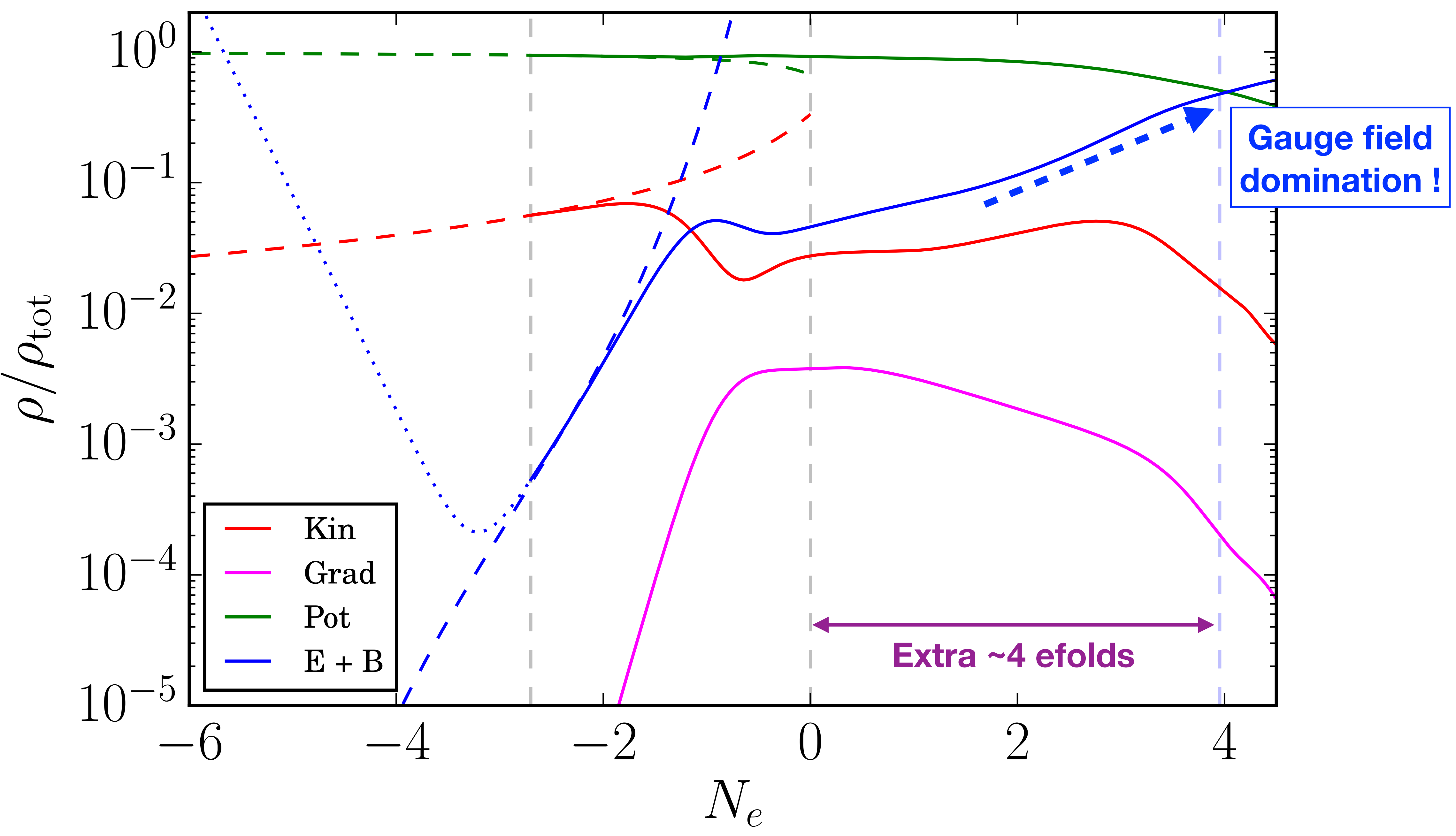
Linear (- - -) **Non-Linear (——)**

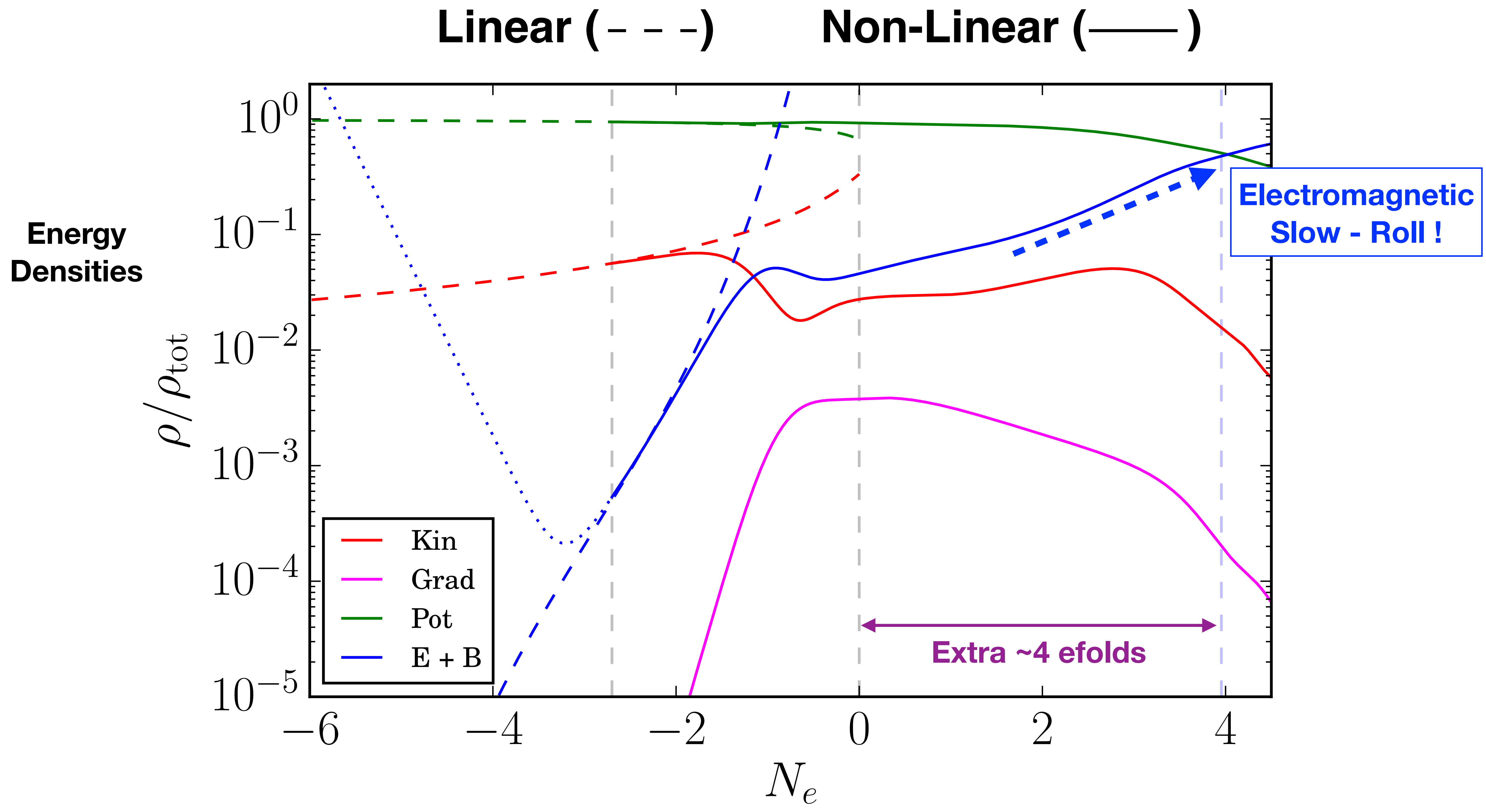
Energy Densities



Gauge field domination !

Extra ~4 efolds

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$



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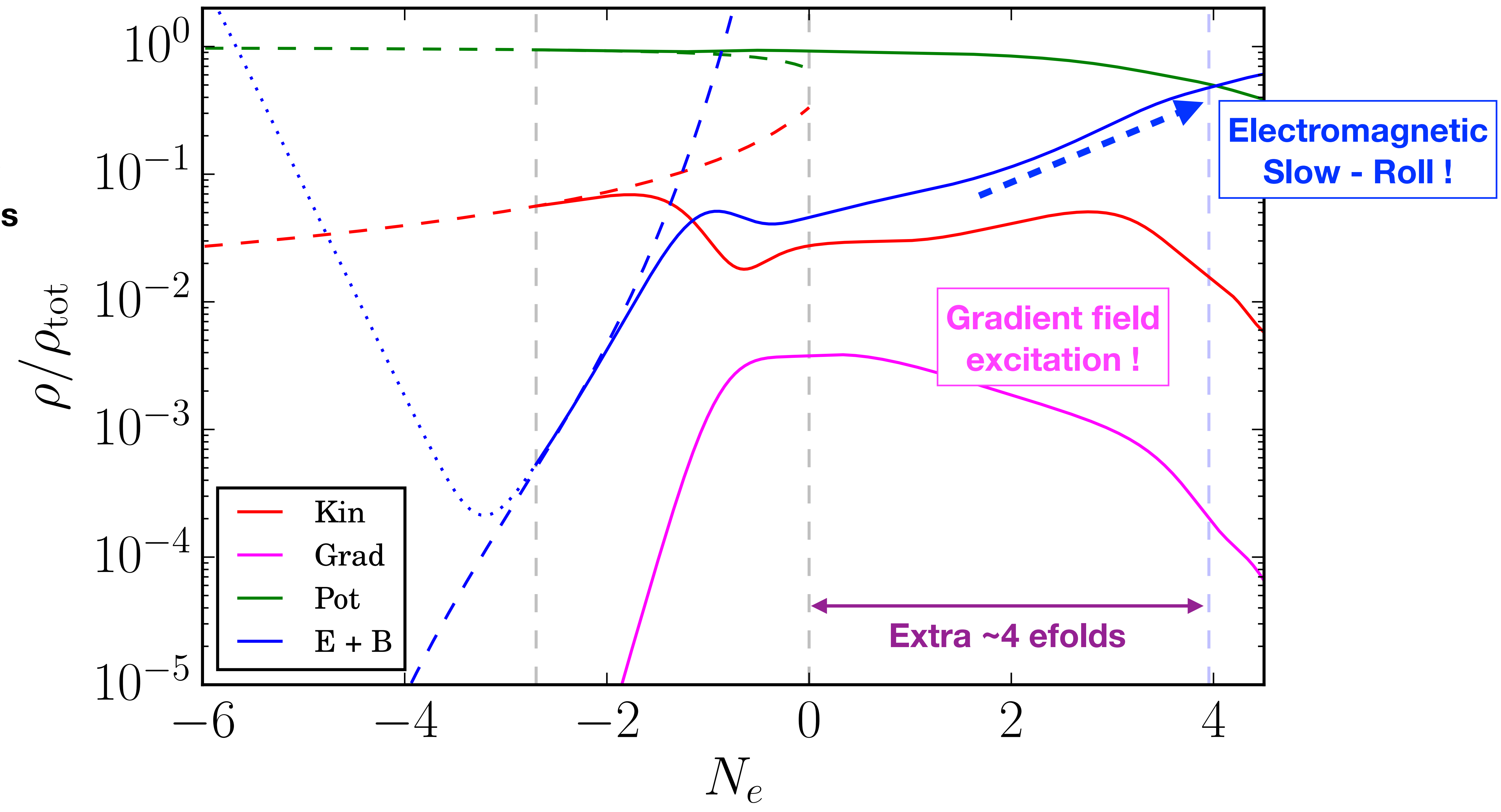
Linear (- - -) **Non-Linear (——)**

Energy Densities



Electromagnetic Slow - Roll !

Gradient field excitation !

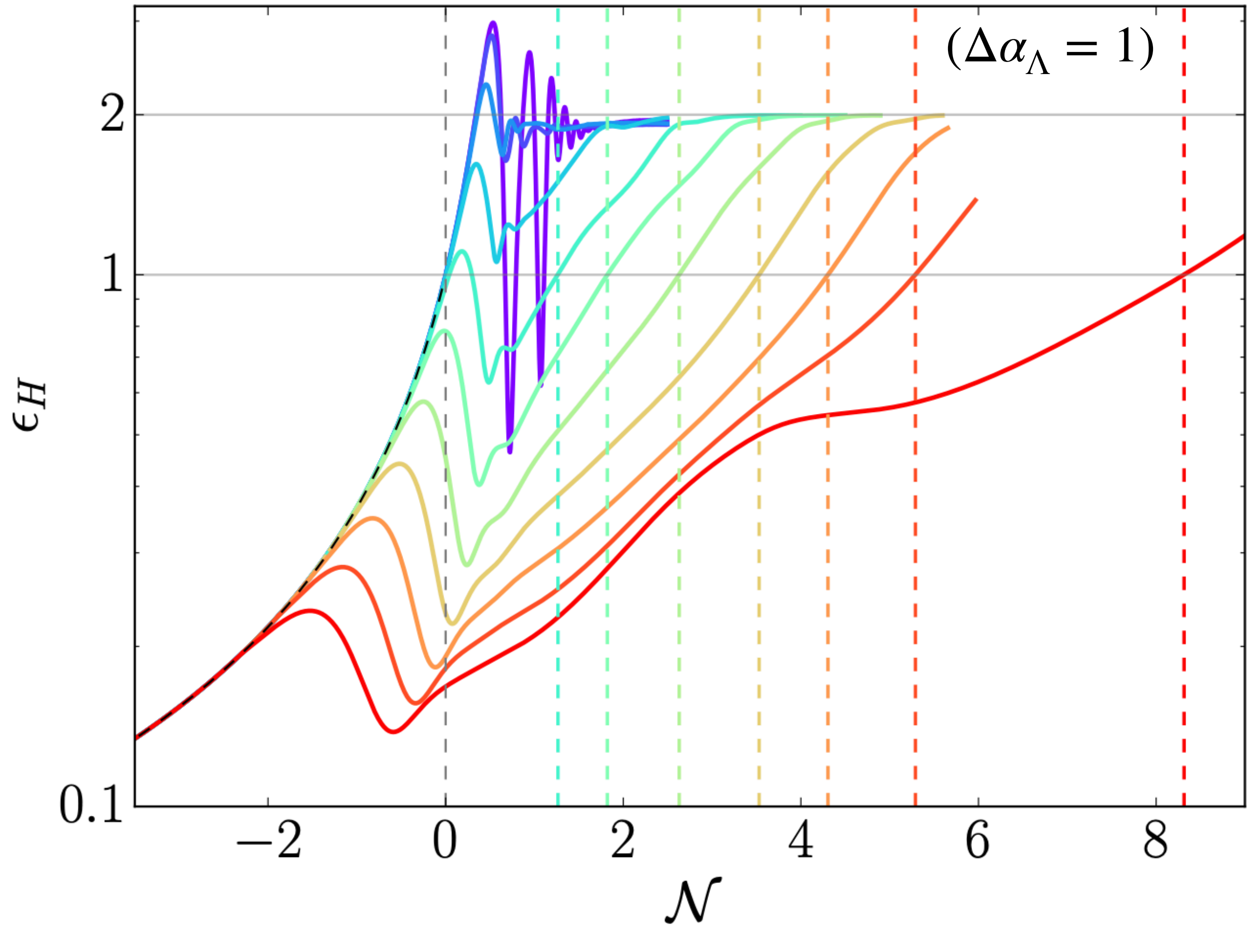
Extra ~4 efolds

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \alpha_\Lambda = X$)

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}$$

(Epsilon Parameter)

$\alpha_\Lambda = 10$ **(purple)** $\ldots$ $\alpha_\Lambda = 20$ **(red)**



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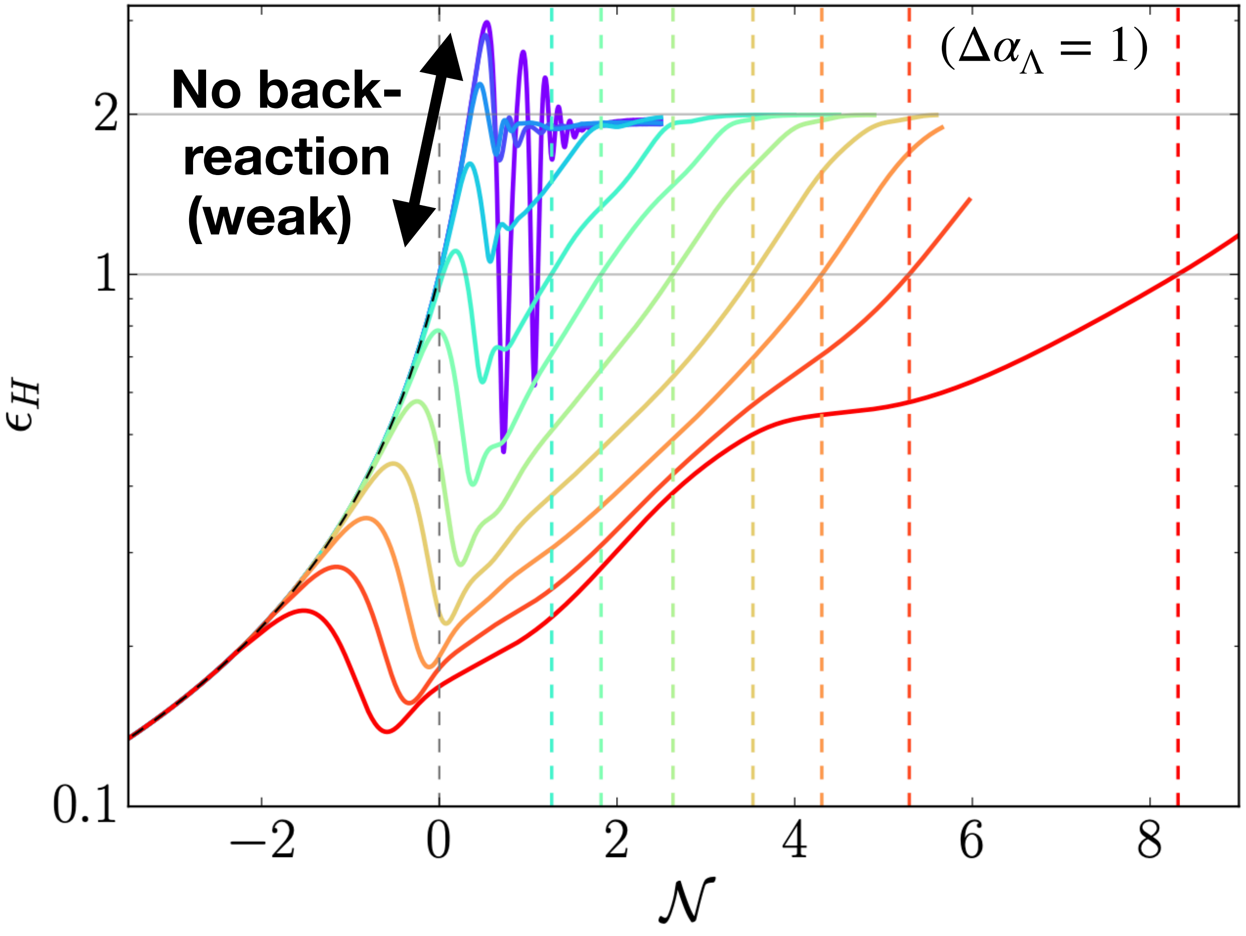

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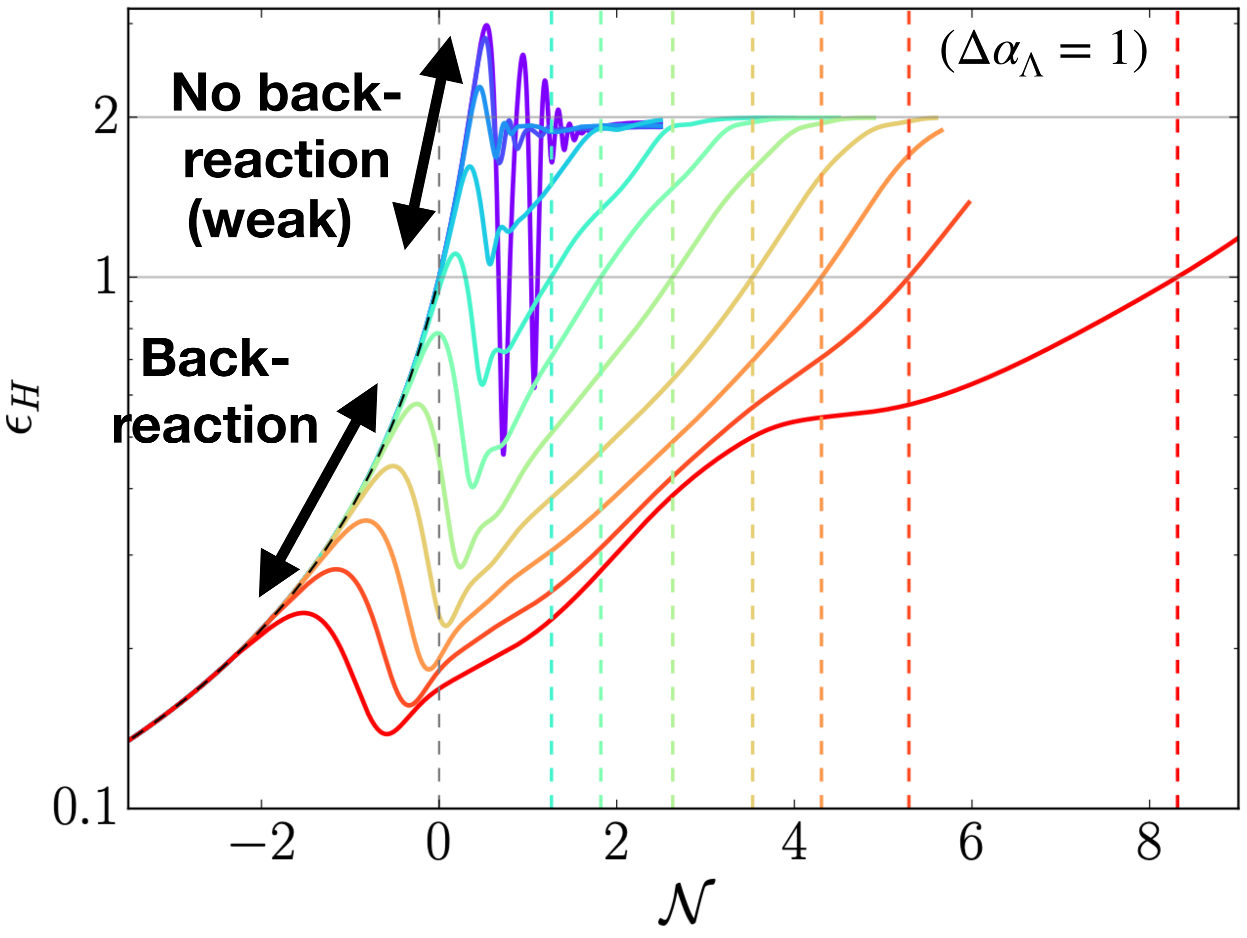

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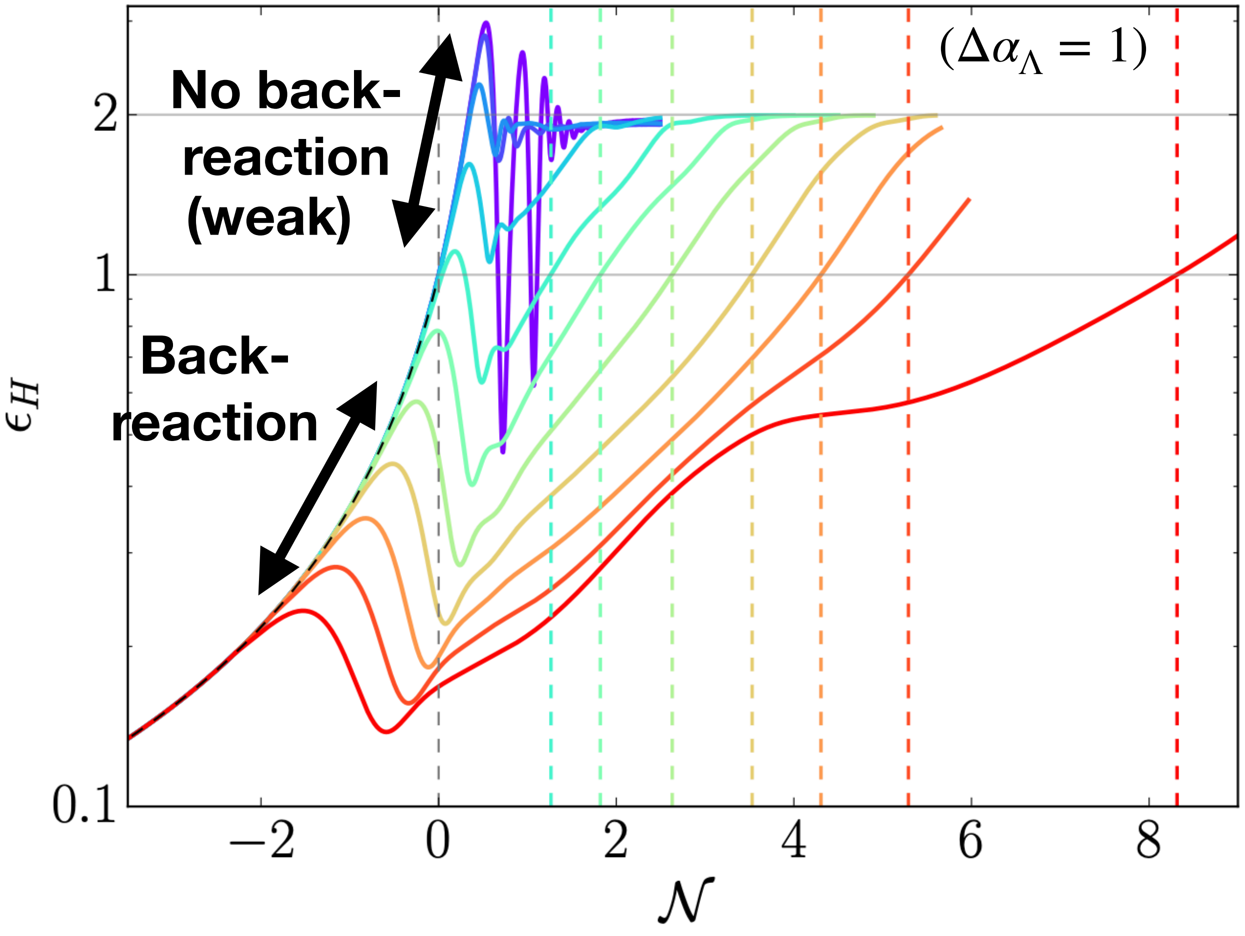
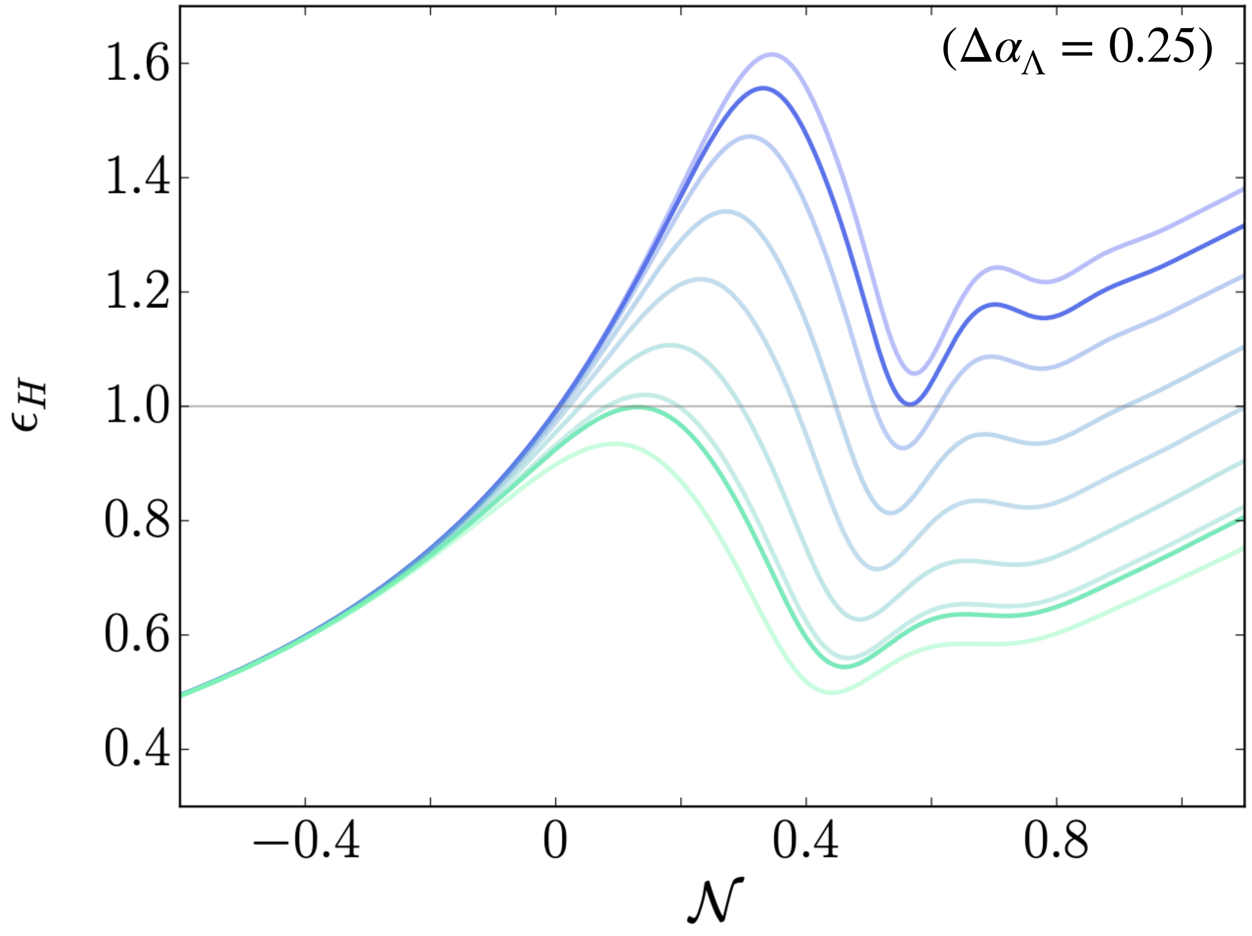


$\alpha_\Lambda = 13.1$ **(blue)** $\ldots$ $\alpha_\Lambda = 14.31$ **(green)**



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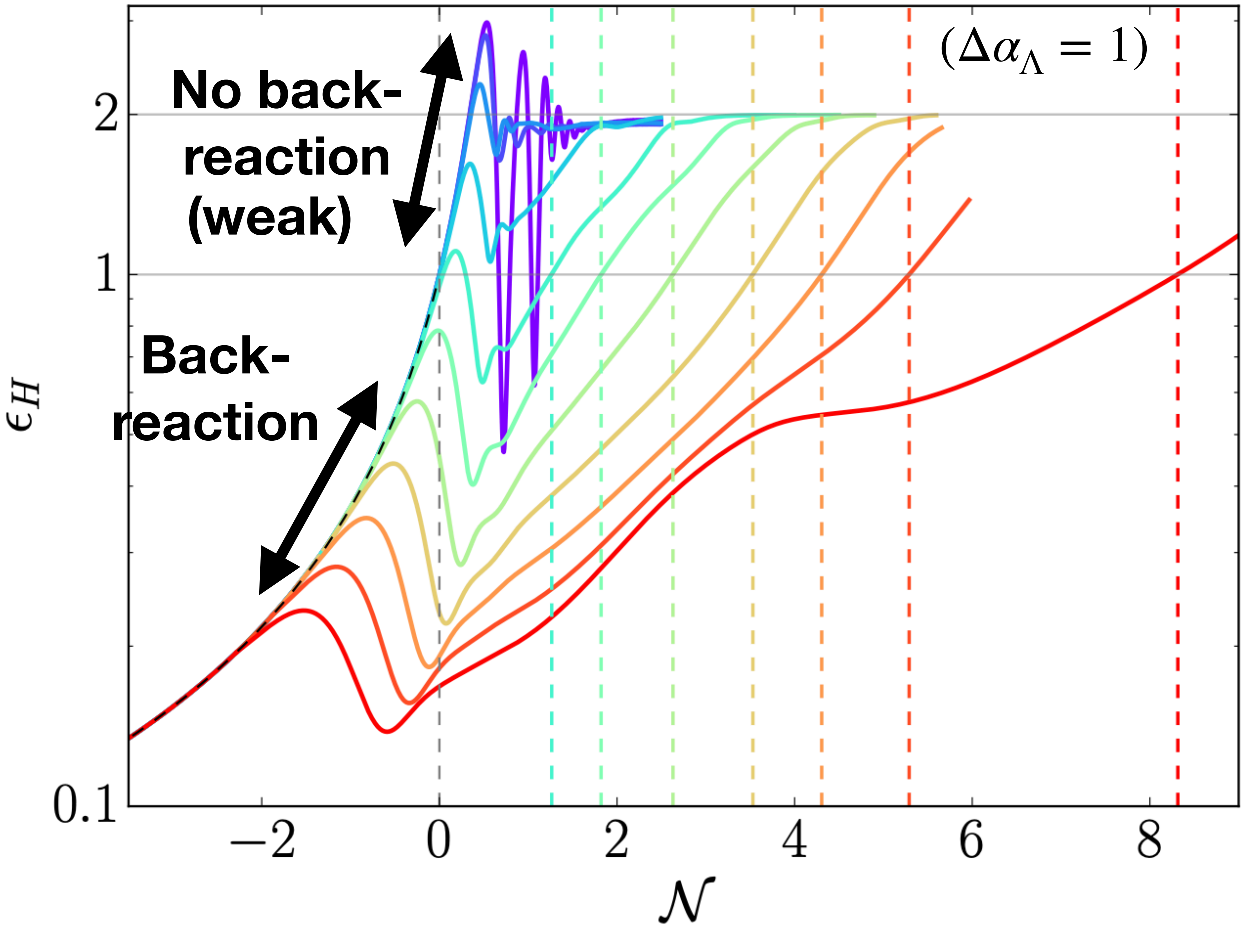
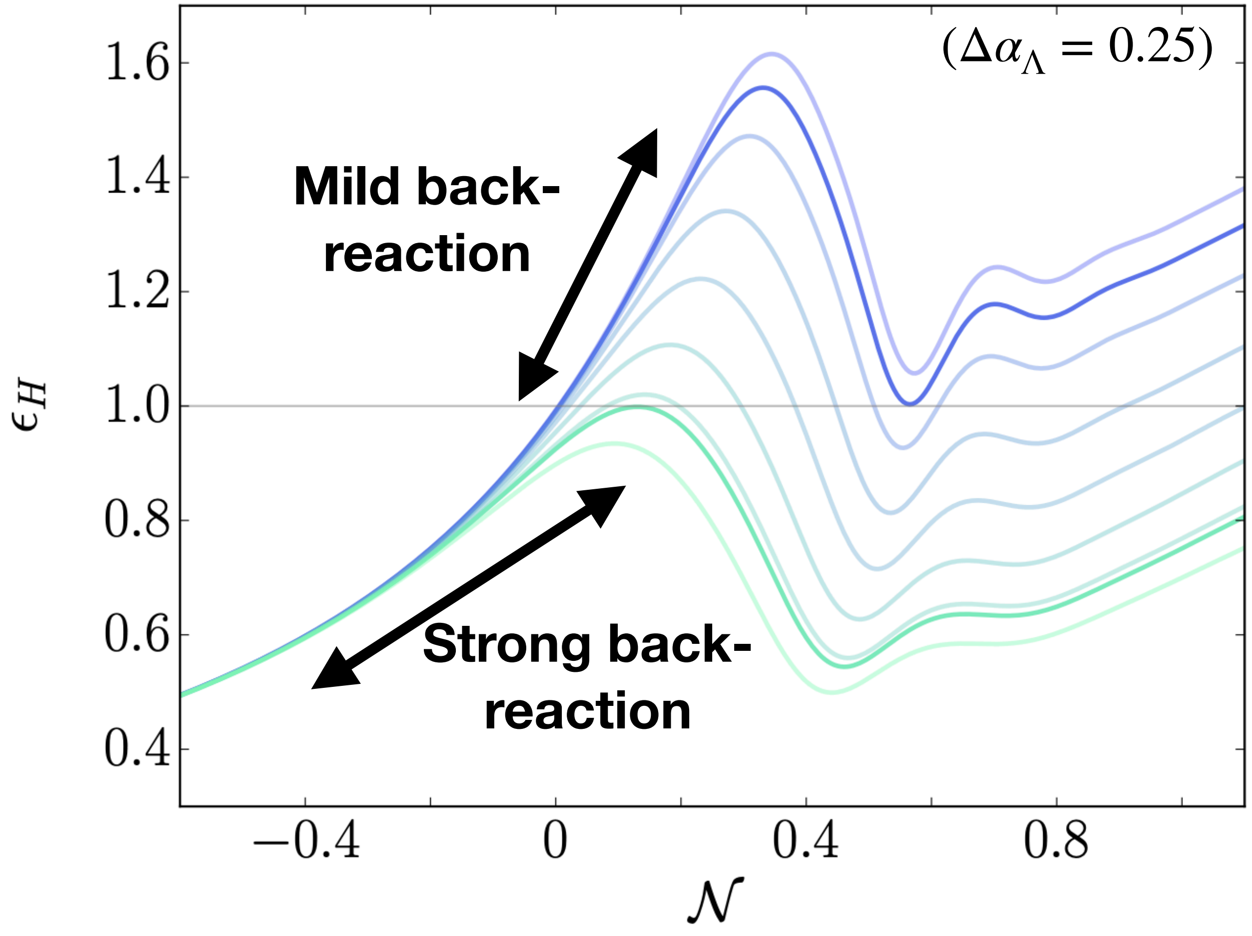


No back-reaction (weak)

Back-reaction

$\alpha_\Lambda = 13.1$ **(blue)** … $\alpha_\Lambda = 14.31$ **(green)**



Mild back-reaction

Strong back-reaction

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = X$)

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}$$

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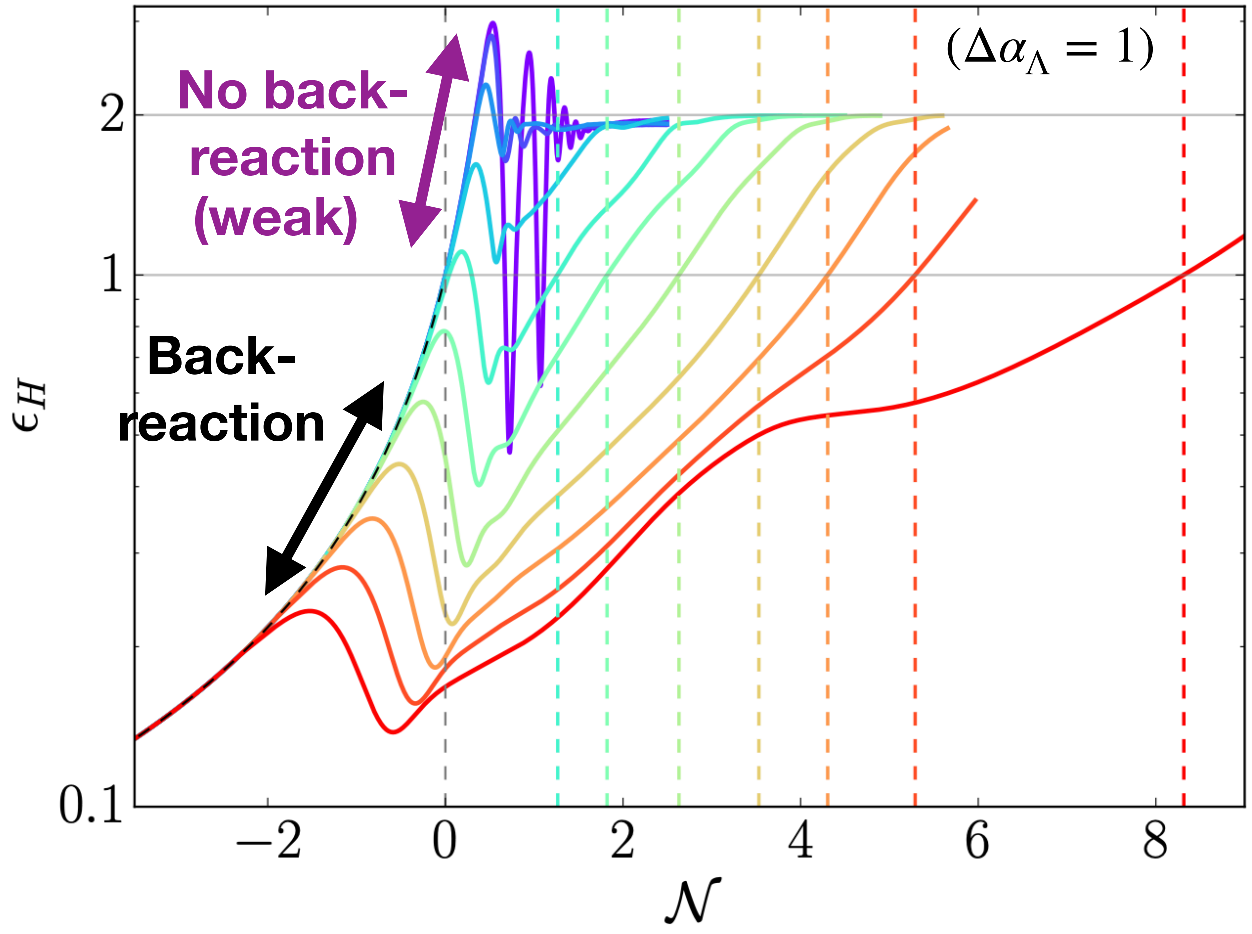

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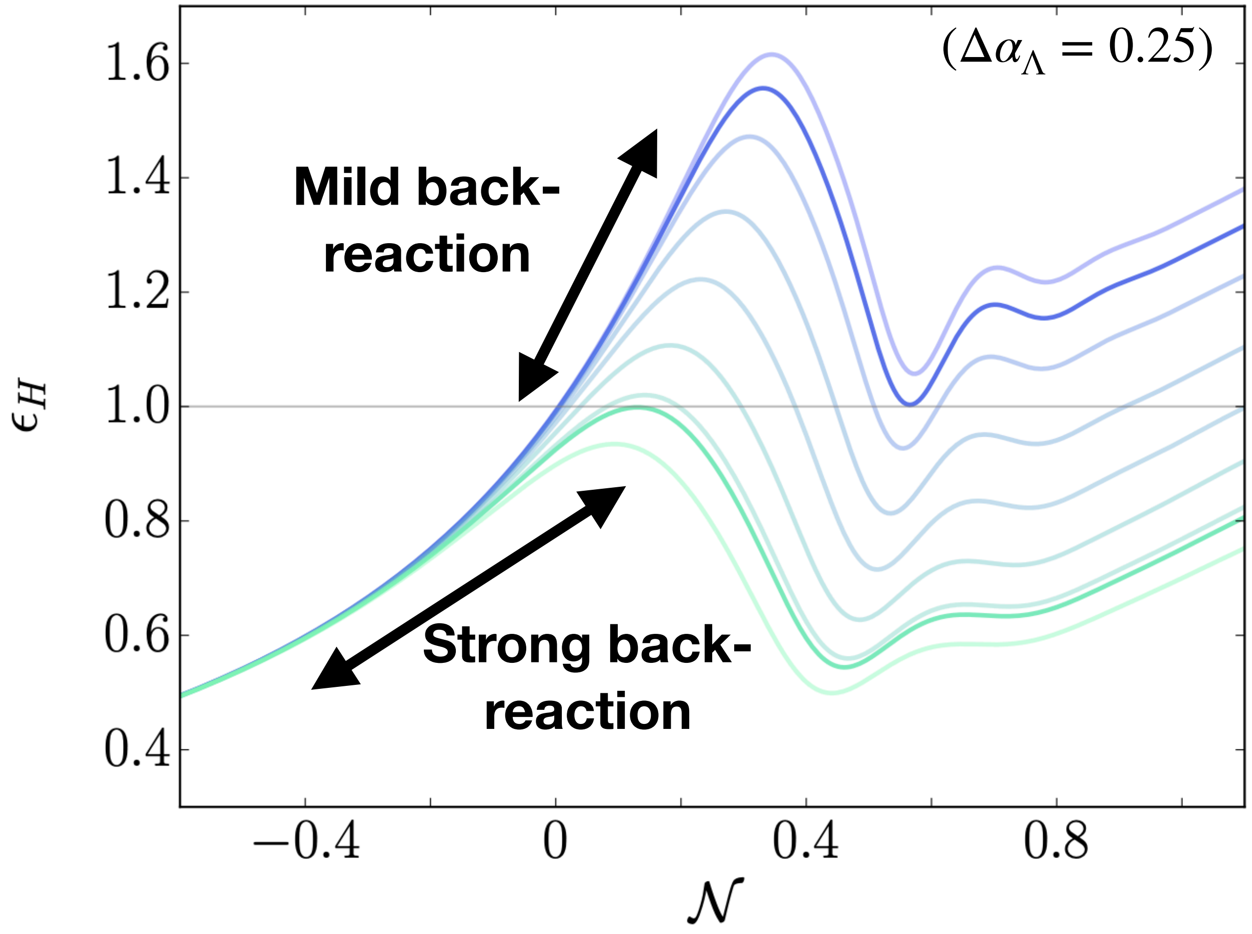

No-to-weak BR: $\alpha_\Lambda \lesssim 13.1$

Mild BR: $13.1 \lesssim \alpha_\Lambda \lesssim 14.31$

Strong BR: $\alpha_\Lambda \gtrsim 14.31$

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = X$)

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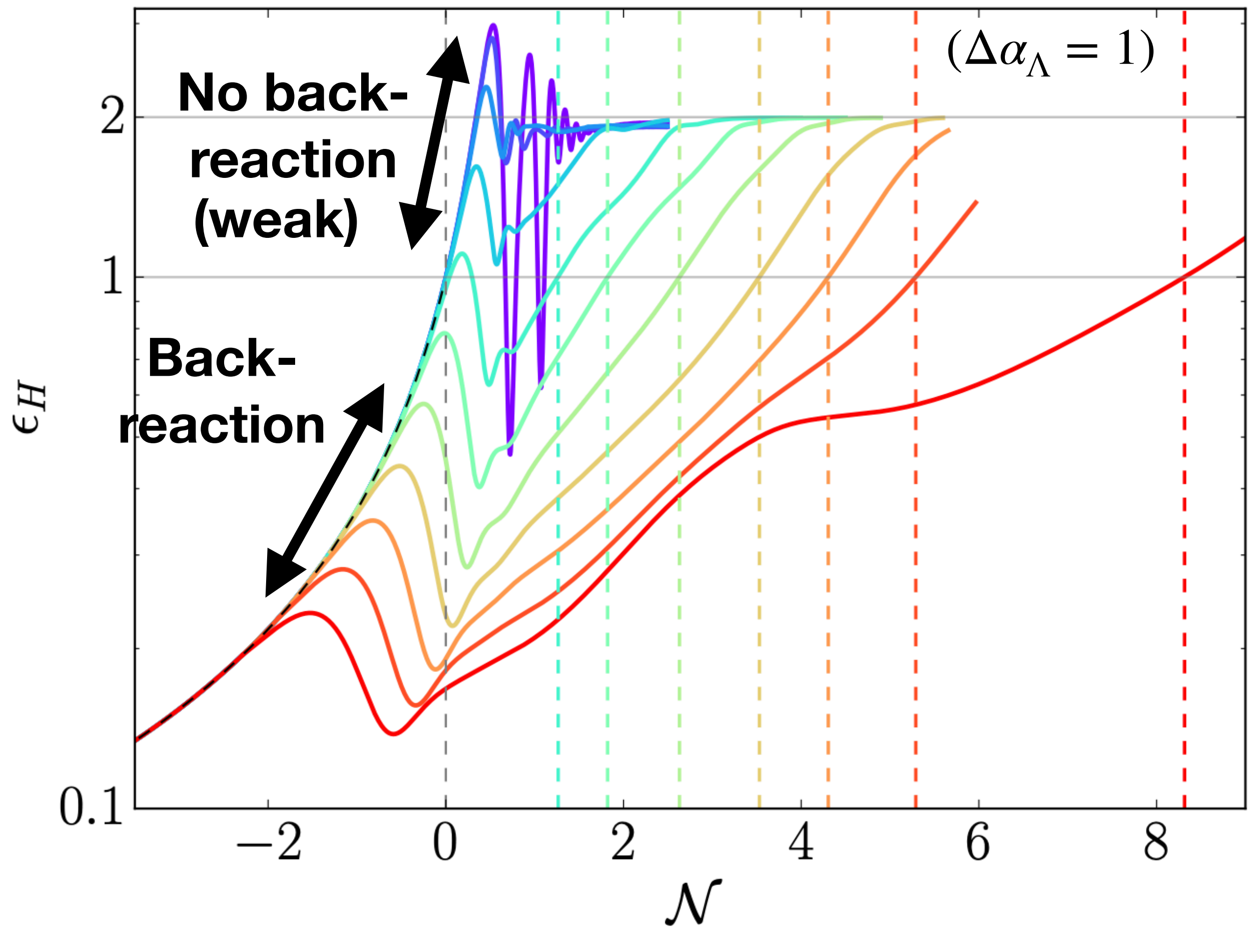

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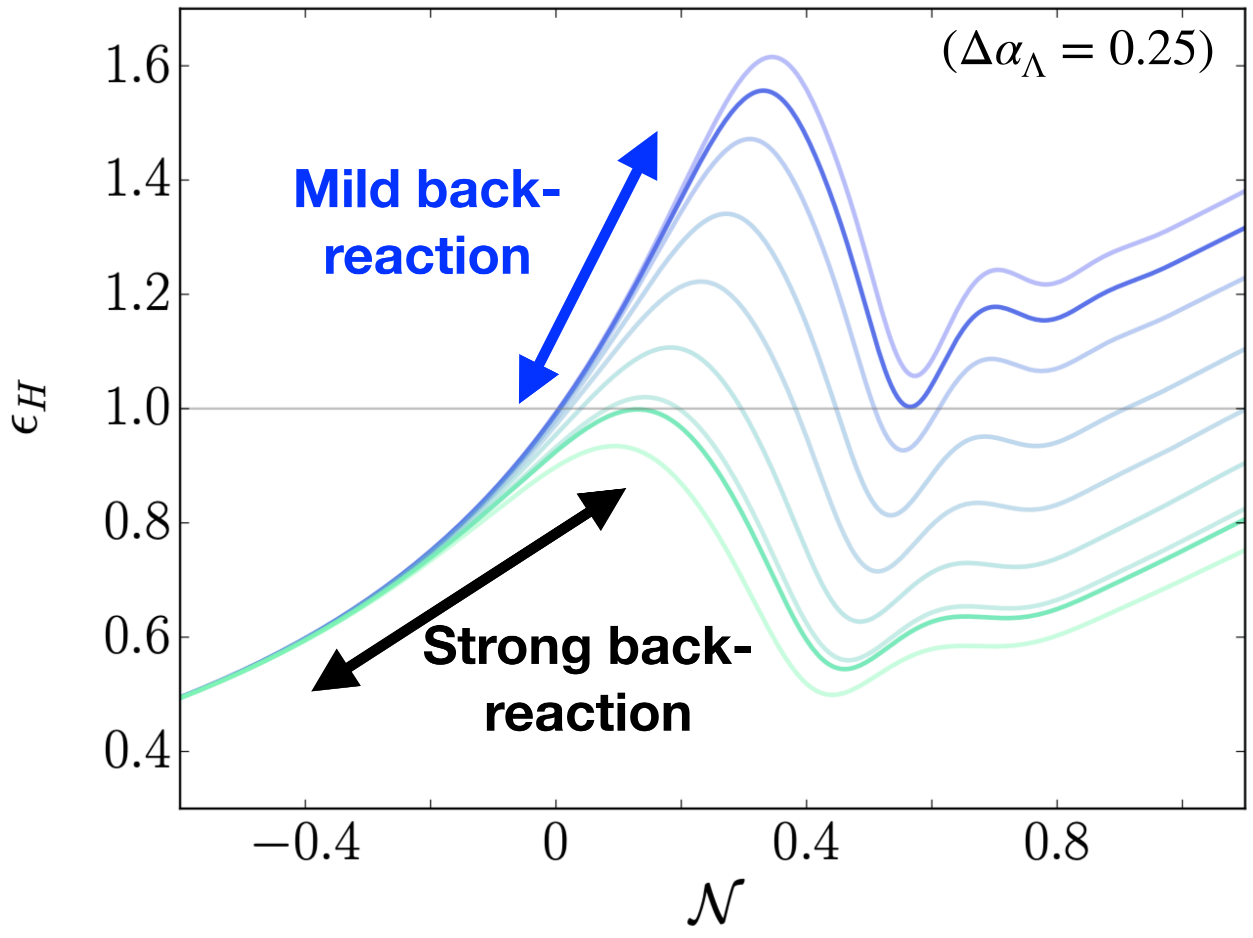

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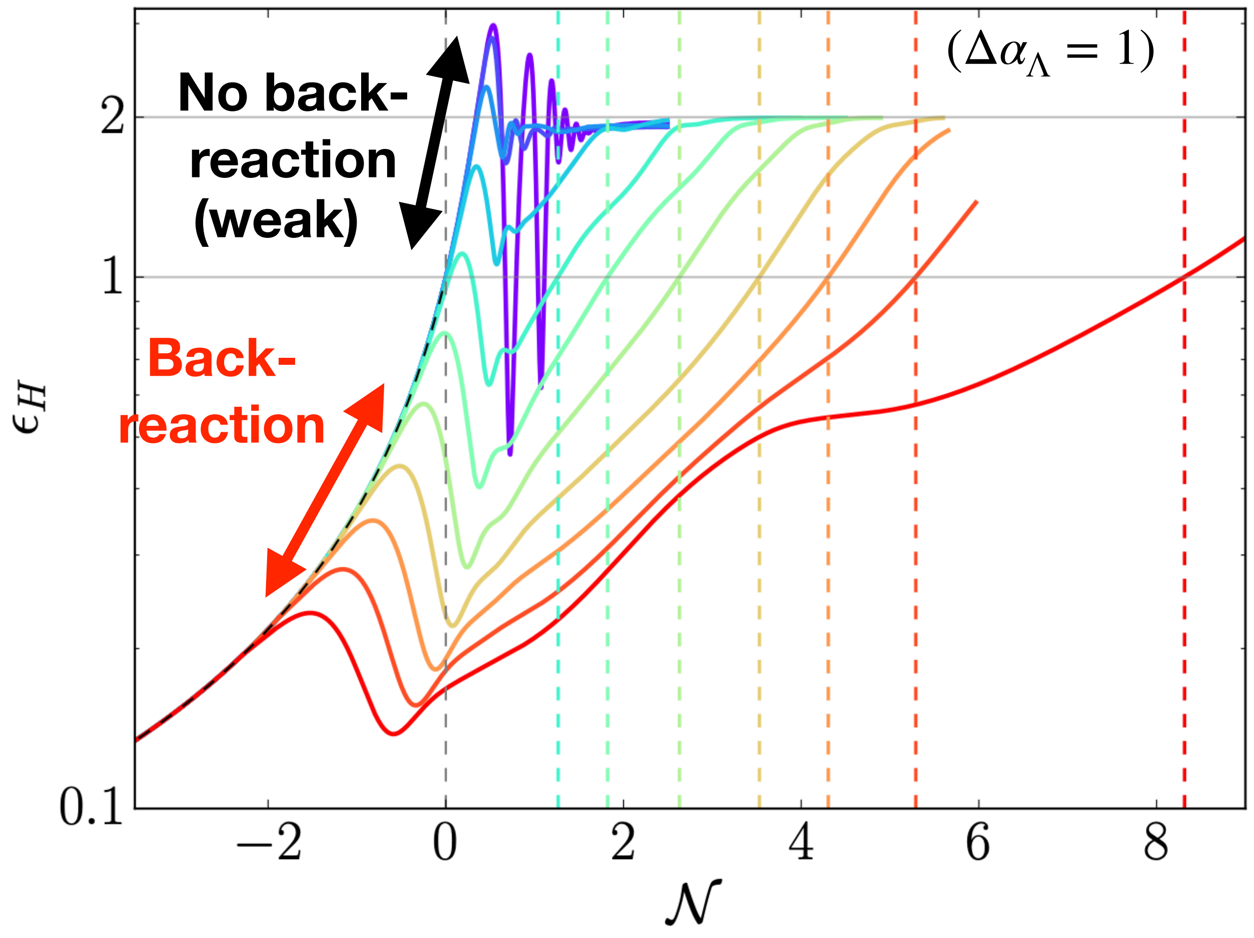
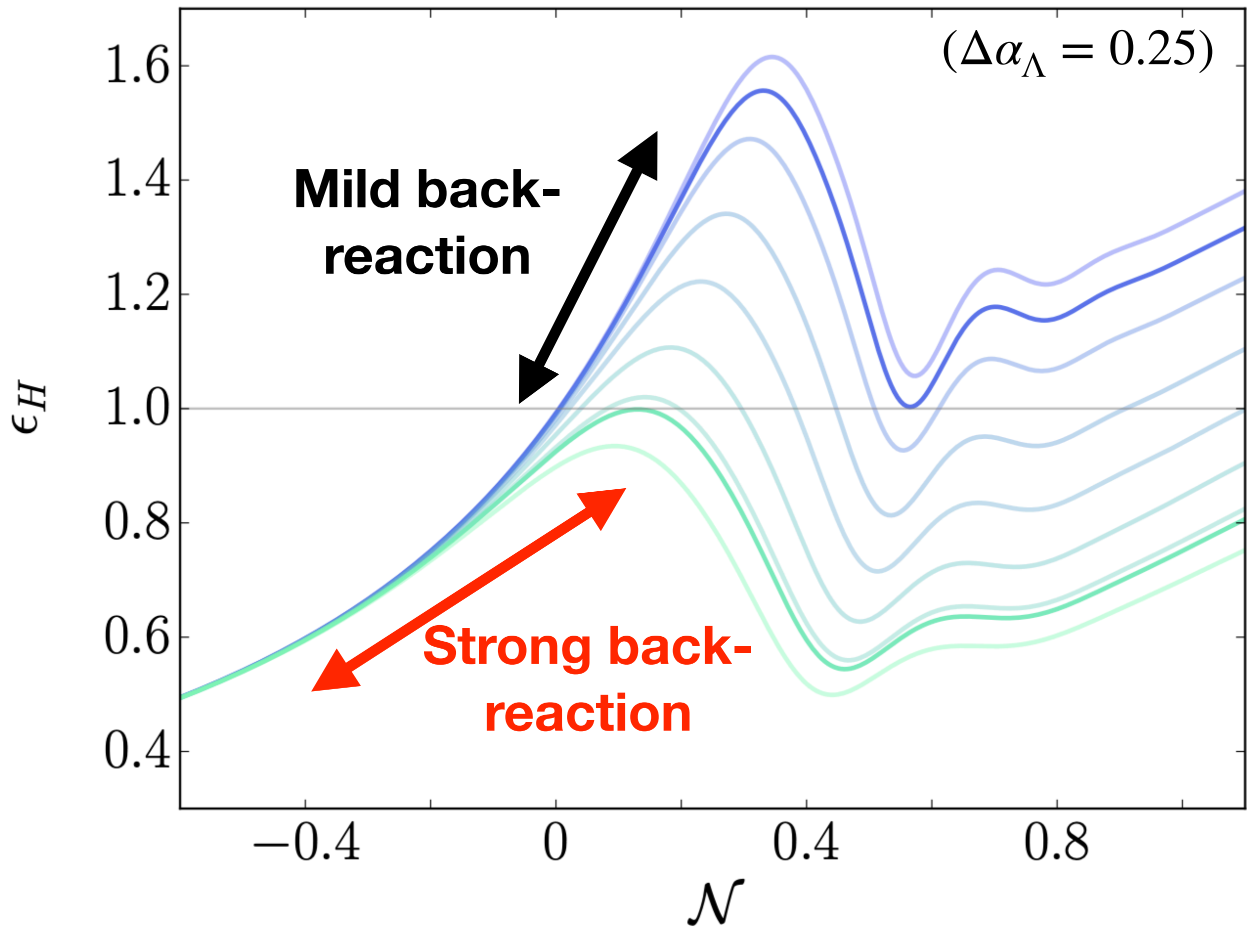


$\alpha_\Lambda = 13.1$ **(blue)** ... $\alpha_\Lambda = 14.31$ **(green)**



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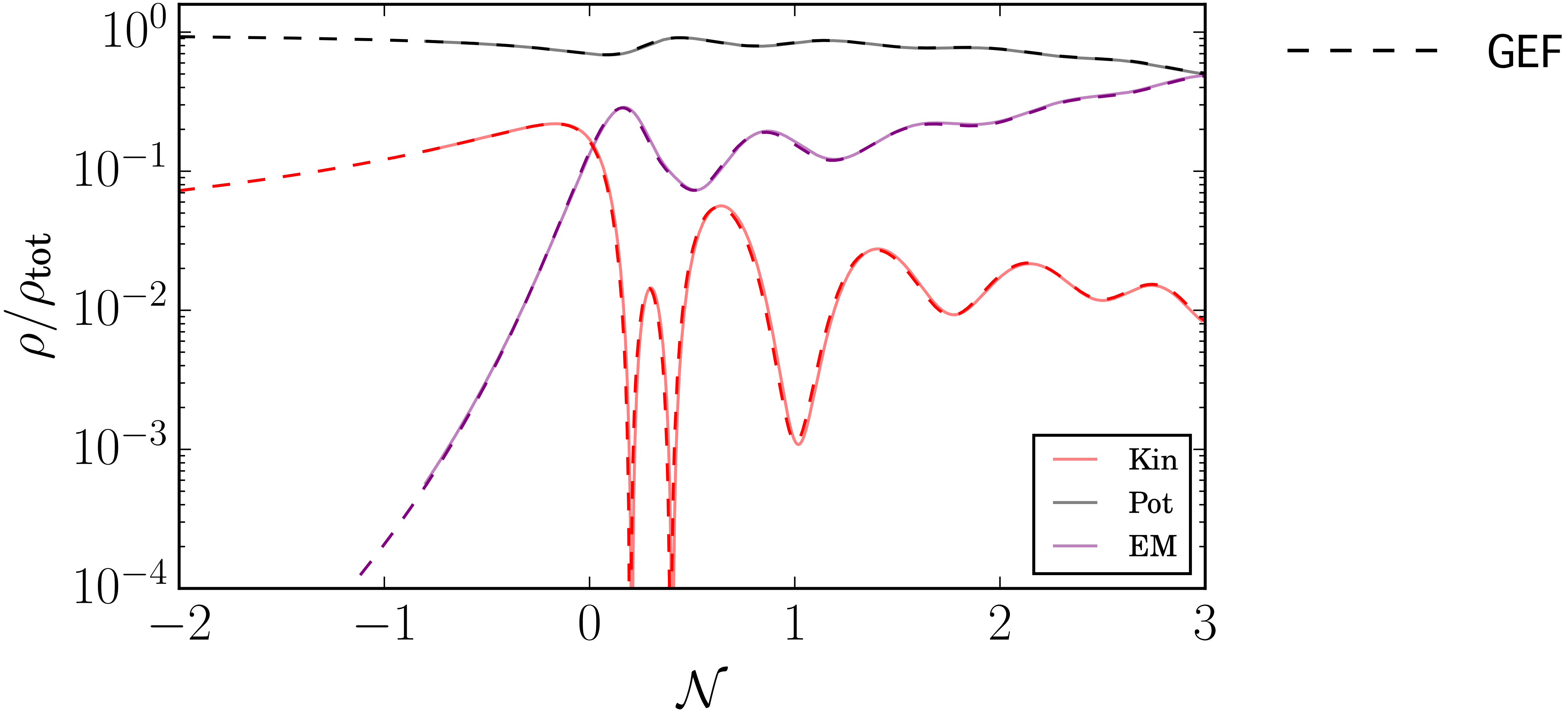
Comparison to Homogeneous Backreaction

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 15\right)$

Comparison to GEF (Homogeneous backreaction)

GEF (O. Sobol, R. von Eckardstein, K. Schmitz)



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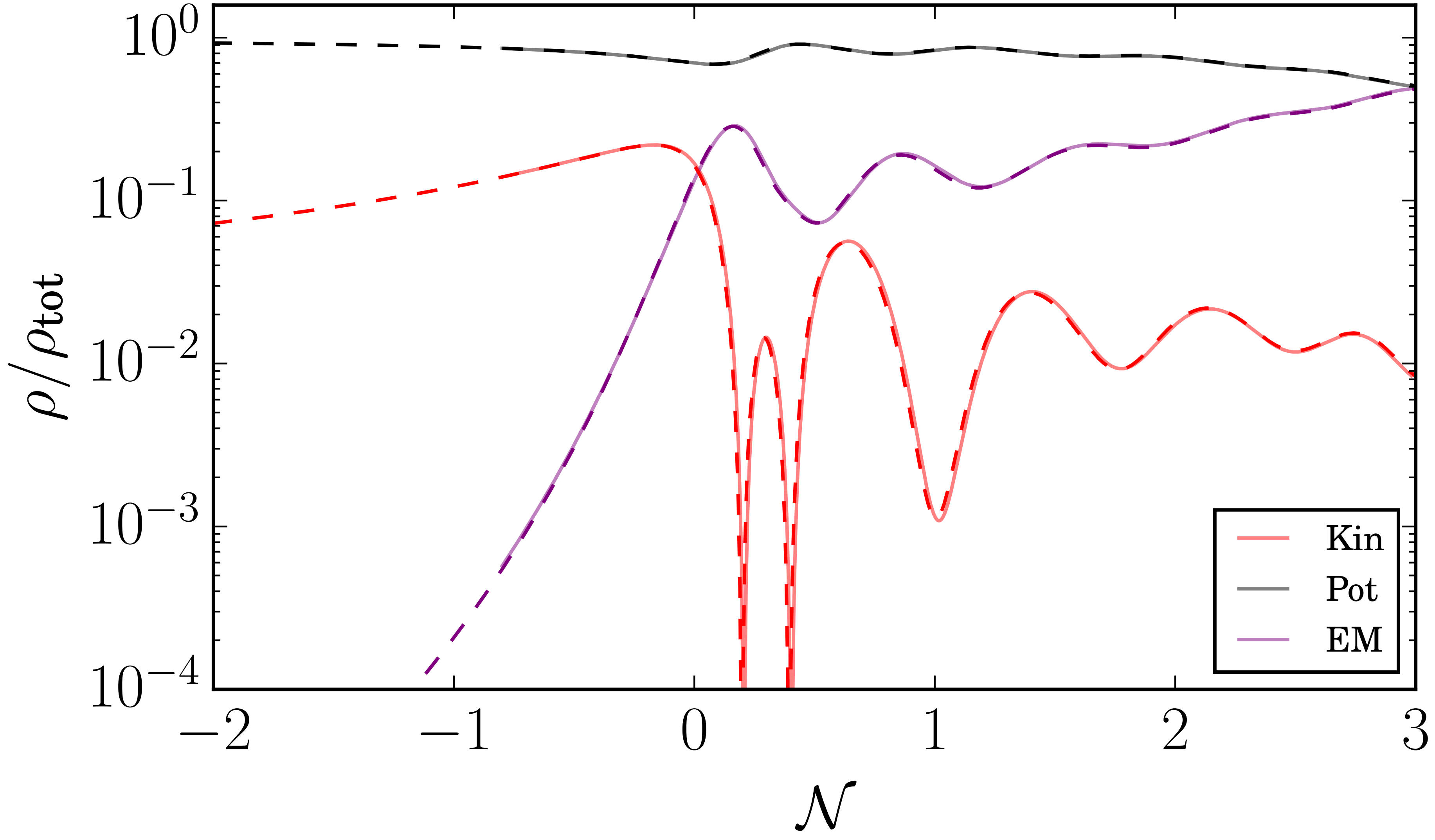

- - - - GEF

——— Lattice*

* Lattice:

$\nabla\phi \longrightarrow 0$

$\vec{E}\cdot\vec{B} \longrightarrow <\vec{E}\cdot\vec{B}>$

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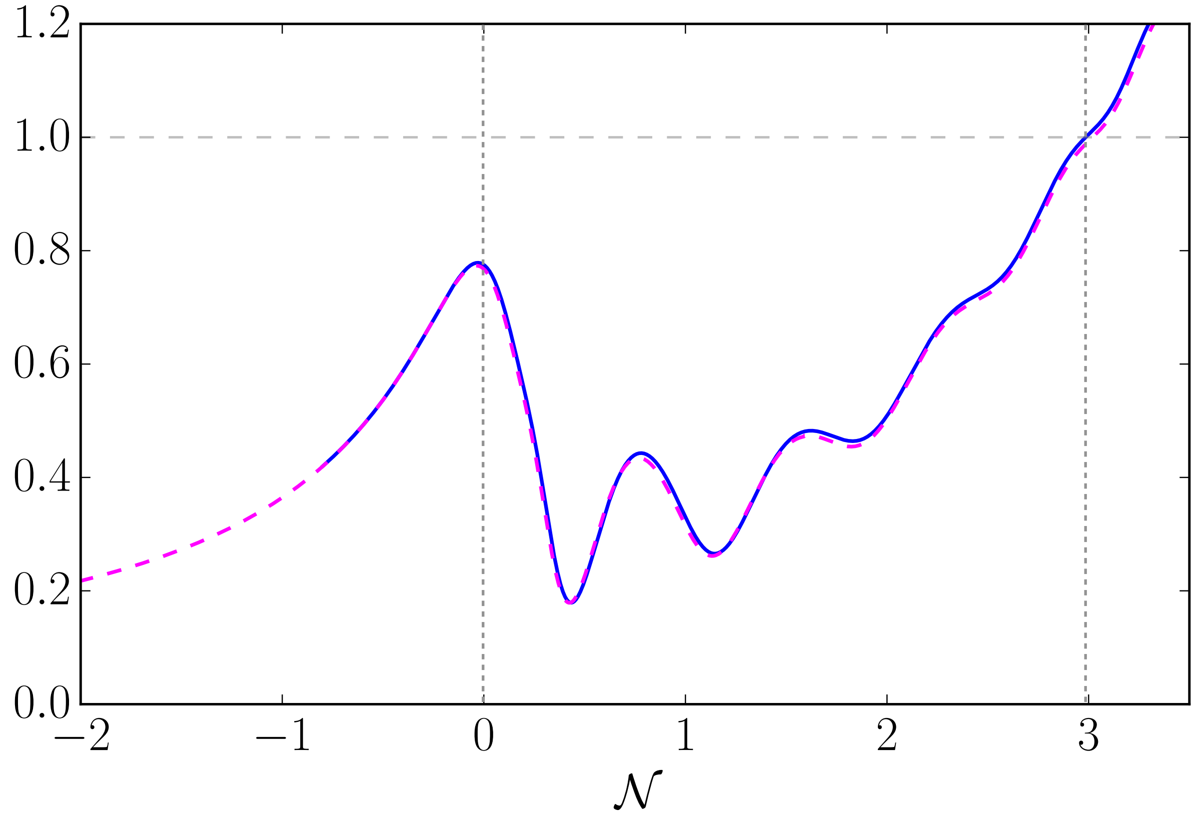


$\Bigg($ ┅┅ GEF

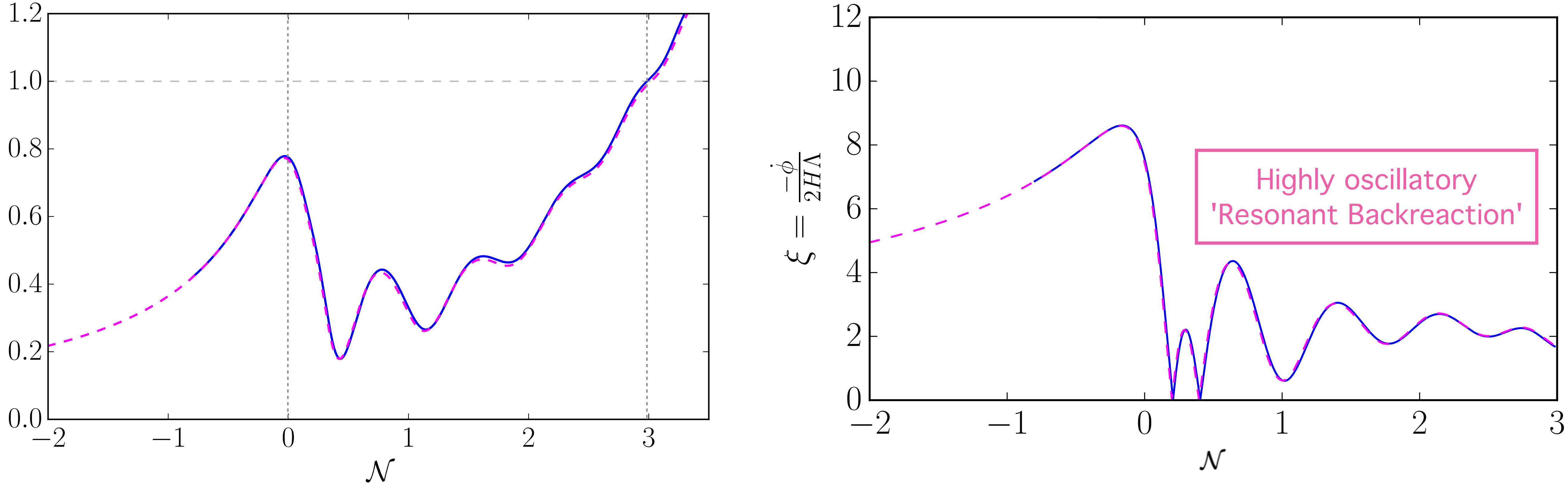
── Lattice*: $\nabla\phi \longrightarrow 0 \ , \ \vec{E}\cdot\vec{B} \longrightarrow < \vec{E}\cdot\vec{B} > \Bigg)$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 15\right)$

Comparison to GEF (Homogeneous backreaction)

GEF (O. Sobol, R. von Eckardstein, K. Schmitz)

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} \qquad\qquad \xi \equiv \frac{|\dot{\phi}|}{2H\Lambda}$$



(- - - - GEF

—— Lattice*: $\nabla\phi \longrightarrow 0\ ,\ \vec{E}\cdot\vec{B} \longrightarrow <\vec{E}\cdot\vec{B}>$)

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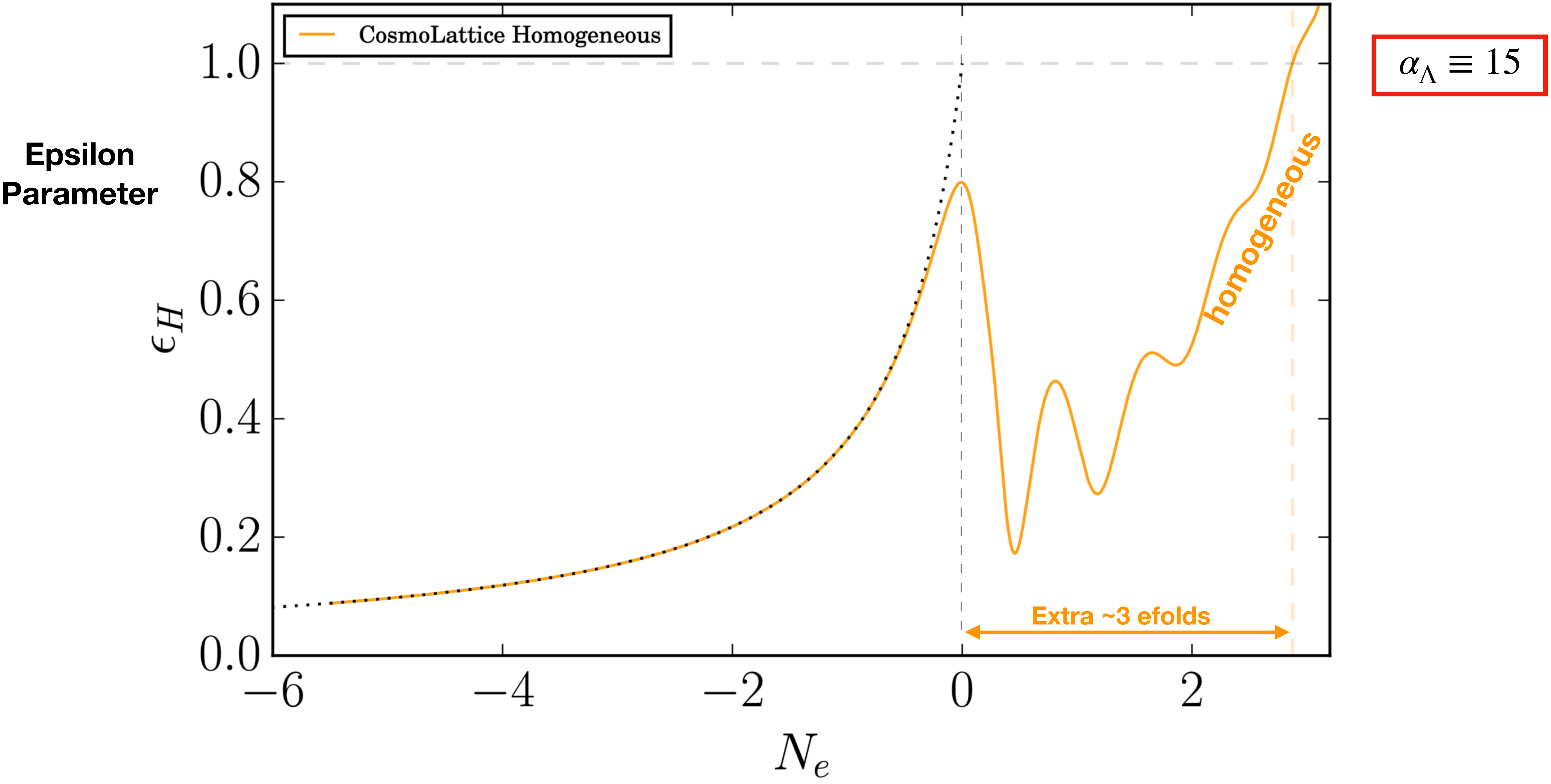
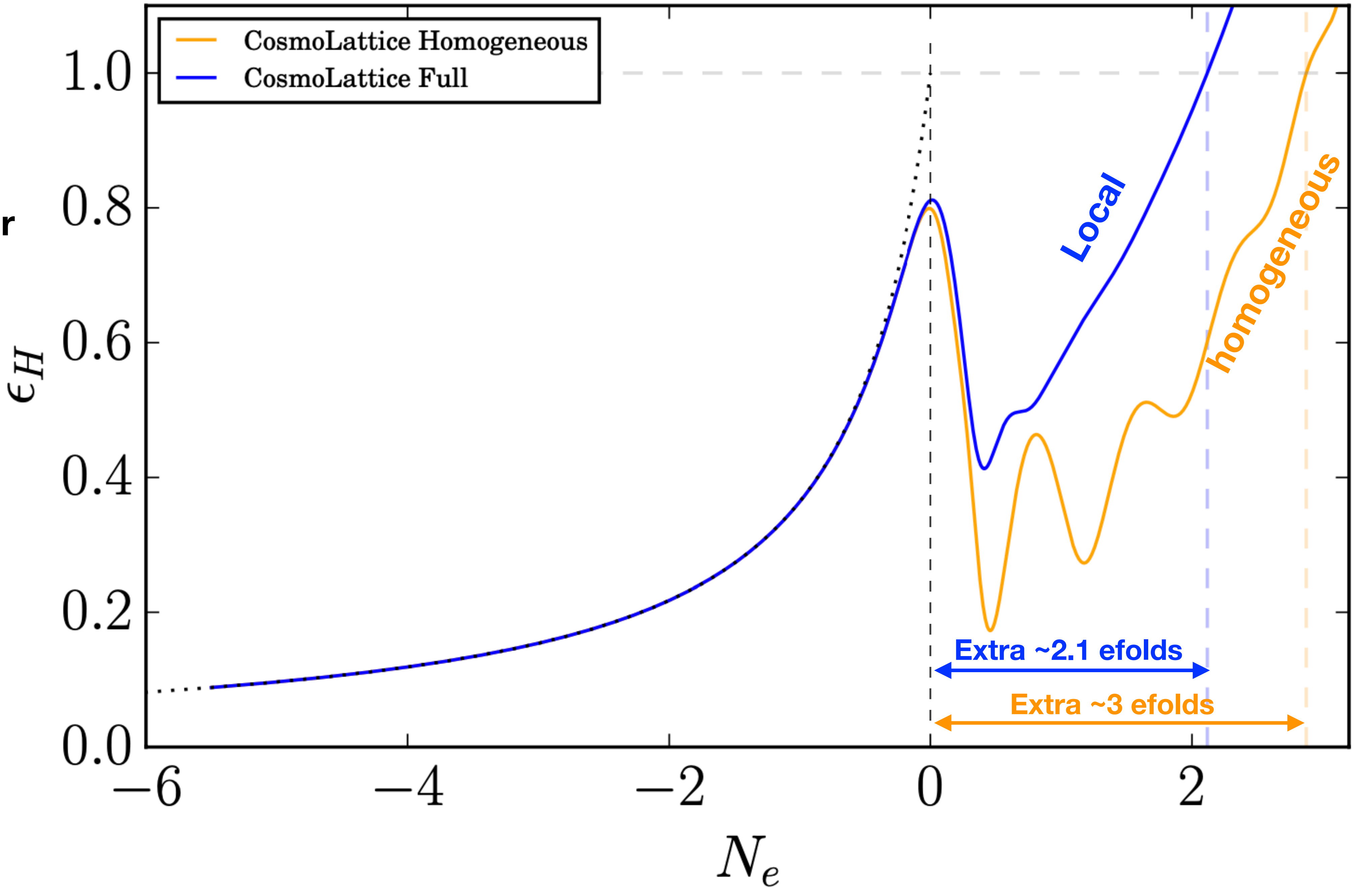
Linear (……..) **Non-Linear (——)**



Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 15$)

Linear (········) **Non-Linear (——)**

Epsilon Parameter

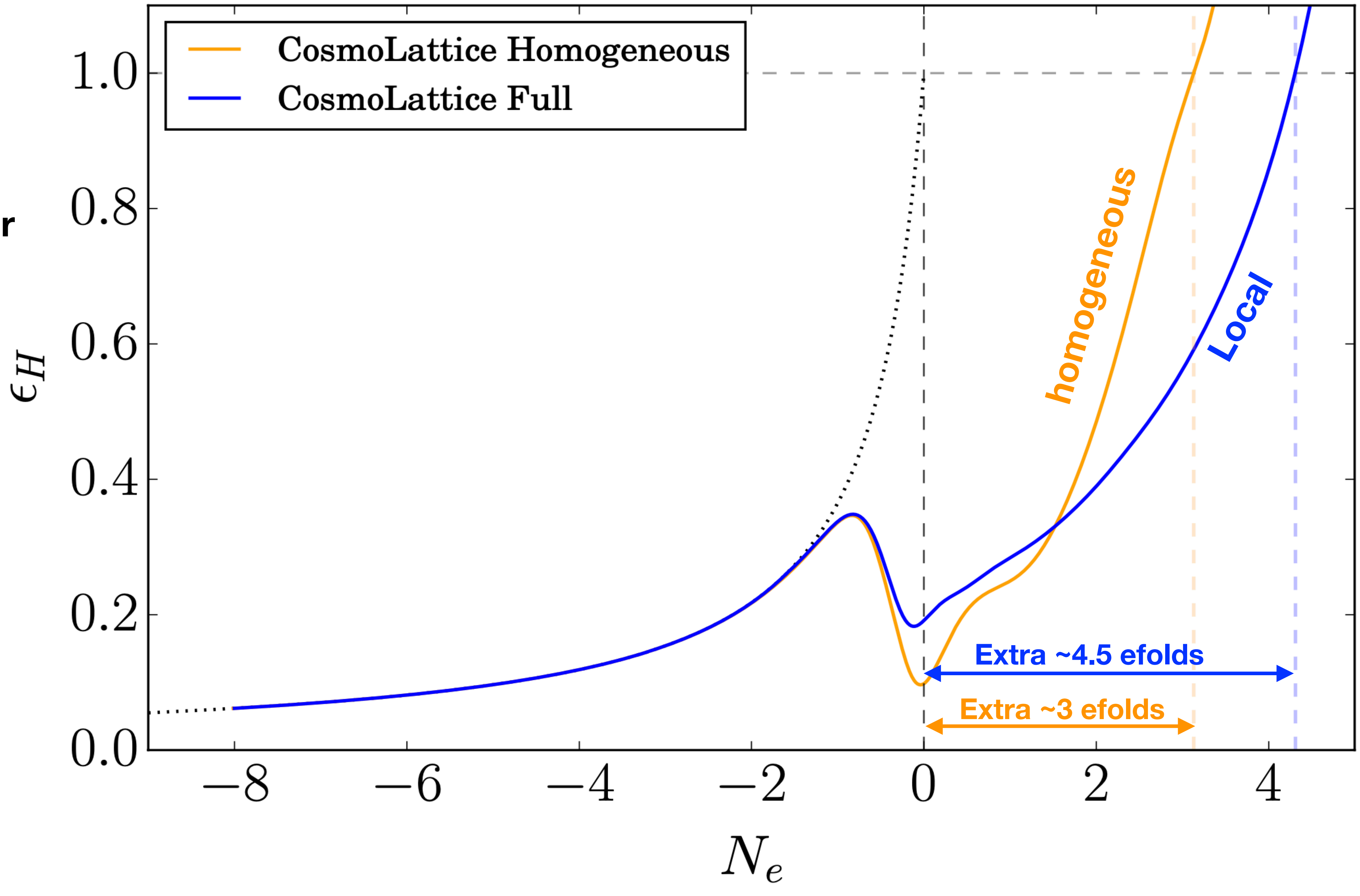


$\alpha_\Lambda \equiv 15$

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

Linear (········) **Non-Linear (——)**

Epsilon Parameter

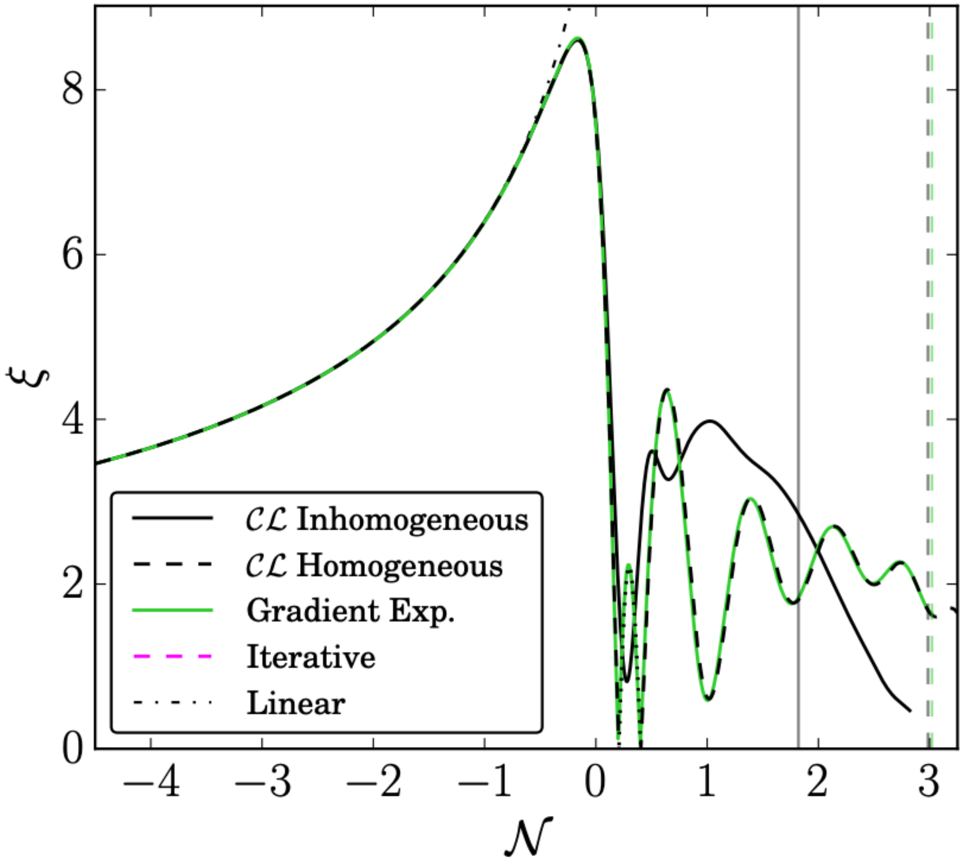


$\alpha_\Lambda \equiv 18$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \;;\; \frac{\phi}{4\Lambda}F\tilde{F} \;;\; \Lambda = \frac{m_p}{\alpha}\right)$

$(\alpha = 15, 18, 20)$

$\alpha_\Lambda = 15$



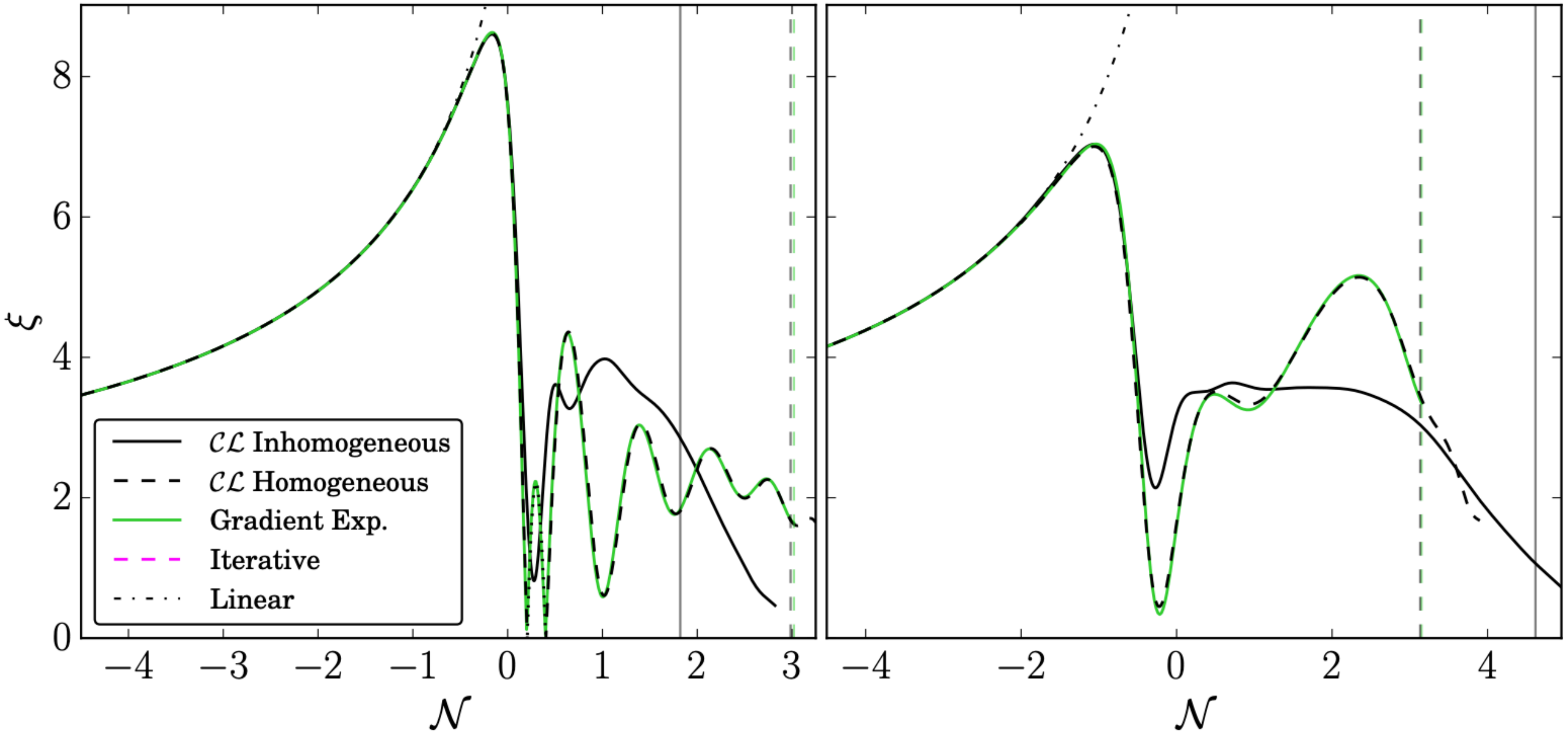
$$\Delta\mathcal{N}_{Hom} \simeq 3$$

$$\Delta\mathcal{N}_{InH} \simeq 1.8$$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \Lambda = \frac{m_p}{\alpha}\right)$

$(\alpha = 15, 18, 20)$

$\alpha_\Lambda = 15$ $\alpha_\Lambda = 18$

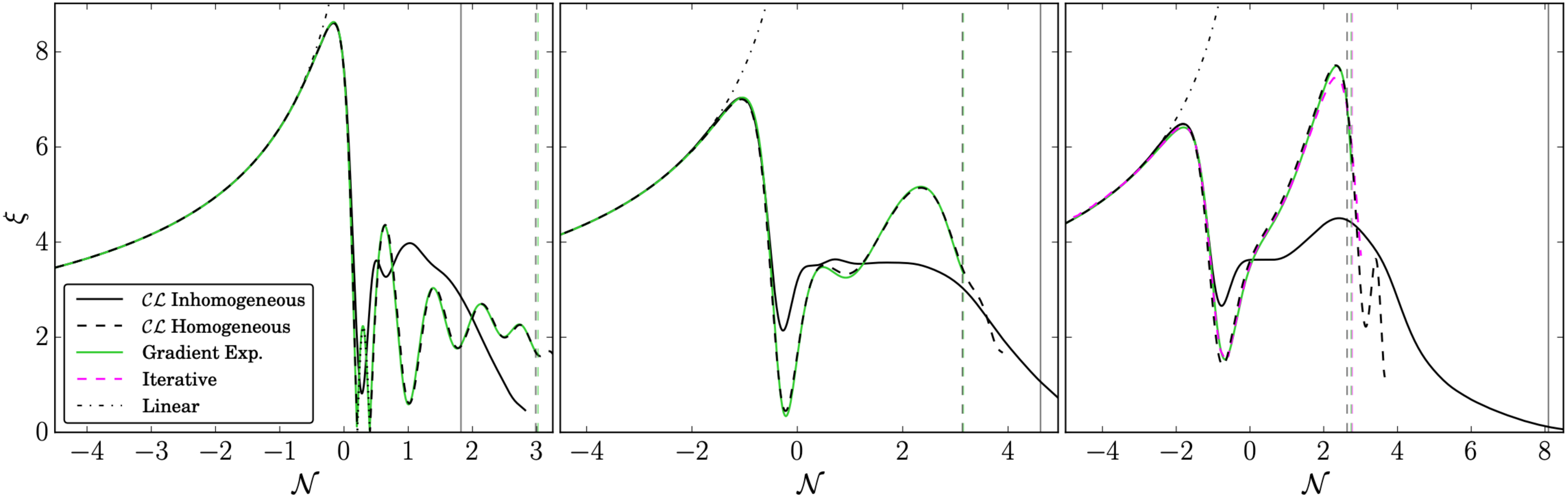


$\Delta\mathcal{N}_{Hom} \simeq 3$ $\Delta\mathcal{N}_{Hom} \simeq 3.1$

$\Delta\mathcal{N}_{InH} \simeq 1.8$ $\Delta\mathcal{N}_{InH} \simeq 4.5$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \Lambda = \frac{m_p}{\alpha}\right)$

$(\alpha = 15, 18, 20)$



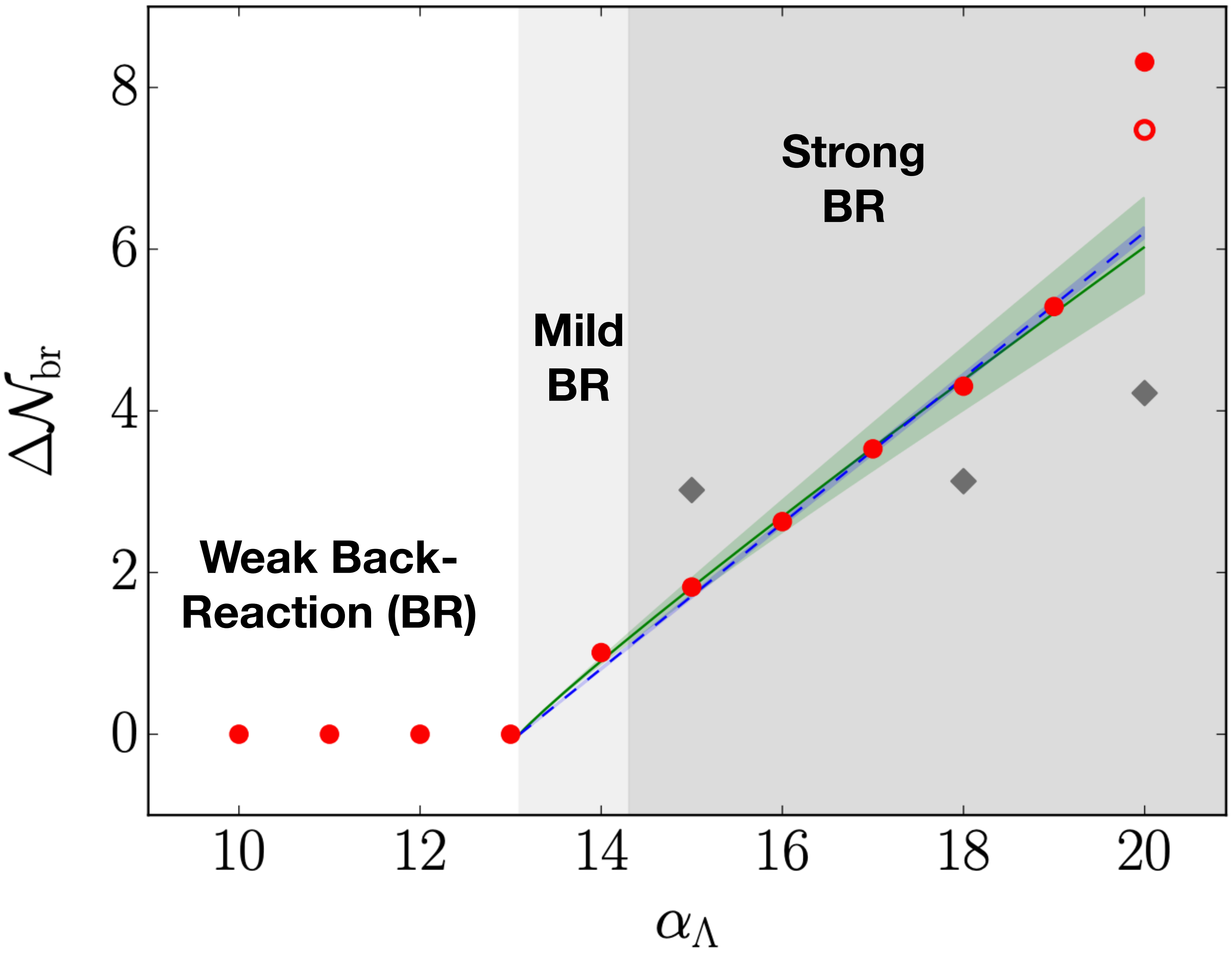
$\alpha_\Lambda = 15$ $\qquad$ $\alpha_\Lambda = 18$ $\qquad$ $\alpha_\Lambda = 20$

$\Delta\mathcal{N}_{Hom} \simeq 3$ $\qquad$ $\Delta\mathcal{N}_{Hom} \simeq 3.1$ $\qquad$ $\Delta\mathcal{N}_{Hom} \simeq 4.1$

$\Delta\mathcal{N}_{InH} \simeq 1.8$ $\qquad$ $\Delta\mathcal{N}_{InH} \simeq 4.5$ $\qquad$ $\Delta\mathcal{N}_{InH} \simeq 8$

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{\alpha}$)

Extra efoldings of Inflation



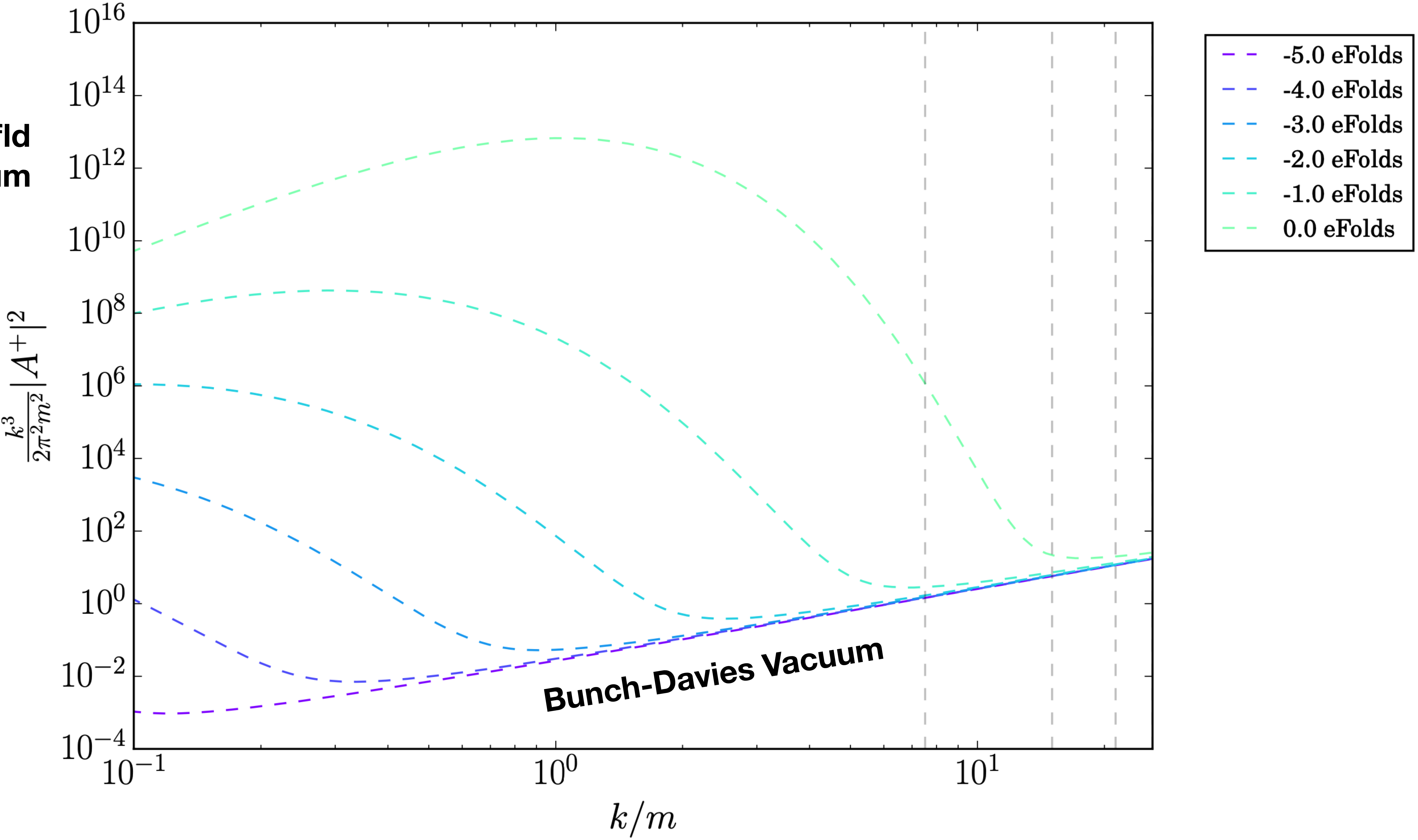
Gauge Amplification

[STRONG Backreaction]

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

Linear regime (≡ ≡ ≡)

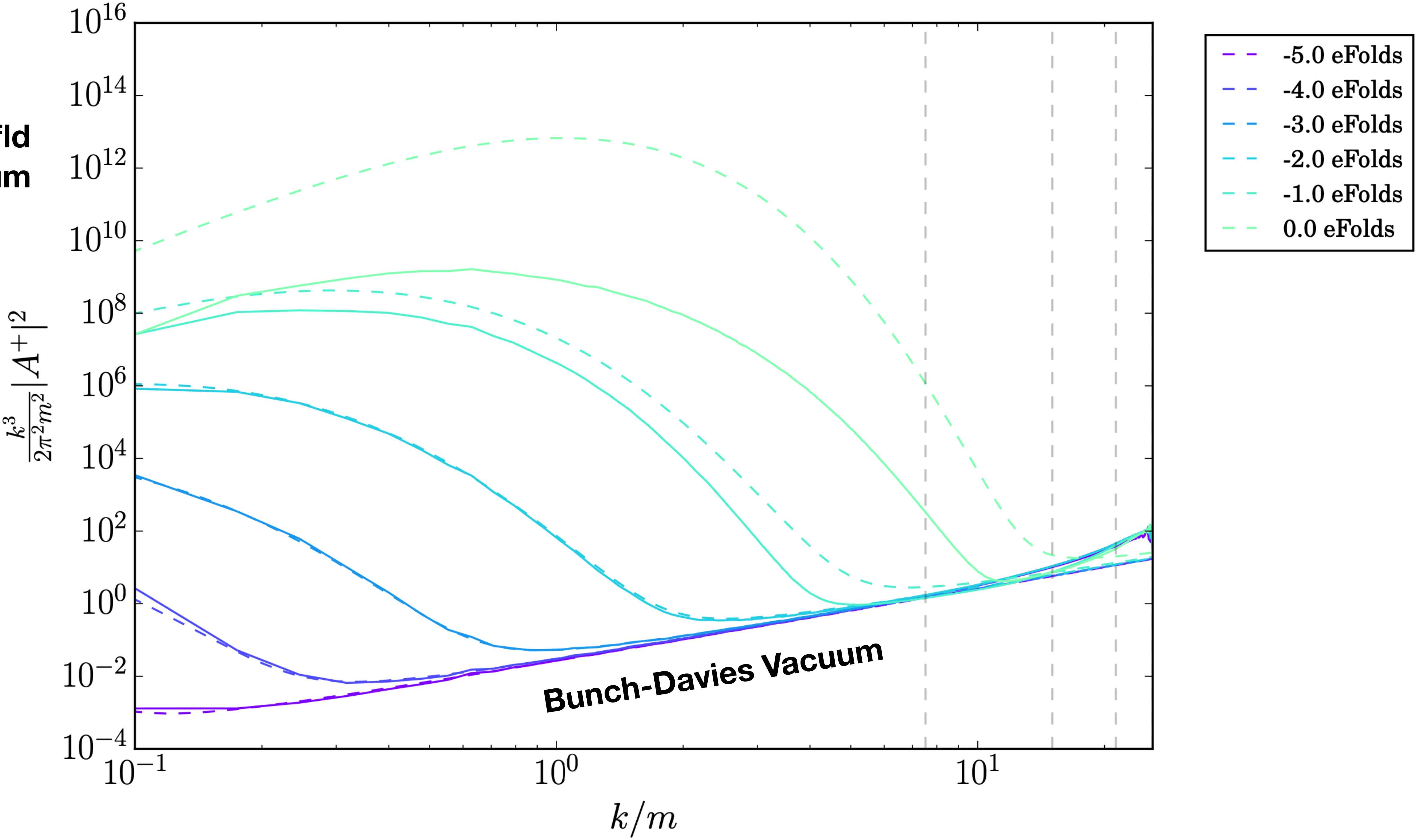
Gauge fld Spectrum



Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

Linear regime (- - -) Non-Linear regime (—)

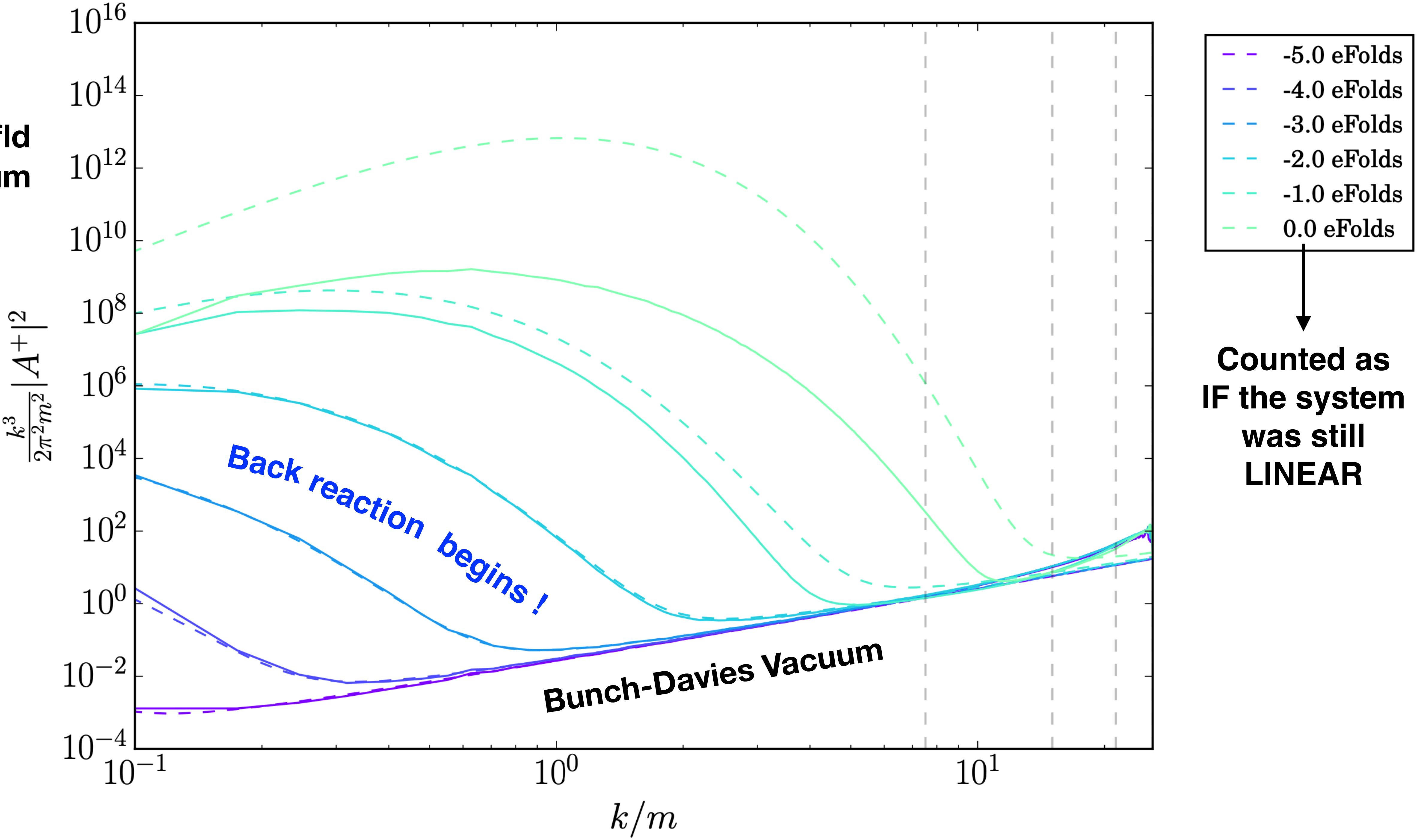
Gauge fld Spectrum



Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2 \ ; \ \frac{\phi}{4\Lambda}F\tilde{F} \ ; \ \alpha_\Lambda = 18$)

Linear regime (– – –)   Non-Linear regime (——)

Gauge fld Spectrum

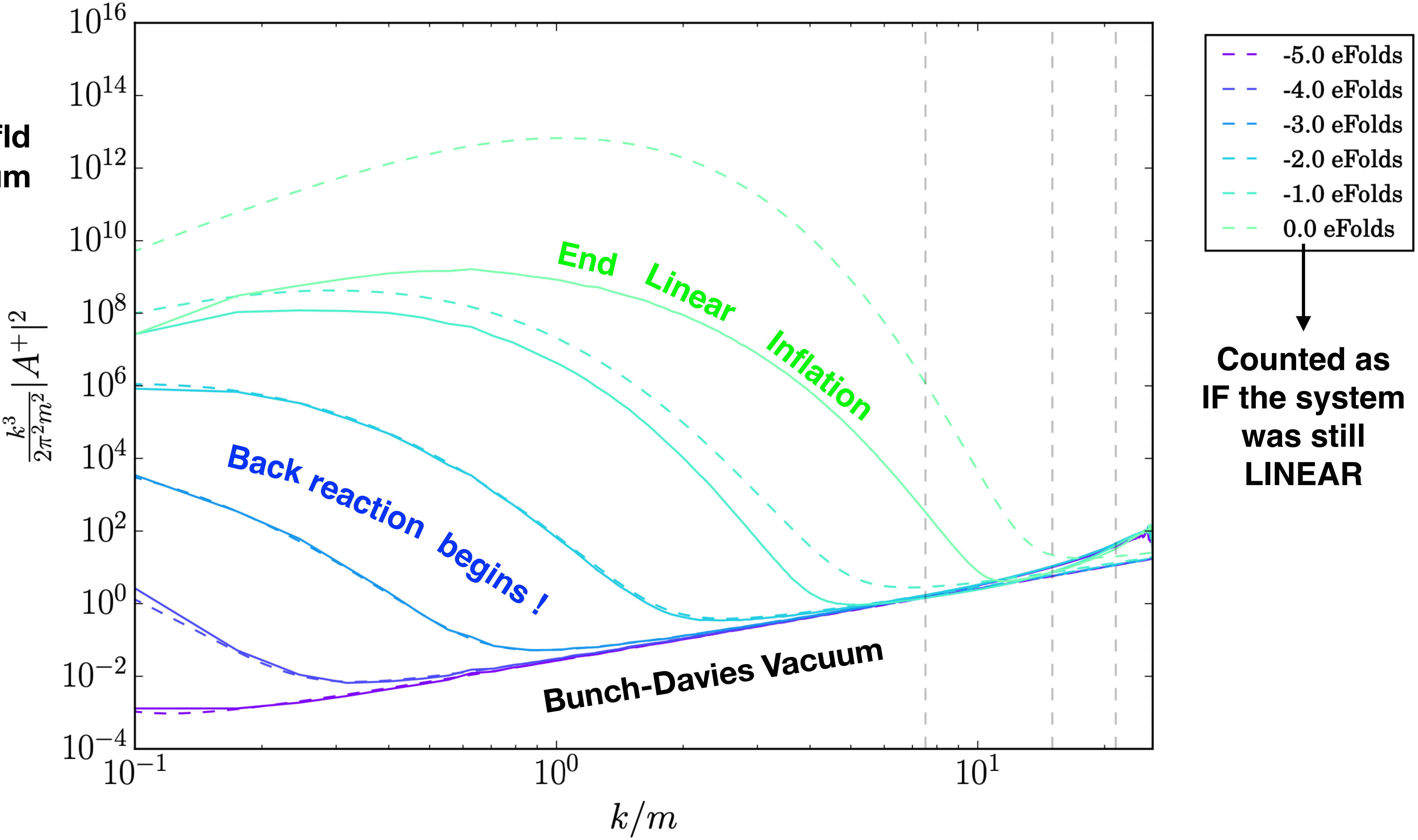


Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18$)

Linear regime (- - -) Non-Linear regime (—)

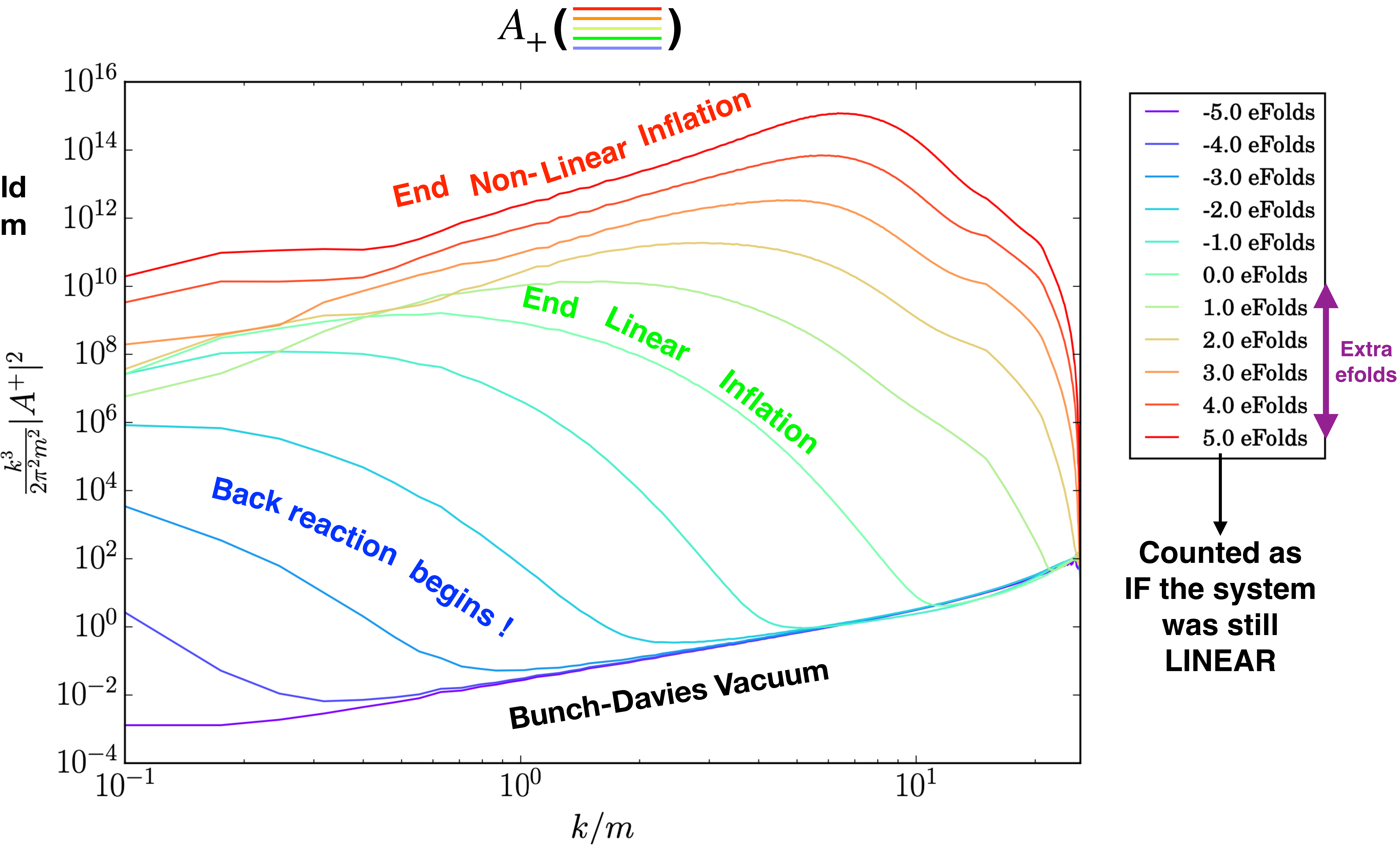
Gauge fld Spectrum



Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2 \ ; \ \frac{\phi}{4\Lambda}F\tilde{F} \ ; \ \alpha_\Lambda = 18$)

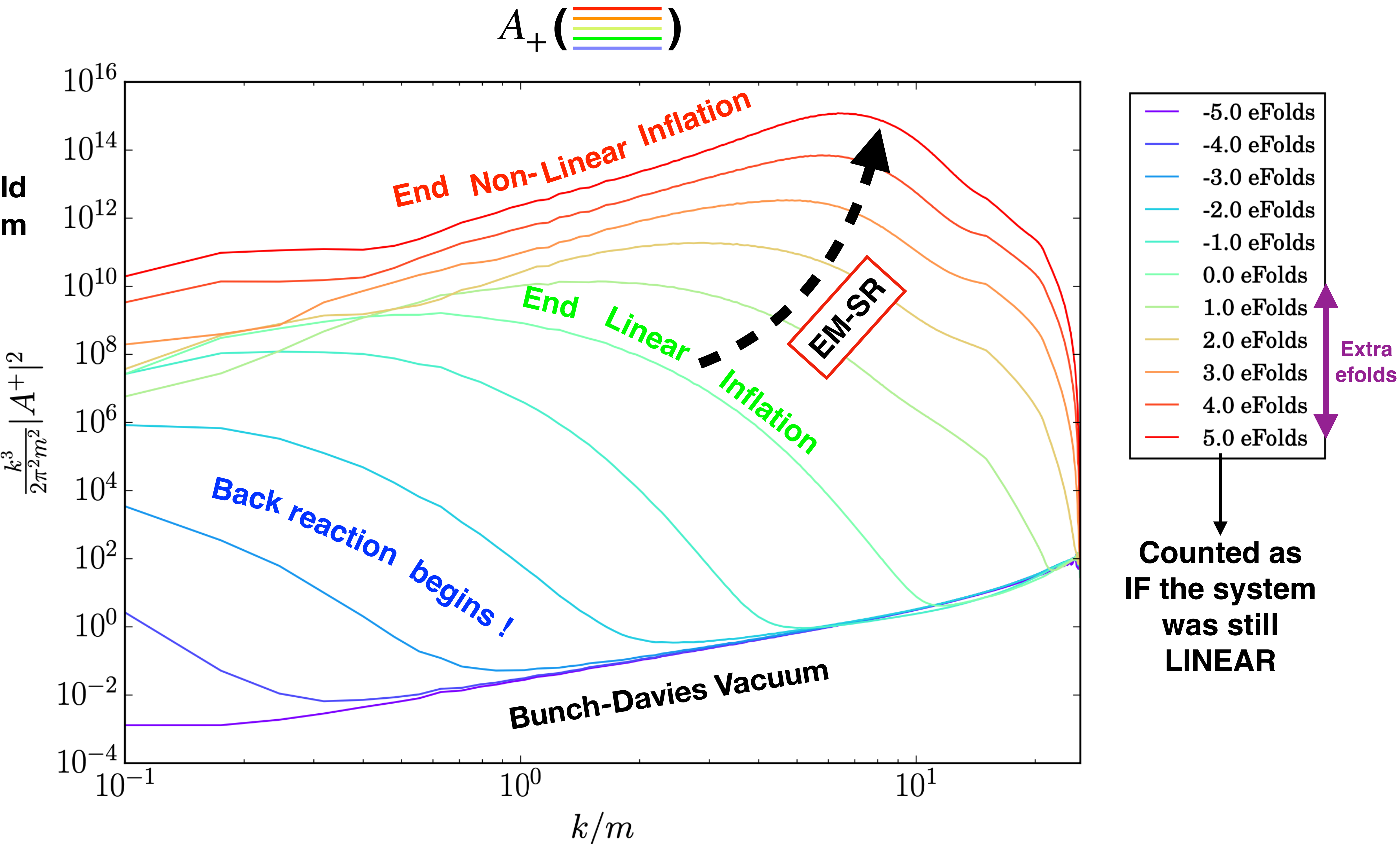
Gauge fld Spectrum



Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18$)

Gauge fld Spectrum



Counted as IF the system was still LINEAR

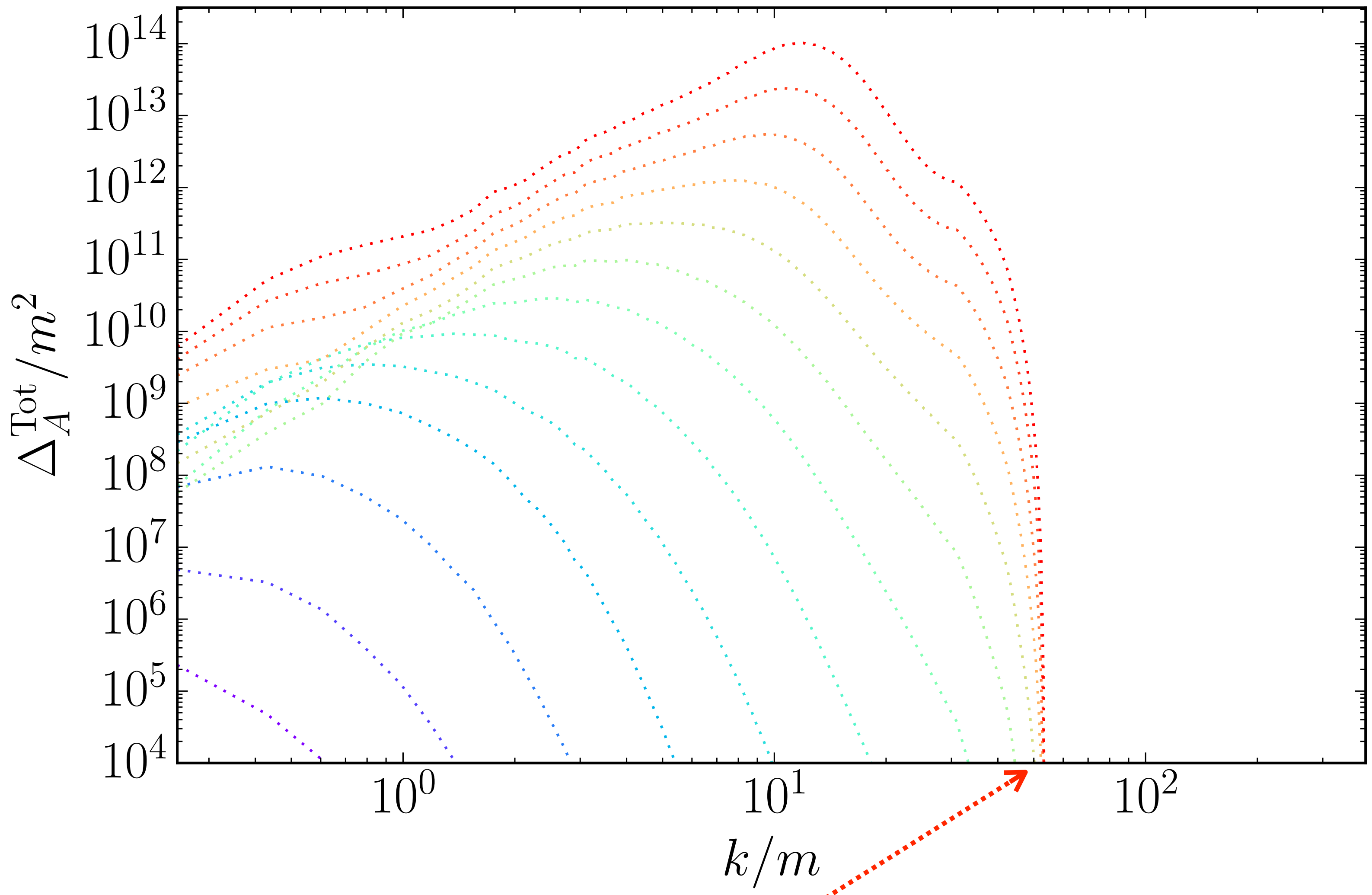
UV sensitivity
(convergence)

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \ ; \ \frac{\phi}{4\Lambda}F\tilde{F} \ ; \ \alpha_\Lambda = 18\right)$

UV sensitivity

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV sensitivity

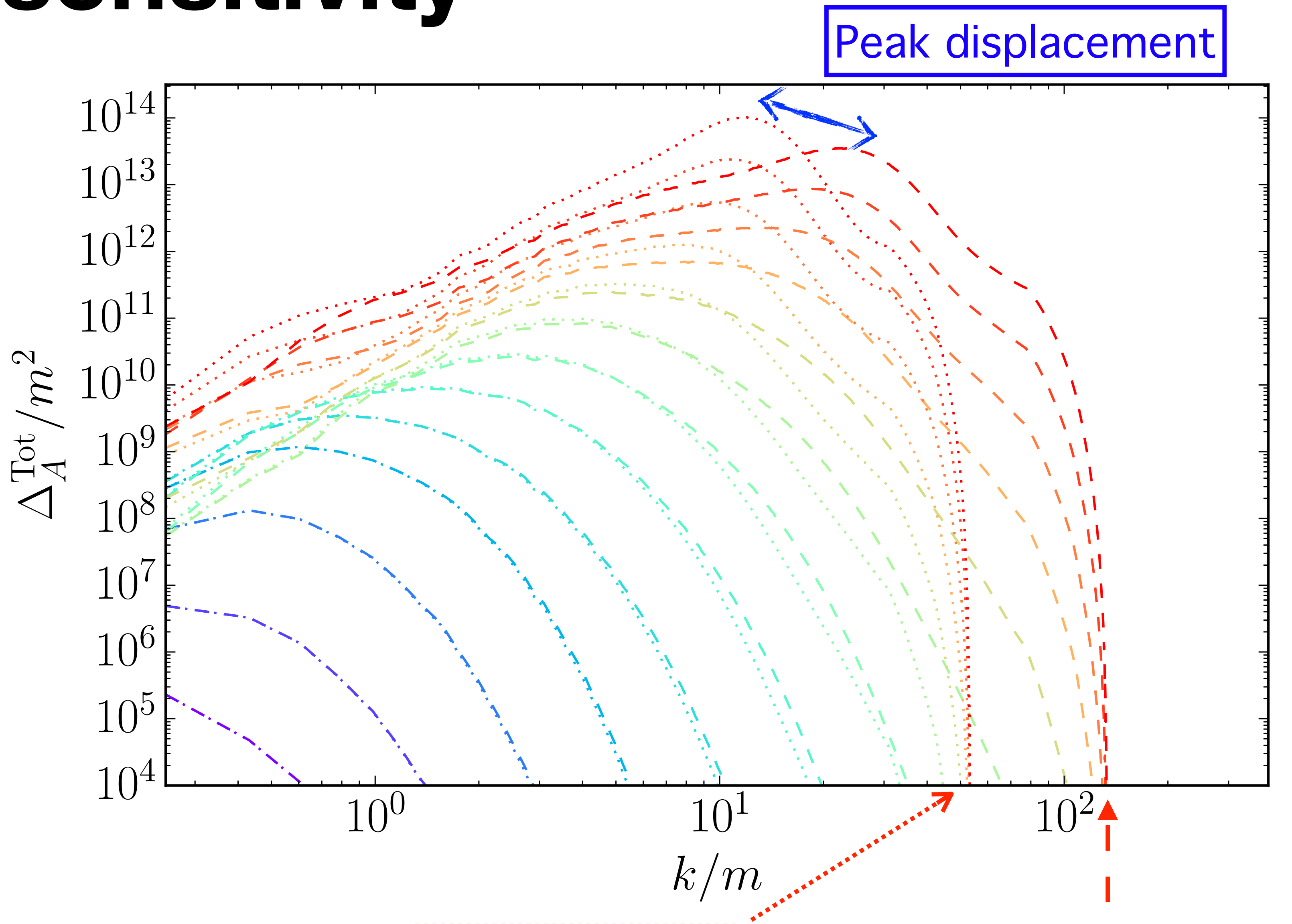


$N = 320$

$k_{\rm UV}/k_{\rm IR} \approx 280$

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

UV sensitivity



$N = 320$

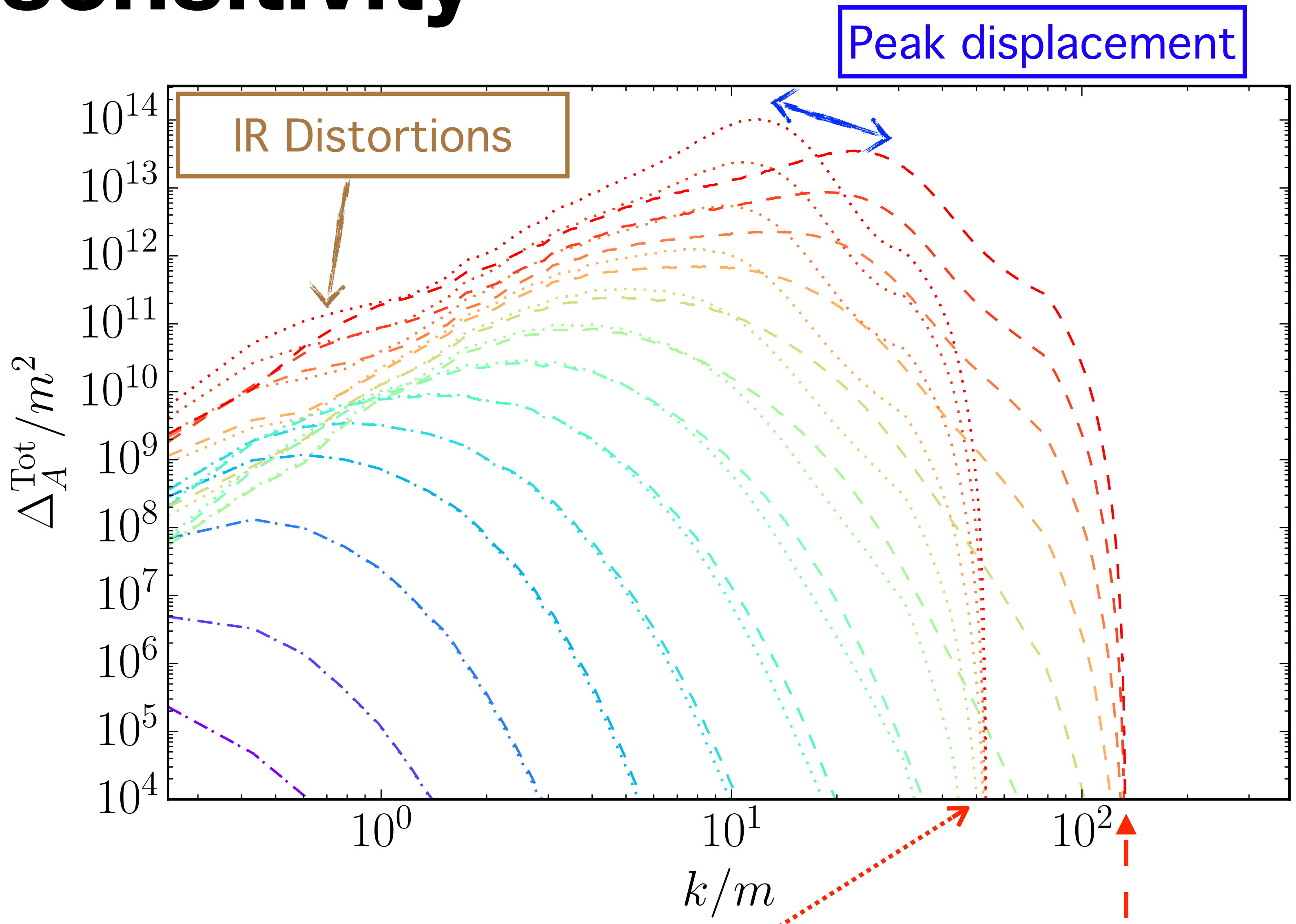
$k_{\mathrm{UV}}/k_{\mathrm{IR}} \approx 280$

$N = 800$

$k_{\mathrm{UV}}/k_{\mathrm{IR}} \approx 690$

Axion-Inflation$\left(V(\phi)=\frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda=18\right)$

UV sensitivity



$N = 320$

$k_{\rm UV}/k_{\rm IR} \approx 280$

$N = 800$

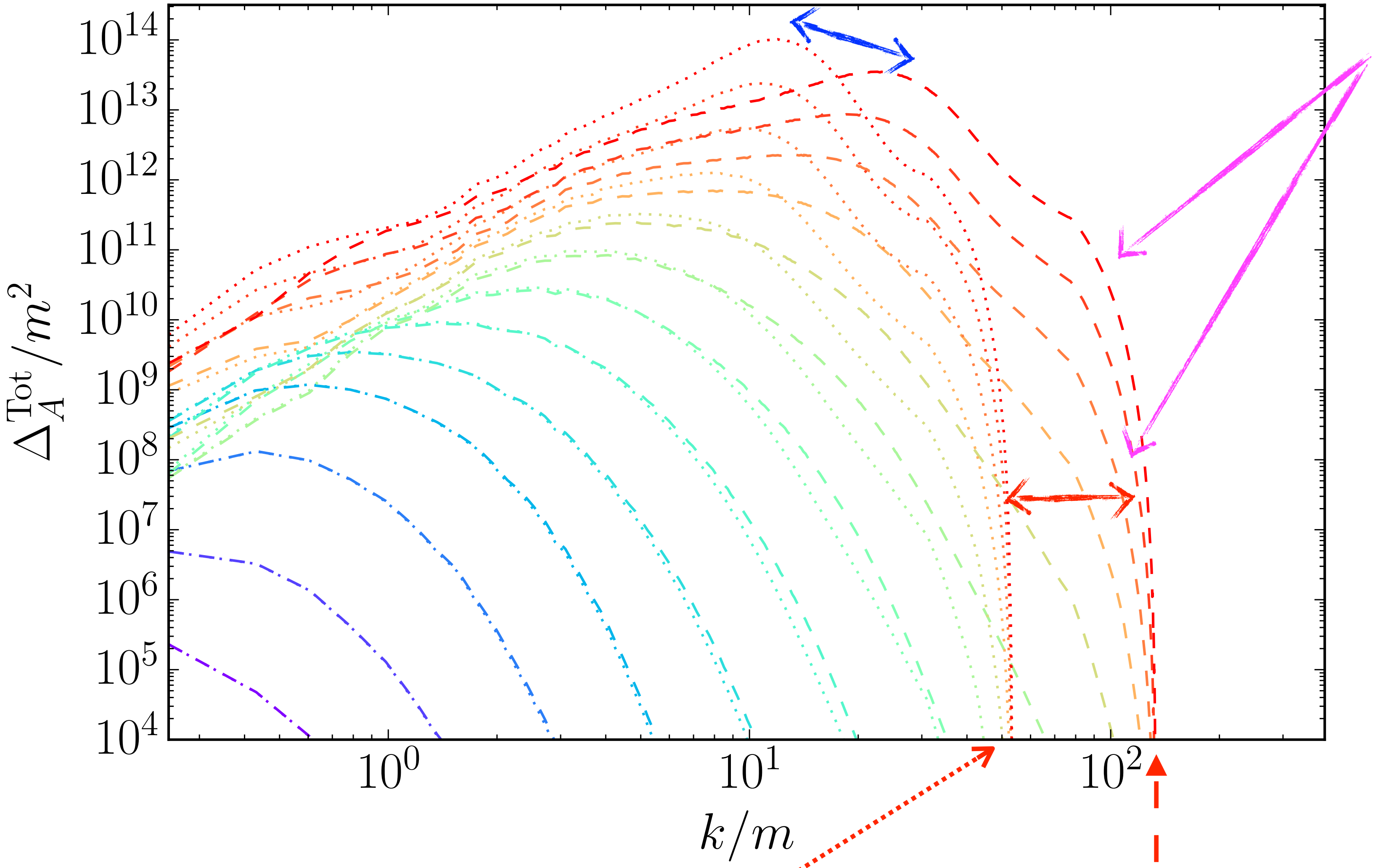
$k_{\rm UV}/k_{\rm IR} \approx 690$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV sensitivity

Peak displacement

More inflation:
Smaller scales
become physical



$N = 320$

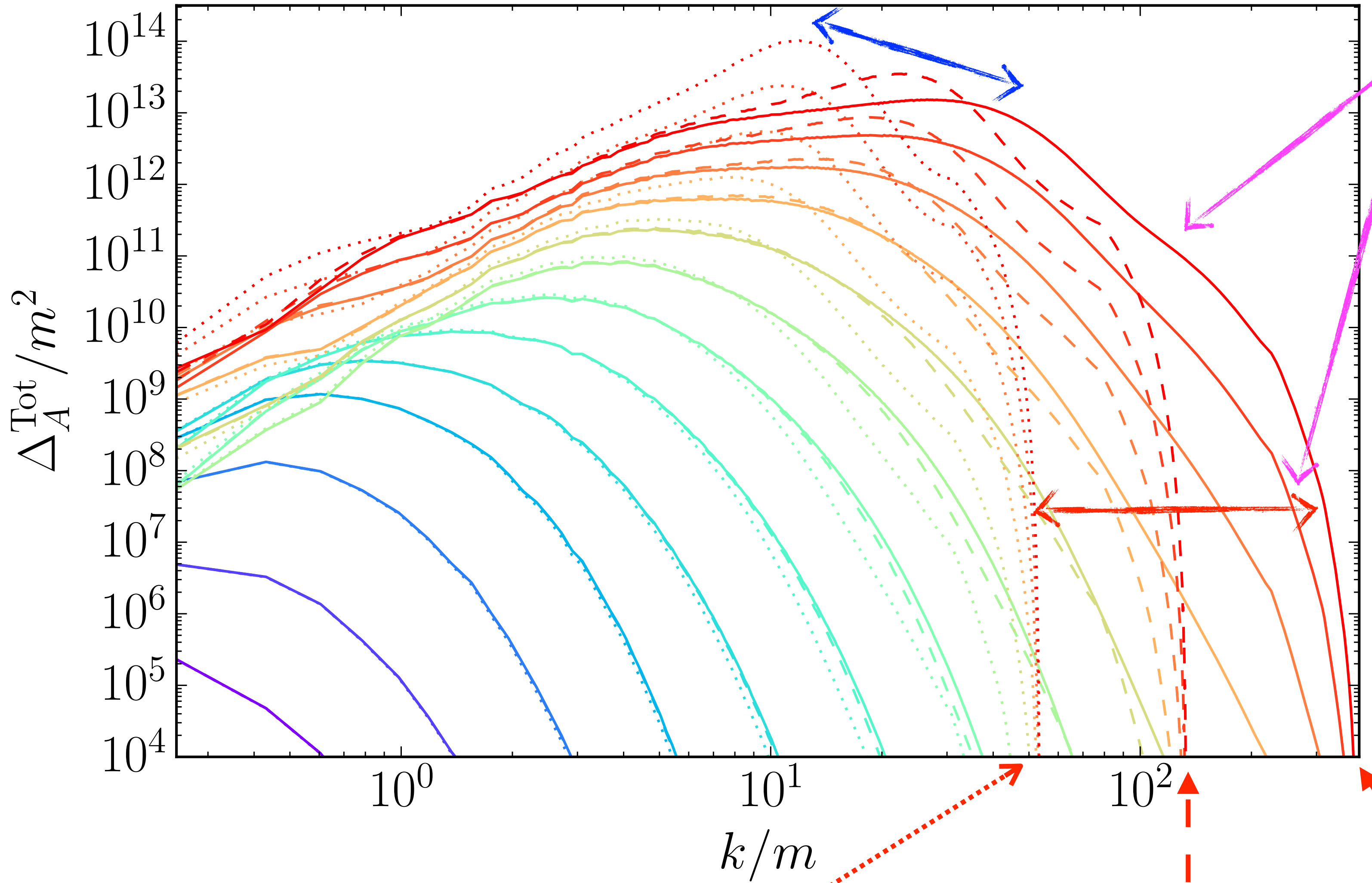
$k_{\rm UV}/k_{\rm IR} \approx 280$

$N = 800$

$k_{\rm UV}/k_{\rm IR} \approx 690$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV sensitivity



Peak displacement

More inflation:
Smaller scales
become physical

$N = 320$

$k_{\rm UV}/k_{\rm IR} \approx 280$

$N = 800$

$k_{\rm UV}/k_{\rm IR} \approx 690$

$N = 2340$

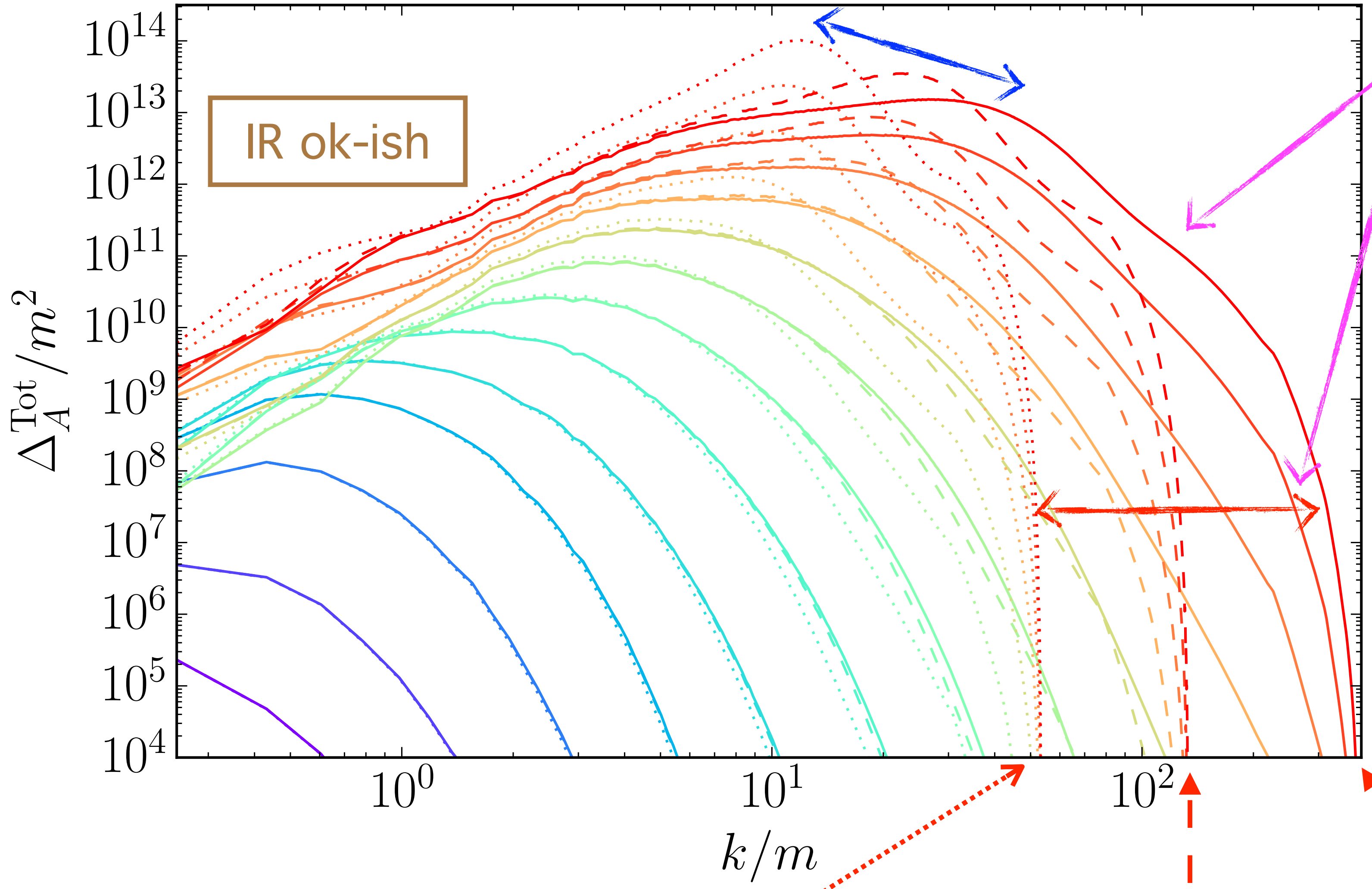
$k_{\rm UV}/k_{\rm IR} \approx 2000$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \alpha_\Lambda = 18\right)$

UV sensitivity

Peak displacement

More inflation:
Smaller scales
become physical



IR ok-ish

Extreme UV dependence!

Simulations in quasi-dS!

$N = 320$

$k_{\rm UV}/k_{\rm IR} \approx 280$

$N = 800$

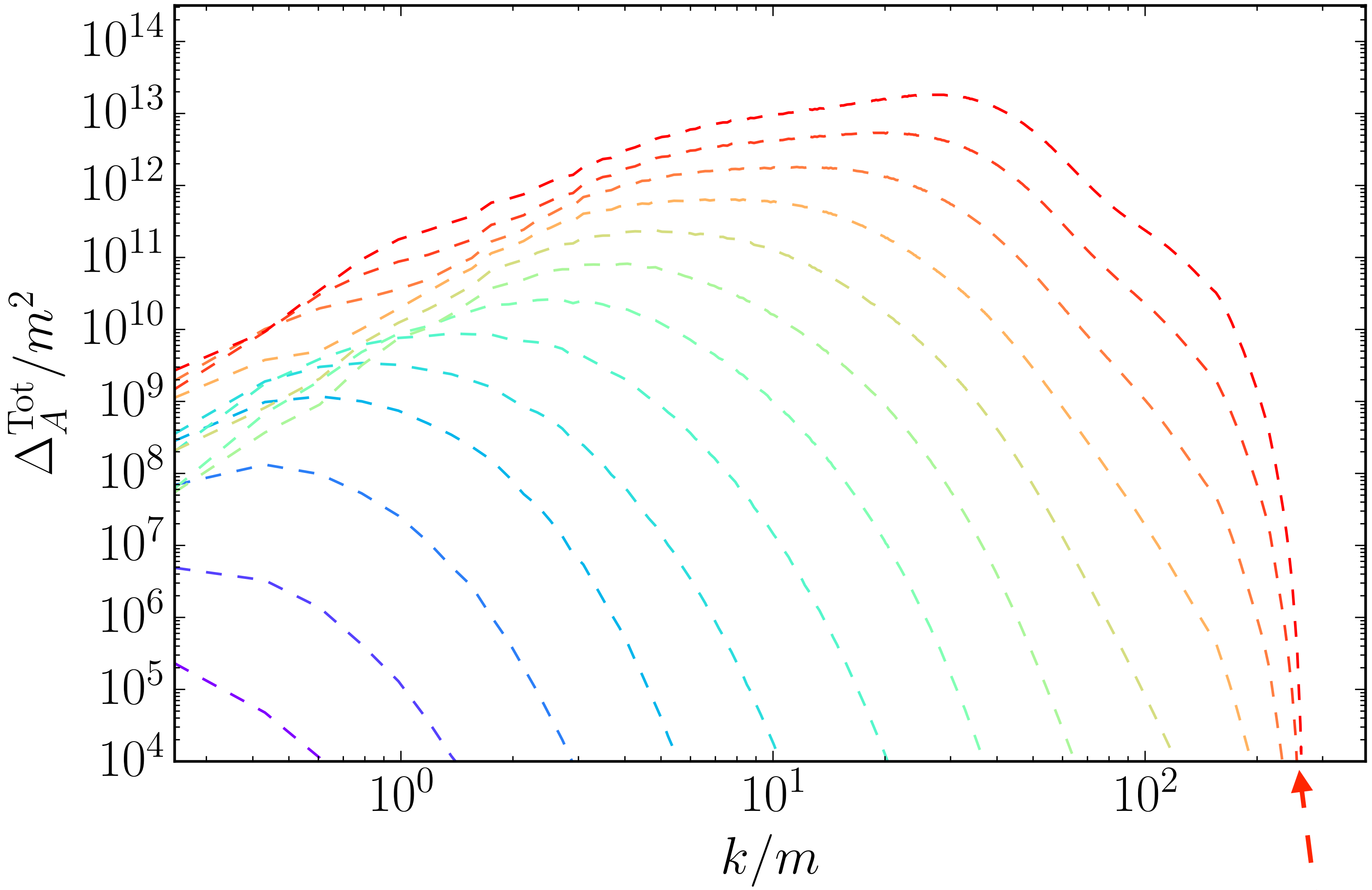
$k_{\rm UV}/k_{\rm IR} \approx 690$

$N = 2340$

$k_{\rm UV}/k_{\rm IR} \approx 2000$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV convergence ?



Extreme UV dependence!

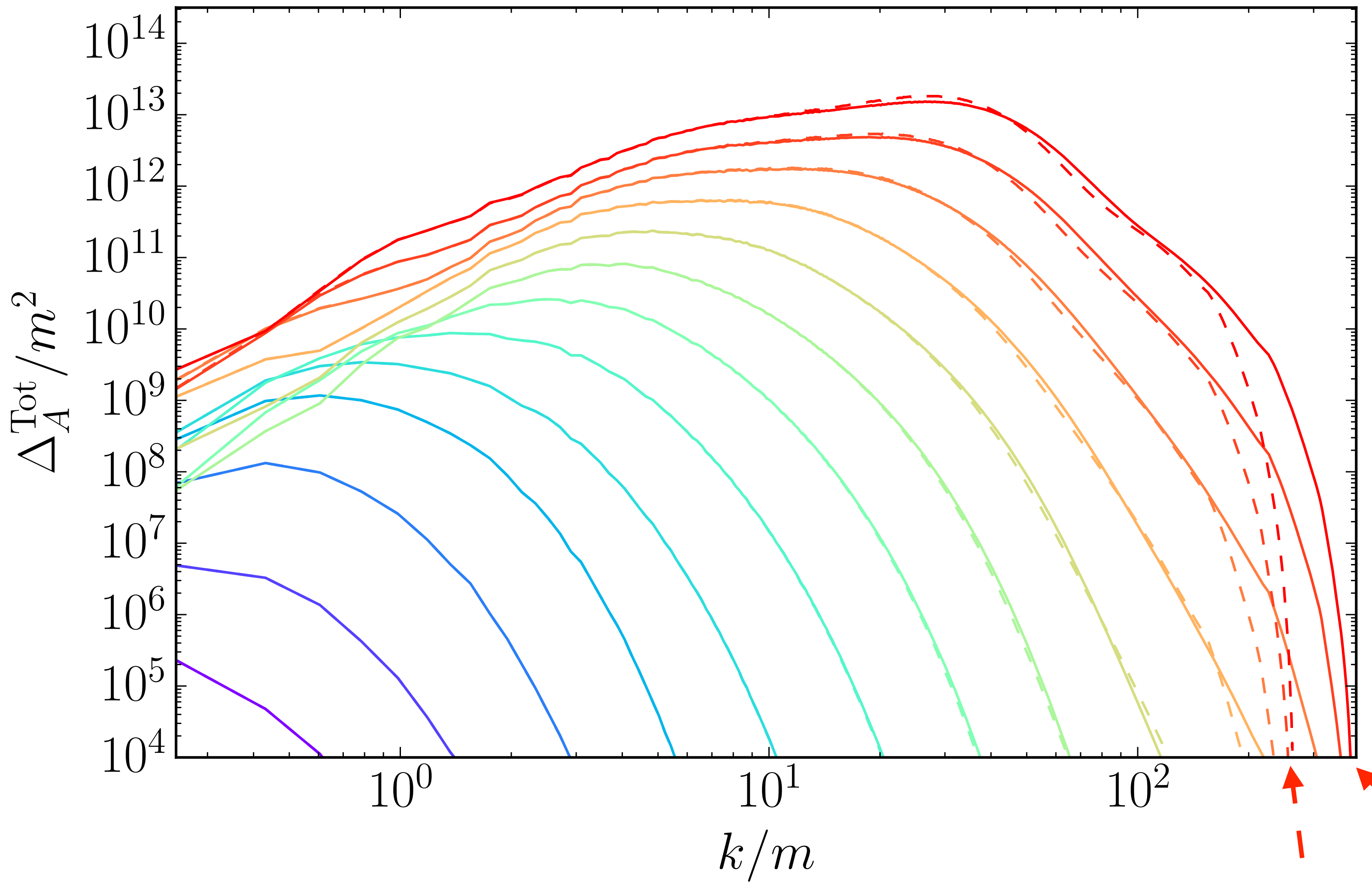
Simulations in quasi-dS!

$N = 1600$

$k_{\rm UV}/k_{\rm IR} \approx 1400$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \alpha_\Lambda = 18\right)$

UV convergence ?



IR = IR

Mid $\simeq$ Mid

UV $\neq$ UV

$N = 1600$

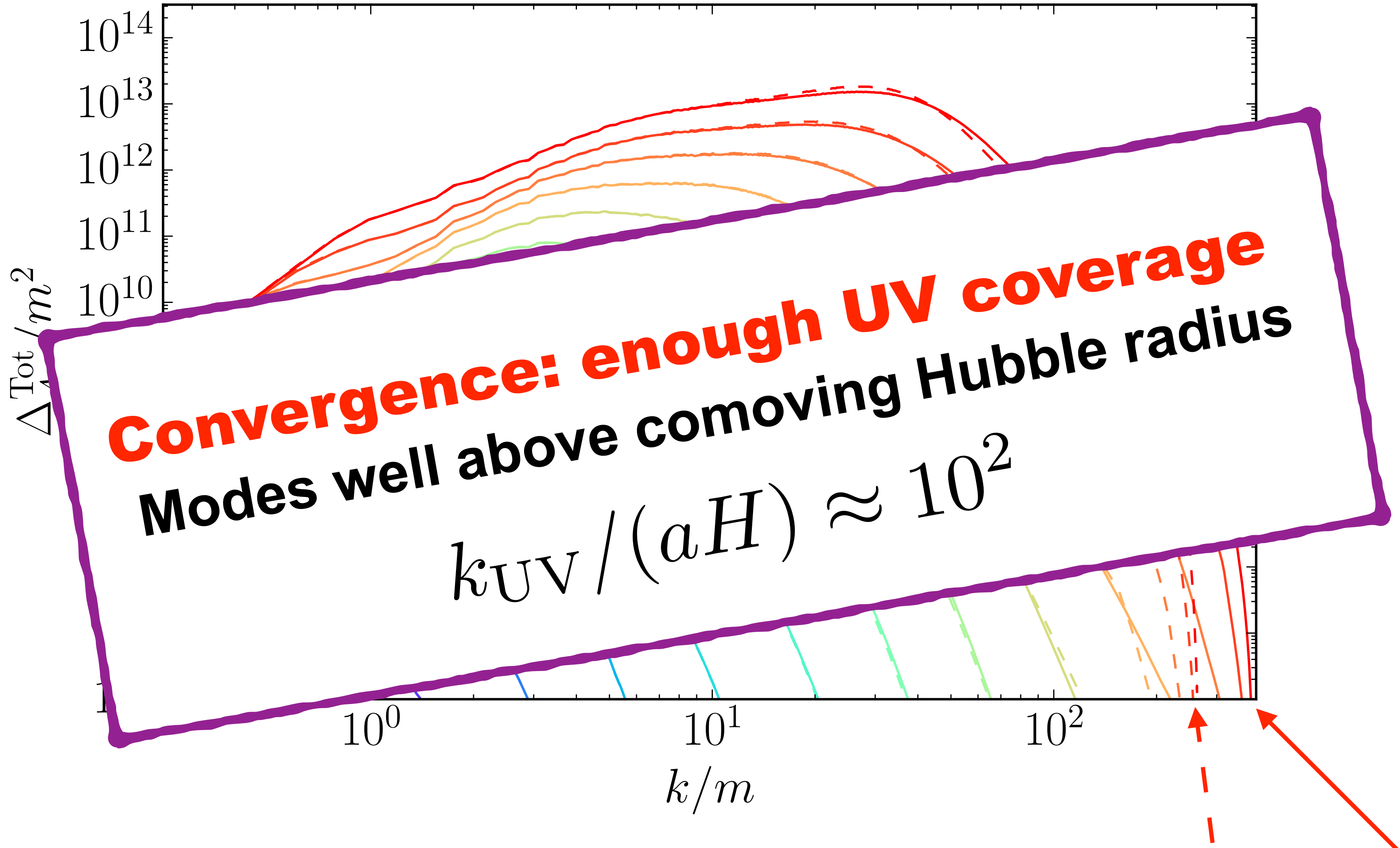
$k_{\rm UV}/k_{\rm IR} \approx 1400$

$N = 2340$

$k_{\rm UV}/k_{\rm IR} \approx 2000$

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV convergence ?



Convergence: enough UV coverage

Modes well above comoving Hubble radius

$$k_{\rm UV}/(aH) \approx 10^2$$

Gigantic lattices !

$N = 1600$

$k_{\rm UV}/k_{\rm IR} \approx 1400$

$N = 2340$

$k_{\rm UV}/k_{\rm IR} \approx 2000$

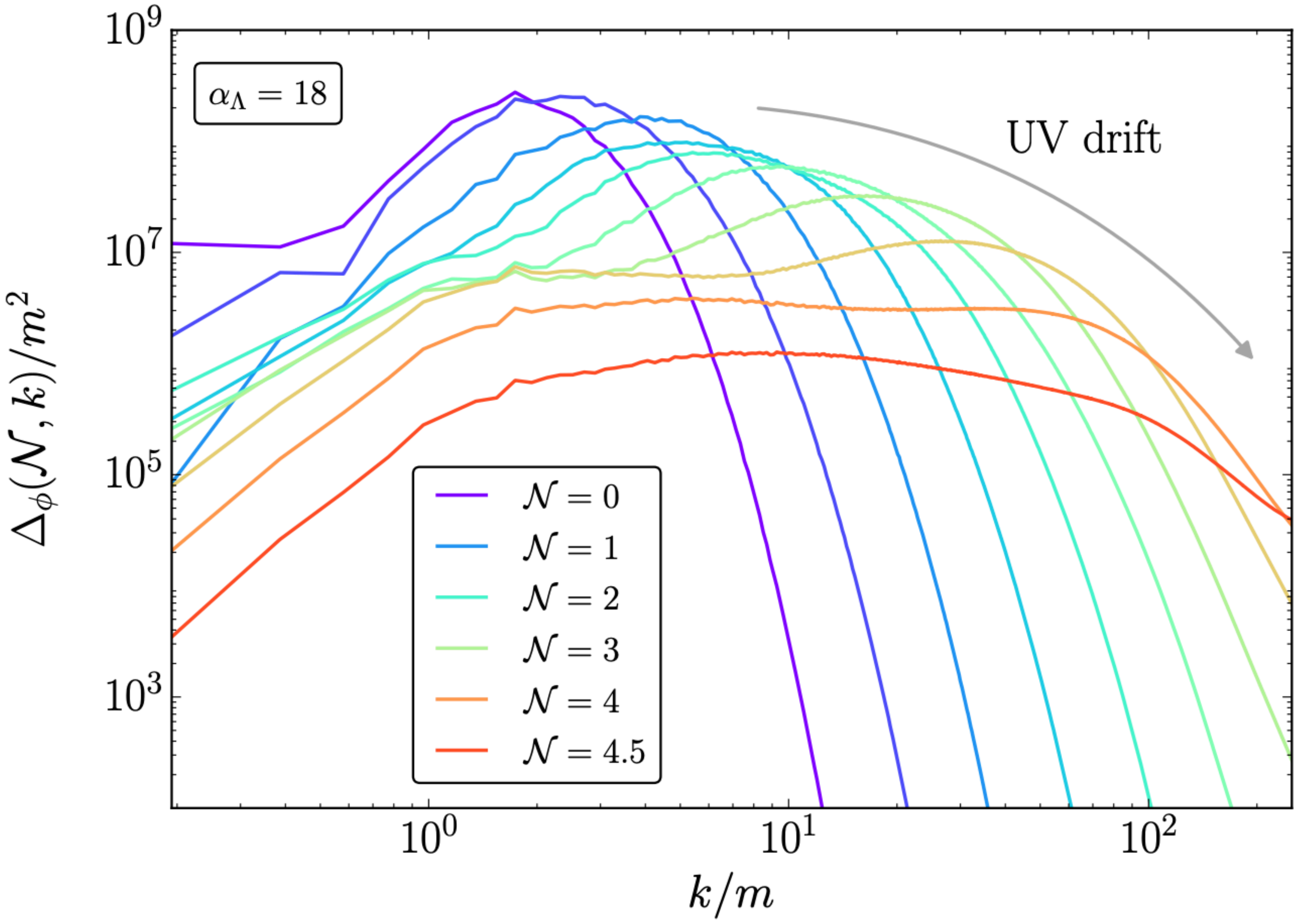
Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

UV convergence ?

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18$)

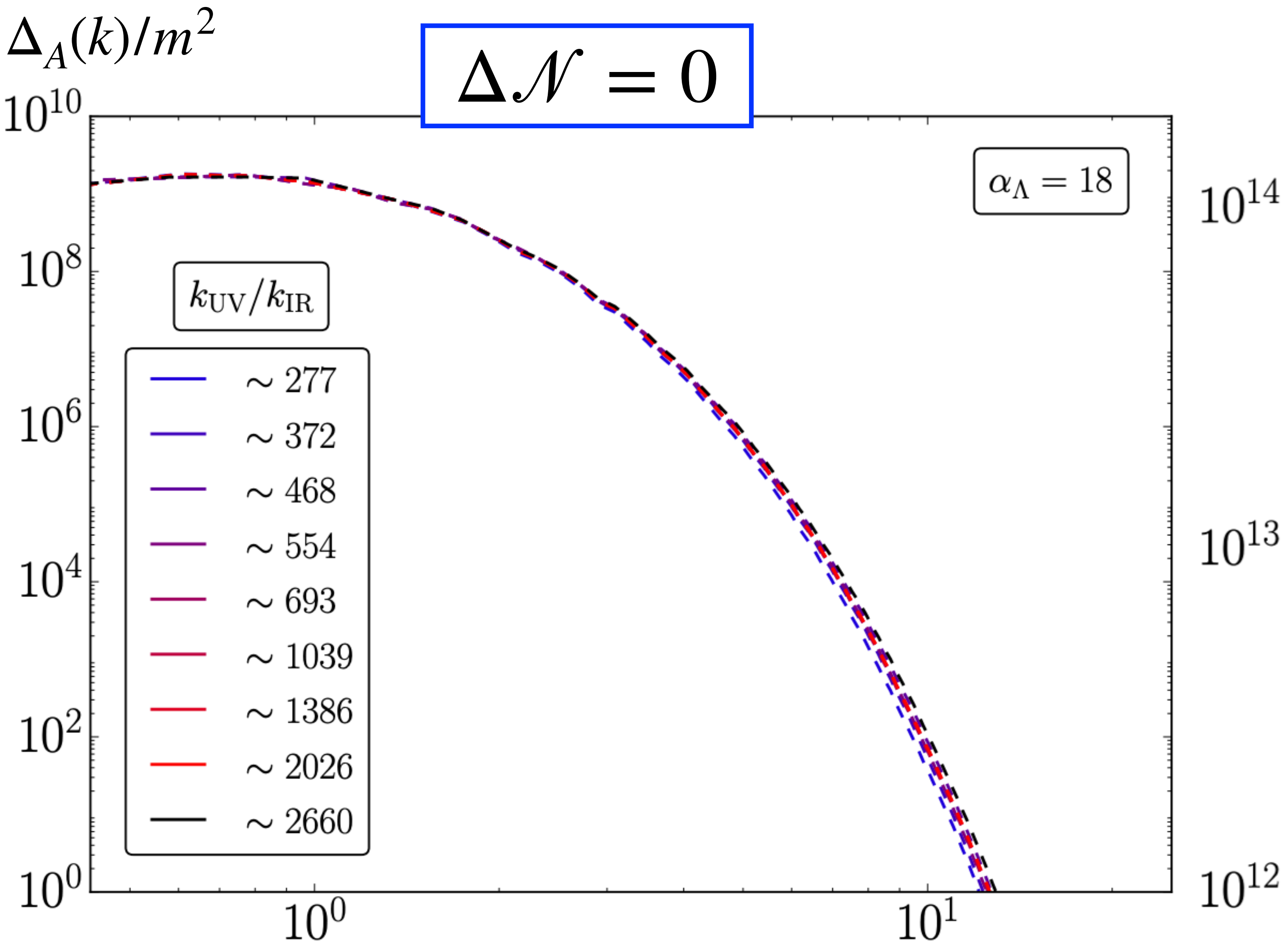
UV convergence ?

UV drift during strong BR



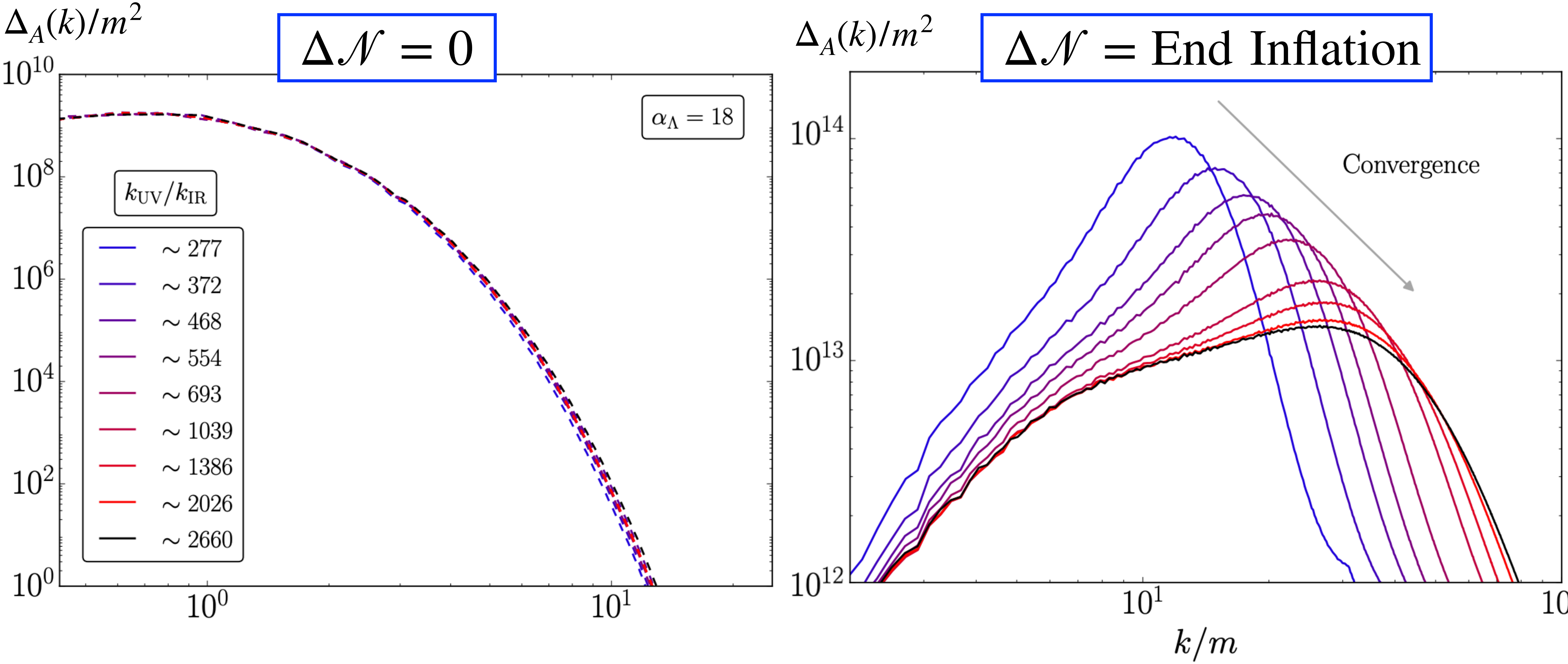
Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

UV convergence ?



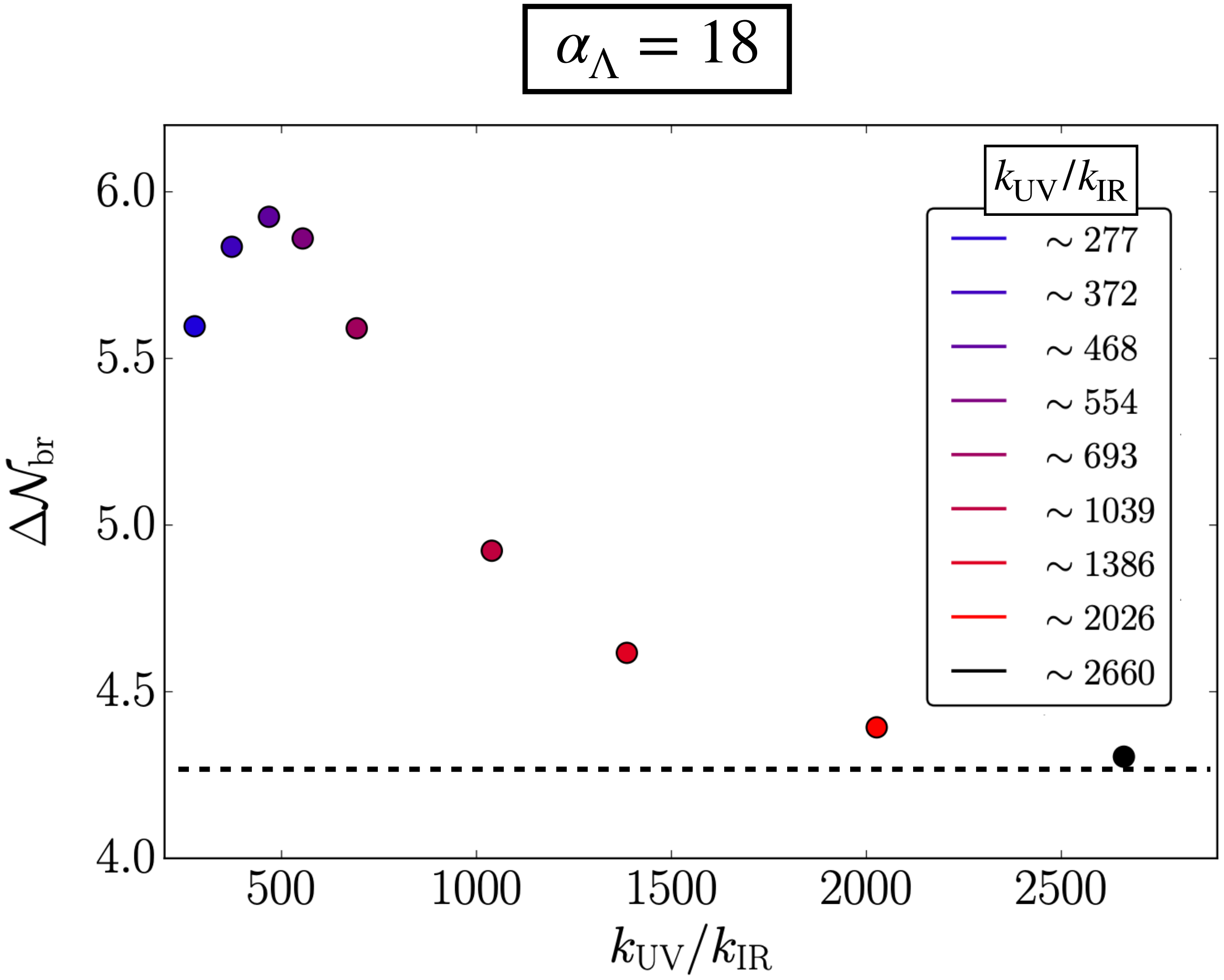
Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \ ; \ \frac{\phi}{4\Lambda}F\tilde{F} \ ; \ \alpha_\Lambda = 18\right)$

UV convergence ?



Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

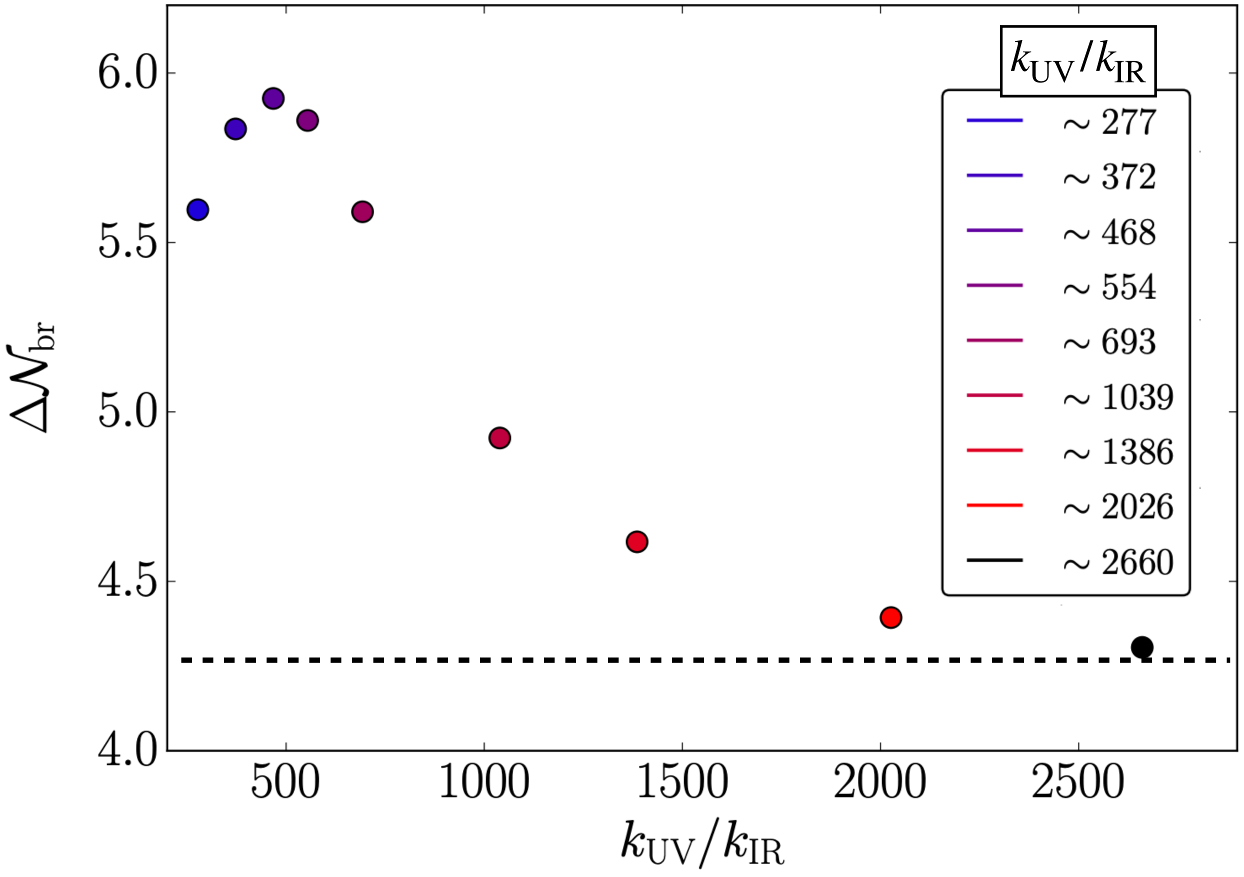
UV convergence ?



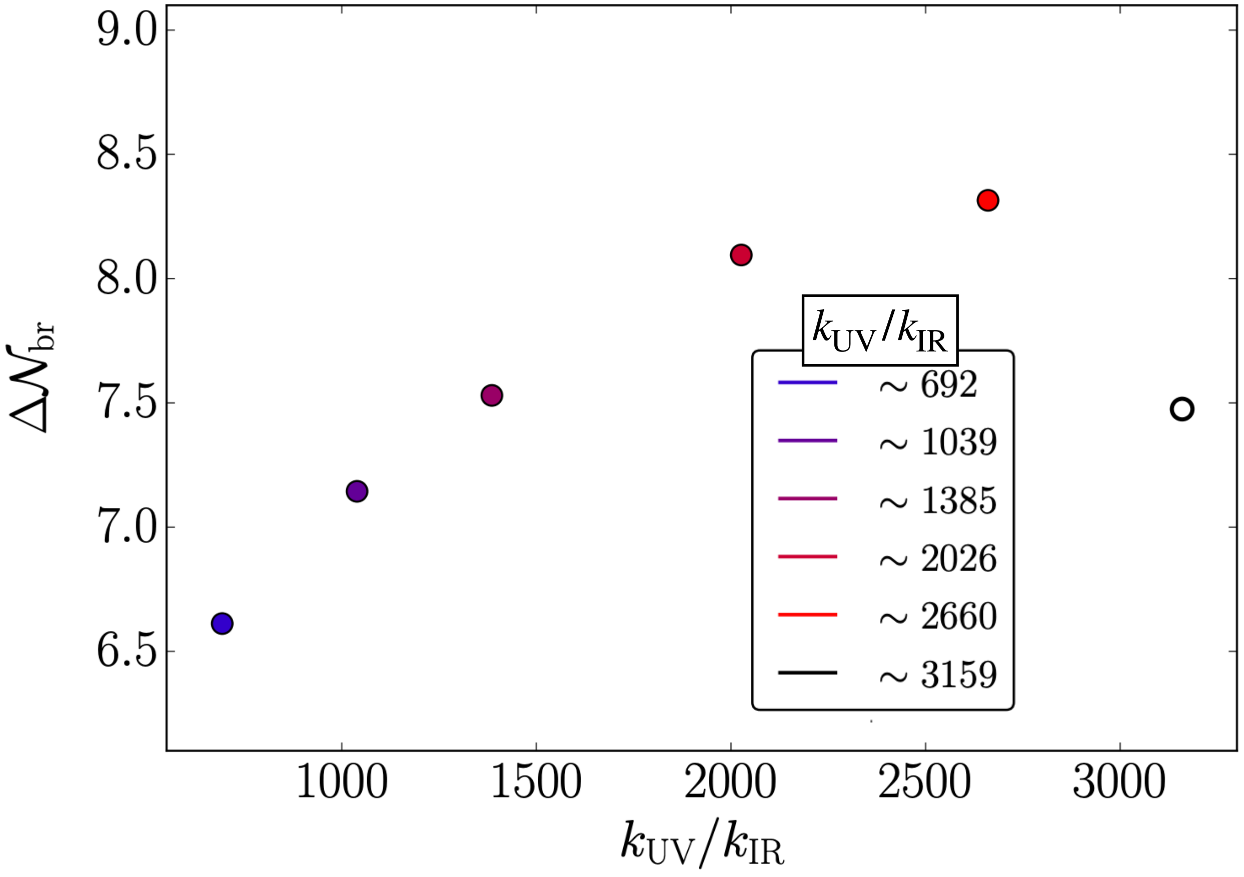
Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

UV convergence ?

$\alpha_\Lambda = 18$

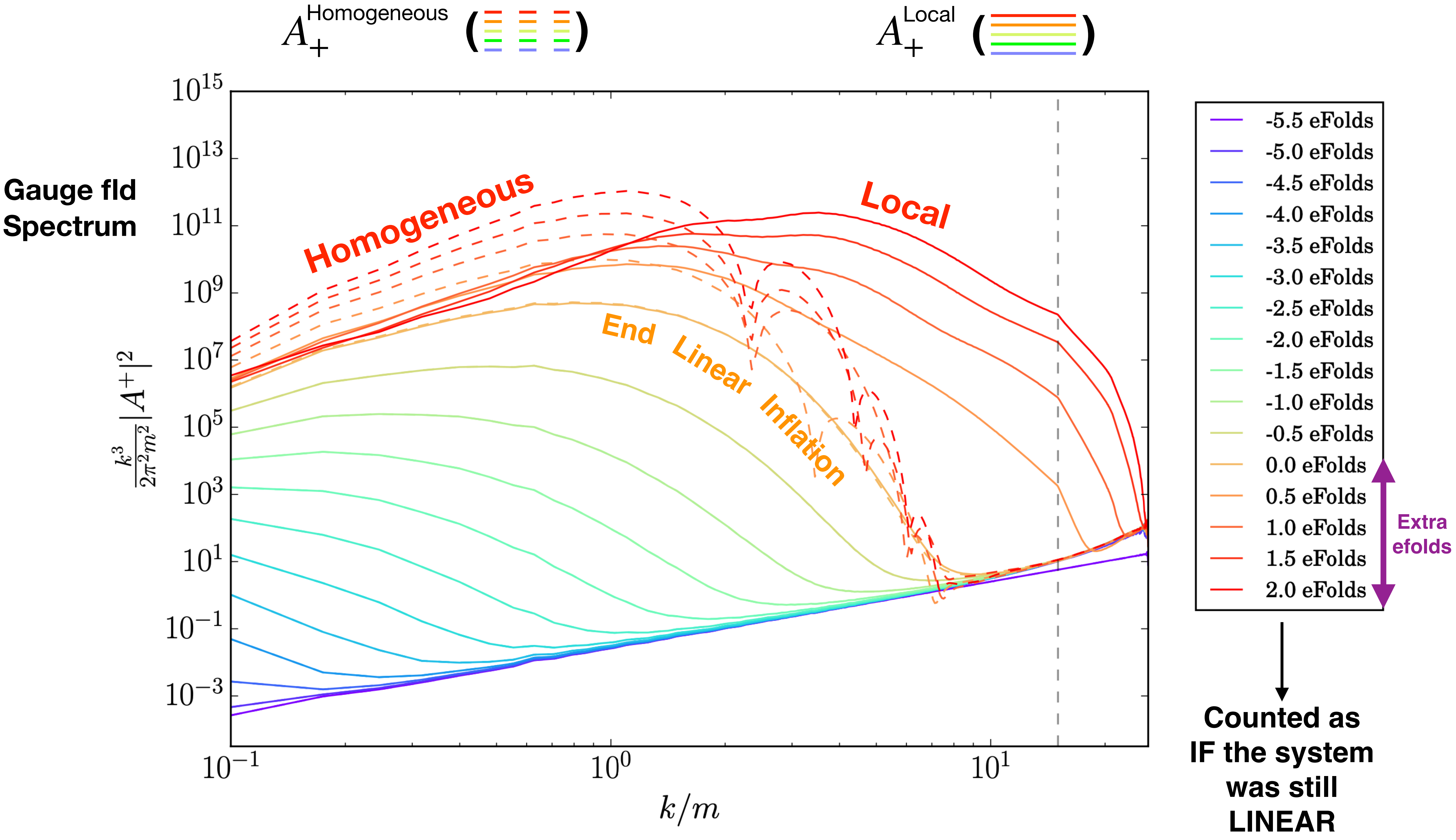


$\alpha_\Lambda = 20$



Comparison to Homogeneous Backreaction (Part II)

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 15\right)$

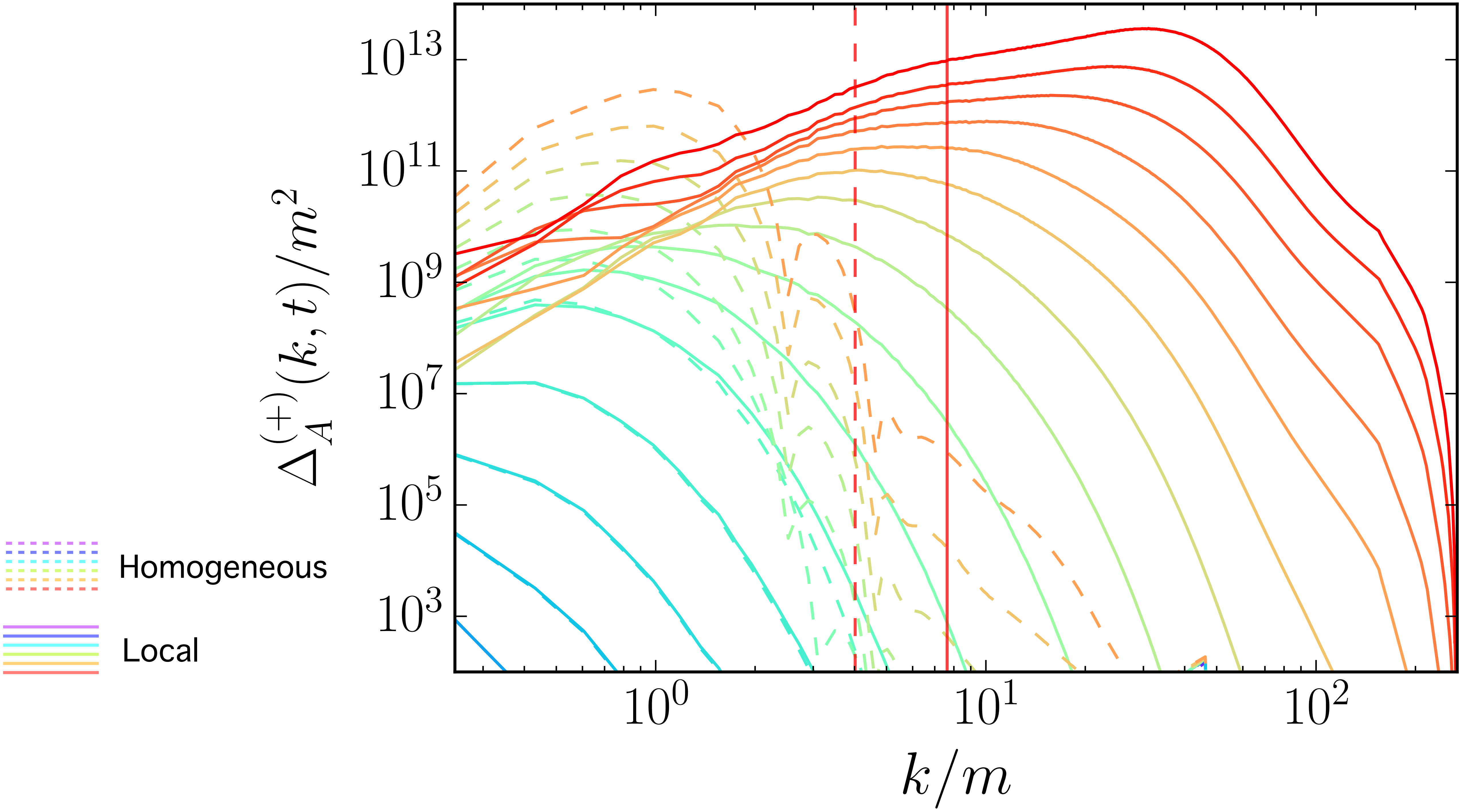


Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 15$**)**

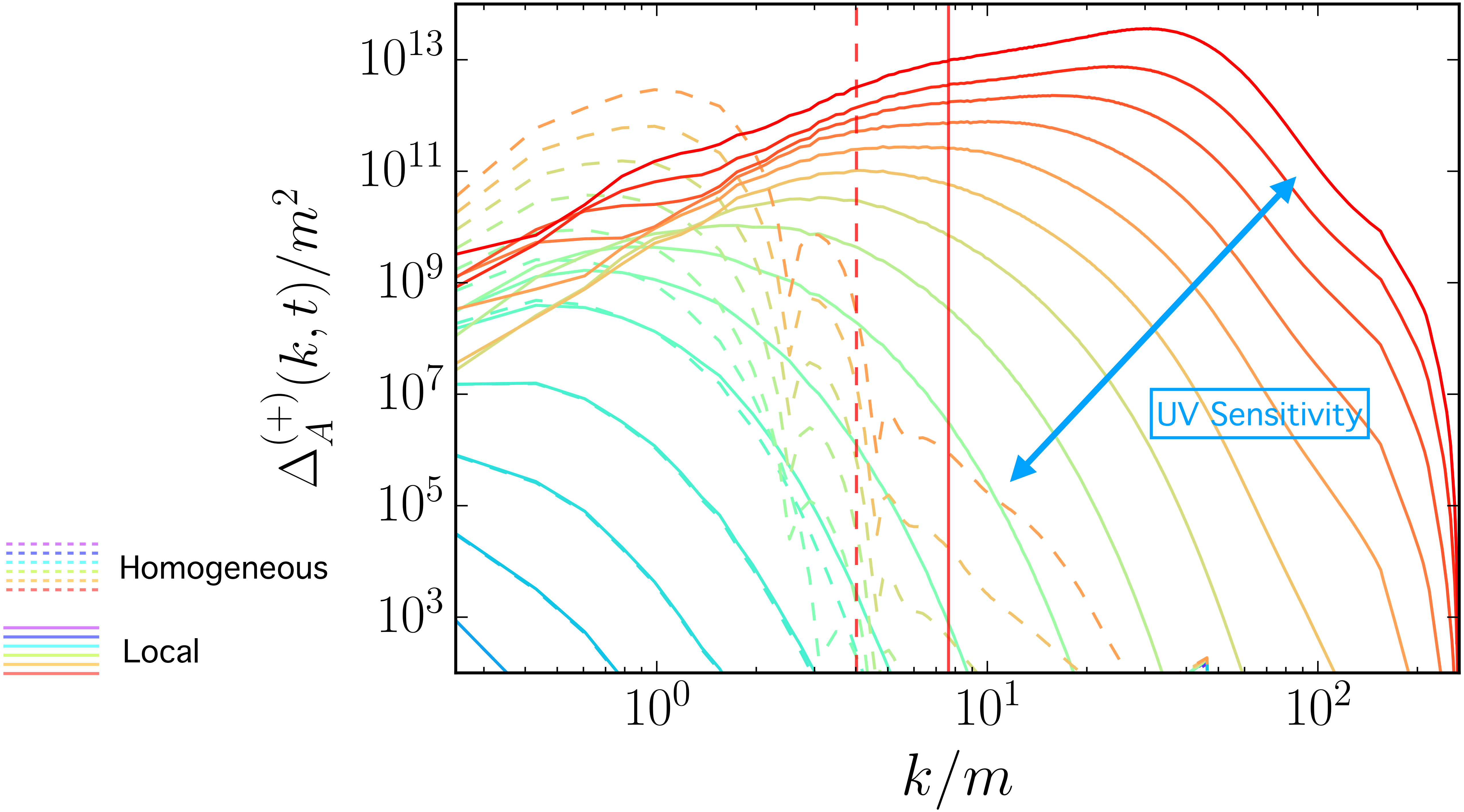
Homogeneous vs Inhomogeneous

ZOOM



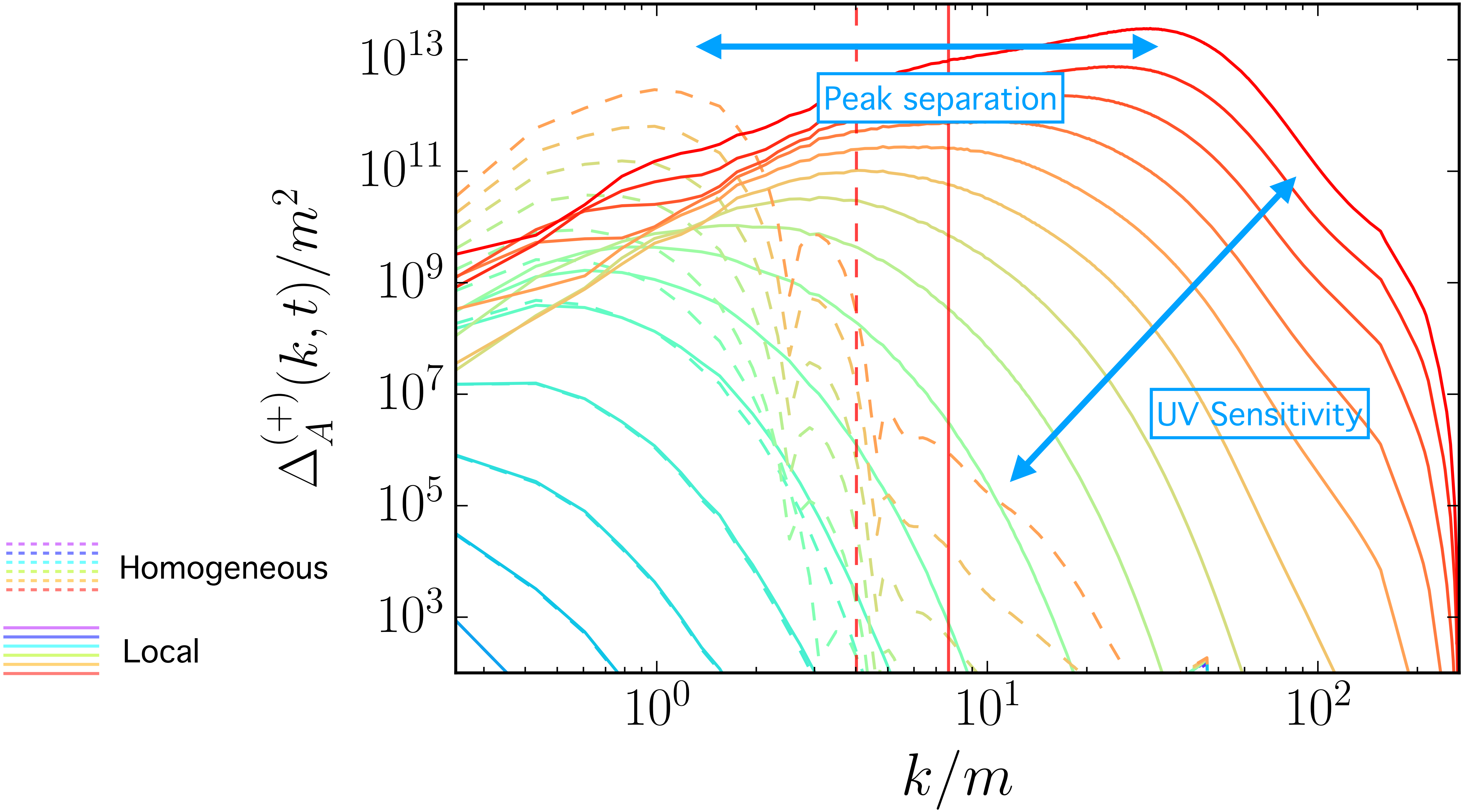
Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 15$)

Homogeneous vs Inhomogeneous



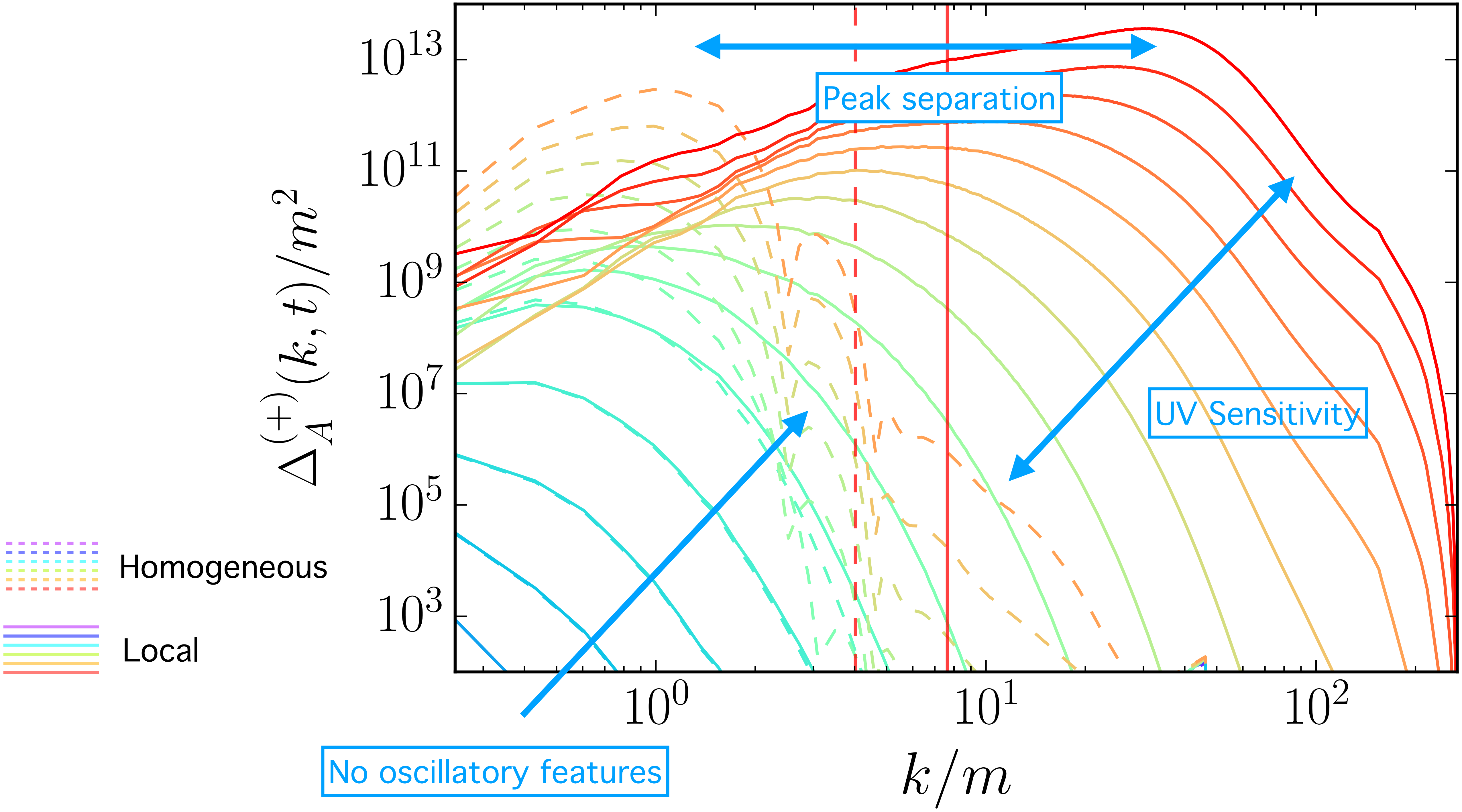
Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 15$)

Homogeneous vs Inhomogeneous



Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 15$)

Homogeneous vs Inhomogeneous



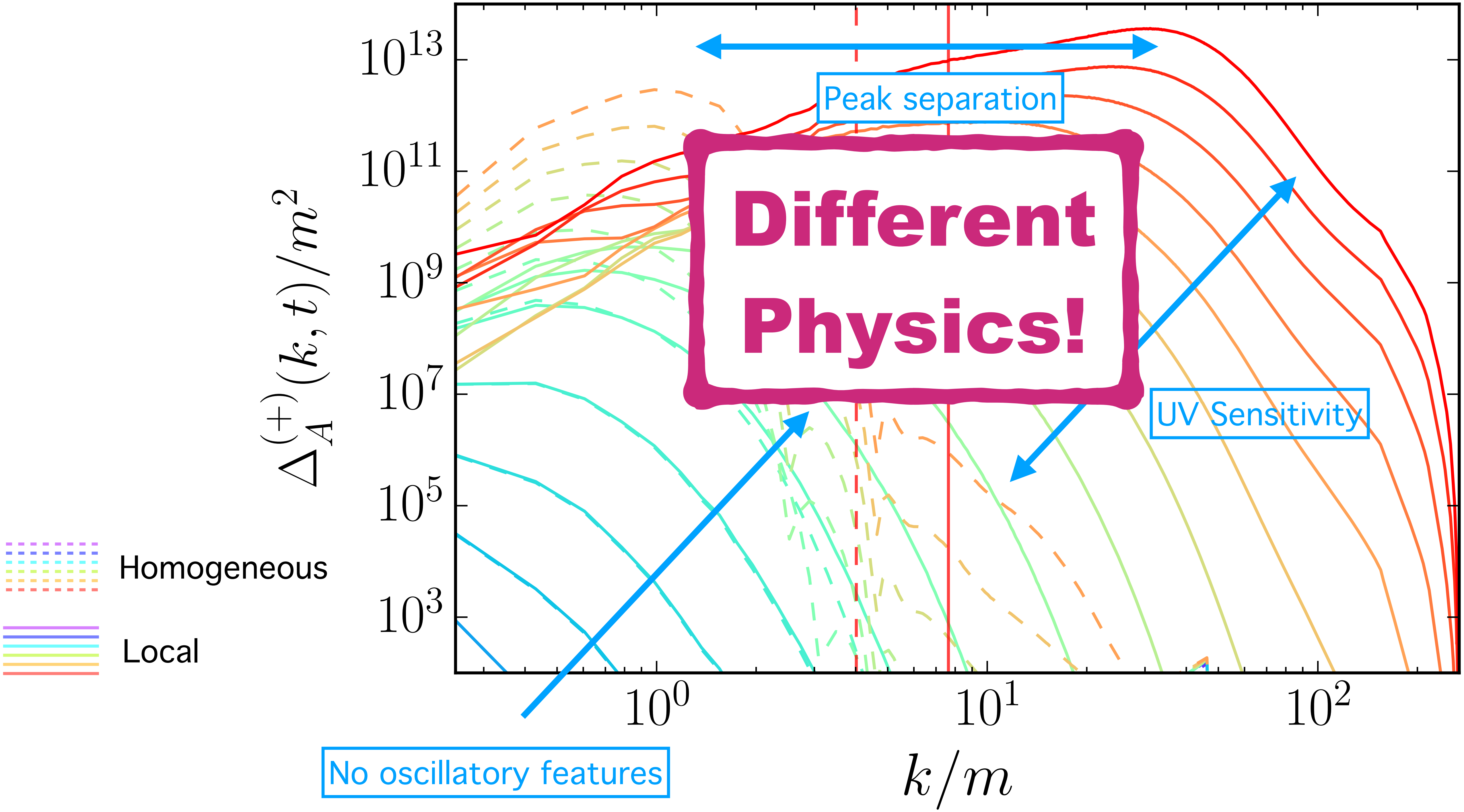
Peak separation

UV Sensitivity

No oscillatory features

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 15$)

Homogeneous vs Inhomogeneous



Peak separation

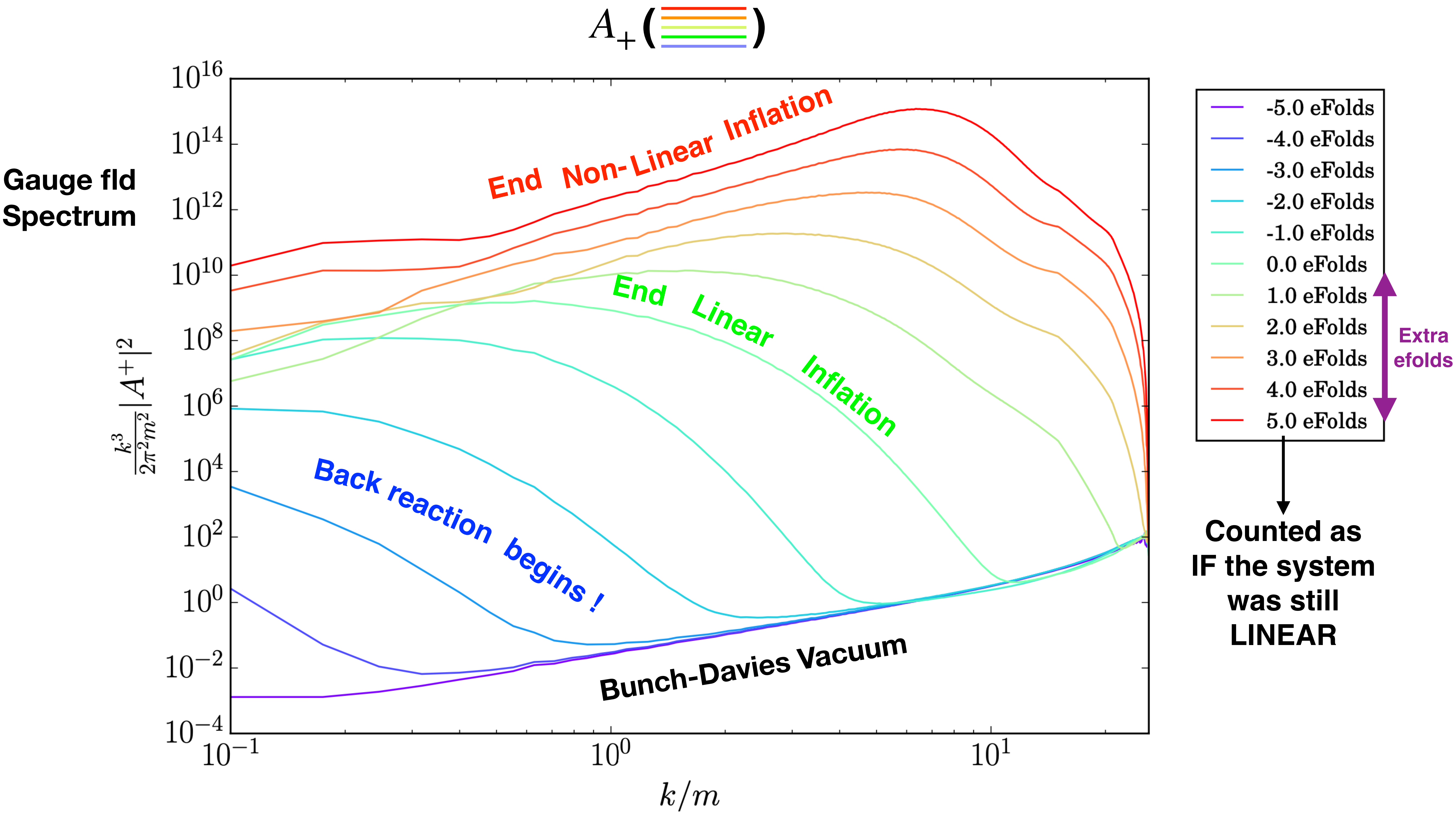
Different Physics!

UV Sensitivity

No oscillatory features

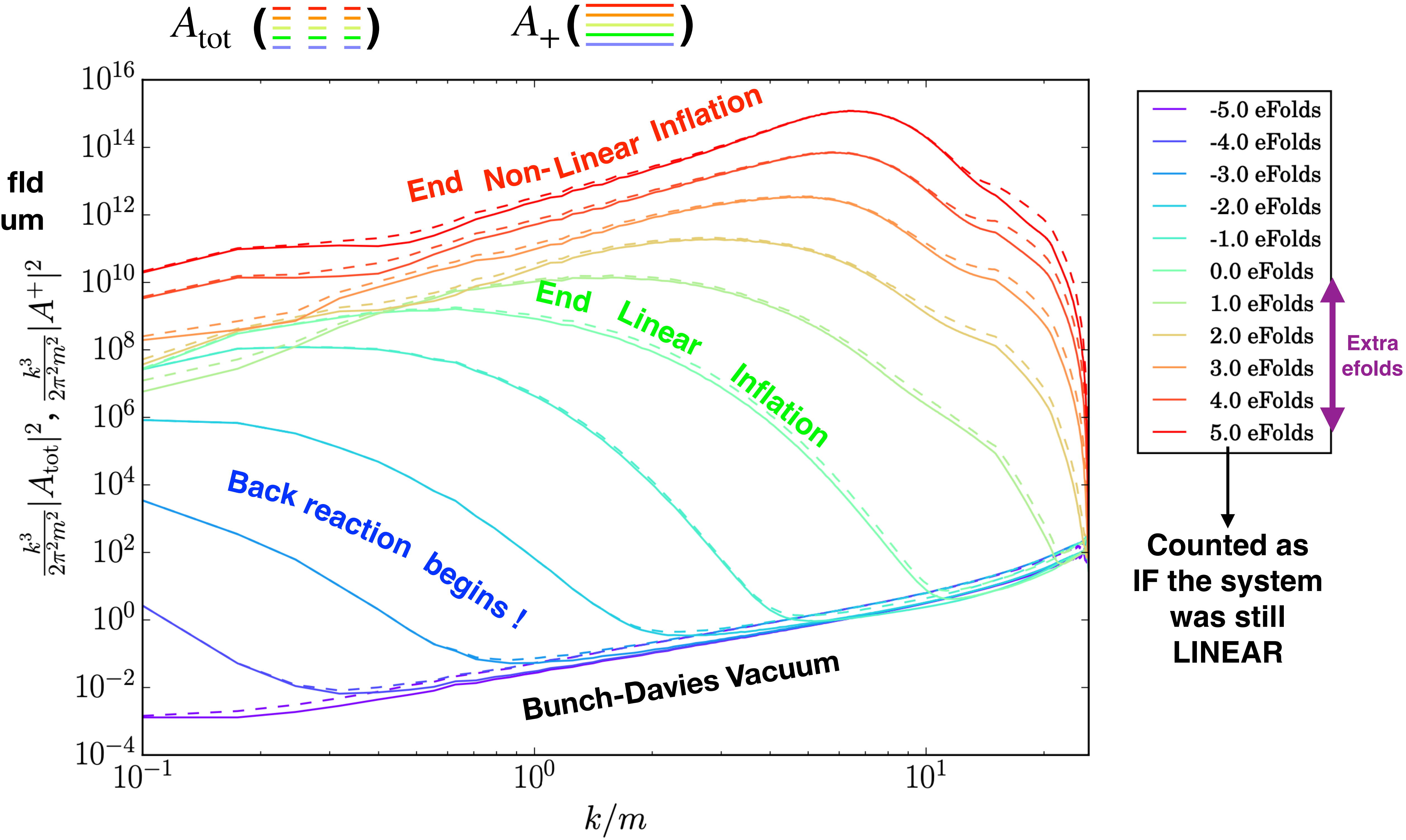
Chirality

Axion-Inflation$\left(V(\phi) = \frac{1}{2} m^2 \phi^2 \; ; \; \frac{\phi}{4\Lambda} F\tilde{F} \; ; \; \alpha_\Lambda = 18 \right)$



Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18\right)$

Gauge fld Spectrum



End Non-Linear Inflation

End Linear Inflation

Back reaction begins !

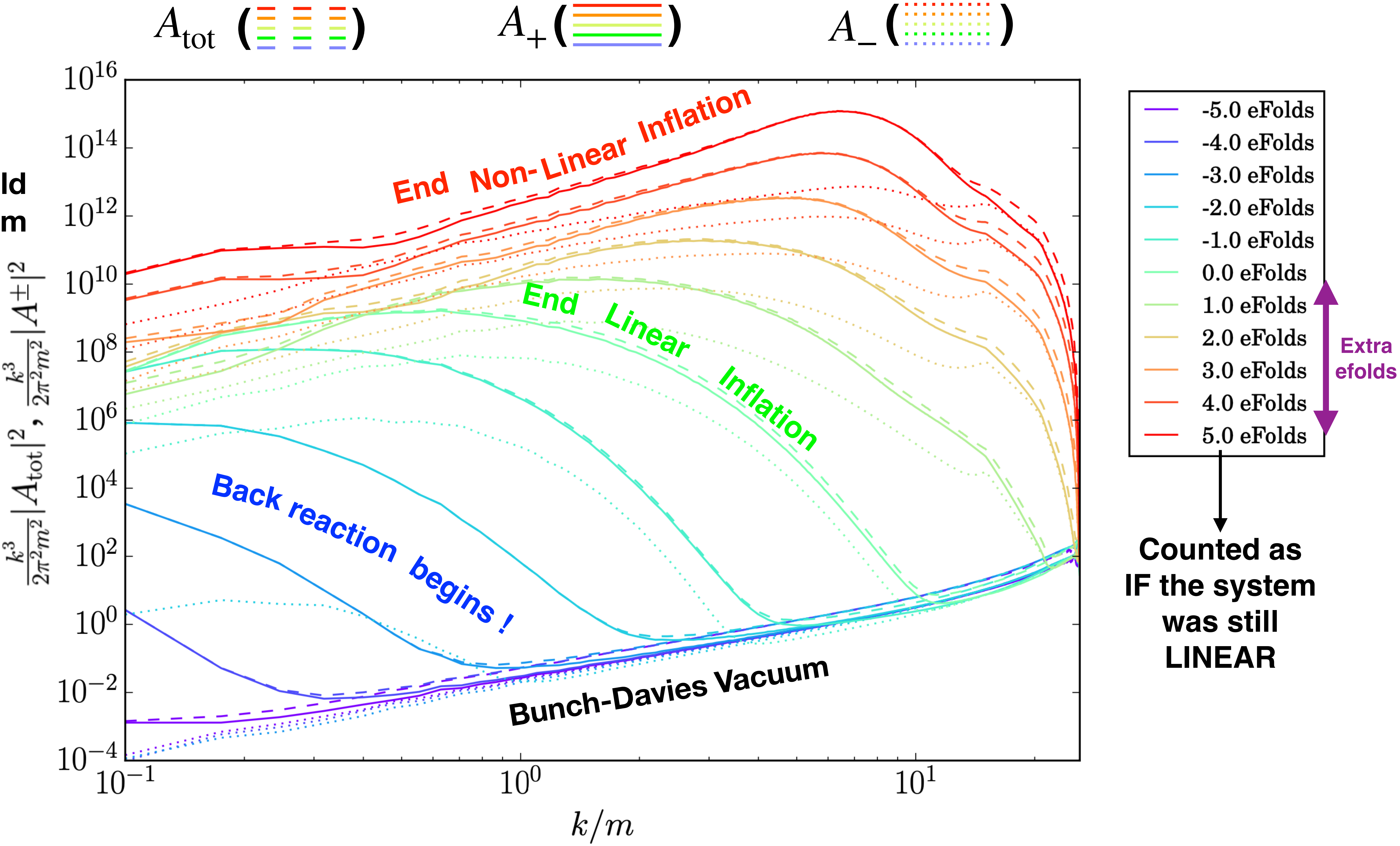
Bunch-Davies Vacuum

Extra efolds

Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \alpha_\Lambda = 18$)

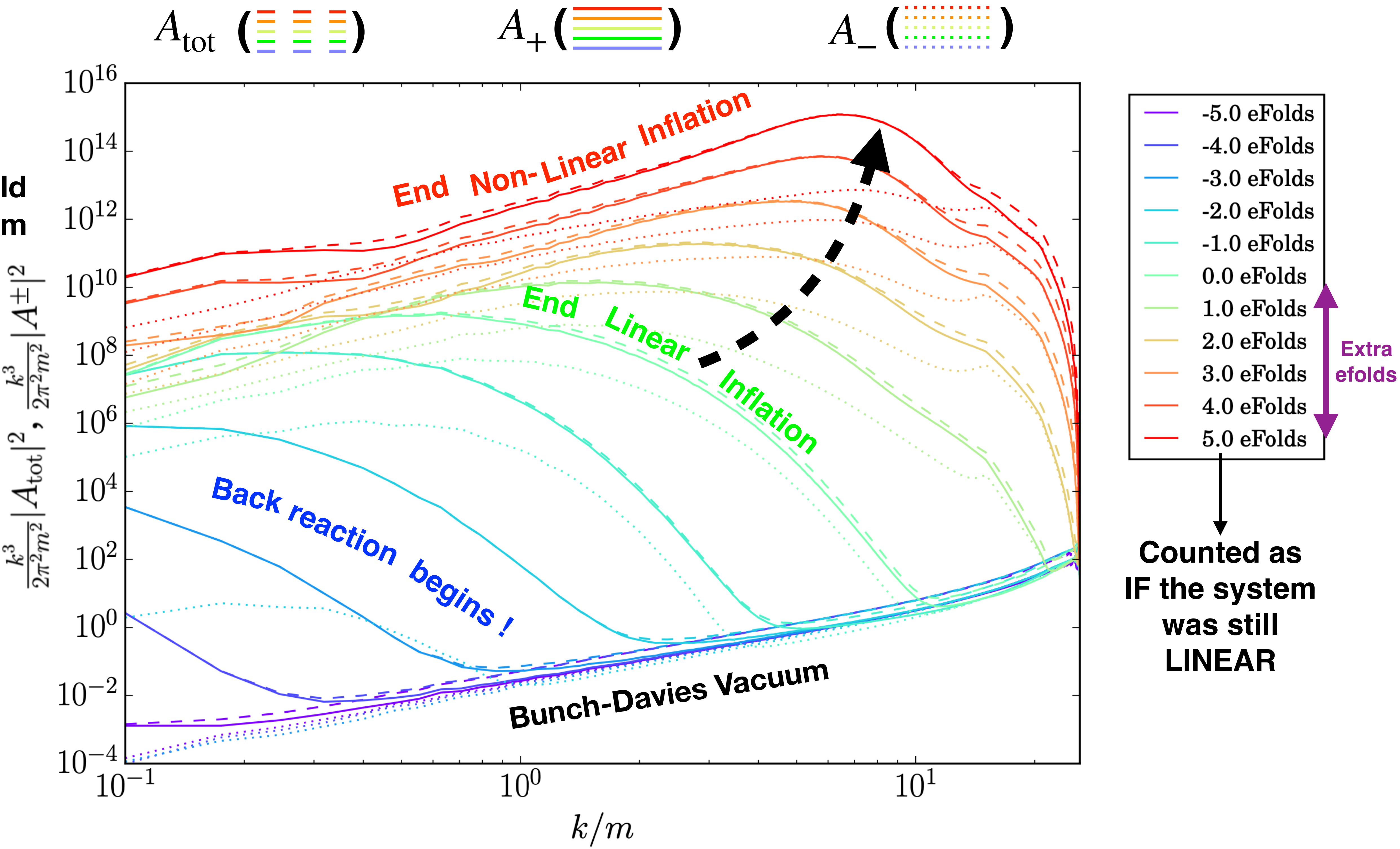
Gauge fld Spectrum



Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

Gauge fld Spectrum



End Non-Linear Inflation

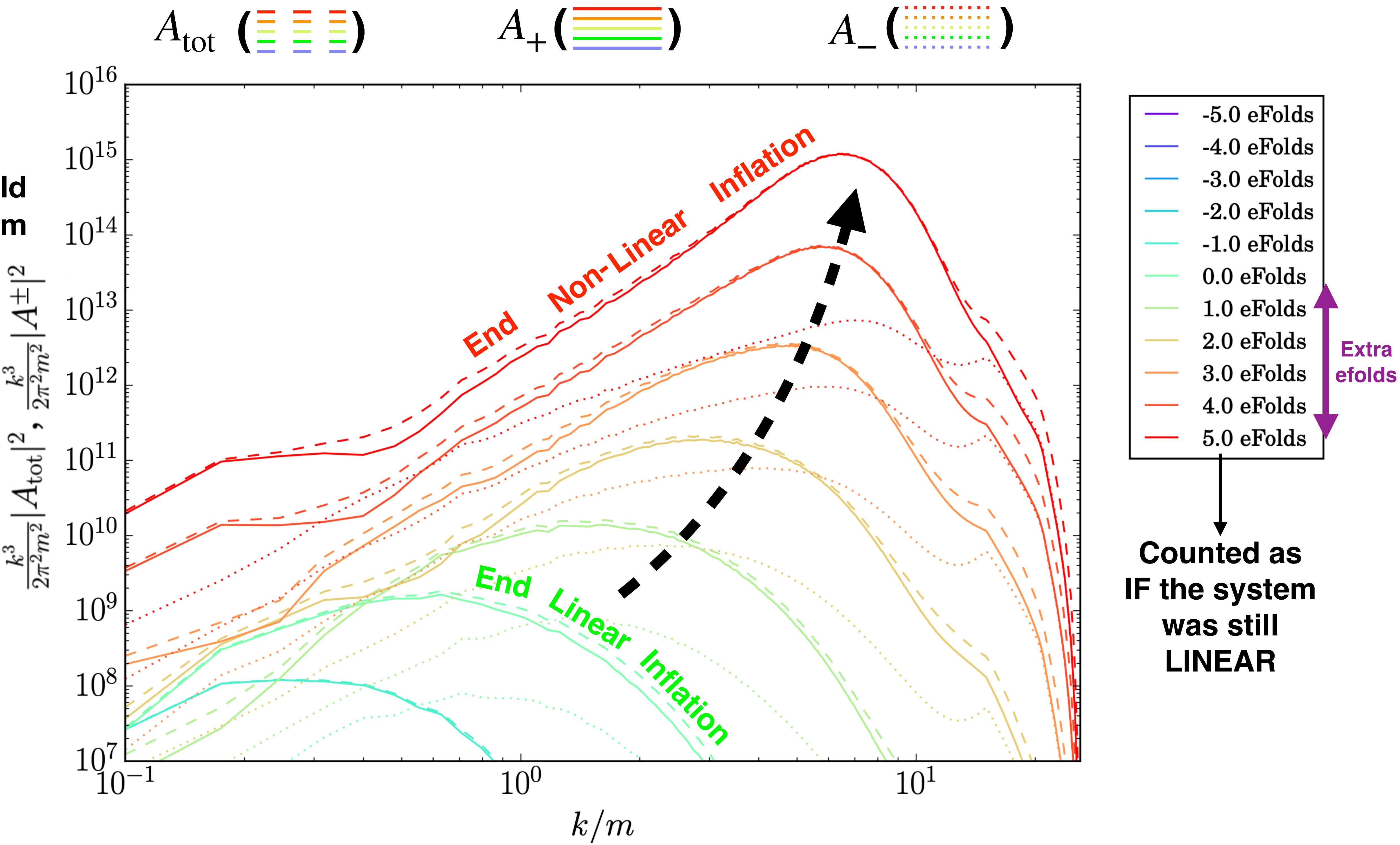
End Linear Inflation

Back reaction begins !

Bunch-Davies Vacuum

Counted as IF the system was still LINEAR

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\alpha_\Lambda = 18$)

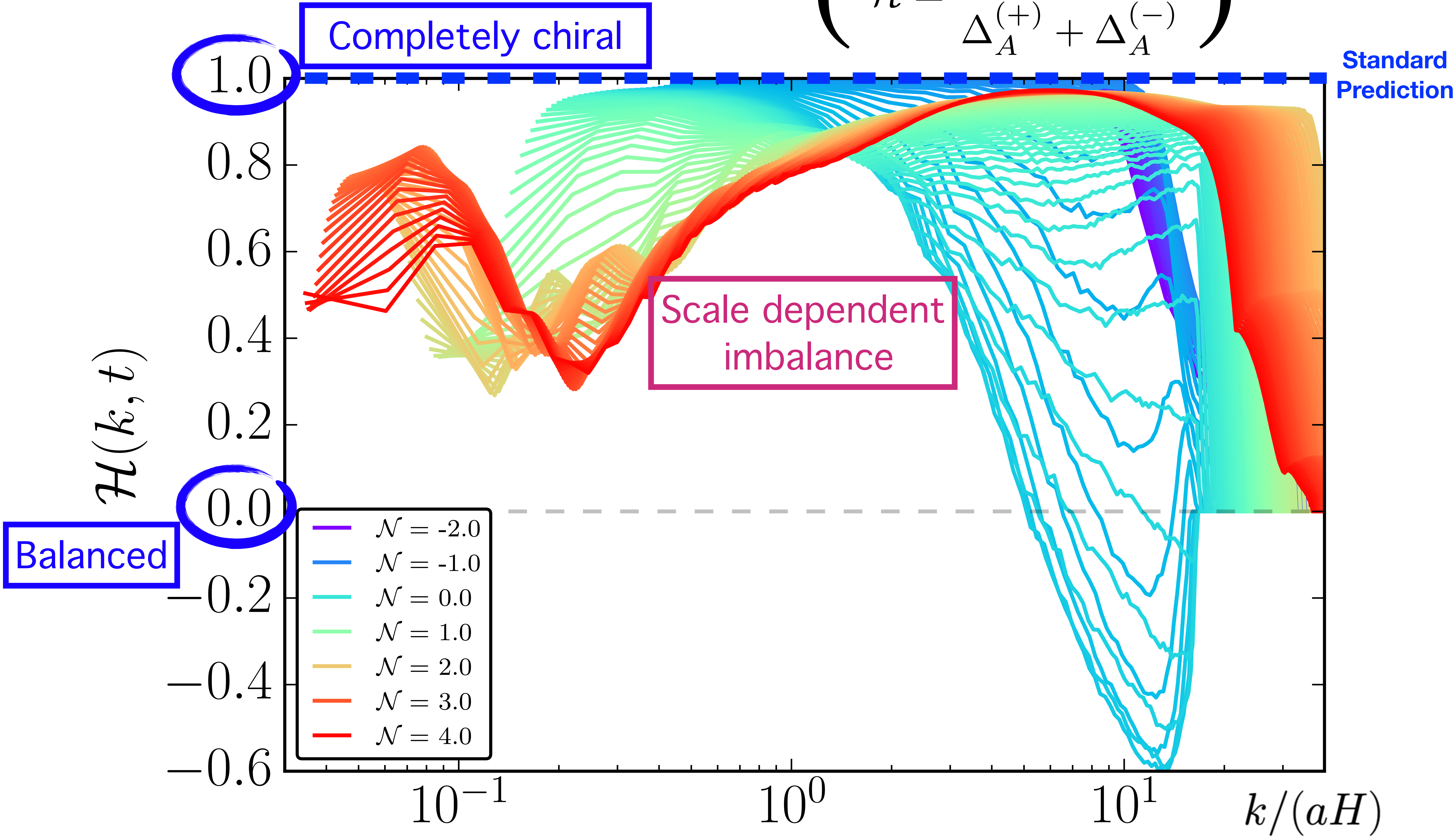


Gauge fld Spectrum

Counted as IF the system was still LINEAR

Axion-Inflation$\left(V(\phi) = \frac{1}{2}m^2\phi^2 \; ; \; \frac{\phi}{4\Lambda}F\tilde{F} \; ; \; \Lambda = \frac{m_p}{18}\right)$

$$\left(\mathcal{H} = \frac{\Delta_A^{(+)} - \Delta_A^{(-)}}{\Delta_A^{(+)} + \Delta_A^{(-)}} \right)$$



Completely chiral

Standard Prediction

Scale dependent imbalance

Balanced

Other Potentials

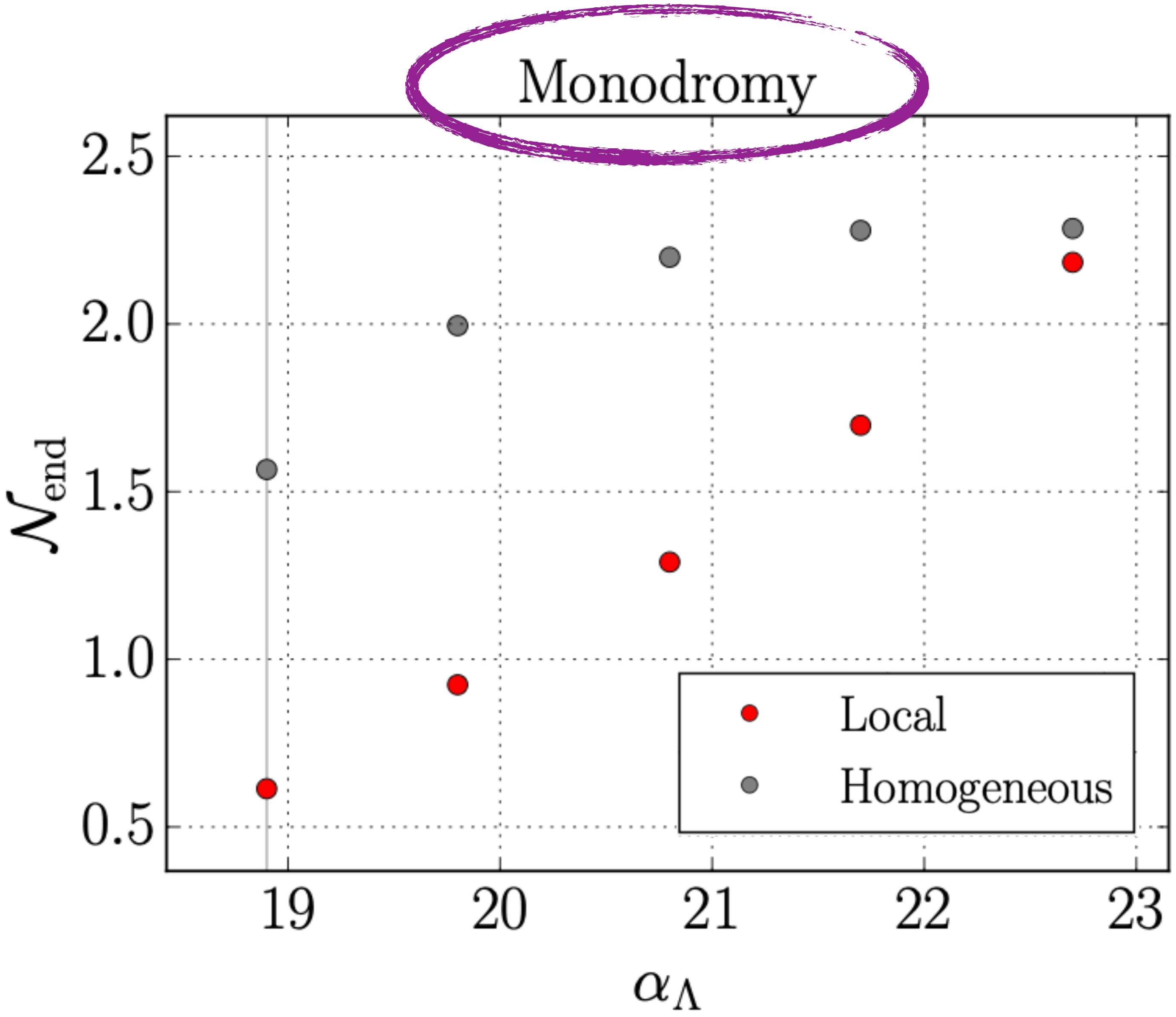
Axion-Inflation($V(\phi) = \xcancel{\frac{1}{2}m^2\phi^2}\ ;\ \frac{\phi}{4\Lambda}F\tilde{F}\ ;\ \Lambda = \frac{m_p}{18}$)

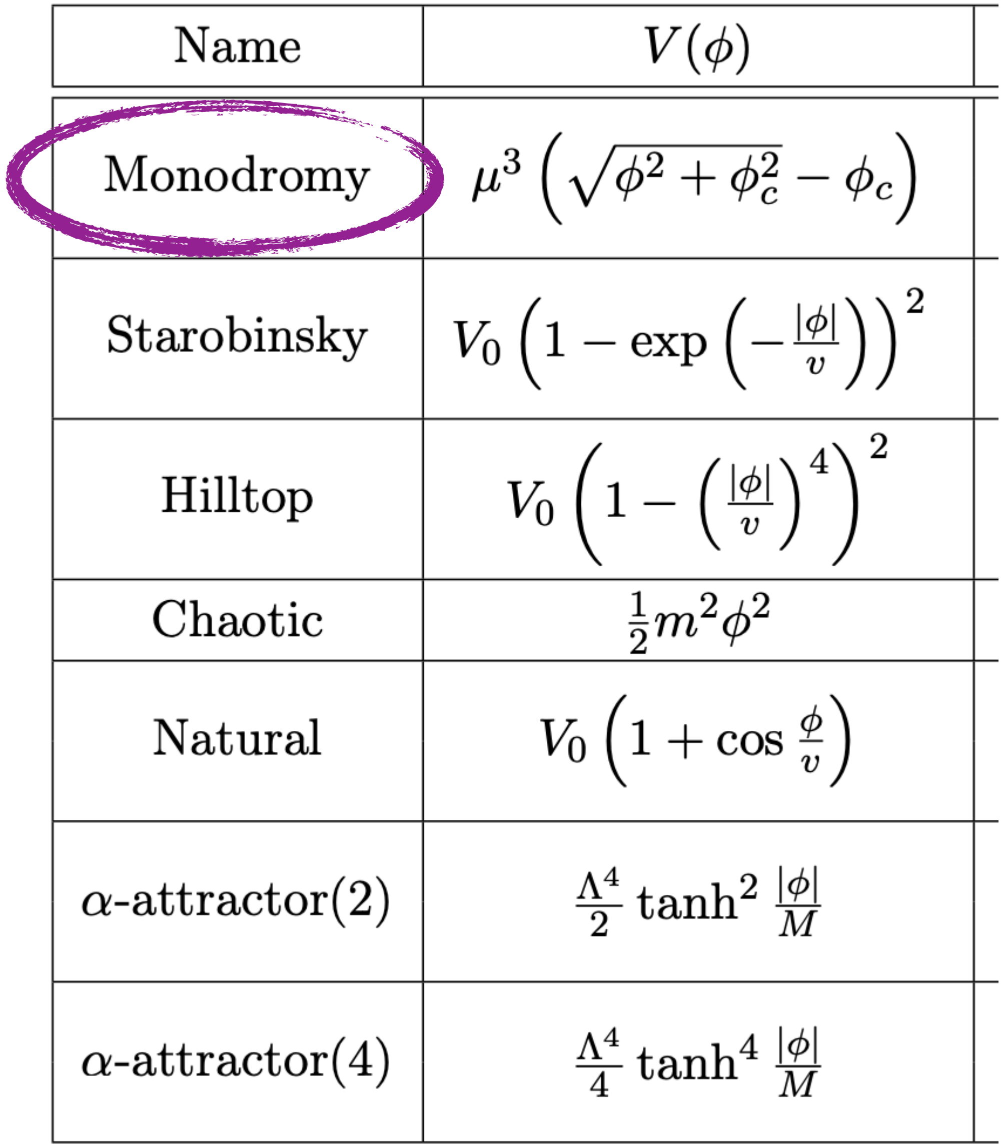
Axion-Inflation($V(\phi) = \cancel{\frac{1}{2}m^2\phi^2}$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{18}$)

Name	$V(\phi)$
Monodromy	$\mu^3\left(\sqrt{\phi^2+\phi_c^2}-\phi_c\right)$
Starobinsky	$V_0\left(1-\exp\left(-\frac{\lvert\phi\rvert}{v}\right)\right)^2$
Hilltop	$V_0\left(1-\left(\frac{\lvert\phi\rvert}{v}\right)^4\right)^2$
Chaotic	$\frac{1}{2}m^2\phi^2$
Natural	$V_0\left(1+\cos\frac{\phi}{v}\right)$
α-attractor(2)	$\frac{\Lambda^4}{2}\tanh^2\frac{\lvert\phi\rvert}{M}$
α-attractor(4)	$\frac{\Lambda^4}{4}\tanh^4\frac{\lvert\phi\rvert}{M}$

Lizarraga et al 2025, Arxiv 2025.19950

Axion-Inflation($V(\phi) = \xcancel{\frac{1}{2}m^2\phi^2}$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{18}$)

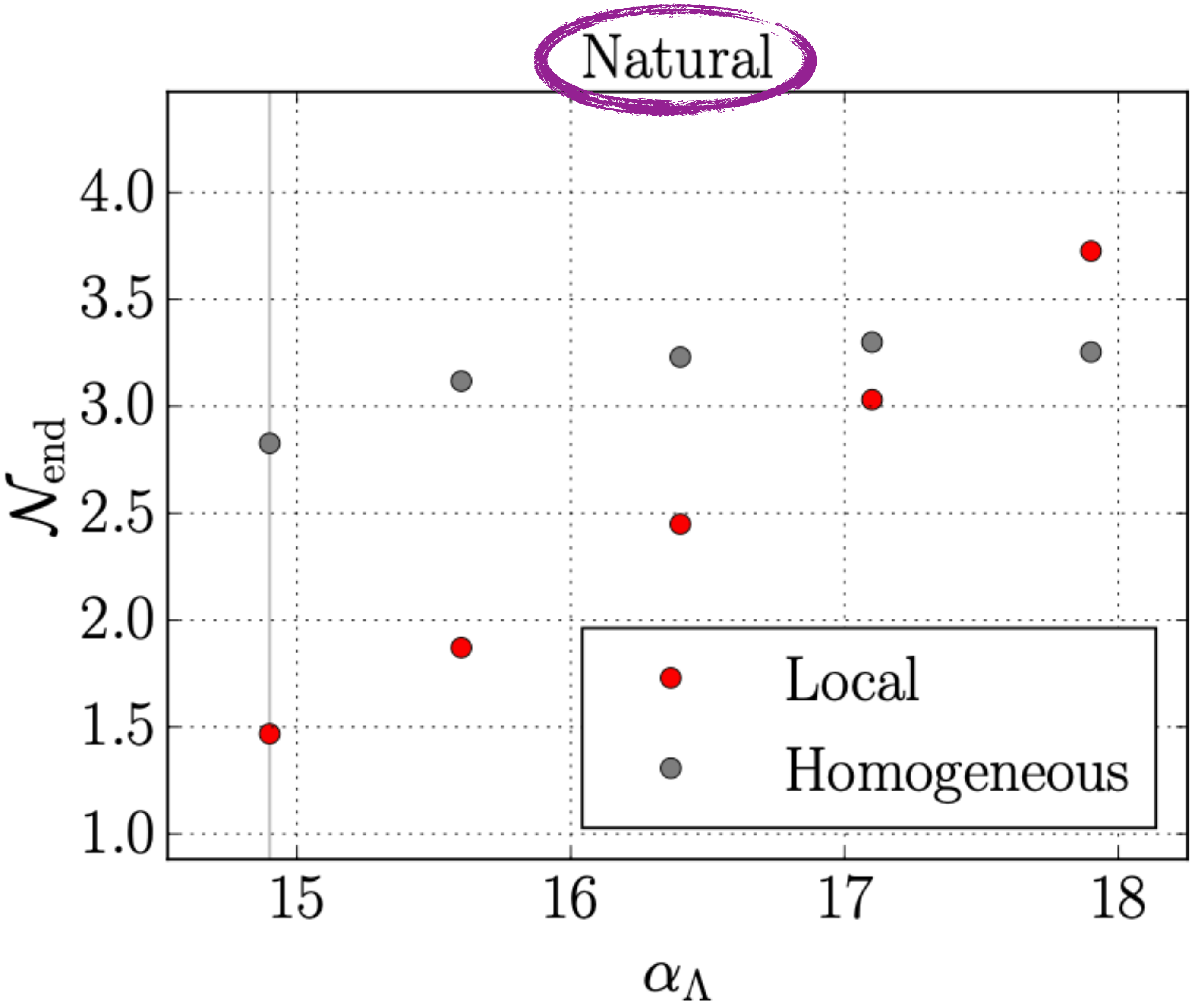


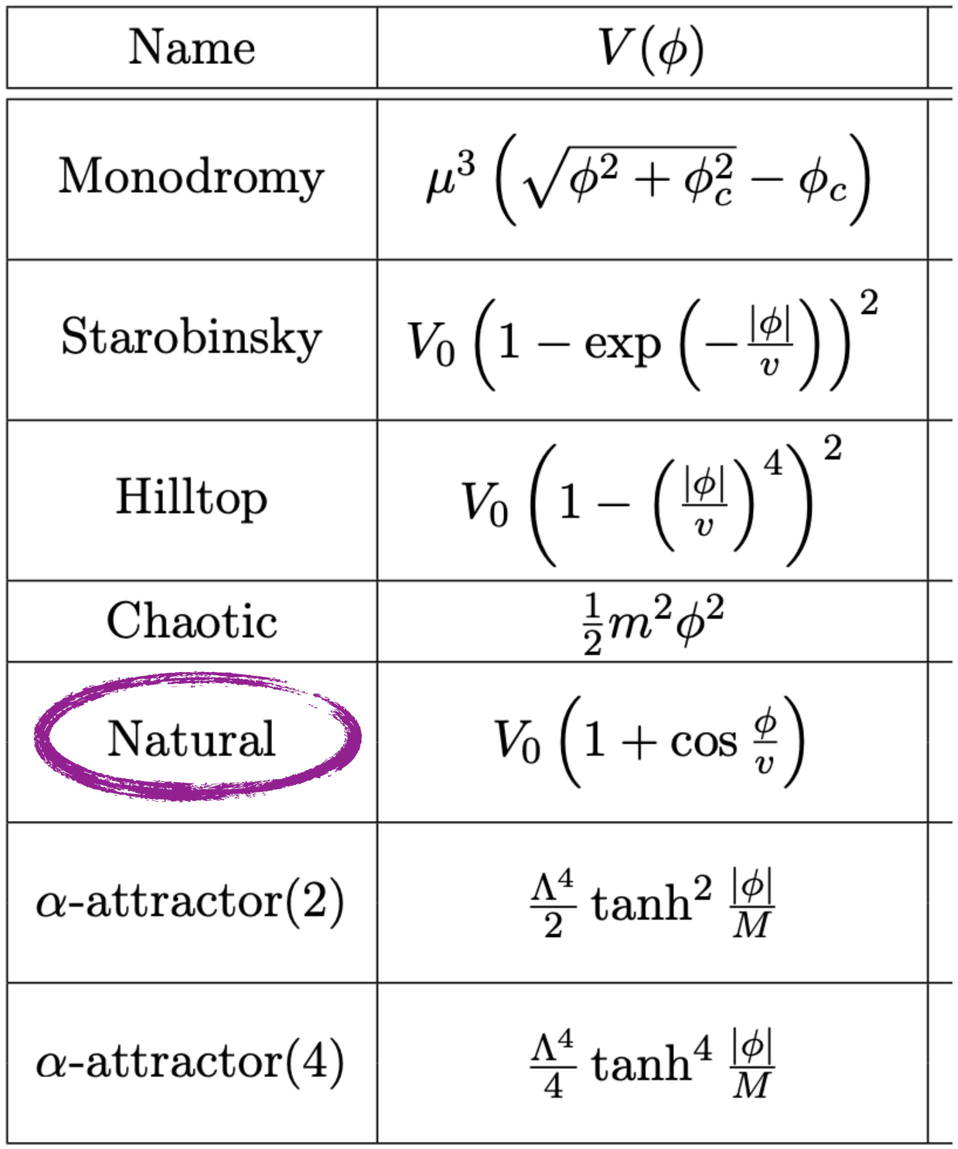


Name	$V(\phi)$
Monodromy	$\mu^3\left(\sqrt{\phi^2+\phi_c^2}-\phi_c\right)$
Starobinsky	$V_0\left(1-\exp\left(-\frac{\lvert\phi\rvert}{v}\right)\right)^2$
Hilltop	$V_0\left(1-\left(\frac{\lvert\phi\rvert}{v}\right)^4\right)^2$
Chaotic	$\frac{1}{2}m^2\phi^2$
Natural	$V_0\left(1+\cos\frac{\phi}{v}\right)$
α-attractor(2)	$\frac{\Lambda^4}{2}\tanh^2\frac{\lvert\phi\rvert}{M}$
α-attractor(4)	$\frac{\Lambda^4}{4}\tanh^4\frac{\lvert\phi\rvert}{M}$

Lizarraga et al 2025, Arxiv 2025.19950

Axion-Inflation($V(\phi) = \cancel{\frac{1}{2}m^2\phi^2}$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{18}$)

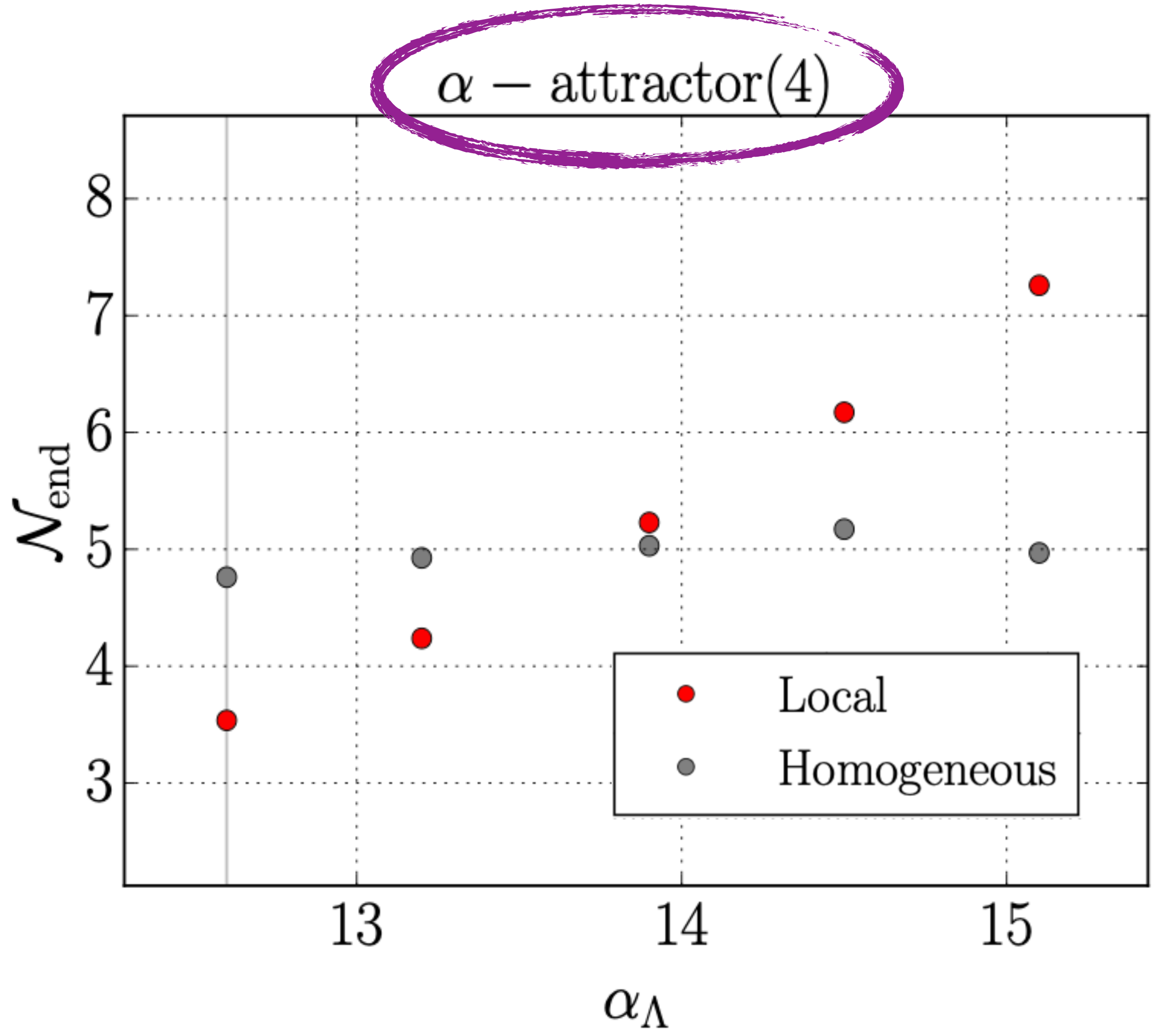


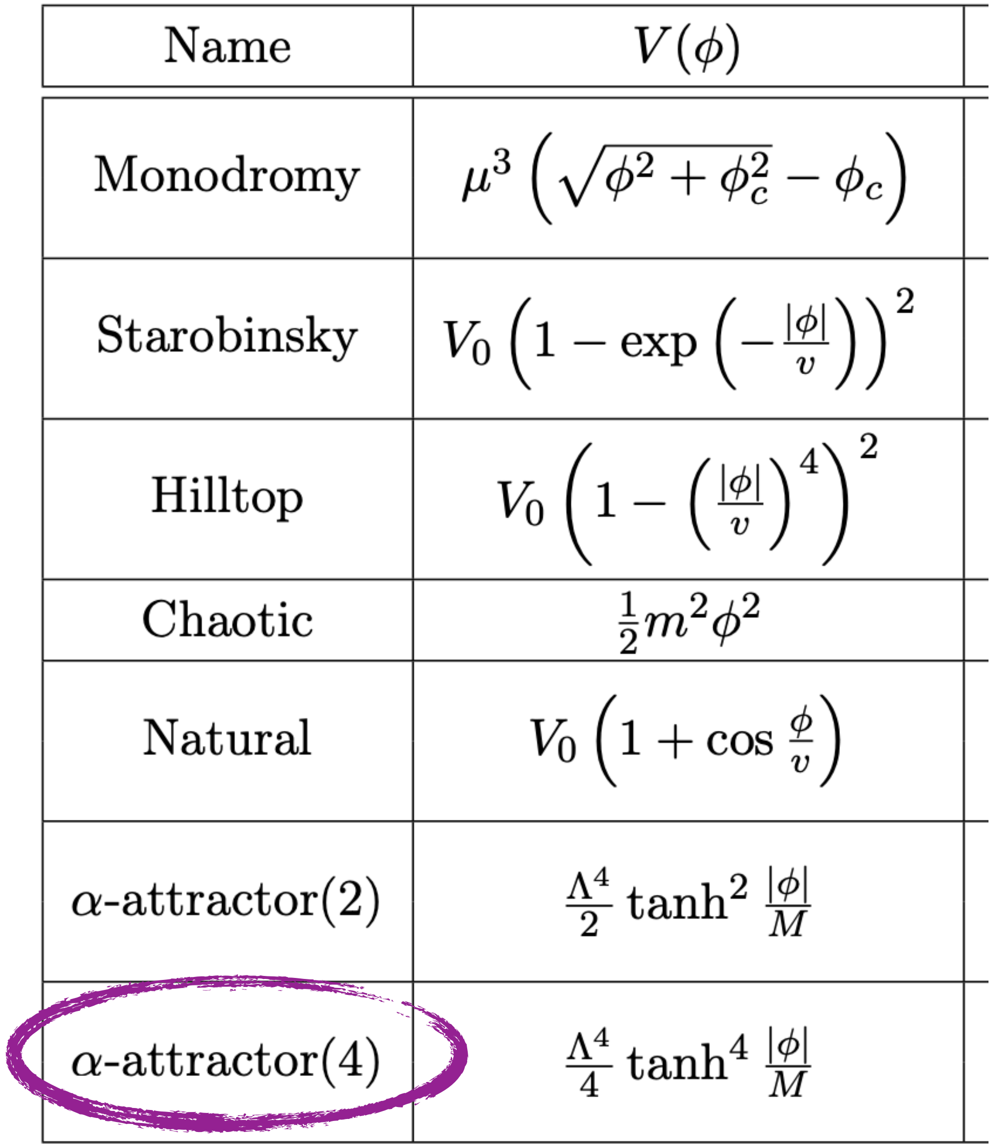


Name	$V(\phi)$
Monodromy	$\mu^3\left(\sqrt{\phi^2+\phi_c^2}-\phi_c\right)$
Starobinsky	$V_0\left(1-\exp\left(-\frac{\lvert\phi\rvert}{v}\right)\right)^2$
Hilltop	$V_0\left(1-\left(\frac{\lvert\phi\rvert}{v}\right)^4\right)^2$
Chaotic	$\frac{1}{2}m^2\phi^2$
Natural	$V_0\left(1+\cos\frac{\phi}{v}\right)$
α-attractor(2)	$\frac{\Lambda^4}{2}\tanh^2\frac{\lvert\phi\rvert}{M}$
α-attractor(4)	$\frac{\Lambda^4}{4}\tanh^4\frac{\lvert\phi\rvert}{M}$

Lizarraga et al 2025, Arxiv 2025.19950

Axion-Inflation($V(\phi) = \cancel{\frac{1}{2}m^2\phi^2}$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{18}$)





Name	$V(\phi)$
Monodromy	$\mu^3\left(\sqrt{\phi^2+\phi_c^2}-\phi_c\right)$
Starobinsky	$V_0\left(1-\exp\left(-\frac{\lvert\phi\rvert}{v}\right)\right)^2$
Hilltop	$V_0\left(1-\left(\frac{\lvert\phi\rvert}{v}\right)^4\right)^2$
Chaotic	$\frac{1}{2}m^2\phi^2$
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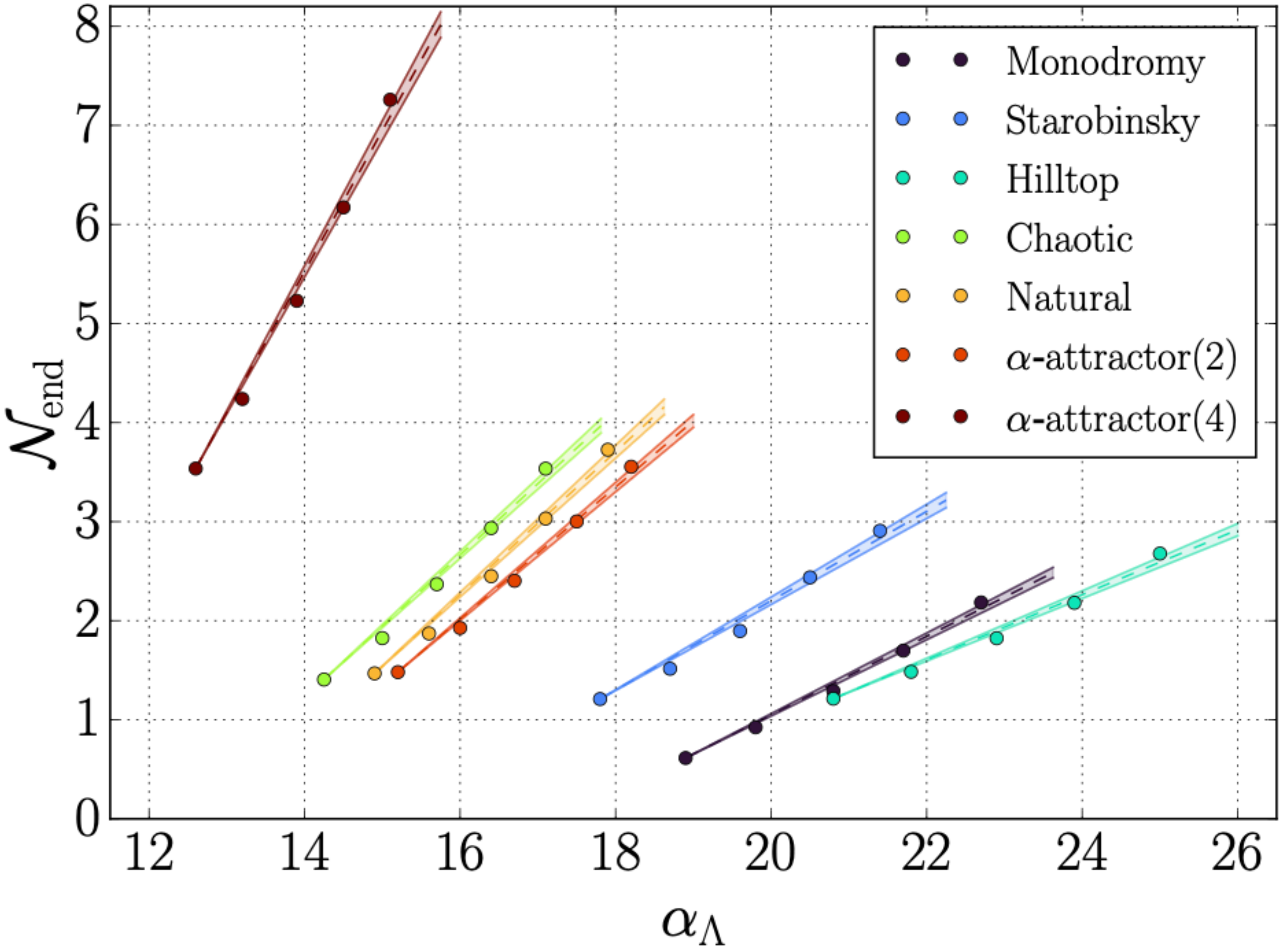
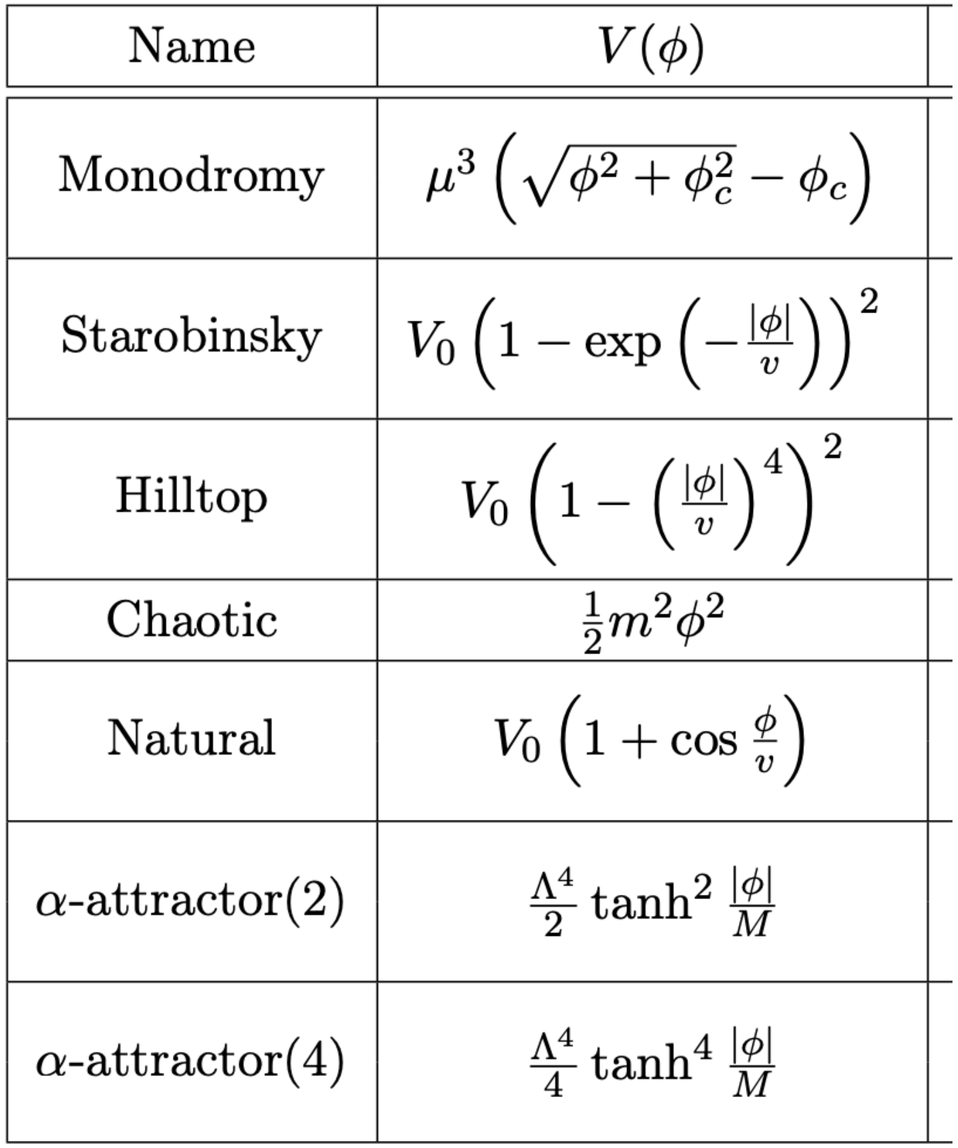
(All potentials)





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Summary

Axion-Inflation($V(\phi) = \frac{1}{2}m^2\phi^2$; $\frac{\phi}{4\Lambda}F\tilde{F}$; $\Lambda = \frac{m_p}{X}$)

$(X = 15, 20, 25)$

Summary

* 'Locality' controls the Scalar/Gauge field excitation

* Linear change in $\xi \equiv \frac{|\dot{\phi}|}{\Lambda H}$: exponential response in A_μ

* Predictions/constraints (PNG, PBH and GWs) depend crucially on ξ : re-assessing real observability !

* Adding Schwinger pair production easy via $\vec{J} = \sigma \vec{E}$

* Other phenomena: BAU, Magnetogenesis, …

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PRL *131 (2023) 15, 151003;* PRD *111 (2025) 6, 063545*

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Work in Progress ...

* Predictions/constraints (PNG, PBH and GWs) depend crucially on ξ : re-assessing real observability !

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Future work ...

* Adding Schwinger pair production easy via $\vec{J} = \sigma \vec{E}$

* Other phenomena: BAU, Magnetogenesis, …

Almost …
the End

If you want to learn
how to "latticesize"
your problems …

Come to some of our CL Schools !

CosmoLattice

1st CL School 2022: Sept 5-8

@Valencia:

Come to some of our CL Schools !

CosmoLattice

2nd CL School 2023: Sept 25-29

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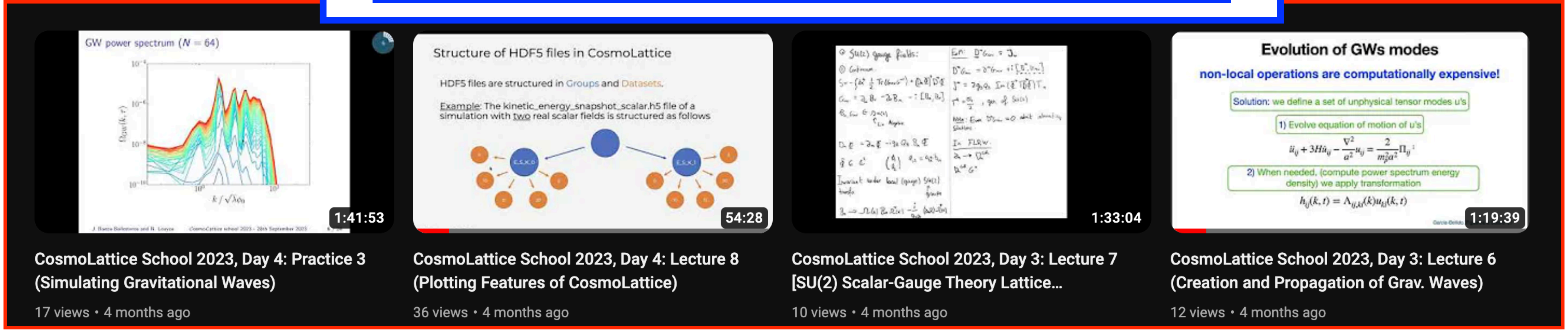
2nd CL School 2023: Sept 25-29

https://www.youtube.com/@CosmoLattice/videos



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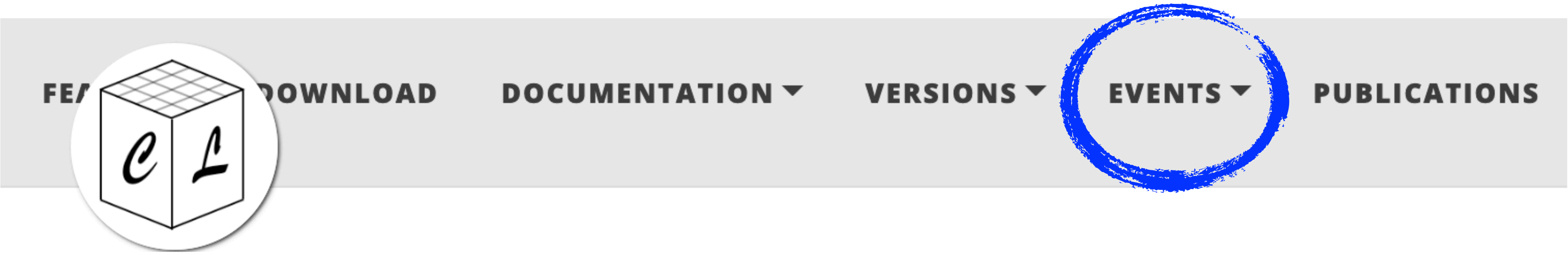
CosmoLattice

3rd CL School 2025: Sept 22-26

Details for next CL School at:

https://cosmolattice.net



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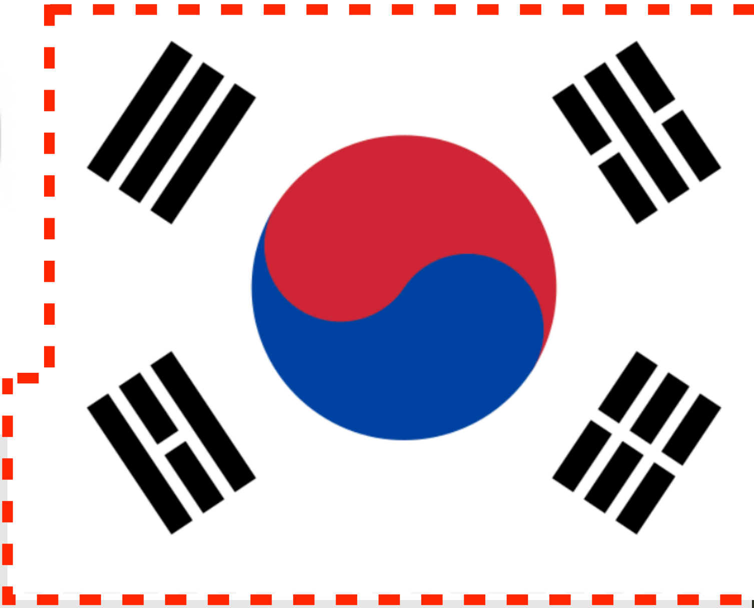
Det hool TBA at:

https

FEA DOWNLOAD NS



Thanks for your attention

Tack för din uppmärksamhet

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Tack för din uppmärksamhet

Back Slides

Constraints

Program Variables

Axion Inflation

Axion Inflation GWs

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

L.Sorbo et al 2006-2012

$$V(\phi) + \frac{\phi}{4\Lambda} F\tilde{F} \quad \Rightarrow \quad A''_{\pm} + \left(k^2 \pm \frac{k\dot{\phi}}{\Lambda H \tau}\right) A_{\pm} = 0$$

Chiral instability

$$A_+ \propto e^{\pi\xi}\,, \quad |A_-| \ll |A_+|$$

$(\xi \propto \dot{\phi})$

A_+ exponentially amplified

Only
one chirality
of gauge field
then… chiral GWs !

Example, GW prediction

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

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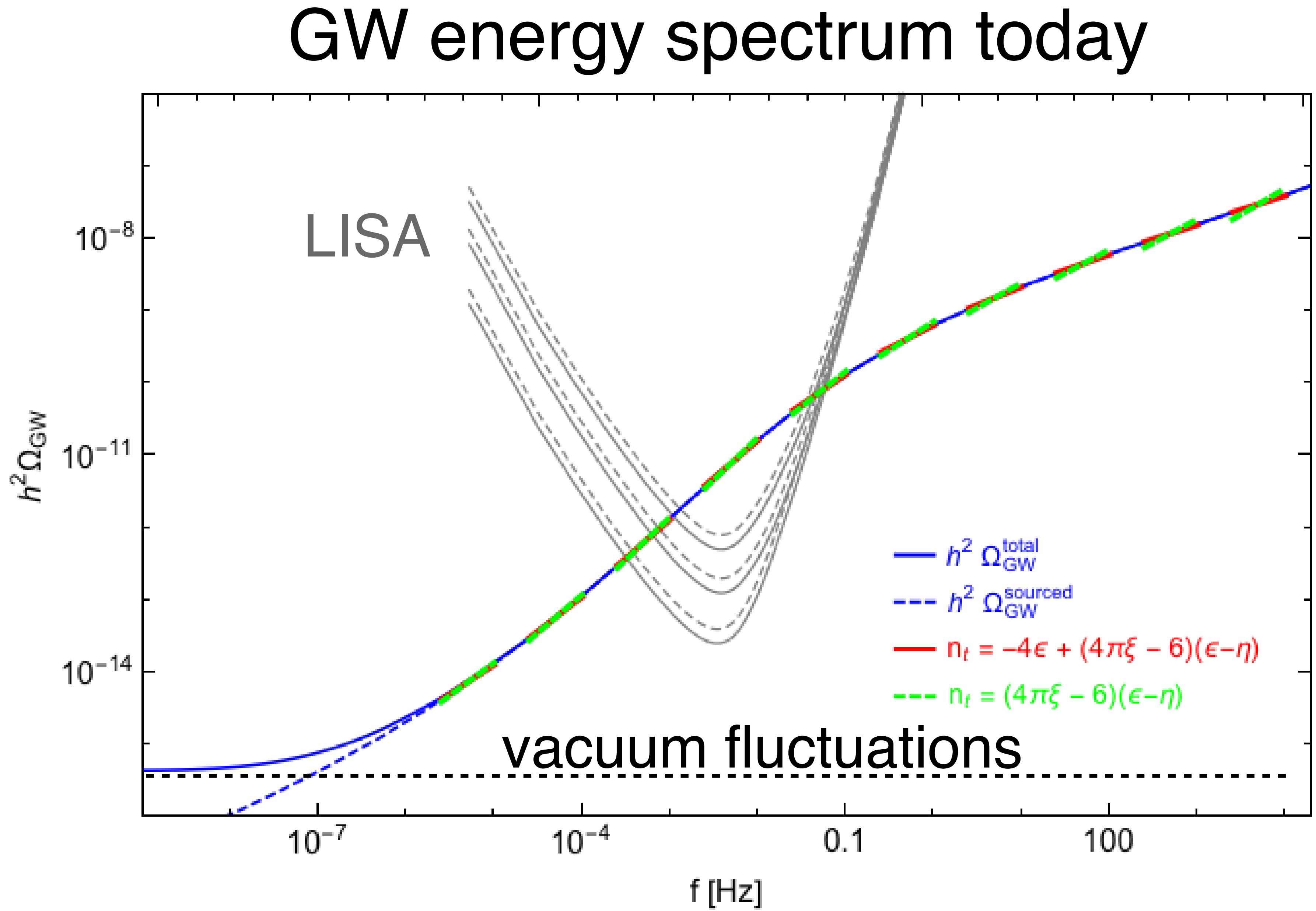
Example, GW prediction

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi^{\mathrm{TT}}_{ij} \propto \{E_i E_j + B_i B_j\}^{TT}$$

well calculated ?

INFLATIONARY MODELS

Axion-Inflation



Blue-Tilted

+ Chiral

+ Non-G

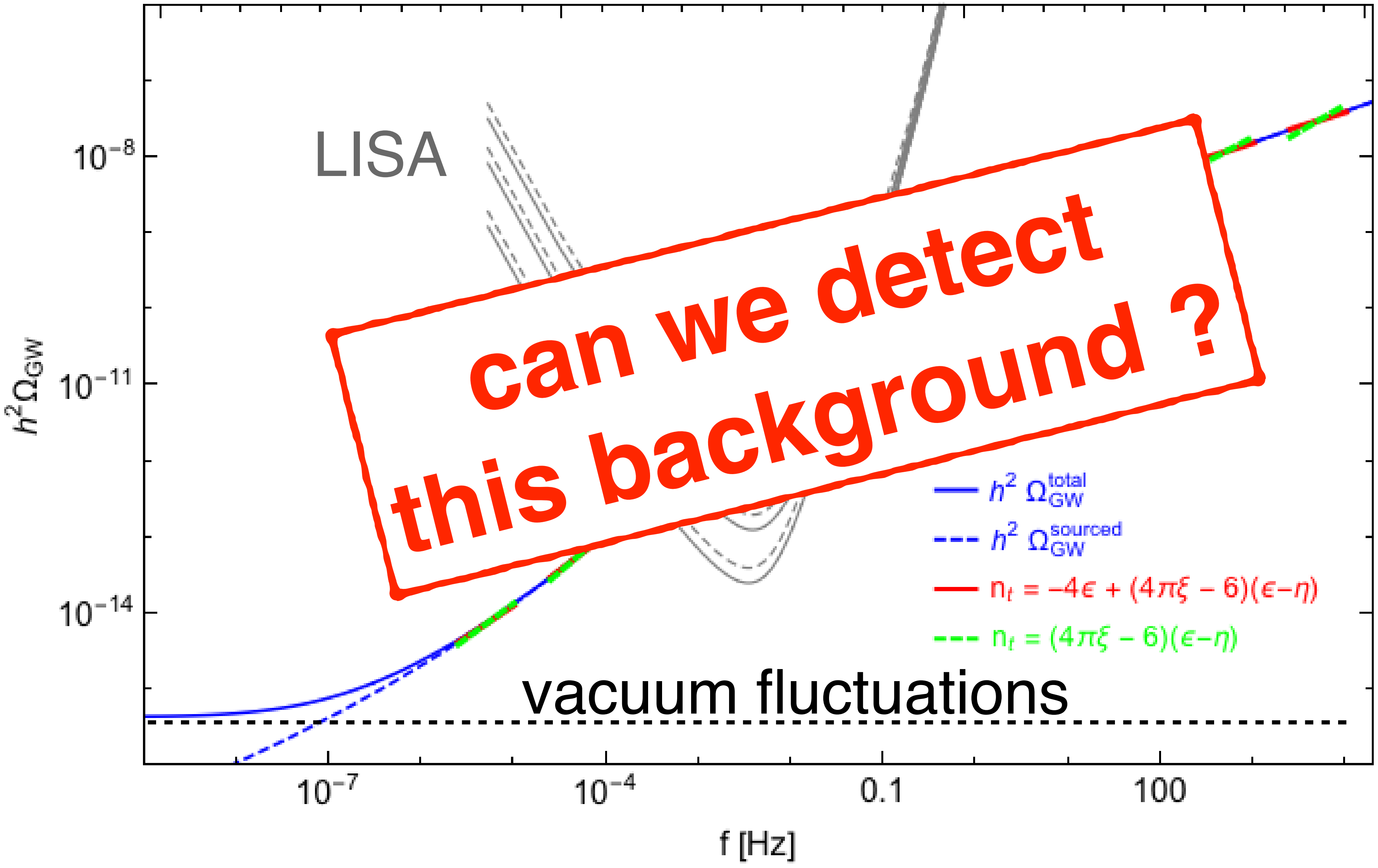
GW background

Bartolo et al ’16, 1610.06481

INFLATIONARY MODELS

Axion-Inflation

GW energy spectrum today



can we detect this background ?

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Bartolo et al ’16, 1610.06481

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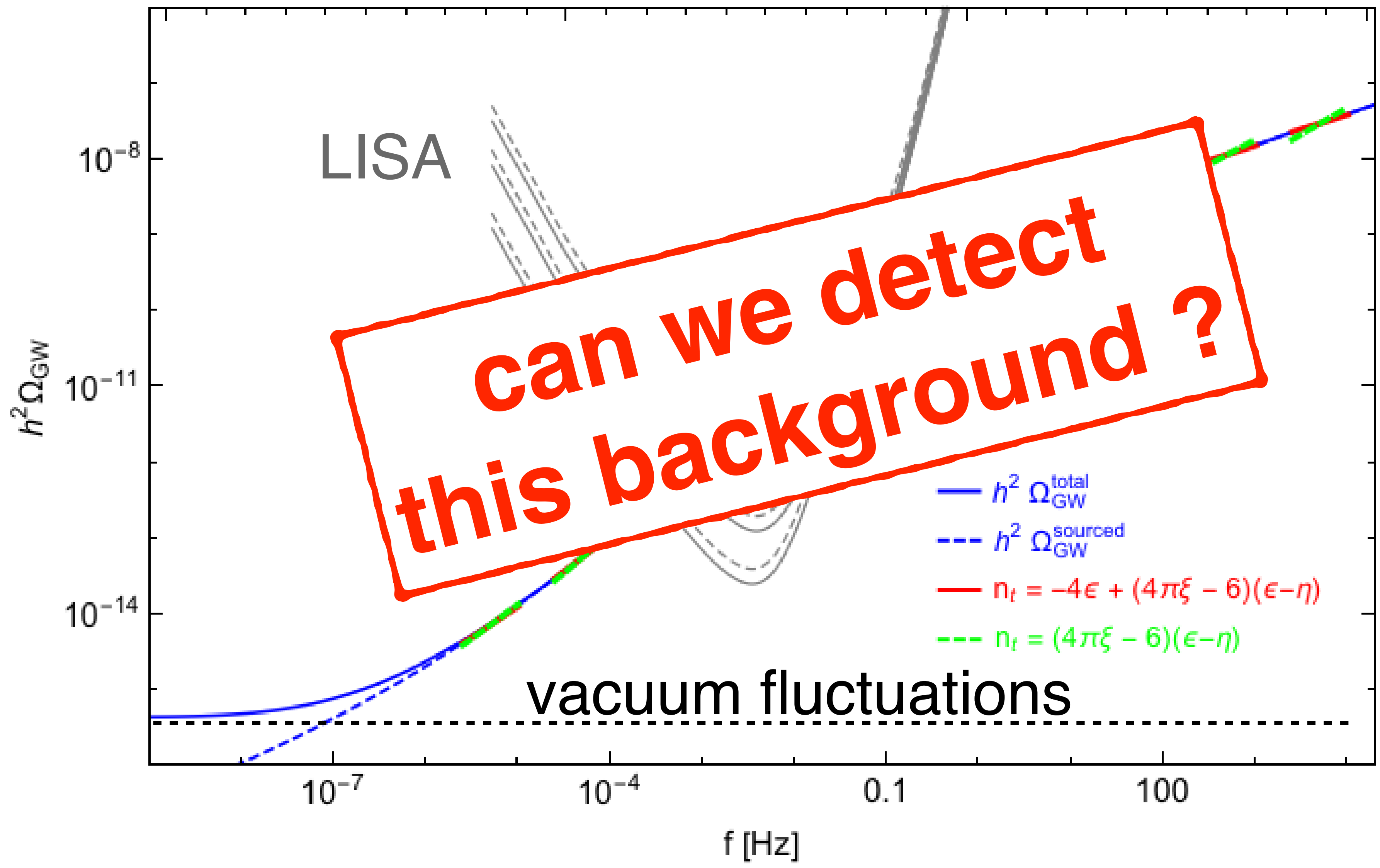

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Blue-Tilted

+ Chiral

+ Non-G

GW background

As $A_+ \propto e^{\dot{\phi}}$, GWs very sensitive to choice of $V(\phi)$ and calculation details

Bartolo et al '16, 1610.06481

Constraints

Energy conservation

- The **first Friedmann equation** is used to check the accuracy of the simulation.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 \simeq \frac{\rho}{3m_p^2}}$$

$$\Big\downarrow$$

$$\Delta_e \equiv \frac{\langle \mathrm{LHS} - \mathrm{RHS} \rangle}{\langle \mathrm{LHS} + \mathrm{RHS} \rangle}$$

Transparencies worked out together with Paco Torrentí

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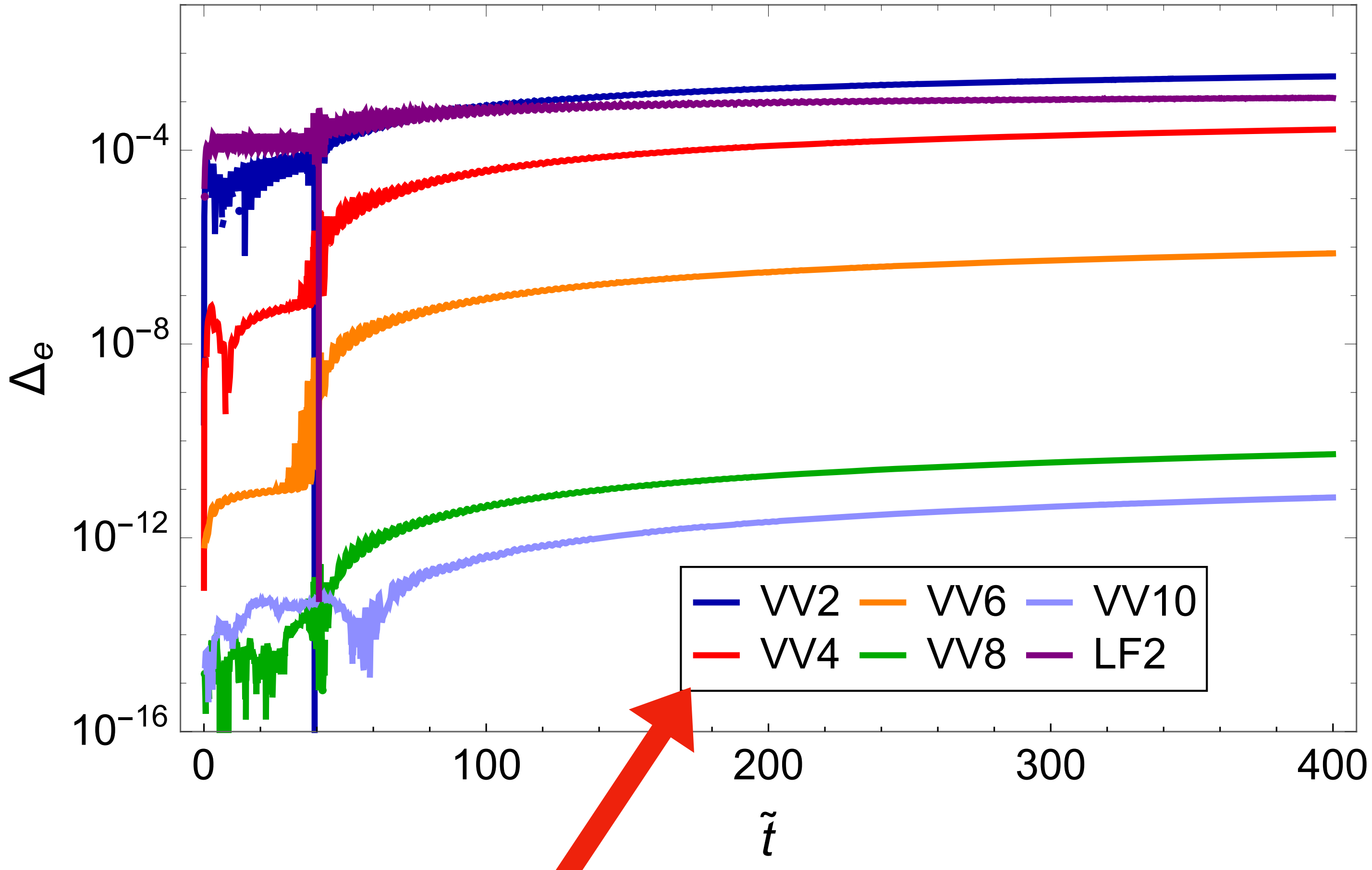

Evolution algorithms:

- **VVn:** Velocity-verlet of accuracy order O(dt^n)
- **LF2:** Staggered leapfrog, accuracy order O(dt^2)

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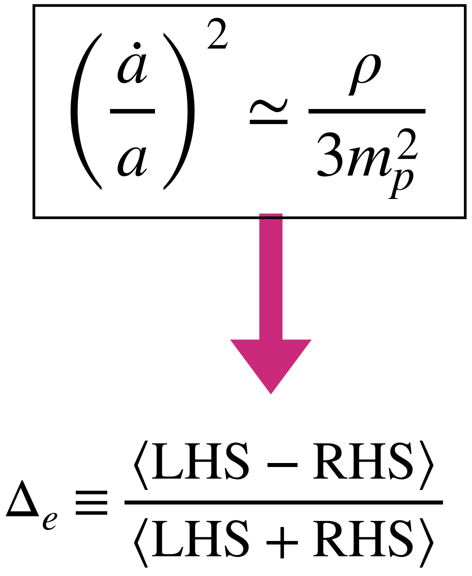

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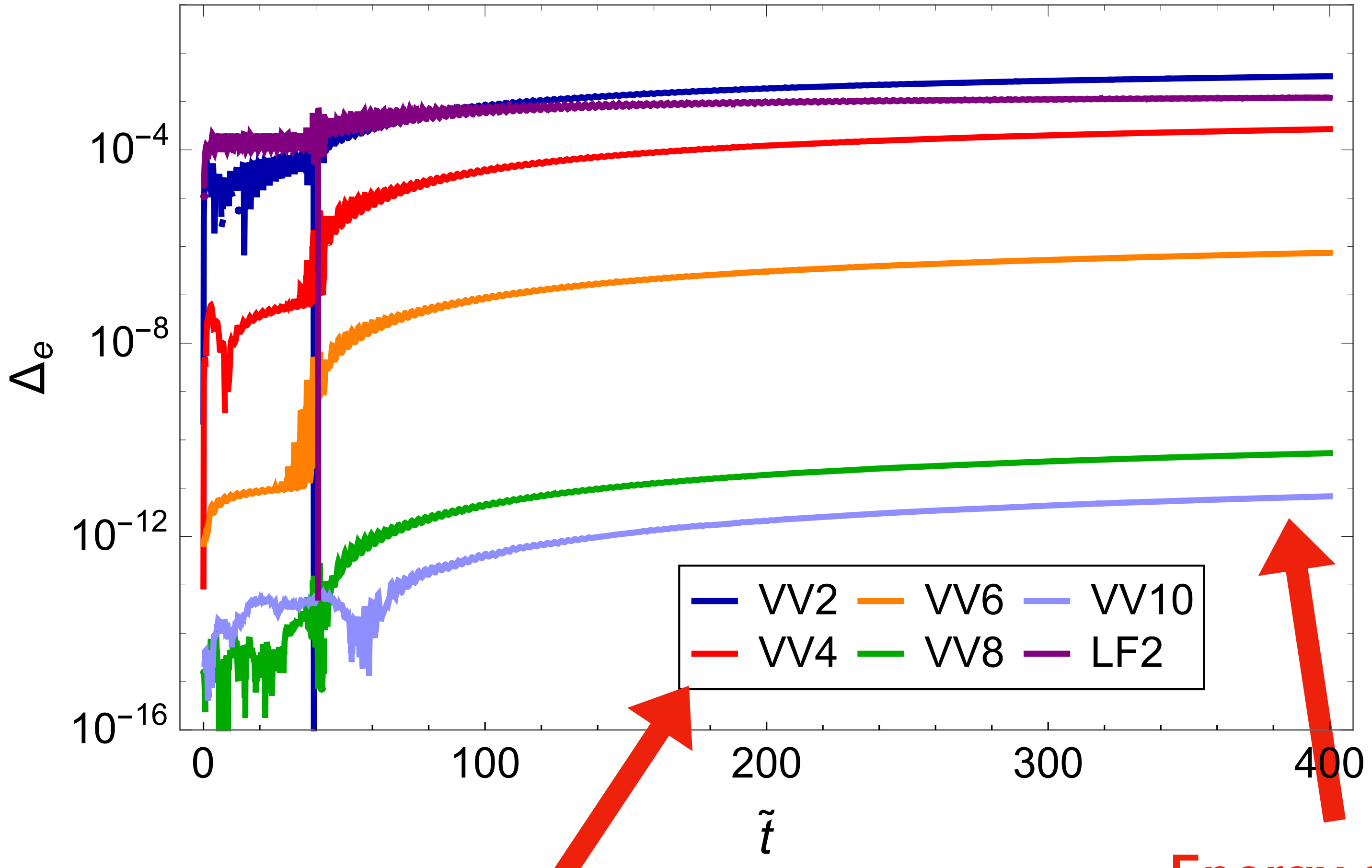

Evolution algorithms:

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Energy conserved up to machine precision for VV10!

Gauge theories: Gauss constraint

- Preservation of U(1) & SU(2) **Gauss constraints** (for all integrators!)

$$\boxed{\begin{aligned} \partial_i F_{0i} &= a^2 J_0^A \\ (\mathscr{D}_i)_{ab}(G_{0i})^b &= a^2 (J_0)_a \end{aligned}}$$

Gauge charges

$$\Delta_g \equiv \frac{\langle\sqrt{(\mathrm{LHS}-\mathrm{RHS})^2}\rangle}{\langle\sqrt{(\mathrm{LHS}+\mathrm{RHS})^2}\rangle}$$

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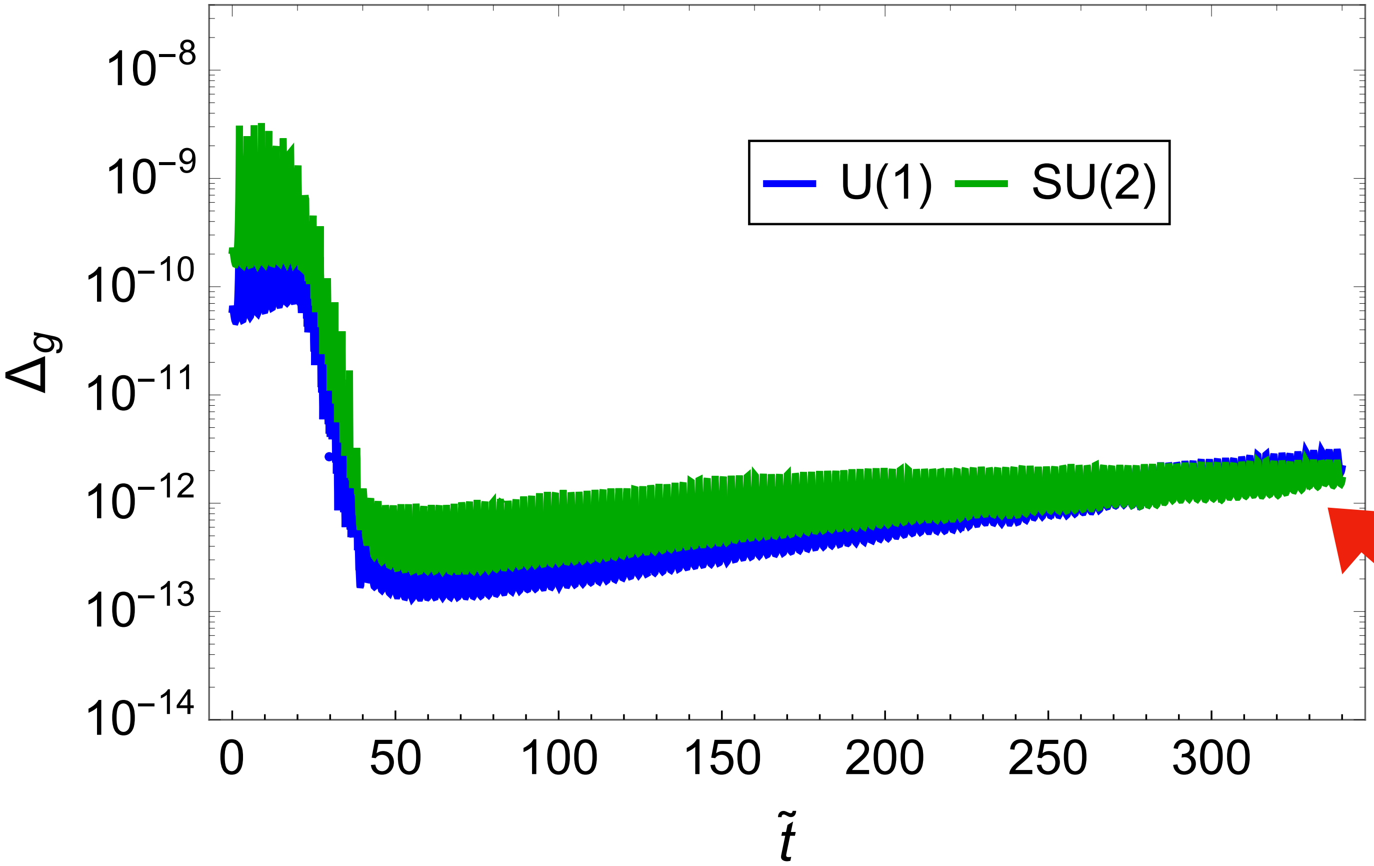

Gauss constraint preserved up to machine precision

Program Variables

CosmoLattice – Program variables

- Equations solved in (dimensionless) ***program variables*:**

Choose:

$$\{\alpha, \omega_*, f_*\}$$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$

$$d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

$$\tilde{\phi} = \frac{\phi}{f_*} \qquad \tilde{\varphi} = \frac{\varphi}{f_*} \qquad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \qquad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

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How do I choose them ?

CosmoLattice – Program variables

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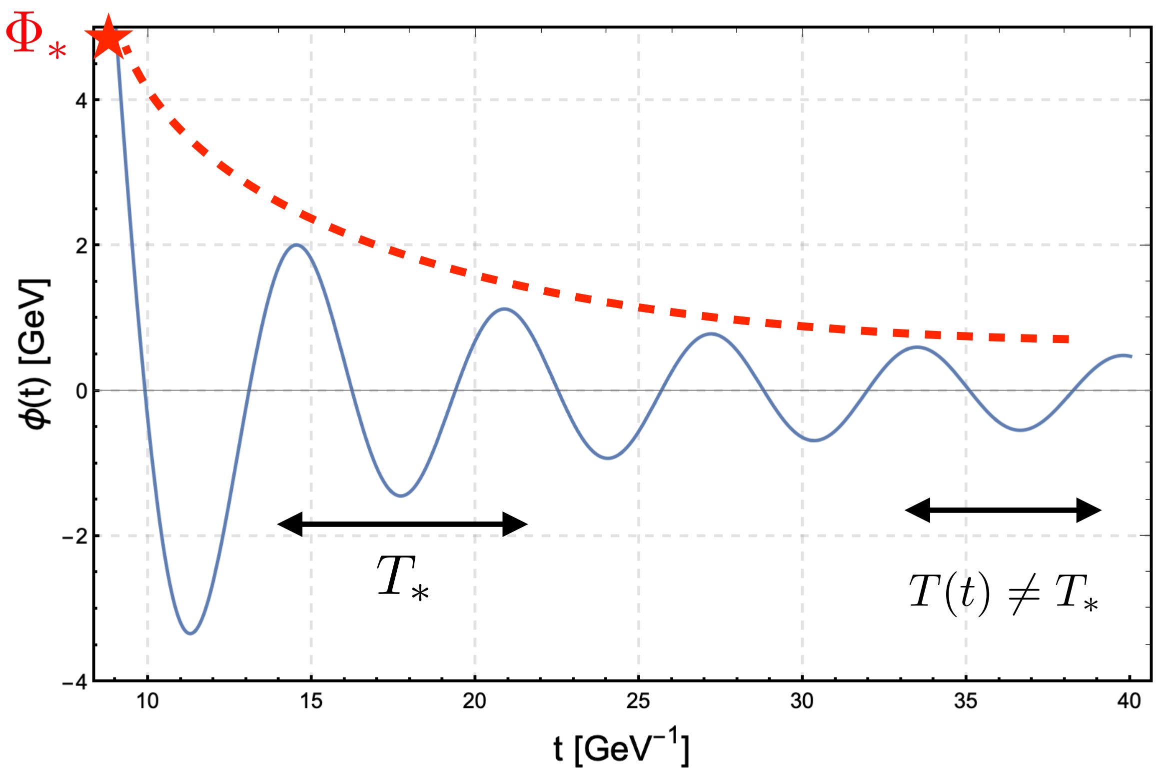
Gauge fields

Example: $\phi(t) \simeq \Phi_* \times f_{\rm osc}(t)$



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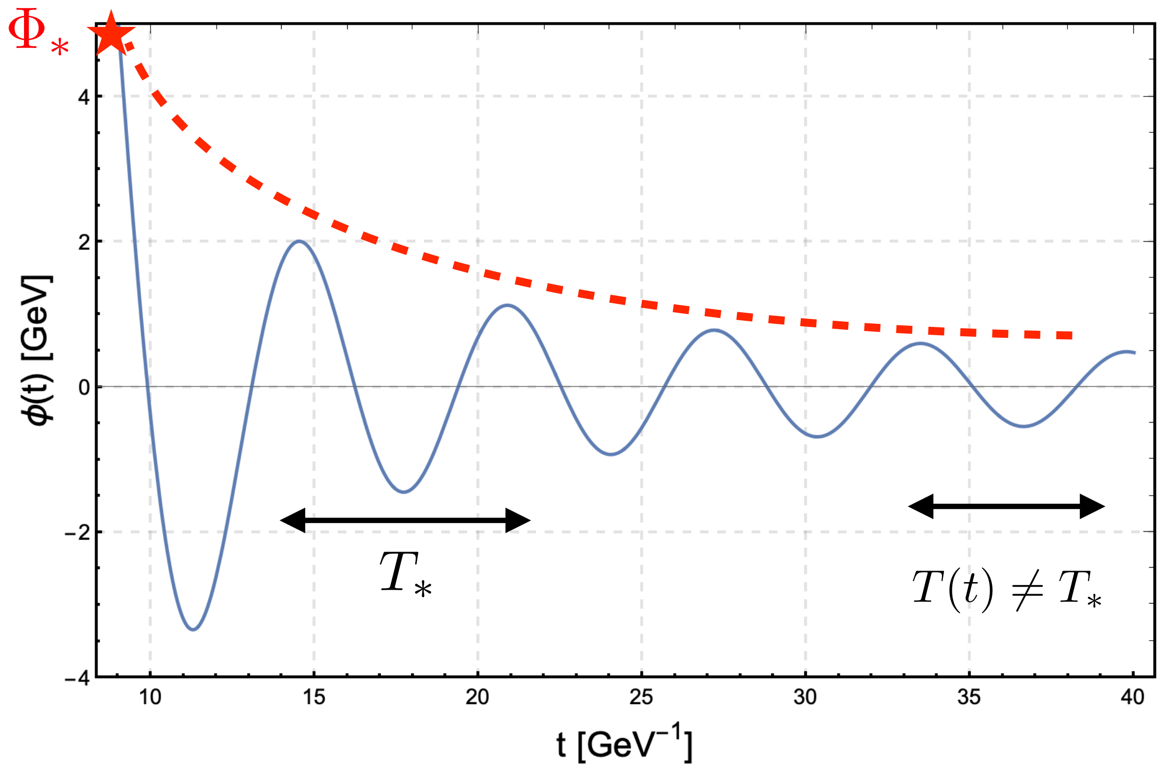

$$\begin{cases} f_* = \Phi_* \\ \omega_* = 1/T_* \\ \alpha \longrightarrow \text{Make period constant in } \tilde{\eta} \end{cases}$$

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Gauge fields

- Write **scalar potential** and first and second derivatives in **one file** (*model.h*)

$$\boxed{\tilde V(\tilde\phi, |\tilde\varphi|, |\widetilde{\Phi}|) \equiv \frac{1}{f_*^2\omega_*^2} V(f_*\tilde\phi, f_*|\tilde\varphi|, f_*|\widetilde{\Phi}|)} \longrightarrow \frac{\partial\tilde V}{\partial\tilde\phi}, \frac{\partial\tilde V}{\partial|\tilde\varphi|}, \frac{\partial\tilde V}{\partial|\tilde\Phi|}, \frac{\partial^2\tilde V}{\partial\tilde\phi^2}, \frac{\partial^2\tilde V}{\partial|\tilde\varphi|^2}, \frac{\partial^2\tilde V}{\partial|\tilde\Phi|^2},$$

CosmoLattice – Program variables

- Equations solved in (dimensionless) ***program variables*:**

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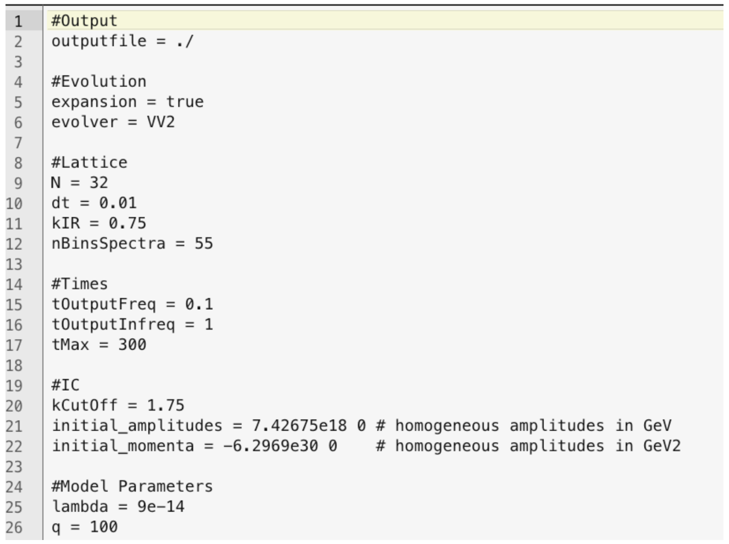
- **Parameters** passed via **one file** (input.txt)

(no need to re-compile !)

→



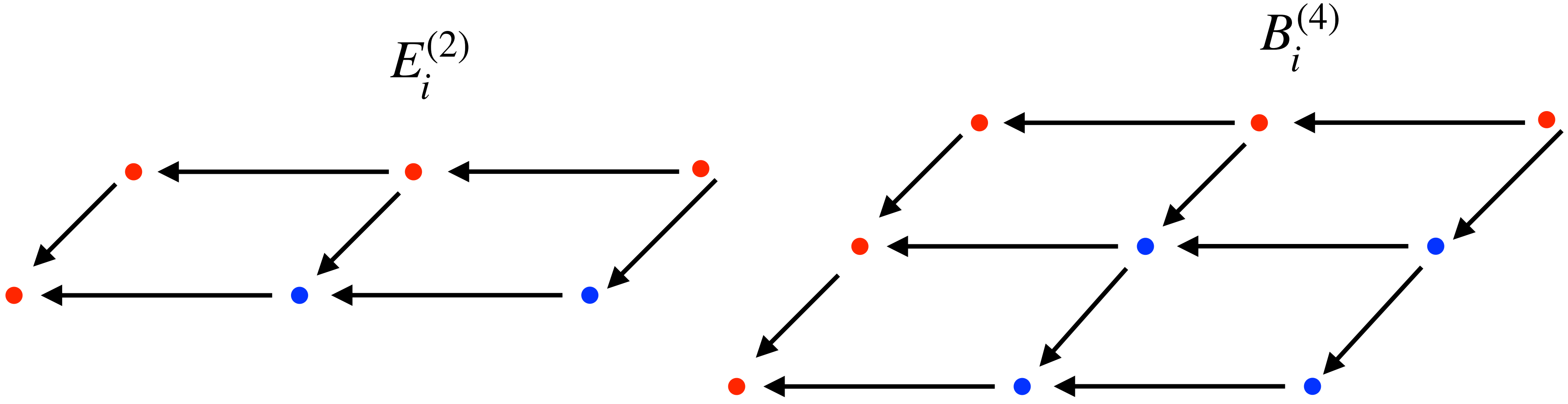
```
#Output
outputfile = ./

#Evolution
expansion = true
evolver = VV2

#Lattice
N = 32
dt = 0.01
kIR = 0.75
nBinsSpectra = 55

#Times
tOutputFreq = 0.1
tOutputInfreq = 1
tMax = 300

#IC
kCutOff = 1.75
initial_amplitudes = 7.42675e18 0 # homogeneous amplitudes in GeV
initial_momenta = -6.2969e30 0    # homogeneous amplitudes in GeV2

#Model Parameters
lambda = 9e-14
q = 100
```

Axion-inflation
extra stuff

Axion-Inflation

PROBLEM: PNG, GW and PBH → Analytical approximations !

Let's "latticize" the system of EOM !



Lattice gauge techniques

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+ \left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \frac{1}{2} E^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)$$

Gauge Fld EoM

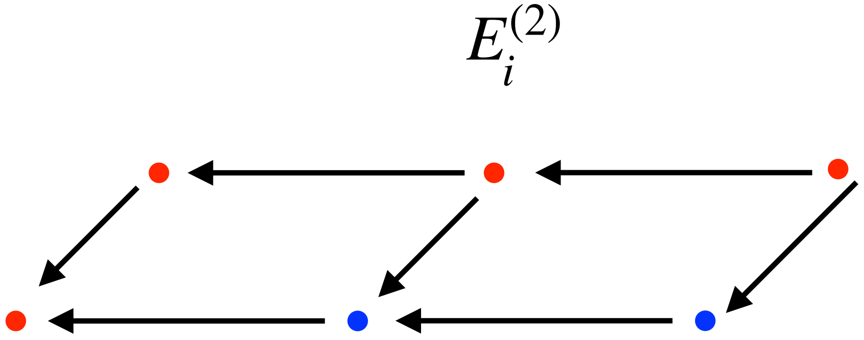
$$\Delta_0^- \left(a_{+\frac{\hat{0}}{2}} E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a} \sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i} B^{(4)}_{i,+i}\right)$$

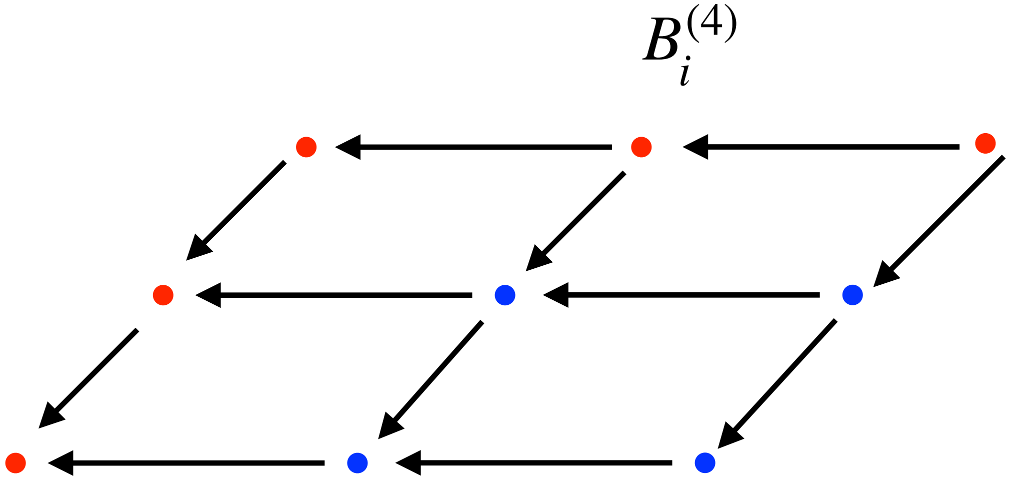
$$+\frac{1}{8\Lambda}(2+dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_\pm\sum_i \left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)}+B^{(4)}_{i,+\hat{0}}\right)_{\pm i} \quad ,$$

(Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018





Lattice gauge techniques

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+\left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)$$

Gauge Fld EoM

$$\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^-B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B^{(4)}_{i,+i}\right)$$

$$+\frac{1}{8\Lambda}(2 + dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)_{\pm i}\ ,$$

(Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018

1. Lattice Gauge Inv: $A_\mu \longrightarrow A_\mu + \Delta_\mu^+\alpha$
2. Cont. Limit to $\mathcal{O}(dx^2)$
3. Lattice Bianchi Identities: $\Delta_i^-(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}) = 0\,,\ldots$
4. Topological Term: $(F_{\mu\nu}\tilde{F}^{\mu\nu})_L = \Delta_\mu^+K^\mu$ (**CS current**)
   $\left[\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu\,\right]$

Exact Shift Sym. on the lattice !

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+\left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}\right)$$

Gauge Fld EoM

$$\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B_{i,+i}^{(4)}\right)$$
$$+\frac{1}{8\Lambda}(2 + dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}\right)_{\pm i}\ ,$$ (Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018

$$\dot{\pi}_\phi = -3H\pi_\phi + \frac{1}{a^2}\vec{\nabla}^2\phi - m^2\phi + \frac{1}{a^3\Lambda}\vec{E}\cdot\vec{B},$$

$$\dot{\vec{E}} = -H\vec{E} - \frac{1}{a^2}\vec{\nabla}\times\vec{B} - \frac{1}{a\Lambda}\pi_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{1}{a\Lambda}\vec{\nabla}\phi\cdot\vec{B}$$ (Gauss Law)

EoM Continuum

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+\left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \underline{\frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)}$$

Gauge Fld EoM

$$\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^-B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B^{(4)}_{i,+i}\right)$$

$$+\frac{1}{8\Lambda}(2+dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i\Delta_i^-E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)_{\pm i} ,$$ (Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018

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$$\dot{\vec{E}} = -H\vec{E} - \frac{1}{a^2}\vec{\nabla}\times\vec{B} - \frac{1}{a\Lambda}\pi_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{1}{a\Lambda}\vec{\nabla}\phi\cdot\vec{B}$$ (Gauss Law)

EoM Continuum

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+\left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}\right)$$

Gauge Fld EoM

$$\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\underline{\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B_{i,+i}^{(4)}\right)}$$

$$+\frac{1}{8\Lambda}(2 + dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}\right)_{\pm i}\ ,$$ (Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018

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$$\dot{\vec{E}} = -H\vec{E} - \frac{1}{a^2}\vec{\nabla}\times\vec{B} - \underline{\frac{1}{a\Lambda}\pi_\phi\vec{B}} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{1}{a\Lambda}\vec{\nabla}\phi\cdot\vec{B}$$ (Gauss Law)

EoM Continuum

LATTICE FORMULATION of $\phi F\tilde{F}$

EoM

Inflaton EoM

$$\Delta_0^+\left(a^3\pi_\phi\right) = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)$$

Gauge Fld EoM

$$\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B^{(4)}_{i,+i}\right)$$

$$+\frac{1}{8\Lambda}(2 + dx\Delta_i^+)\sum_{\pm}\sum_{j,k}\left\{\underline{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}}\right\}$$

$$a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda}\sum_{\pm}\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)_{\pm i}\ ,$$ (Gauss Law)

DGF, Shaposhnikov 2017
Canivete, DGF 2018

$$\dot{\pi}_\phi = -3H\pi_\phi + \frac{1}{a^2}\vec{\nabla}^2\phi - m^2\phi + \frac{1}{a^3\Lambda}\vec{E}\cdot\vec{B},$$

$$\dot{\vec{E}} = -H\vec{E} - \frac{1}{a^2}\vec{\nabla}\times\vec{B} - \frac{1}{a\Lambda}\pi_\phi\vec{B} + \underline{\frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{E}}$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{1}{a\Lambda}\vec{\nabla}\phi\cdot\vec{B}$$ (Gauss Law)

EoM Continuum

LATTICE FORMULATION of $\phi F\tilde{F}$

Lattice Formulation

EoM

$$\begin{aligned}
\Delta_0^+\left(a^3\pi_\phi\right) &= a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} \\
&\quad + \frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right), \\
\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) &= -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B^{(4)}_{i,+i}\right) \\
&\quad + \frac{1}{8\Lambda}(2 + dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\} \\
a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} &= -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)_{\pm i}, \quad \text{(Gauss Law)}
\end{aligned}$$

Expansion

$$\left(\Delta_0^+ a_{-\hat{0}/2}\right)^2 = \frac{a^2}{3m_{\rm pl}^2}\rho_L\,,$$

$$\Delta_0^-\Delta_0^+ a_{+\hat{0}/2} = -\frac{a_{+\hat{0}/2}}{6m_{\rm pl}^2}(\rho_L + 3p_L)_{+\hat{0}/2}$$

LATTICE FORMULATION of $\phi F\tilde{F}$

Lattice Formulation

EoM

$$\begin{aligned}
\Delta_0^+\left(a^3\pi_\phi\right) &= a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} \\
&\quad +\frac{1}{\Lambda}\sum_i \frac{1}{2}E^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)}+B^{(4)}_{i,+\hat{0}}\right)\,, \\
\Delta_0^-\left(a_{+\frac{\hat{0}}{2}}E_{i,+\frac{\hat{0}}{2}}\right) &= -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2\Lambda}\left(\pi_\phi B_i^{(4)} + \pi_{\phi,+i}B^{(4)}_{i,+i}\right) \\
&\quad +\frac{1}{8\Lambda}(2+dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)E^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\} \\
a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^- E_{i,+\frac{\hat{0}}{2}} &= -\frac{1}{4\Lambda}\sum_\pm\sum_i\left(\Delta_i^\pm\phi_{+\frac{\hat{0}}{2}}\right)\left(B_i^{(4)}+B^{(4)}_{i,+\hat{0}}\right)_{\pm i}\,, \quad \text{(Gauss Law)}
\end{aligned}$$

Expansion

$$\left(\Delta_0^+ a_{-\hat{0}/2}\right)^2 = \frac{a^2}{3m_{\rm pl}^2}\rho_L\,,$$

$$\Delta_0^-\Delta_0^+ a_{+\hat{0}/2} = -\frac{a_{+\hat{0}/2}}{6m_{\rm pl}^2}(\rho_L+3p_L)_{+\hat{0}/2}$$

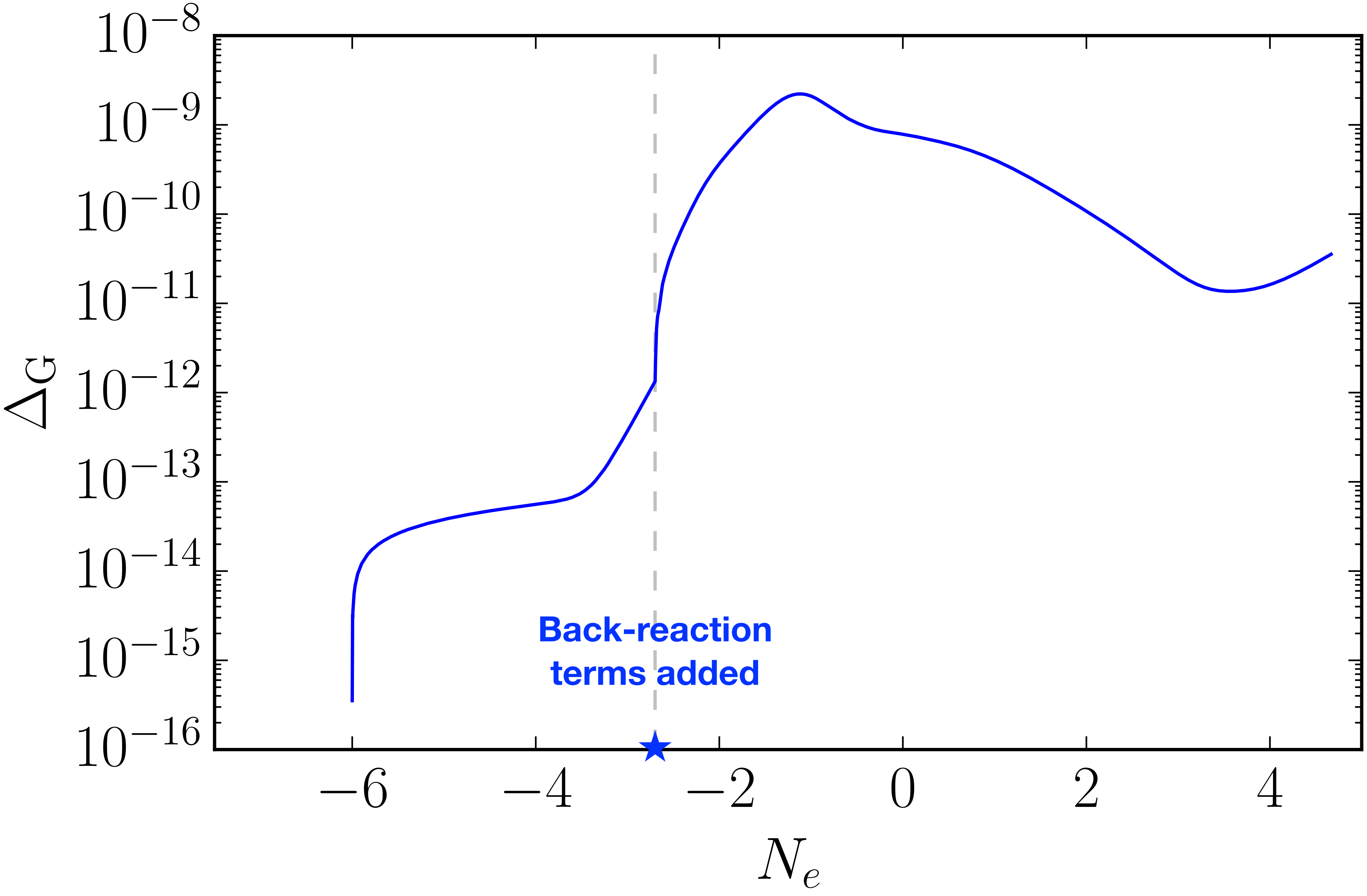
$$\rho_L = \bar{H}^{\rm kin} + \frac{1}{a^2}\frac{1}{2}(\bar{H}^{\rm grad}_{-\hat{0}/2}+\bar{H}^{\rm grad}_{+\hat{0}/2}) + \frac{1}{2}(\bar{H}^{\rm pot}_{-\hat{0}/2}+\bar{H}^{\rm pot}_{+\hat{0}/2}) + \frac{1}{a^2}\frac{1}{2}(\bar{H}^{E}_{-\hat{0}/2}+\bar{H}^{E}_{+\hat{0}/2}) + \frac{1}{a^4}\bar{H}^B\,,$$

$$(\rho_L+3p_L)_{+\hat{0}/2} = 2(\bar{H}^{\rm kin}+\bar{H}^{\rm kin}_{+\hat{0}}) - 2\bar{H}^{\rm pot}_{+\hat{0}/2} + \frac{2}{a^2_{+\hat{0}/2}}\bar{H}^E + \frac{1}{a^4_{+\hat{0}/2}}(\bar{H}^B+\bar{H}^B_{+\hat{0}})\,,$$

$$\left(\begin{array}{lll}
\bar{H}^{\rm kin} = \left\langle\frac{1}{N^3}\sum_{\vec{n}}\frac{\pi_\phi^2}{2}\right\rangle & \bar{H}^{\rm grad} = \left\langle\frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}\sum_i(\Delta_i^+\phi_{+\frac{\hat{0}}{2}})^2\,,\right. & \bar{H}^{\rm pot} = \left\langle\frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}m^2\phi^2_{+\frac{\hat{0}}{2}}\right\rangle \\
\bar{H}^E = \left\langle\frac{1}{N^3}\sum_{\vec{n}}\sum_i\frac{1}{2}E^2_{i,+\frac{\hat{0}}{2}}\right\rangle & \bar{H}^B = \left\langle\frac{1}{N^3}\sum_{\vec{n}}\sum_{i,j}\frac{1}{4}(\Delta_i^+A_j-\Delta_j^+A_i)^2\right\rangle &
\end{array}\right)$$

Gauss Constraint

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{a\Lambda} \vec{\nabla}\phi \cdot \vec{B}$$



Hubble Constraint

$$\pi_a^2 = \frac{a^2}{3m_{pl}^2}(K_\phi + G_\phi + V + K_A + G_A) \quad ; \quad \pi_a \equiv \dot{a}$$