Day of Open Doors

Nordita
December 5, 2024
Matthew de Courcy-Ireland
(WINQ)

WINQ Wallenberg Initiative on Networks and Quantum Information

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  dynamical systems, stochastic dynamics, machine learning

- **Niccolò Zagli**
  dynamical systems, stochastic processes, statistical mechanics of nonequilibrium systems

- **Henri Riihimäki**
  topology and algebra of networks, network dynamics, network neuroscience

- **Hanlin Sun**
  network dynamics, higher-order networks, critical phenomena on networks

Kolmogorov Modes and Linear Response of Jump-Diffusion Models: Applications to Stochastic Excitation of the ENSO Recharge Oscillator

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Article https://doi.org/10.1038/s41467-023-37019-5

The dynamic nature of percolation on networks with triadic interactions

Received: 9 October 2022
Accepted: 24 February 2023
Published online: 10 March 2023
Check for updates

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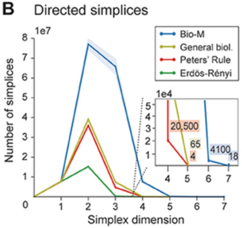
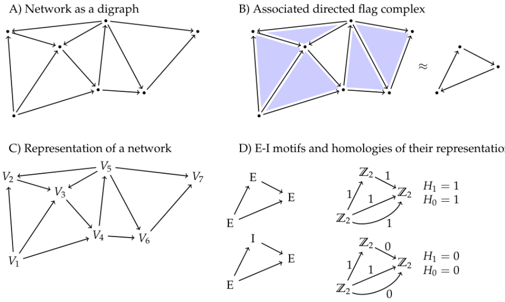
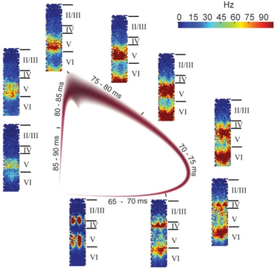
Percolation establishes the connectivity of complex networks and is one of the most fundamental critical phenomena for the study of complex systems. On simple networks, percolation displays a second-order phase transition; on multiplex networks, the percolation transition can become discontinuous. However, little is known about percolation in networks with higher-order interactions. Here, we show that percolation can be turned into a fully fledged



Complex brain networks have intricate higher–dimensional combinatorics…

that organise into higher–dimensional structures during network dynamics.





My research, and possible thesis projects, is about mathematical foundations for analysing these structures

- Topology and algebra of networks
- Discrete dynamical systems
- Computational analysis

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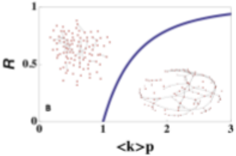
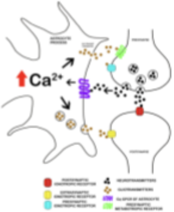
Key words



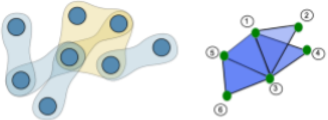
Percolation theory

Percolation theory evaluates the network robustness (macroscopic connectivity) under random failure of nodes or edges. It is an important critical phenomenon defined on networks, and ***it is closely related to dynamic processes such as epidemic spreading.***



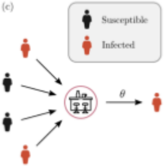
Triadic regulatory interactions

Triadic regulatory interactions are a general type of higher-order interaction that takes place when a node regulate (positively or negatively) the interaction between two other nodes. It widely exists in brain networks, ecological networks, social networks, climate networks, etc. ***It can drive the co-evolution between network structure and dynamics.***



Higher-order network structure

Higher-order networks, including hypergraphs and simplicial complexes, etc., are generalize network structures that encode interactions beyond pair-wise



Dynamic processes on networks

Network allows dynamic processes such as epidemic spreading, social contagion, opinion dynamics, diffusion, etc. take place

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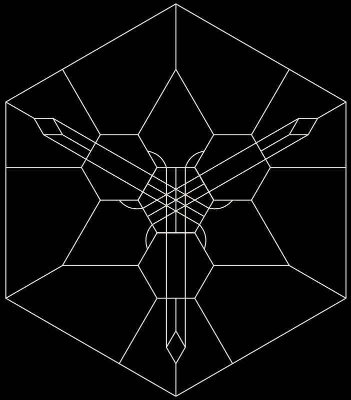
General research questions: *How does (higher-order) network structure and network dynamics affect each other?*

In real-world complex networks,

- **the network dynamics is strongly affected by network structure** (the spread of virus depends on the network structure of individual contacts),
- **the network structure can be altered by network dynamics** (the individual contacts structure can be changed due to the pandemic).
- **Understand the interplay between network dynamics and network structures can provide insights to many real-world problem**: (e.g. predict and mitigate epidemic spreading, understanding pathological brain network and activities)

- Develop percolation theory on higher-order networks
- Epidemic modelling with higher-order networks
- Coevolution of network structure and network dynamics: a general framework for the interplay between network structure and network dynamics

Graphs and networks coming from

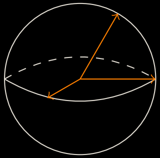


$$x^2 + y^2 + z^2 = xyz_{+k}$$

Markoff equation

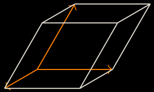
Example: volume of a box

- a,b,c unit vectors in $\mathbb{R}^3$ span a box of volume V

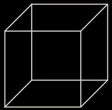


- Volume versus angles?

$$\left.\begin{aligned} x &= 2a\cdot b \\ y &= 2b\cdot c \\ z &= 2c\cdot a \end{aligned}\right\} x^2+y^2+z^2 = xyz + \underbrace{4(1-V^2)}_{k}$$



- Special cases
  $k=0$ orthogonal vectors
  $k=4$ linearly dependent vectors



Markoff and Fricke

If A and B are 2×2 matrices, $\det(A) = \det(B) = 1$, then

$$\left.\begin{aligned} x &= \mathrm{tr}(A) \\ y &= \mathrm{tr}(B) \\ z &= \mathrm{tr}(AB) \\ k &= 2 + \mathrm{tr}(ABA^{-1}B^{-1}) \end{aligned}\right\} x^2 + y^2 + z^2 = xyz + k$$

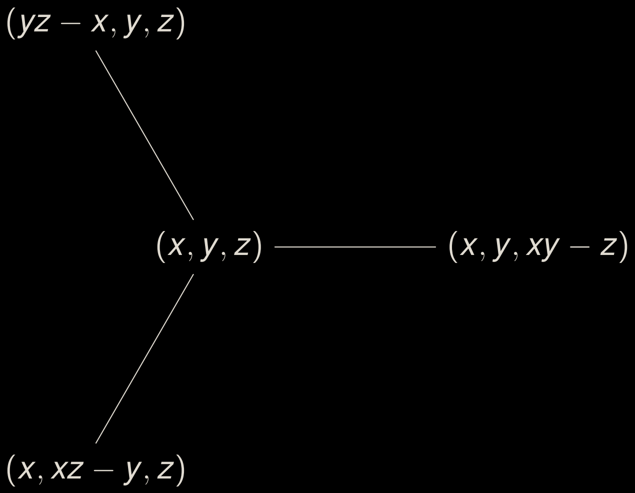
(Fricke's trace identity)

- Special cases

$$A = B \qquad\qquad A = B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$(\mathrm{tr}\,A)^2 = \mathrm{tr}(A^2) + 2 \qquad\qquad \cos(2\theta) = 2(\cos\theta)^2 - 1$$

Markoff graphs



$$x \mapsto yz - x$$

preserves

$$x^2 + y^2 + z^2 - xyz$$

Dynamics

$$\vdots$$

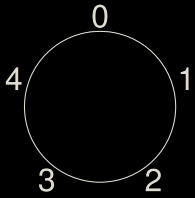
$$\begin{pmatrix} yz - x \\ z(yz-x) - y \\ z \end{pmatrix} \to \begin{pmatrix} yz - x \\ z(yz-x) - y \\ (yz-x)\big(z(yz-x)-y\big) - z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} yz - x \\ y \\ z \end{pmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix}$$

$$\begin{pmatrix} yz - x \\ y \\ y(yz-x) - z \end{pmatrix} \qquad \cdots$$

tetrahedral symmetry: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ x \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \\ z \end{pmatrix}, \begin{pmatrix} -x \\ -y \\ z \end{pmatrix}$

Modulo p



clock arithmetic with p hours

xy mod 5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$4 \equiv -1 \bmod 5$

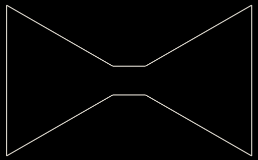
2 and 3 mod 5 are $\sqrt{-1}$

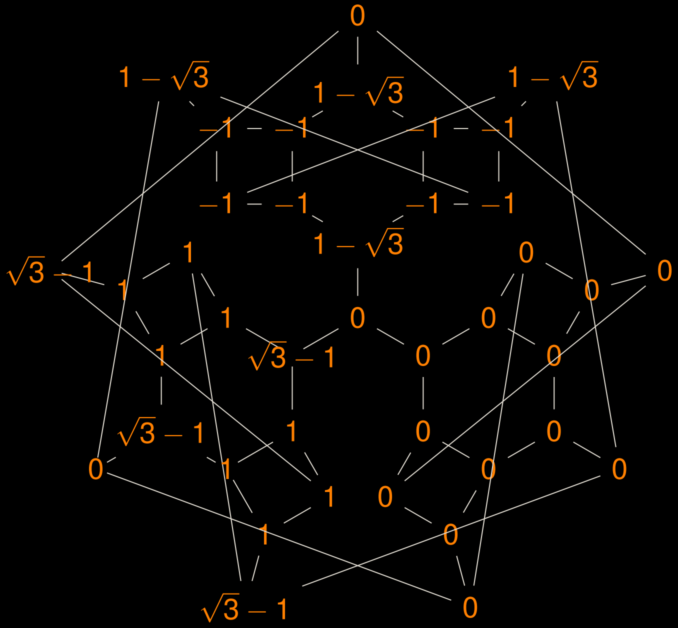
xy mod 7	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

no $\sqrt{-1}$ mod 7

Adjacency operator

- $$Af\begin{pmatrix}x\\y\\z\end{pmatrix} = f\begin{pmatrix}yz-x\\y\\z\end{pmatrix} + f\begin{pmatrix}x\\xz-y\\z\end{pmatrix} + f\begin{pmatrix}x\\y\\xy-z\end{pmatrix}$$
- $Af = 3f$ for constant function $f = 1$
- multiplicity of eigenvalue 3 $\iff$ number of connected components
  other eigenvalues far from 3 $\iff$ robustly connected, no bottleneck





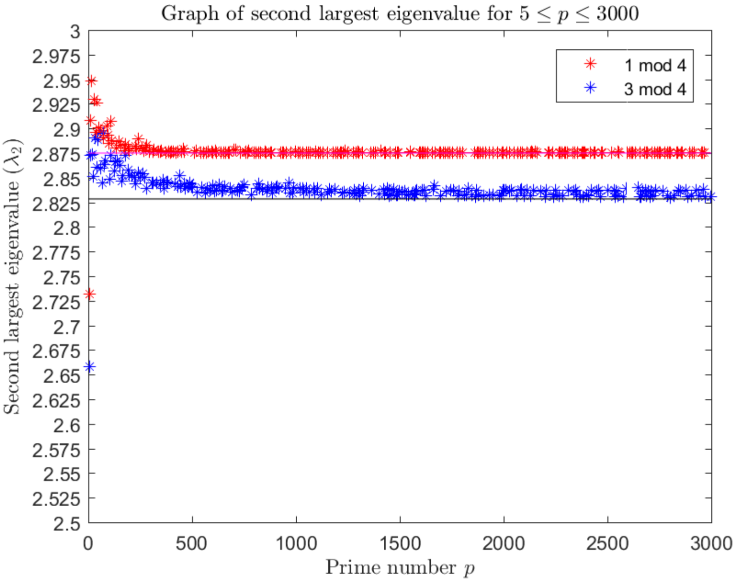
Markoff graph for $p = 5$

$$Af = (1+\sqrt{3})f$$

localized eigenfunction!

possible project:

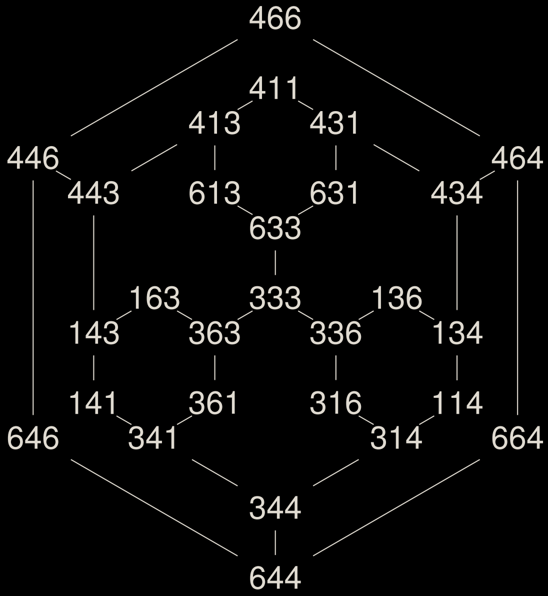
localization vs delocalization



Thanks to Seungjae Lee!

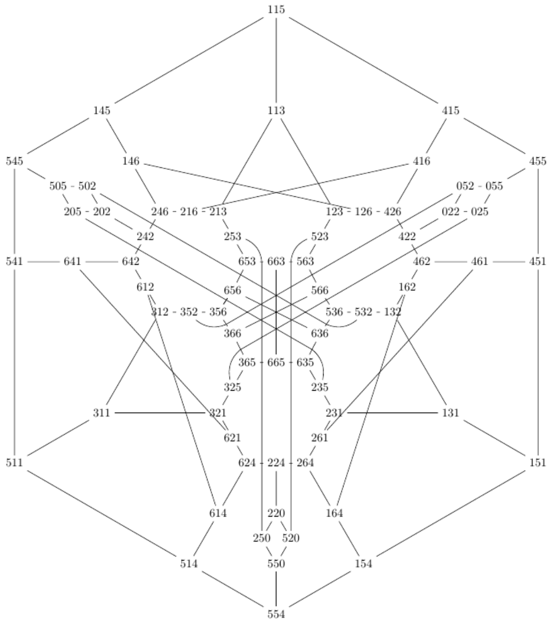
Largest planar example

(for $k = 0$)



$x^2+y^2+z^2 = xyz \bmod 7$

$$x^2 + y^2 + z^2 = xyz + 1 \bmod 7$$



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