Algebras and regions, the ABJ anomaly, and Noether's theorem

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Based on works with Marina Huerta, Diego Pontello, Valentin Benedetti, and Javier Magán

Plan of the talk:

Local and non local operators in QFT. Non uniqueness of the algebra for a region.

Haag duality violations / generalized symmetries. (HDV)

Entropic order parameters. “Certainty relation”.

Global symmetries versus HDV: an obstruction to Noether’s theorem.

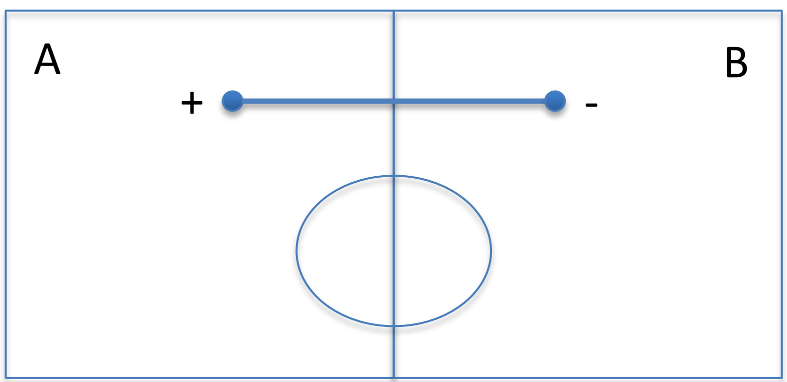
Classification of some simple possibilities: non compact/free, anomalous.

ABJ anomaly as a symmetry that mixes HDV sectors: anomaly quantization, anomaly matching, and the absence of a Noether current.

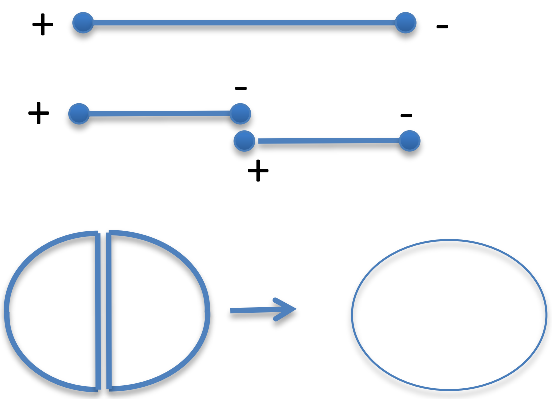
Entanglement entropy of a region → Localization

Operators are localized in relativistic QFT (not particles)
This is more serious in gauge theories: charges particles have longe range tails

Local operators: neutral, charged operators non local



Not in A nor in B, but in the product?

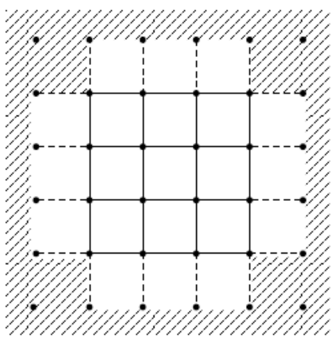
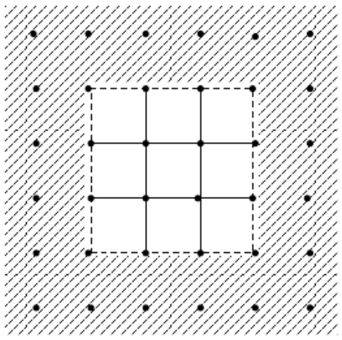
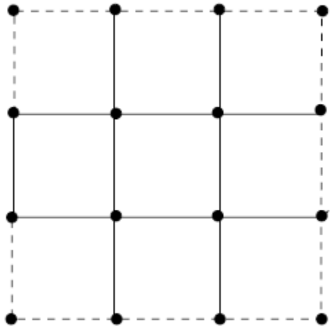


Depends on details of the boundary

Algebraic approach for localization
Lattice to see the details

How to define EE in a gauge theory?
Is there a partition of the Hilbert space into a tensor product?

“Remarks on the EE for gauge fields”, HC, M. Huerta, A. Rosabal (2013)

. Algebras/ states is the natural setting of this problem.

. In the lattice there are many choices of algebras. Some with center, some without center (allowing a tensor product decomposition between inside and outside).

. This is NOT peculiar of gauge theories: problem of defining the “region” in terms of operator content.

. This (UV) problem disappear in the continuum limit (both for gauge and non gauge theories): Relative entropy quantities are unique and well defined (entropy is not). Algebras in the continuum do not have center.

However, there are algebra/ region “problems” or “features”, and non uniqueness of algebras that are macroscopic and physical for some particular theories in the continuum. These features encode the idea of symmetries (in a generalized sense) in QFT

(not a thing of the continuum: these same macroscopic features also appear in lattice models)

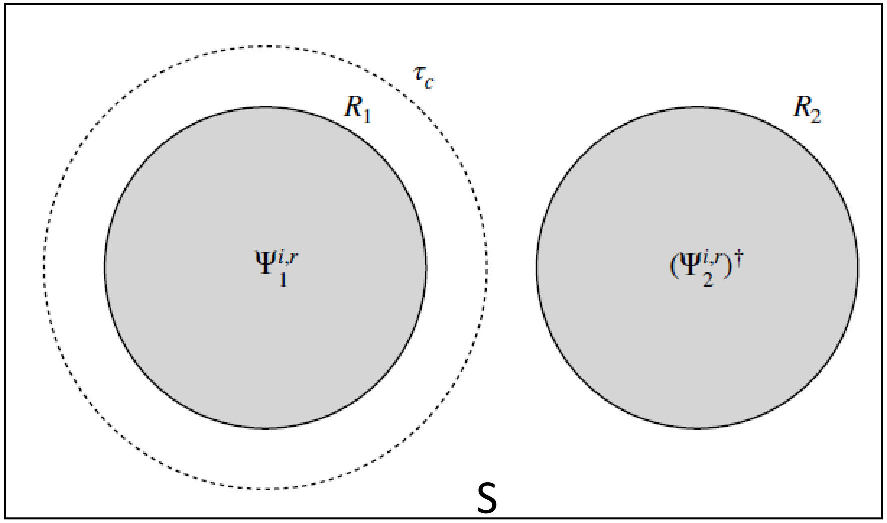
Example: global symmetries, regions with non trivial homotopy groups π_0 or π_{d-2}

$\breve{\mathcal{O}} = \dot{\mathcal{F}}/G$ Take only operators invariant under the group G: neutral operators

Non local operators:

Charge-anticharge operators $\mathcal{I}_r = \sum_i \psi_1^{i,r} (\psi_2^{i,r})^\dagger$

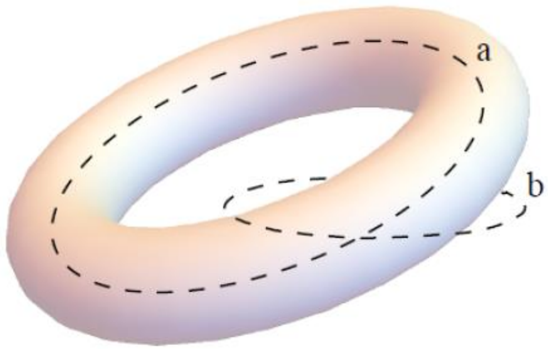
Twists $\tau_c = \sum_{h \in c} \tau_h$



$$
\begin{aligned}
(\mathcal{O}_{\rm add}(R_1 R_2))' &= \mathcal{O}_{\rm add}(S) \vee \{\tau_c\}\,, \\
(\mathcal{O}_{\rm add}(S))' &= \mathcal{O}_{\rm add}(R_1 R_2) \vee \{\mathcal{I}_r\}
\end{aligned}
$$

Two natural algebras for the same region

Example: Regions with non trivial homotopy groups π_1 or π_{d-3}



Gauge theories: Wilson loops $W_r \equiv \mathrm{Tr}_r \mathcal{P}\, e^{i \oint_C dx^\mu A^r_\mu}$

Not all Wilson loops are non locally generated in a ring-like region: for charged representations the loops can be broken in Wilson lines ended in charged operators

$$\phi_r(x)\, P e^{i\int_x^y dx^\sigma A_\sigma}\, \phi^\dagger_r(y)$$

Non local classes of WL labelled by representations of the center of the gauge group

Dual operators are the 't Hooft loops labelled by elements of the center ´t Hooft (1978)

For SU(N) the dual groups of non local operators are Z_N

For the free Maxwell field these are exponentials of the magnetic and electric fluxes for any charges

$$W_q = e^{i\,q\,\Phi_B} \quad T_q = e^{i\,q\,\Phi_E} \qquad W_q W_{q'} = W_{q+q'}\,, \quad T_g T_{g'} = T_{g+g'} \qquad W_q T_g = e^{i\,q\,q}\, T_g W_q$$

For QED: $A = U(1)\,, B = Z\,, d \neq 4\,,\qquad A = U(1)\times\mathbb{Z}\,, B = \mathbb{Z}\times U(1)\,, d = 4$

Operator algebras and regions

QFT: R $\longrightarrow$ $\mathcal{A}(R)$ (net of von Neumann algebras)

Two different meanings of the world “local”:

1) $\mathcal{A}(R) \equiv \bigvee_{B \text{ ball}, \cup B = R} \mathcal{A}(B)$ Additivity: operators constructed from local degrees of freedom

2) $\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$ Causality. $\mathcal{A}'$ Commutant. R’: points spatially separated from R, causal complement

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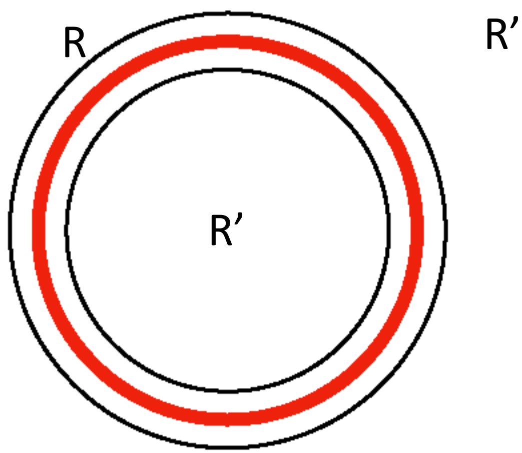

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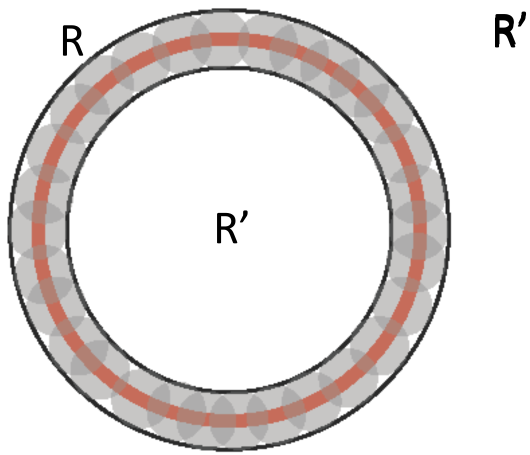

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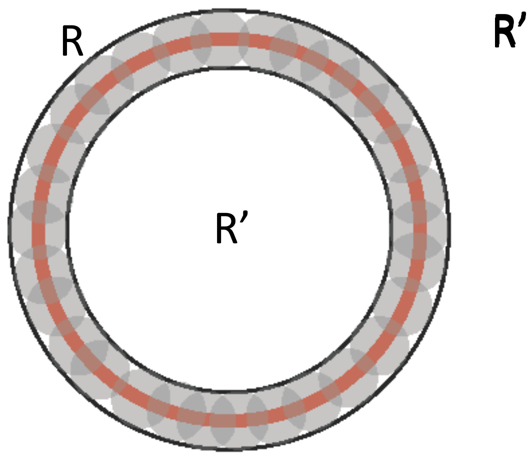


"Complete theory"

The local observable algebras formed additiviely with local degrees of freedom are the maximal ones compatible with causality.

$\mathcal{A}(R) = (\mathcal{A}(R'))'$ Haag duality of the additive algebra (for any R)

Non complete theories?

H.C., M. Huerta, J. Magán, D. Pontello (2020)

$$\mathcal{A}_{\max}(R) \equiv (\mathcal{A}(R'))'$$

The maximal algebra compatible with causality corresponds to the smallest one for the complementary region

$$\mathcal{A}_{\max}(R) = \mathcal{A}(R) \vee \{a\}$$

$$\mathcal{A}_{\max}(R') = \mathcal{A}(R') \vee \{b\}$$

a,b, dual ''non local'' operators.
cannot choose the net to satisfy
simultaneously duality and additivity

$$\begin{array}{ccccc} \mathcal{A}_{\max}(R) \equiv \mathcal{A}(R) \vee \{a\} & \supset & \mathcal{A}(R) & & \\ \updownarrow ' & & \updownarrow ' & & \\ \mathcal{A}(R') & \subset & \mathcal{A}(R') \vee \{b\} \equiv \mathcal{A}_{\max}(R') & & \end{array}$$

Haag duality fails for R iff it fails for R' : non local operators come in dual sets.

Dual non local operators cannot commute (all) to each other

Remarks:

“Non local operator” is notion relative to a certain region. An operator (Wilson loop) can be non local in a region (ring) and is always additive on a different region (ball that contains the ring).

In the standard HEP literature this subject is called “Generalized symmetries” (Gaiotto, Kapustin, Seiberg, Willett (2015))
Usual description do not use Haag duality violation but continues by putting the QFT in compact topologically non trivial spacetimes, ussually Euclidean description.

Notion of symmetry as unitaries that commute with the Hamiltonian too large for QFT (in general many non local unitaries), and too restrictive (does not contain localized transformations of the agebras). Haag duality violations more general notion to describe and classify features of QFT.

Entropic order parameters:

Two algebras for the same region → two states for the same algebra → relative entropy

Conditional expectation: $\varepsilon : \mathcal{A}_{\text{max}} \to \mathcal{A}_{\text{add}}$

Uses: lift a state from the subalgebra to the algebra $\omega_{\mathcal{A}_{\text{add}}} \to \omega_{\mathcal{A}_{\text{add}}} \circ \varepsilon$

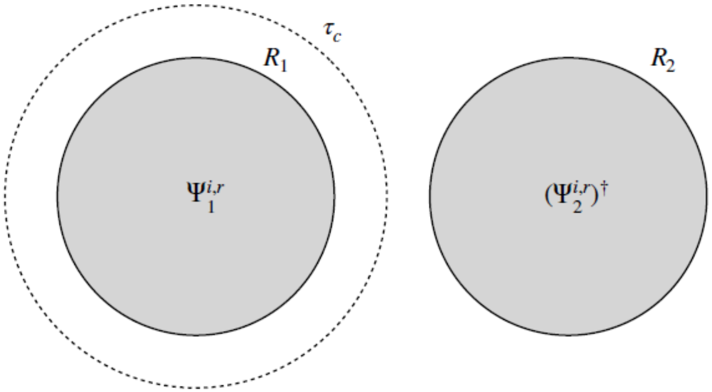
Entropic order parameter $\boxed{S_{\mathcal{A}_{\text{max}}}(\omega|\omega \circ \varepsilon)}$

Relative entropy is a measure of distinguishability between two states in the same algebra

Relative entropy between the vacuum and the vacuum with the non local operators set to zero expectation value.

It is a measure of the statistics (expectation values) of non local operators.

It is function of the geometry of the region only



Complementarity diagram

$$\begin{array}{ccc} \mathcal{A}_{\text{add}}(R) \vee \{a\} & \xrightarrow{\varepsilon} & \mathcal{A}_{\text{add}}(R) \\ \updownarrow \prime & & \updownarrow \prime \\ \mathcal{A}_{\text{add}}(R') & \xleftarrow{\varepsilon'} & \mathcal{A}_{\text{add}}(R') \vee \{b\} \end{array}$$

Dual conditional expectations are unique in continuum QFT

For a pure global state we have the entropic certainty relation (a cousin of entropy is equal for commutant algebras, and a stronger form for the uncertainty relations)

$$S_{\mathcal{A}_{\max}(R)}(\omega|\omega \circ \varepsilon) + S_{\mathcal{A}_{\max}(R')}(\omega|\omega \circ \varepsilon') = \log \lambda \qquad (\log |G|)$$

Index of the inclusion $\mathcal{A}_{\text{add}}(R) \subset \mathcal{A}_{\max}(R)$ it is the same for R': dual symmetries have the same size

(Jones index, R. Longo)

H. Casini, M. Huerta, J. Magan, D. Pontello e-Print: 1905.10487 [hep-th]

J.Magan, D. Pontello e-Print: 2005.01760 [hep-th]

S. Hollands e-Print: 2009.05024 [quant-ph]

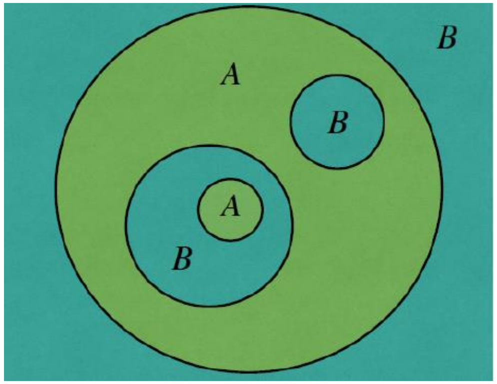
Feng Xu e-Print: 1812.01119 [math-ph]

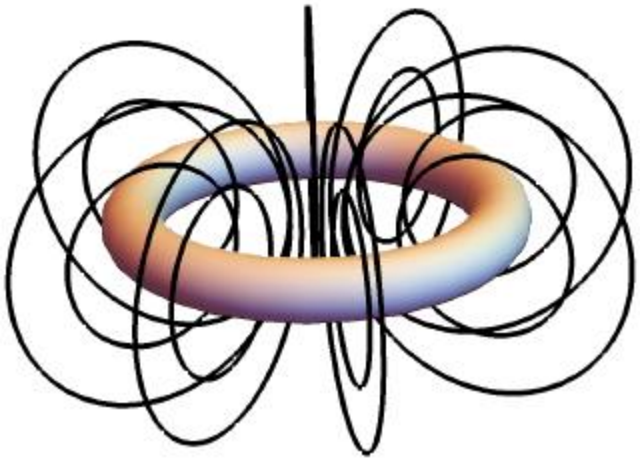
Uses

H.C., M. Huerta, J. Magán, D. Pontello (2019,2020)

Topological contributions to the entropy

$$S_{\mathcal{A}_{\max}(AB)} = I_{\mathcal{F}}(A,B) - I_{\mathcal{O}}(A,B) = n_{\partial}\, \log|G|$$





Heuristics of area law:

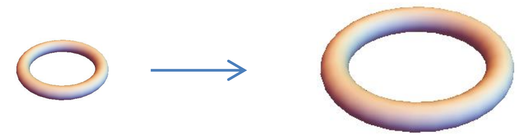
an area worth of decoupled loops with approx constant expectation value

$\longrightarrow$ the relative entropy for the exterior of the orange ring approaches Log|G| exponentially in the number of loops (then exponentially in the area)

$\longrightarrow$ The relative entropy in the ring has an area law because of the certainty relation

Theories with an order parameter with an area law must have HDV

Phase identification for gauge theories



Wilson loops and 't Hooft loops relative entropies $S_W(\lambda R)$ and $S_T(\lambda R)$ for scaled regions, as $\lambda \to 0$ or $\lambda \to \infty$

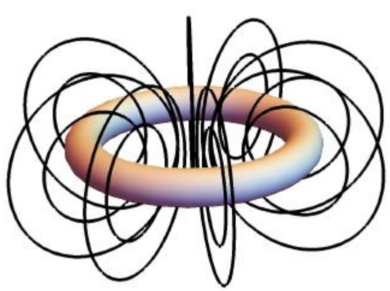
Saturation: $S_W(\lambda R)$ and $S_T(\lambda R)$ dualy converge to $\log|G|$ or 0 independently of shape

Exponentially fast in λ: **Higgs/confinent phases**

Power law in λ (power depends on shape): Electric and magnetic **Coulomb phases**

$S_W(\lambda R)$ and $S_T(\lambda R)$ converge to intermediate values between $\log|G|$ and 0 depending on the shape of R

Conformal phase



Global symmetry + HDV: obstructions to Noether's theorem

V. Benedetti, H.C., J. Magan (2022)

A global internal symmetry, by definition, always keeps the local algebras.
But can it change the non local operators between themselves? Many examples where it does.

A Noether current for a continuous symmetry allows the construction of additive local charges:

$$\tau_\lambda(R,Z) = e^{i\,\lambda\, Q(R,Z)}\,, \qquad Q(R,Z) = \int d^dx\, \beta(x^0)\, \alpha(\vec{x})\, j_0(x)$$

An additive operator cannot change non local classes

If a continuos symmetry has a Noether current there cannot be charged non-local classes
If a continuous symmetry changes the non local classes of a region it cannot have a Noether current

Some examples of violations of the Noether theorem:

All known examples have “charged” non local sectors

Free graviton
No stress tensor.
Non local classes have Lorentz indices
Poincare symmetry mixes classes.

Two free Maxwell fields
No current for rotation symmetry.
Symmetry mixes Wilson loops.

Weinberg-Witten theorem

Free and massless. Why?

Duality symmetry Maxwell field
Symmetry mixes electric and magnetic fluxes.

Maxwell field for dimension $d\neq4$
Derivatives of free scalar $d>2$
No dilatation current.
Symmetry mixes classes with dimensionfull labels.

Known counterexamples for DI implies CI

ABJ anomaly → Interacting case

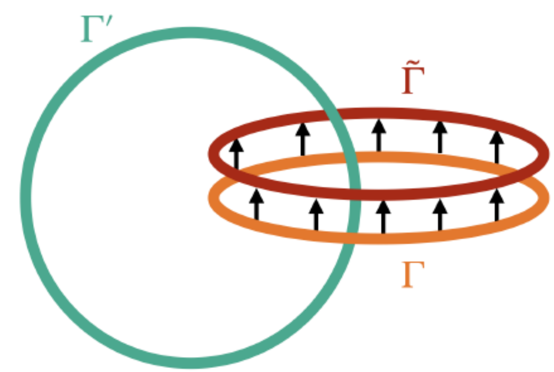
An interpretation of Weinberg-Witten theorem:

V. Benedetti, H.C, J. magan (2021)

"There is no Noether current corresponding to a charged particle with helicity h=1 or greater, and there is no stress tensor for a theory with a massless particle with helicity h=2 or greater"

A (free) graviton in d=4 has closed two forms:

$$
\begin{aligned}
A_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, a^{\alpha\beta}\,, \qquad a^{\alpha\beta} = -a^{\beta\alpha}\,, \\
B_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, (x^\alpha b^\beta - x^\beta b^\alpha)\,, \\
C_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, c^{\alpha\beta\gamma} x_\gamma\,, \qquad c^{\alpha\beta\gamma} = -c^{\beta\alpha\gamma} = -c^{\alpha\gamma\beta}\,, \\
D_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, (x^\alpha d^{\beta\gamma} x^\gamma - x^\beta d^{\alpha\gamma} x^\gamma + \frac{1}{2} d^{\alpha\beta} x^2)\,, \qquad d^{\alpha\beta} = -d^{\beta\alpha}
\end{aligned}
$$



There is a $\mathbb{R}^{20}$ group of generalized symmetries. The classes have Lorentz índices and transform non trivially as a linear representation of Poincare, conformal, and duality symmetries.

→ no stress tensor, no dilatation current, no duality current

ABJ (Adler, Bell, Jackiw) anomaly:

Massless fermions coupled to electromagnetic field, chiral symmetry $\psi \to e^{-i\alpha\gamma^5}\psi$

$j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ would be Noether current, clasically conserved

$\partial_\mu j_5^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ quantum non conservation

Necessary to explain neutral pion decay into photons in QCD

Is this still a symmetry, and if so in what sense?
Usually considered an "anomalous symmetry", not a «normal» symmetry, but not explicitly broken, different from spontaneously broken.

It has quite peculiar features:

1) If it is considered a continuous symmetry it does not have a conserved Noether current

2) However, if it is not considered a symmetry, still the Goldstone theorem works (pions)

3) Anomaly quantization: general anomaly coefficient is proportional to an integer (from Atiyah-Singer index theorem in the path integral computation, or WZW term in effective chiral model)

4) Anomaly matching (the coefficient of the anomaly matches between the UV and IR)

What type of actions of a continuous 1-parameter symmetry on non local classes?

Classes and dual classes non invariant under a continuous symmetry both must form a continuum.
Lets assume Abelian group for the non local classes.

"One dimensional case": fusion $a_1 + a_2$ $b_1 + b_2$ Commutation relations $a\, b = e^{i\, a\, b}\, b\, a$

$\longrightarrow$ $a(\lambda) = e^{\lambda}\, a \qquad b \to e^{-\lambda}\, b$

Implies dual classes are non-compact and continuous (group R). Example: dilatation symmetry for free Maxwell field d≠4

"Two dimensional case": $a = (a_1, a_2) \qquad b = (b_1, b_2) \qquad a\, b = b\, a\, e^{ia\cdot b}$

$\longrightarrow$ $a \to M(\lambda)\, a\,, \qquad b \to (M(\lambda)^T)^{-1}\, b$

Apart from dilatation examples as the above one, there is rotation group U(1), for non compact sectors

$A = \mathbb{R}\times\mathbb{R} \qquad B = \mathbb{R}\times\mathbb{R}$ Example: rotation between two free Maxwell fields, duality for Maxwell field

Reason for free massless models: non-compact sectors (or continuous dual sectors), independently of any global symmetry, lead to free massless theories: $\Phi_F = \int_{\Sigma_F} F\,, \qquad \Phi_G = \int_{\Sigma_G} G \qquad [\Phi_F, \Phi_G] = i$

It is necessary that the cross correlator is a "linking number term", a massless term that cannot renormalize

$$\langle F(x)G(0)\rangle \;=\; \int \frac{d^dp}{(2\pi)^{d-1}}\,\theta(p^0)\,\delta(p^2)\,e^{ipx}\,(P^{(k)}\tilde{*})(p) \qquad \Box\,\langle F(x)G(0)\rangle = 0$$

V. Benedetti, H.C, J. Magán (2022)

Additionaly,

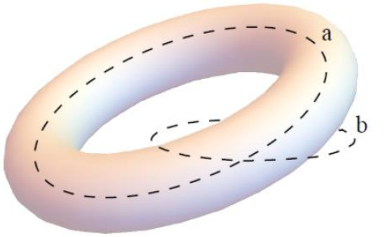
there is only one possibility of U(1) group action that allows for compact sectors (and then for interacting theories)

"ABJ anomalous case"

$$A = \mathbb{Z} \times U(1) \qquad\qquad B = U(1) \times \mathbb{Z}.$$

$$(a_1, a_2) \to (a_1, a_2 + \lambda\, a_1)\,, \qquad (b_1, b_2) \to (b_1 - \lambda\, b_2, b_2)$$

$$a_1, b_2 \in \mathbb{Z}\,, \qquad a_2 \equiv a_2 + 2\pi\,, b_1 \equiv b_1 + 2\pi$$



In this case we would have:

1) A continuous global U(1) symmetry without Noether current (it changes classes)

2) "Quantization" of the group action: the compatibility of the cycle of the symmetry group $\lambda \equiv \lambda + \lambda_0$ with the one of the non-local sectors implies $\lambda_0 = 2\pi\, n$ (anomaly quantization)

3) If the non local operators exist in the IR limit then the symmetry has to exist in the IR: there must be massless local excitations charged under the symmetry and the rates of group action and group of non local operators must match between different scales (anomaly matching)

The ABJ anomaly as an ordinary internal U(1) symmetry that, however, messes with the Haag duality violations of the theory. This clarifies and unifies the origin of its main features in a perspective based on ordinary symmetry ideas

How does the anomalous chiral symmetry changes non local classes?

$$\partial_\mu j^\mu = \frac{\beta}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Take electric charges with minimal charge q_0 the (non local) WL charge have range $q \in [0, q_0)$

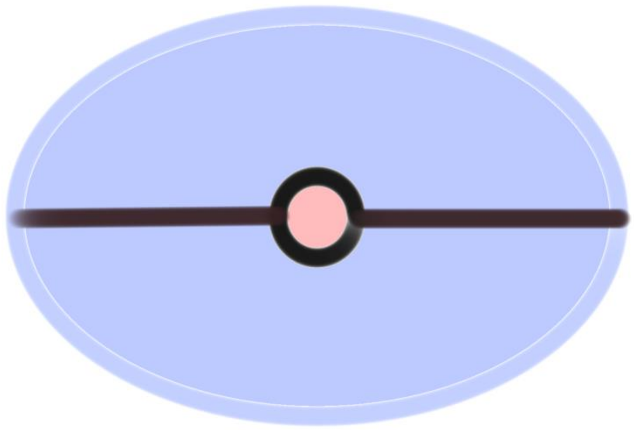
The non local TL have integer charges given by the Dirac quantization condition $g = \frac{2\pi}{q_0} k$

The current is normalized to have minimal charge equal to 1. Parameter $\lambda \in [0, 2\pi)$

Transformation with $\lambda(x)$ changes the action by $\delta S = \int \lambda(x)\, \partial_\mu j^\mu = \frac{\beta}{4} \int \lambda(x)\, F_{\mu\nu} \tilde{F}^{\mu\nu}$

$$\longrightarrow \quad \nabla E = \beta \, (\nabla \lambda) \cdot B$$

Monopole boundary conditions (TL) get mixed with WL boundary conditions - Witten effect



$$(g, q) \to (g, q + \beta \, \delta\lambda \, g)$$

$$\longrightarrow \quad \beta \times (2\pi) \times \left(\frac{2\pi}{q_0} \right) = n \, q_0$$

$$\longrightarrow \quad \partial_\mu j^\mu = n \, \frac{q_0^2}{16 \, \pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

QED then has the minimal possible value n=1 (given that the minimal gauge invariant chiral charge involves two electrons). QED with 1 chiral fermion does not make sense: no gauge invariant local chirally charged operator (gauge and chiral charge are the same)

Final remarks

Algebraic description the natural setting for talking about information quantitites in QFT

The algebraic approach to QFT gives a surprinsing and clear picture of what is going on on the ABJ anomaly

Classification of posible HDV?

Inequalities for entropic order parameters and the renormalization group?