Majorization, Wigner negativity, and QFT

Esko Keski-Vakkuri

Helsinki Institute of Physics
and University of Helsinki

Informational Foundations of QFT, Nordita, May 23 2024

Jan de Boer, Giuseppe Di Giulio, E. K-V, Erik Tonni, arXiv:24??.nnnn
Raúl Arias, J. de Boer, G. Di Giulio, EKV, E. Tonni, Phys. Rev. Res. 5 (2023) 4, 043082

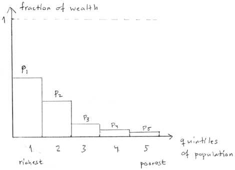
Outline

1. Background Motivation
   - Entanglement as a resource
   - Wealth distribution, majorization, monotonicity, resource theories
   - Side tour: new monotones
   - Resource theory for magic and Wigner negativity
   - Continuous majorization in quantum phase space

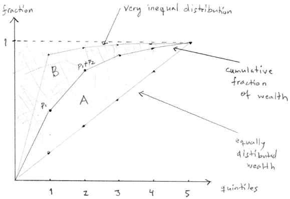
2. Our work: summary only (caveats: not published yet)
   - N-mode Gaussian states, Gaussian channels
   - Including Wigner negativity

- Consider a bipartite quantum system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice can act by local operations in A, Bob in B. They can also call each other and communicate classically.
- Local Operations and Classical Communication: LOCC operations
- correlations between A, B can be classical or quantum
- quantum correlations $\sim$ quantum entanglement
- Alice and Bob cannot produce AB-entangled states, someone else who has access to $\mathcal{H}_{AB}$ must create them. Alice and Bob can then operate on what is accessible to them.
- From the point of view of Alice & Bob, entanglement is a resource which enables operations which would otherwise be impossible for them. *E.g.* teleportation.
- The diminishing of entanglement under LOCC can be quantified by monotones such as entanglement entropy.
- LOCC also produces a partial order among quantum states called majorization.

- Consider a country, divide the population to n equal size parts, each having the fraction p_i of total wealth, $i = 1, \ldots, n$. Let $i = 1$ label the richest part, $i = n$ the poorest part. Thus $p_1 \geq p_2 \geq \cdots \geq p_n$.



- We can visually compare wealth distributions of different countries by Lorentz curves: consider the cumulative fraction of wealth $S_k = \sum_{i=1}^{k} p_i$ at $k = 1, 2, \cdots, n$. Points (k, S_k) span the Lorentz curve



- Areas $A\&B$ above and below the curve define the Gini index = $\frac{A}{A+B}$. Smaller value = more equality

- Distribution $p = (p_1, p_2, \cdots, p_n)$ is more unequal than $q = (q_1, q_2, \cdots, q_n)$ iff $\sum_{i=1}^k p_i \geq \sum_{i=1}^k q_i \; \forall k$.
- We say that p majorizes q, denoted $p \succ q$ iff the above condition on partial sums is true.
- Equivalently: the Lorentz curve of p is above the Lorentz curve of q.

Theorem (Hardy-Littlewood-Pólya)

Let $p, q \in \mathbb{R}^n$ probability vectors. The following are equivalent:

1. $p \succ q$ (p majorizes q)
2. there exists a doubly stochastic* matrix S such that $Sp = q$ $\;(^* : S_{ij} \geq 0, \sum_i S_{ij} = \sum_j S_{ij} = 1)$
3. $\sum_{i=1}^n f(p_i) \geq \sum_{i=1}^n f(q_i)$ for all convex functions $f : [0,1] \to \mathbb{R}$.

Let ρ, σ be a pair of states of a quantum system, λ, μ their ordered eigenvalue vectors. We say that ρ majorizes σ, $\rho \succ \sigma$ iff $\lambda \succ \mu$.
This defines a partial order among quantum states with

$$\mathrm{diag}(1, 0, \ldots, 0) \succ \cdots \succ \rho \succ \cdots \succ \frac{1}{d}\,\mathrm{diag}(1, 1, \ldots, 1)$$

Let $|\psi\rangle, |\chi\rangle \in \mathcal{H}_{AB}$ be two pure states in a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define

$$|\psi\rangle \succ |\chi\rangle \Leftrightarrow \mathrm{tr}_B[|\psi\rangle\langle\psi|] = \rho_A \succ \sigma_A = \mathrm{tr}_B[|\chi\rangle\langle\chi|]$$

If it is possible to convert a pure state $|\chi\rangle$ to $|\psi\rangle$ by LOCC transformations, $|\chi\rangle \underset{\text{LOCC}}{\longrightarrow} |\psi\rangle$, entanglement can only decrease. Then entanglement entropy decreases $S_{EE}(\sigma_A) \geq S_{EE}(\rho_A)$.

Nielsen's theorem

$|\chi\rangle \underset{\text{LOCC}}{\longrightarrow} |\psi\rangle \Leftrightarrow |\psi\rangle \succ |\chi\rangle \Leftrightarrow \rho_A \succ \sigma_A$.

Majorization partial order thus gives thus a classification of bipartite entanglement among pure states!
Not only entanglement entropy decreases in LOCC:
A quantifier $T(\rho)$ is Schur concave if $\rho \succ \sigma \Rightarrow T(\rho) \leq T(\sigma)$.
For example, Rényi entropies $S^{(\alpha)}(\rho) = \frac{1}{1-\alpha} \ln \mathrm{tr}\, \rho^\alpha$ are Schur concave $\forall \alpha > 0$.
Related concept: $T(\rho)$ is concave if

$$T(\lambda\rho + (1-\lambda)\sigma) \geq \lambda T(\rho) + (1-\lambda) T(\sigma) \;\; \forall \lambda \in [0,1]$$

Concave $\Rightarrow$ Schur concave. $S^{(\alpha)}(\rho)$ concave for $\alpha \in (0,1)$.

Related concept: $T(\rho)$ is concave if

$$T(\lambda\rho + (1-\lambda)\sigma) \geq \lambda T(\rho) + (1-\lambda)T(\sigma) \;\; \forall \lambda \in [0,1] \tag{1}$$

Concave $\Rightarrow$ Schur concave. $S^{(\alpha)}(\rho)$ concave for $\alpha \in (0,1)$.
Recipe: Let $f : [0,1] \to \mathbb{R}$ be smooth, concave. Then

$$T_f(\rho) = \mathrm{tr}\, f(\rho)$$

is concave in the sense of (1).

Vidal's theorem

Let $T_f(\rho)$ be as in the above, and $T_f(U\rho U^\dagger) = T_f(\rho)$ when U is unitary. Then $T_f(\rho)$ defines an entanglement monotone.

(Here I am shortcutting details about the construction, *e.g.* for mixed vs. pure states). Entanglement monotones have the property that they are non-increasing under LOCC transformations (more generally than Schur concave quantifiers).

A quantum channel $\mathcal{E}$ can be represented as an operator-sum

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \tag{2}$$

where the Kraus operators E_i satisfy $\sum_i E_i^\dagger E_i = \mathbb{1}$, or equivalently by the Stinespring dilation theorem

$$\mathcal{E}(\rho) = \mathrm{tr}_E[U(\rho \otimes \rho_E)U^\dagger] \tag{3}$$

where U is unitary and ρ_E is an initial state of an environment E.
A channel is unital if it satisfies $\mathcal{E}(\mathbb{1}) = \mathbb{1}$.

Uhlmann's theorem of majorization

If $\mathcal{E}$ is a unital channel, then $\rho \succ \mathcal{E}(\rho)$.

BEGIN SIDE TOUR: new sequences of monotones [Arias, de Boer, Di Giulio, EKV, Tonni, Phys. Rev. Res. 5 (2023) 4, 043082]

- Define moments of shifted modular Hamiltonian

$$M_n(\rho, b_n) = \langle (K + b_n)^n \rangle_\rho = \mathrm{tr}[\rho(-\log\rho + b_n)]^n$$

  which are concave for $b_n \geq n - 1$.

- Von Neumann entropy is $M_1(\rho, 0)$.
- By Vidal's theorem they define an infinite sequence of entanglement monotones. In particular,

$$\rho \succ \sigma \Rightarrow M_n(\sigma, b_n) \geq M_n(\rho, b_n)$$

- Can convert Rényi entropies to these monotones, by

$$M^{(n)}(\rho; b_n) = e^b (-1)^n \frac{d^n}{d\alpha^n} k_\alpha(\rho; b)|_{\alpha=1, b=b_n} ,$$

where

$$k_\alpha(\rho; b) = \exp\left\{-\alpha b + (1-\alpha) S^{(\alpha)}(\rho)\right\}$$

- The inequalities can be rewritten in terms of cumulants of $-\log \rho$.
- The first cumulant = von Neumann entropy $S(\rho)$, the second cumulant is capacity $C(\rho)$, and so on
- E.g. $\rho \succ \sigma \Rightarrow M_2(\sigma, 1) \geq M_2(\rho, 1)$ becomes a lower bound on entropy production depending on decrease of capacity:

$$S(\sigma) - S(\rho) \geq \frac{C(\rho) - C(\sigma)}{S(\rho) + S(\sigma) + 2}$$

- A generalization of this approach gives monotones for other resource theories: e.g. for that of thermodynamics
- Free operations: thermal operations (towards equilibrium) with fixed point with $\sigma_* = \gamma_\beta \equiv \exp(-\beta H)/Z$
- A relative majorization with fixed point γ_β: "thermomajorization"
- Construct and infinite sequence of relative monotones, expand inequalities with familiar quantities
- Leads to *e.g.* to a finite-dimension correction to the Clausius inequality:

$$S(\beta) - S(\rho) \geq \frac{1}{k_B T}[\langle H\rangle_\beta - \langle H\rangle_\rho] + \frac{C(\rho||\gamma_\beta)}{2 + 2\beta(E_{\max} - F(\beta))}\ , \tag{4}$$

$\langle H\rangle_\rho$ non-equilibrium energy of ρ, $\langle H\rangle_\beta$ for thermal γ_β, $S(\rho)$ non-eq. entropy of ρ, and thermal entropy $S(\beta) = S(\gamma_\beta)$, $E_{\max}$ max energy eigenvalue, $F(\beta)$= thermal free energy

- Explored majorization for CFT states, monotonicity of c-functions from C, M_2 (cf. entropic c-functions from alpha-entropies by Casini & Huerta, (accidental ?) monotonicity without SSA) END SIDE TOUR

Resource theories are a general framework to characterize what happens to a resource or feature (e.g. entanglement) under operations of interest

They consist typically of

- $\{free\ states\}$ (in resource theory of entanglement: separable states)
- $\{free\ operations\}$ mapping free states to free states (entanglement: LOCC)
- $\{resource\ states\} = \neg\{free\ states\}$ (entangled states)
- $\{monotones\}$ (ent. entropy, ent. Rényi entropies)
- a *partial order* associated with free operations (majorization)

Magic a/k/a non-stabilizerness

- In DV quantum computing, Gottesman-Knill theorem says that all circuits that consist only of "easy gates"(Pauli X,Y,Z, Hadamard H, phase S, CNOT), and certain types of state preparations, measurements, and classically conditioned control can be efficiently simulated with a classical computer.
- Easy operations generate Clifford group
- Note: CNOT creates entanglement, not enough for quantum advantage. Need "hard gates" such as T or Toffoli gates
- Or: use Clifford operations together with a supply of magic states. The latter are then a resource for quantum advantage.
- The set of non-magical states reachable by Clifford operations, measurements, classical conditioning are called stabilizer states.

Resource theory of magic

- free states = stabilizer states
- free operations = Clifford operations + measurements, classical conditioning
- resource states = magic states
- monotones = mana, stabilizer Rényi entropy, ...
- partial order = majorization of discrete Wigner functions

Following Koukoulekidis and Jennings, npj Quantum Information (2022):

Let $\mathcal{H} = \mathrm{sp}\{|0\rangle, \ldots, |d-1\rangle\} = \mathrm{sp}\{|k\rangle \mid k \in \mathbb{Z}_d\}$, shift X and clock Z operators

$$X|k\rangle = |k+1\rangle \; ; \; Z|k\rangle = \omega^k |k\rangle \; ; \; \omega = e^{2\pi i/d}$$

Discrete phase space $\Gamma_d = \{z \equiv (p,q) \in \mathbb{Z}_d \times \mathbb{Z}_d\}$, def. discrete displacement operator

$$D_z = \tau^{qp} X^q Z^p \; ; \; \text{phase } \tau = -\omega^{1/2} \, ,$$

one can generalize to n copies $\mathcal{H}^n$ wtih $z \in \Gamma_d^n, \; D_z = D_{(p,q)}^{\otimes n}$. They form the Heisenberg-Weyl group

$$HW_d^n = \{ \tau^k D_z \; : \; k \in \mathbb{Z}_d, \; z \in \Gamma_d^n \} \, .$$

The set of unitary operators that normalize HW_d^n form the Clifford group C_d^n. Define pure stabilizer states

$$STAB_0 = \{U|0\rangle\langle 0|U^\dagger \; : \; U \in C_d^n\}$$

and full set of stabilizer states $STAB = \{\sum_i p_i |\psi_i\rangle\langle\psi_i| \; : \; |\psi_i\rangle \in STAB_0 \; \forall \, i\}$. Observe that $U \, : \, STAB \to STAB$ for all $U \in C_d^n$.

For any $z \in \Gamma^d$ define the phase point operator

$$A_z = \frac{1}{d} \sum_{y \in \Gamma_d} \omega^{y^T \Omega z} D_y$$

and now for any quantum state ρ its discrete Wigner distribution

$$W_\rho(z) = \frac{1}{d} \operatorname{tr}(A_z \rho) = \frac{1}{d^2} \sum_{y \in \Gamma_d} \omega^{y^T \Omega z} \operatorname{tr}(D_z \rho) \ .$$

Normalization

$$\sum_{z \in \Gamma_d} W_\rho(z) = 1$$

and one can think of W as a d^2-dimensional vector (with components labelled by $z \in \Gamma_d \equiv \mathbb{Z}_d \times \mathbb{Z}_d$).

For stabilizer states, all components of W are non-negative: W is a probability vector,
for non-stabilizer states (=magic states) W has at least one negative component: W is a quasi-probability vector.

Now for a resource theory of magic, choose

- free operations: Clifford unitaries, Pauli measurements, classical conditioning
- Consider here qudits with d=odd prime. Then in circuits with above free operations, all states with a positive Wigner function admit an efficient classical simulation (Eisert-Mari, Veitch et. al)
- free states $\mathcal{F} = \{\rho \ : \ W_\rho(z) \geq 0 \, \forall \, z \in \Gamma_d\}$
- resource states: states with Wigner negativity
- quantum channels $\mathcal{E}$ are dual to states, so they also admit a Wigner representation $W_\mathcal{E}(y|z)$. Under the action of the channel, Wigner functions transform by

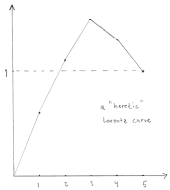
$$W_{\mathcal{E}(\rho)}(y) = \sum_{z \in \Gamma_d} W_\mathcal{E}(y|z) W_\rho(z) \, .$$

- if the channel involves free operations, the kernel is bistochastic: $\sum_y W_\mathcal{E}(y|z) = \sum_z W_\mathcal{E}(y|z) = 1$
- Thus positivity of Wigner functions is preserved, and input Wigner function majorizes output Wigner function:

$$W_\rho \succ W_{\mathcal{E}(\rho)}$$

- majorization of states is ”Wigner majorization"
- What if the input state is a resource state with Wigner negativity?

- Consider a quasiprob. vector $p = (p_1, \ldots, p_n)$ with $p_1 \geq p_2 \geq \cdots \geq p_n$: most negative components are at the end
- The Lorentz curve is "heretic": it will overshoot 1 and come down to 1 only at the end

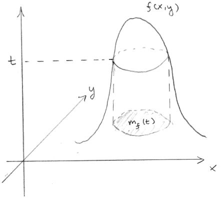


- Koukoulekidis and Jennings extend the notion of majorization to quasiprobabilities: $p \succ q$ it the Lorenz curve of p is above that of q (even if heretic curves)
- Thus also input resource states (Wigner) majorize output states under free operations, Wigner negativity can only decrease
- Monotone: note that $\sum_z |W_\rho(z)| \geq 1$ with $>$ iff ρ is a resource state
- def. mana $M(\rho) = \log(\sum_z |W_\rho(z)|)$, can only decrease under free operations
- KJ use this set-up for magic distillation protocols, and compute estimates for distillation rates

Does any of this apply to QFTs?

- One can discretize, *e.g.* $\mathbb{Z}_3$ parafermion CFT with $c = 4/5$ to 3-state Potts model, so now a qudit system (see *e.g.* White, Cao, Swingle (2021), . . ., lots of "magic activity")
- Alternatively, perhaps more natural to replace qudits by Fock space: move to continuous variable quantum information
- Our starting point: Z. Van Herstraeten, M. G. Jabbour, N. J. Cerf, "Continuous majorization in quantum phase space", Quantum **7**, 1021 (2023).
- This paper considers applying continuous majorization of probability distributions in $\mathbb{R}^{2n}$ to Wigner functions in (continuous) phase space.

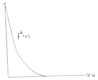
Kong-Ming Chong 1974, Harry Joe 1987: Consider probability distributions $f : \mathbb{R}^{2n} \to [0, \infty)$; $f(x) \geq 0 \; \forall \; x \in \mathbb{R}^{2n}$, $\int d^{2n}x \, f(x) = 1$.
Define level-function $m_f : \mathbb{R}_+ \to \mathbb{R}_+$, $m_f(t) = \mathrm{Vol}(\{x \in \mathbb{R}^{2n} \mid f(x) \geq t\})$: gives the volume of the domain that contains the elements x such that $f(x) \geq t$. Note $m_f(t)$ is non-increasing.



If two distributions f, g have $m_f = m_g$, we say f, g are level-equivalent
Define the decreasing arrangement $f^{\downarrow}$ of a function

$$f^{\downarrow}(u) = \inf \left\{ s \in \mathbb{R}^+ \;\middle|\; m_f(s) \leqslant u \right\} = m_f^{-1}(u) \, .$$

The decreasing arrangement is the continuous analogue of a rearranged probability vector $(p_1, p_2, \ldots, p_n)$ (with $p_1 \geq p_2 \geq \ldots \geq p_n$).



Previously we considered partial sums $S_k = p_1 + p_2 + \ldots + p_k$. The continuous version is the integral

$$I_f(v) \equiv \int_0^v f^{\downarrow}(u)du \, .$$

Points $(v, I_f(v))$ give the continuous version of the Lorentz curve of a probability distribution. We define

Definition: Continuous majorization

Given two probability distributions f, g, we say that f *majorizes* g, $f \succ g$ iff

$$\int_0^v f^{\downarrow}(u)du \geq \int_0^v g^{\downarrow}(u)du \;\; \forall \, v \geq 0 \, .$$

Most ordered: delta distribution, least ordered: e.g. gaussian distribution with variance $\sigma \to \infty$.

Van Herstraeten, Jabbour and Cerf consider the state space bult by N pairs of annihilation/creation operators $a_i, a_i^\dagger$, $i = 1, \ldots, N$ corresponding to N pairs of quadrature operators q_i, p_i. For every state ρ one can define its Wigner function $W_\rho(x)$ over the $2N$-dimensional quantum phase space $x = (q, p) \in \Gamma_N = \mathbb{R}^{2N}$. The Wigner function is unit normalized,

$$\int_{\Gamma_N} W(x) = 1 \, .$$

A subset of states $\mathcal{W}_+$ contains all the states whose Wigner function is non-negative everywhere. For these states the Wigner function is a probability distribution, so we can apply continuous majorization. We then say that

Definition: Wigner majorization

Given two states $\rho, \sigma \in \mathcal{W}_+$, we say that ρ *Wigner majorizes* σ iff $W_\rho \succ W_\sigma$.

VHJC consider the one-mode case $N = 1$. Their main points are:

- all pure states are level-equivalent with the Fock space vacuum $|0\rangle$
- Wigner entropy and Wigner Rényi entropies are monotonic under Wigner majorization
- Conjecture: the (equivalence class of) vacuum Wigner majorizes (the equivalence class of) every other Wigner positive state.
- They test the conjecture with the family $\rho = (1 - p_1 - p_2)|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2|$ and find the conjecture works.

Our work: main results (so far)

- A simple criterion for the N-mode case: ρ Wigner majorizes σ if and only if $\det \gamma_\rho \leq \det \gamma_\sigma$, where γ_ρ is the covariance matrix of ρ
- A proof of the VHJC conjecture for the convex hull ofl N-mode Gaussian states
- In the one-mode case, Wigner majorization is equivalent with usual majorization (and not when $N > 1$)
- For Gaussian channels, for Gaussian input states always a majorization relation between the input and output, but the relation can be both ways
- For Wigner negative input, output cannot majorize input
- Gaussian channels $\mathcal{E}$ characterized by matrix pair X, Y: Wigner functions map

$$W_{\mathcal{E}(\rho)}(x) = \int dy\, k_{\mathcal{E}}(x,y) W_\rho(y')$$

  with a channel-associated kernel $k_{\mathcal{E}}$ satisfying

$$\int dx\, k_{\mathcal{E}}(x,y) = 1\,;\ \int dy\, k_{\mathcal{E}}(x,y) = \frac{1}{\det X}\,.$$

- Can see a big difference to the discrete case: the kernel is generally not bistochastic (only "column stochastic")

results (continued)

- Two candidates for extension of the concept of Wigner majorization to all bosonic states
- Both versions imply the monotonicity of logarithmic Wigner negativity $M(\rho) = \log(\int_\Gamma dx|W_\rho(x)|)$
- For channels with $\det X = 1$ (classical mixing channel), the input state (even a Wigner negative one) always Wigner majorizes the output state
- Generalizes to all random displacement channels, even all random Gaussian unitary channels (need to double check): a version of Uhlmann's theorem for Wigner majorization in phase space. (Note: since these channels are unital, also original Uhlmann applies: input majorizes output also in the usual sense)
- For all bosonic states and all bosonic channels, between the input and output state an inequality

$$(\det X)^{\alpha-1} \int dx|W_{out}(x)|^\alpha \leq \int dx|W_{in}(x)|^\alpha \quad \forall \alpha \geq 1\,,$$

  generalizing the monotonicity of Wigner negativity (but with a channel-dependent correction term).
- CV case is more involved than discrete. Has "more hierarchy": Gaussian states, convex hull of Gaussian states, Wigner positive states, "stellar rank"
- For resource theory of Wigner negativity, free operations should be Wigner positive, but those harder to classify than Gaussian operations. Perhaps we can have some progress in that ...

THANK YOU!