Encapsulation Structure and Dynamics in Hypergraphs

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Encapsulation structure and dynamics in hypergraphs

Timothy LaRock[3,1] and Renaud Lambiotte[1,2]

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UNIVERSITY OF OXFORD MATHEMATICAL INSTITUTE

In collaboration with Prof. Renaud Lambiotte

Multi-way/higher-order interactions

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F. Battiston, G. Cencetti, I. Iacopini et al. / Physics Reports 874 (2020) 1–92 5

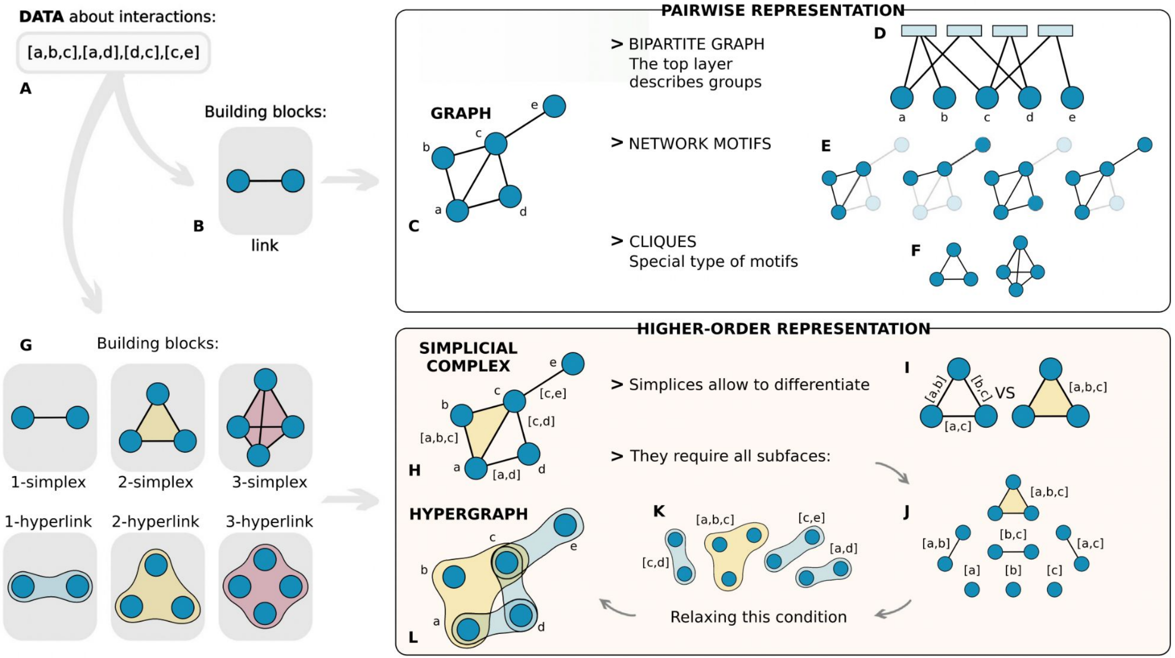
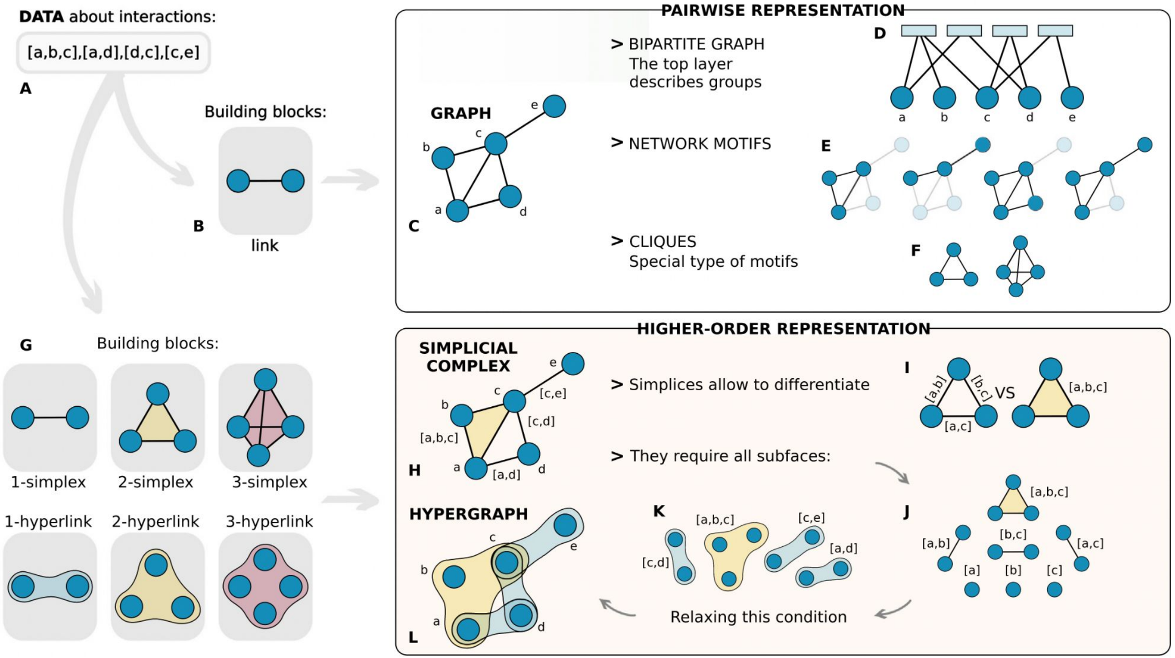


Fig. 1. Representations of higher-order interactions. A set of interactions of heterogeneous order (A) can be represented using only pairwise

Multi-way/higher-order interactions

Interactions occur between sets of nodes of arbitrary size.

F. Battiston, G. Cencetti, I. Iacopini et al. / Physics Reports 874 (2020) 1–92 5



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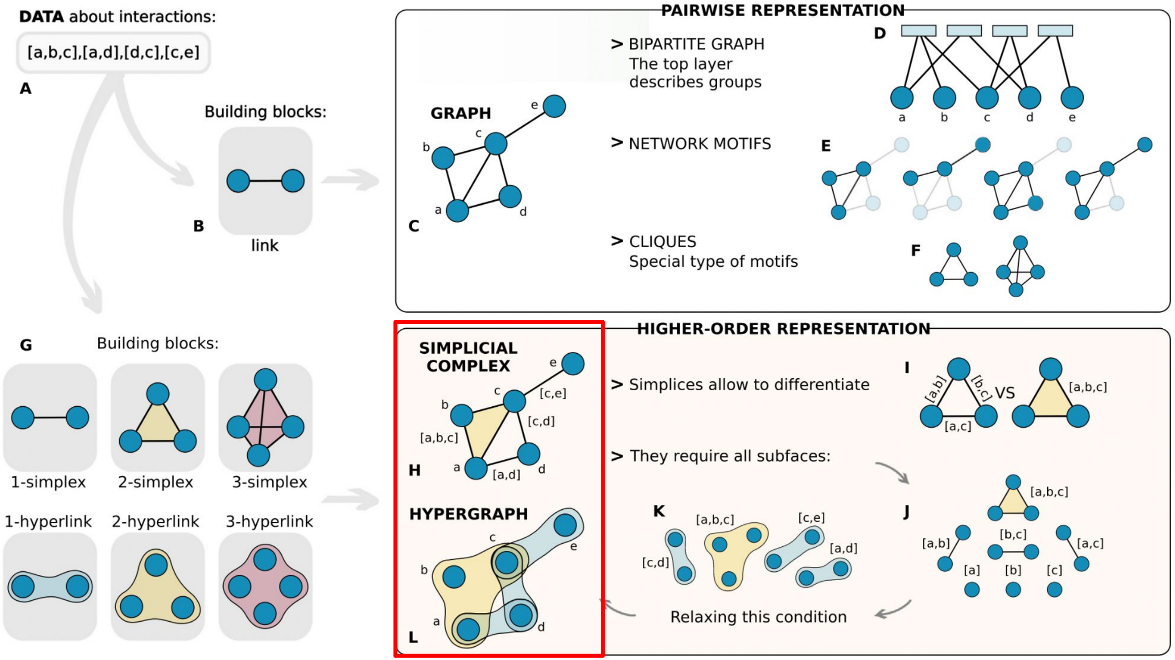

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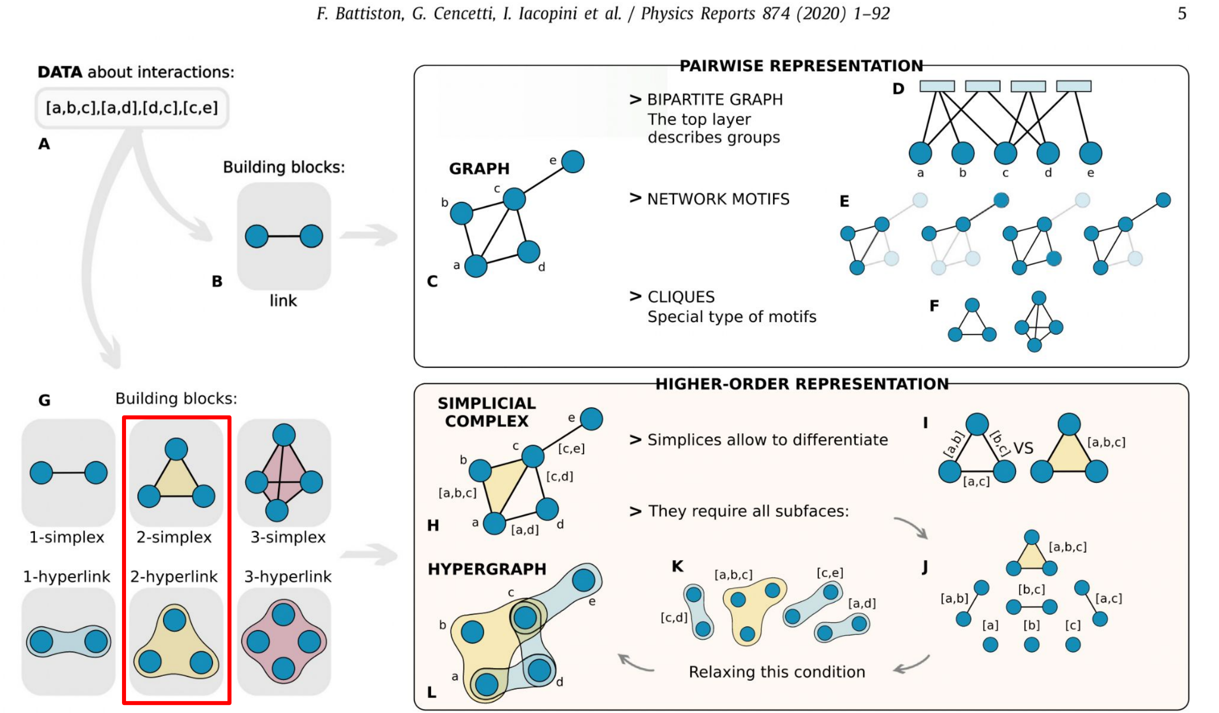

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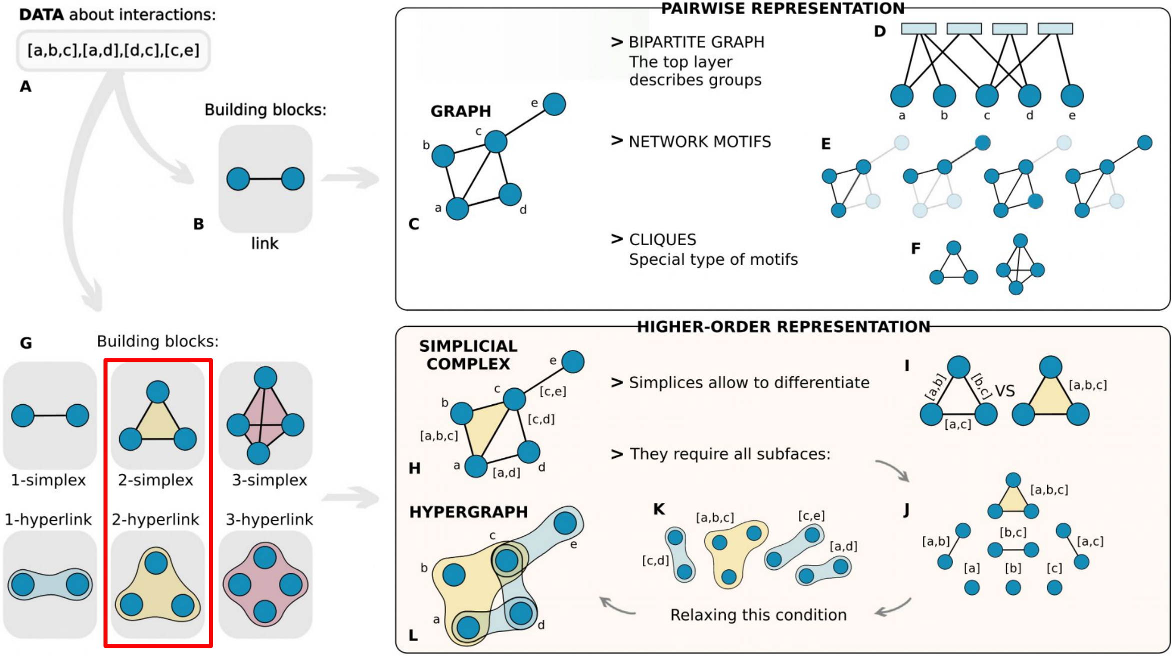

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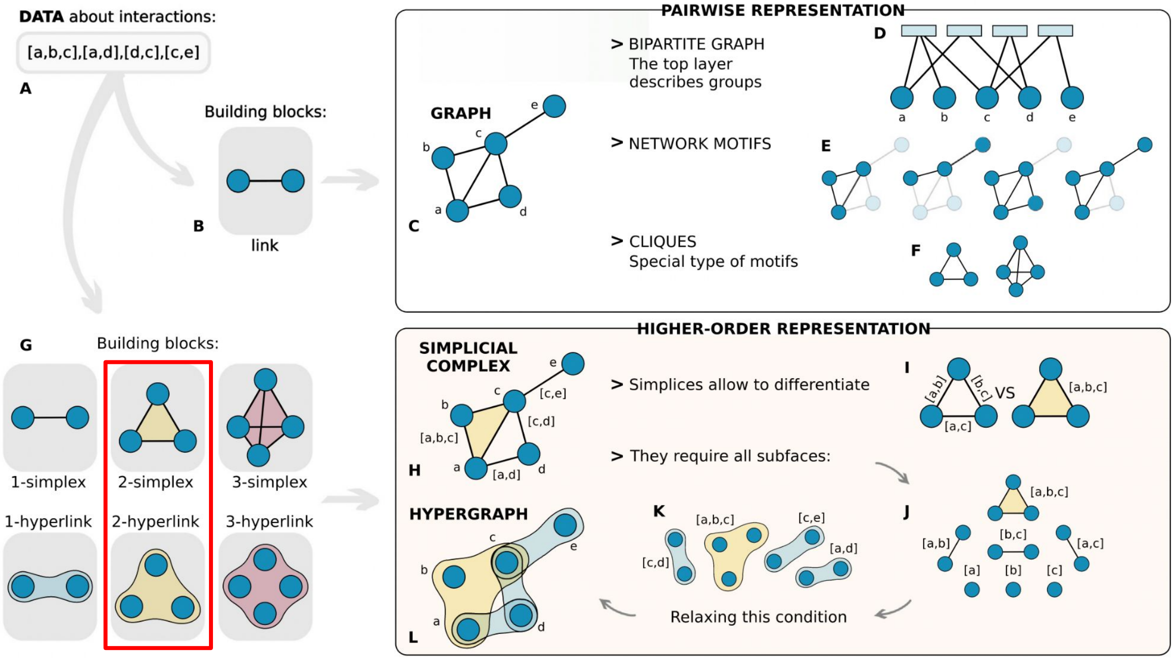
Our goals: (1) Characterize hypergraph structure based on ***encapsulation***, then (2) study a contagion process driven by encapsulation where activation occurs **on hyperedges**.



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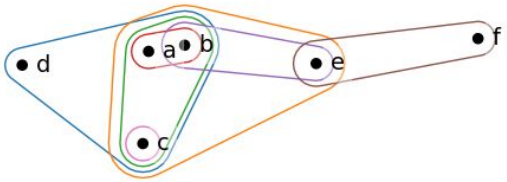
Encapsulation, Overlap, and Line Graphs of Hypergraphs

Hypergraph (Euler diagram)



Encapsulation, Overlap, and Line Graphs of Hypergraphs

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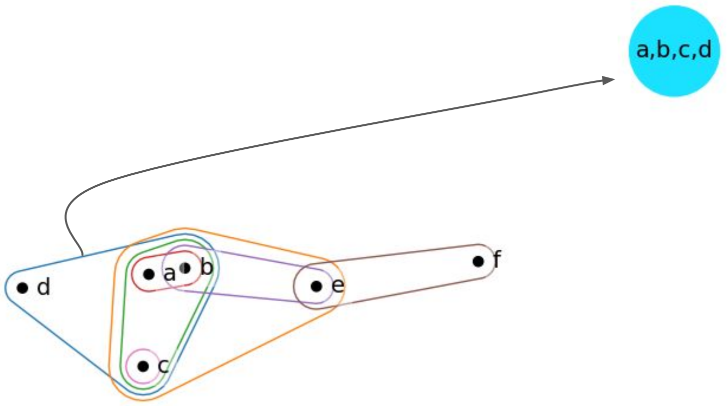

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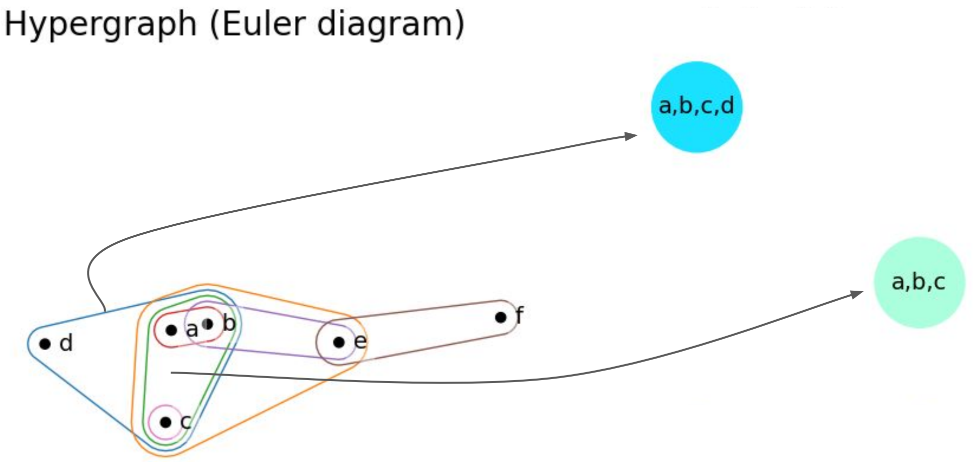

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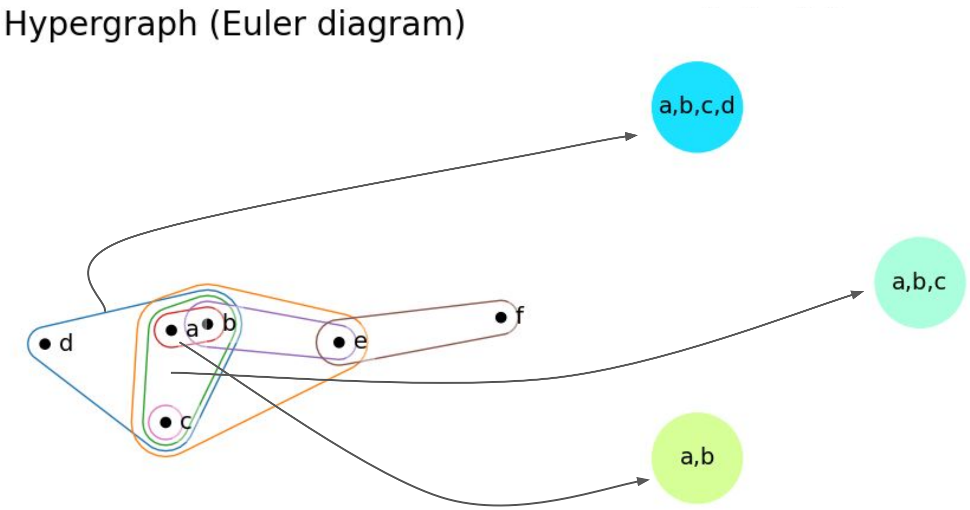

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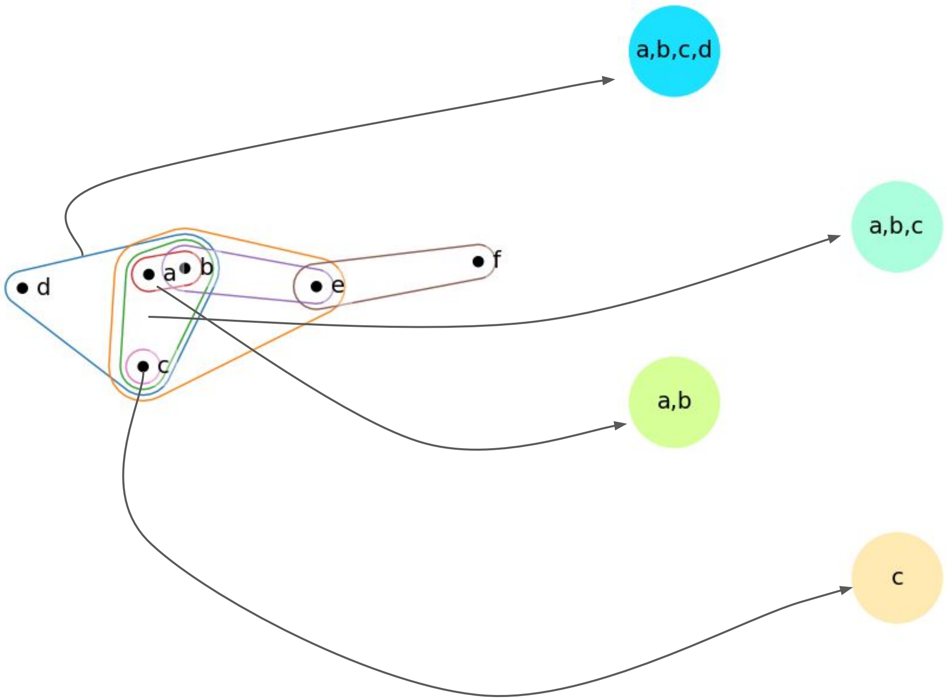

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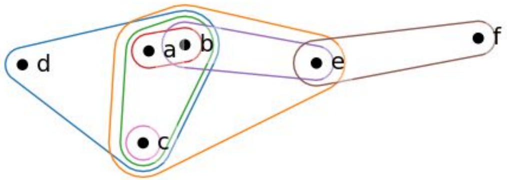

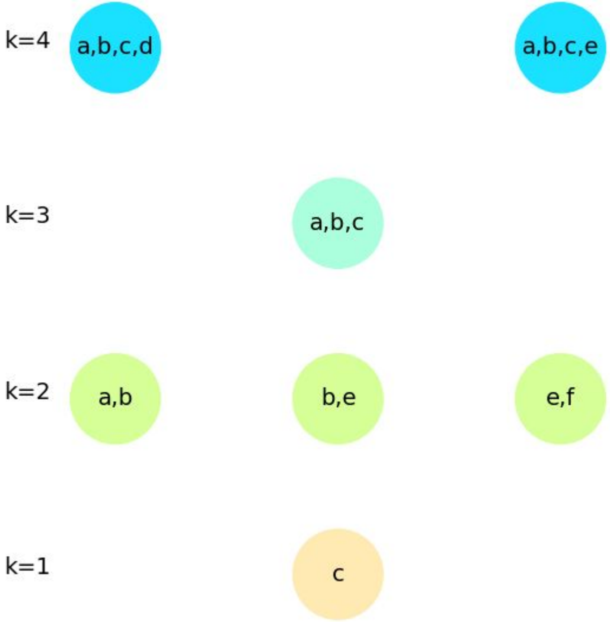

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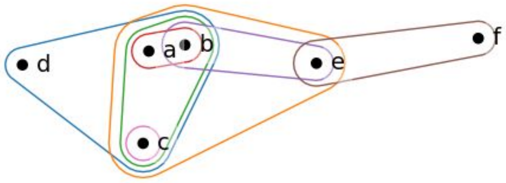
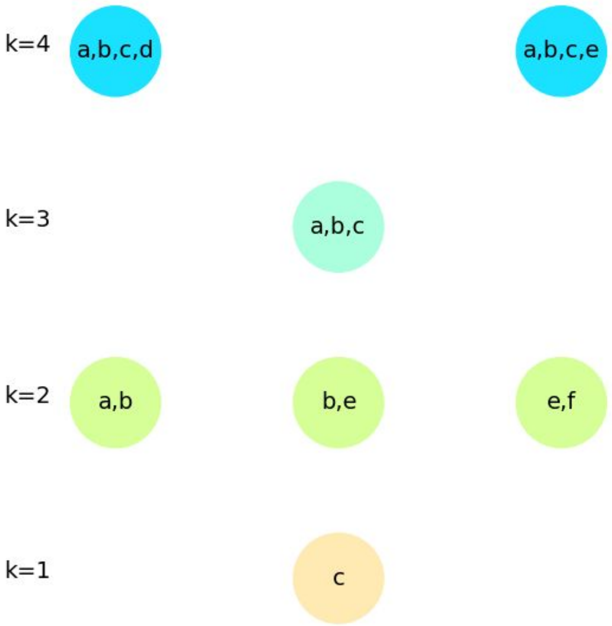
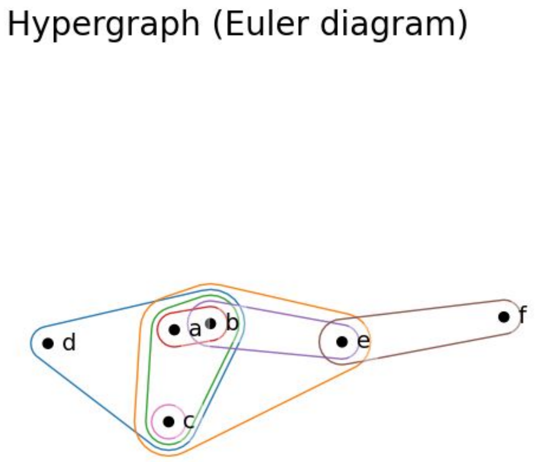
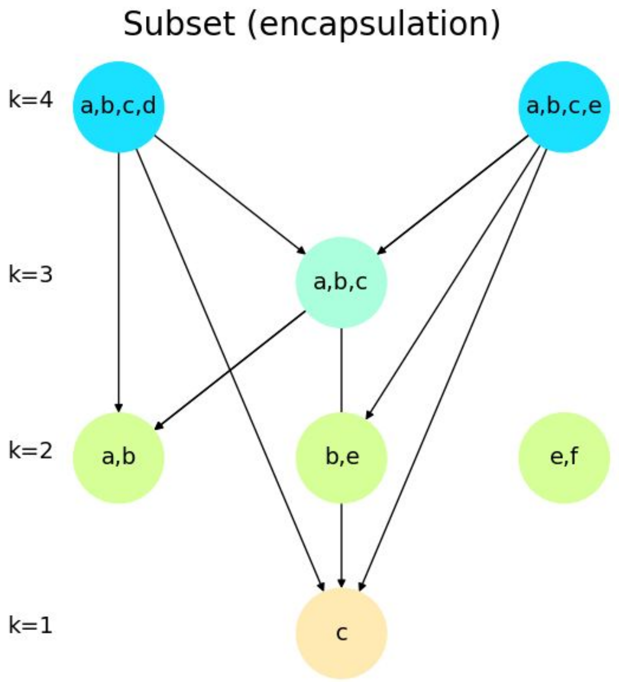




Construct the ***line graph*** of a hypergraph by treating hyperedges as nodes and drawing edges between them based on some relation.

Encapsulation, Overlap, and Line Graphs of Hypergraphs





Encapsulation DAG

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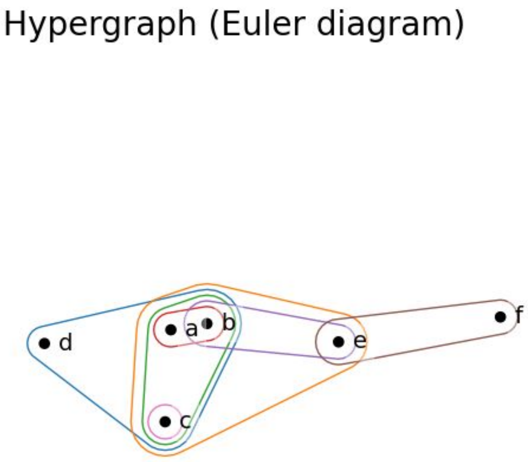

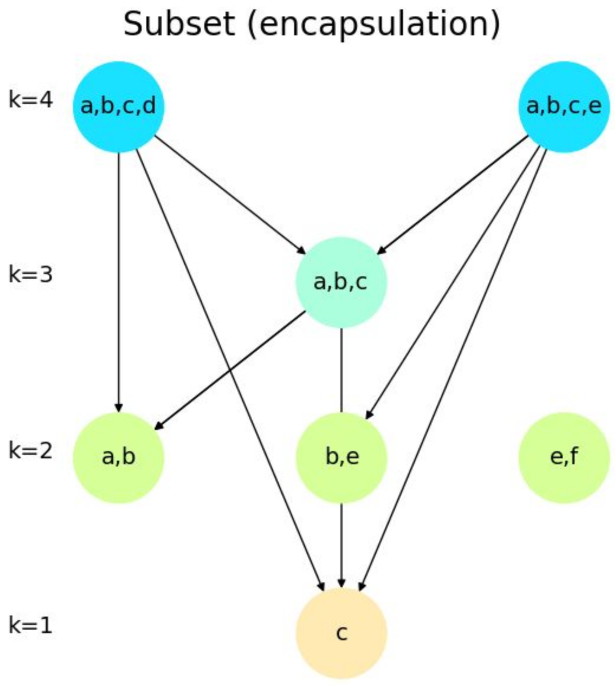

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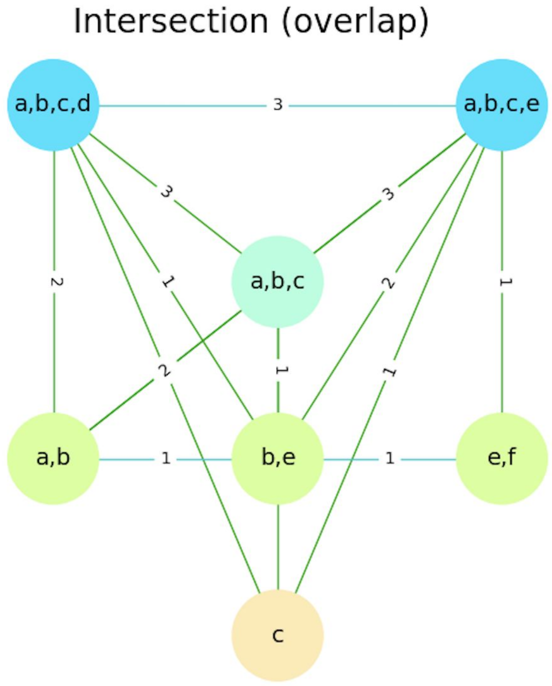


Overlap Line Graph

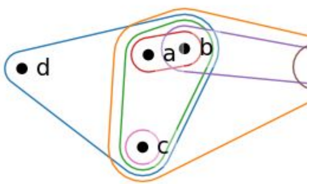
Construct the ***line graph*** of a hypergraph by treating hyperedges as nodes and drawing edges between them based on some relation.

Encapsulation, Overlap, and Line Graphs of Hypergraphs

Hypergraph (Euler diagram)

Subset (encapsulation)

Intersection (overlap)



Aksoy et al. *EPJ Data Science* (2020) 9:16

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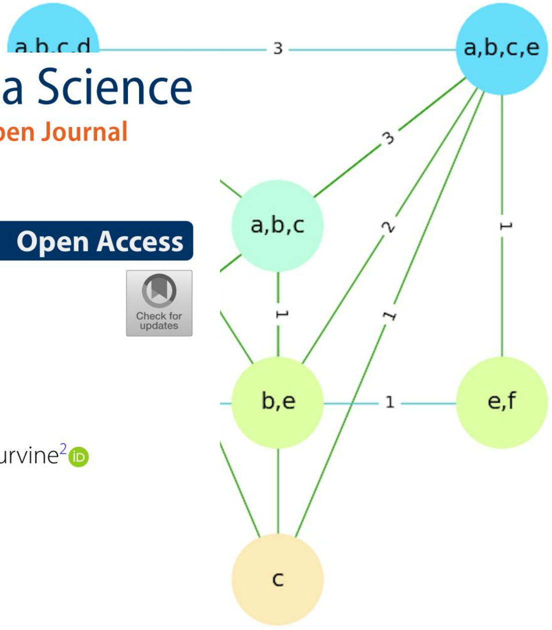
Hypernetwork science via high-order hypergraph walks

Sinan G. Aksoy[1*], Cliff Joslyn[2], Carlos Ortiz Marrero[1], Brenda Praggastis[2] and Emilie Purvine[2]



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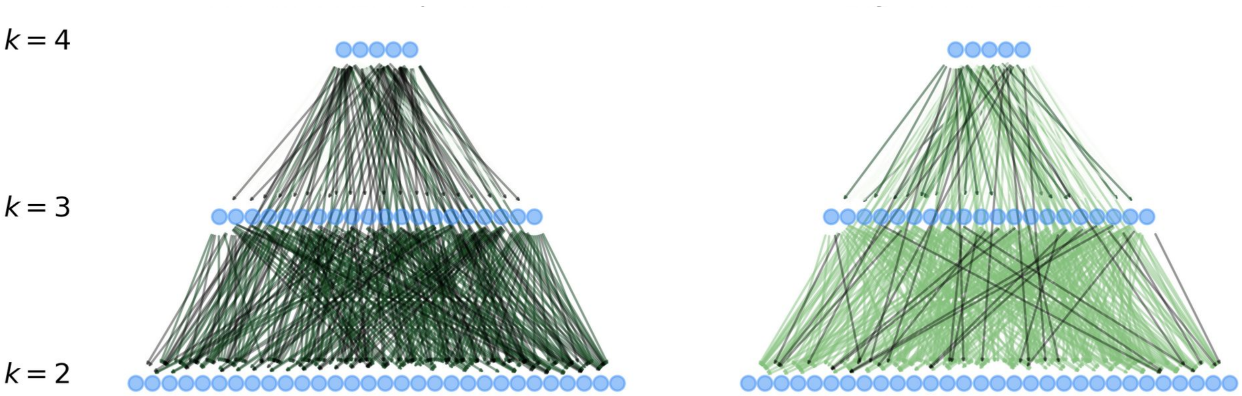
Why encapsulation DAGs?



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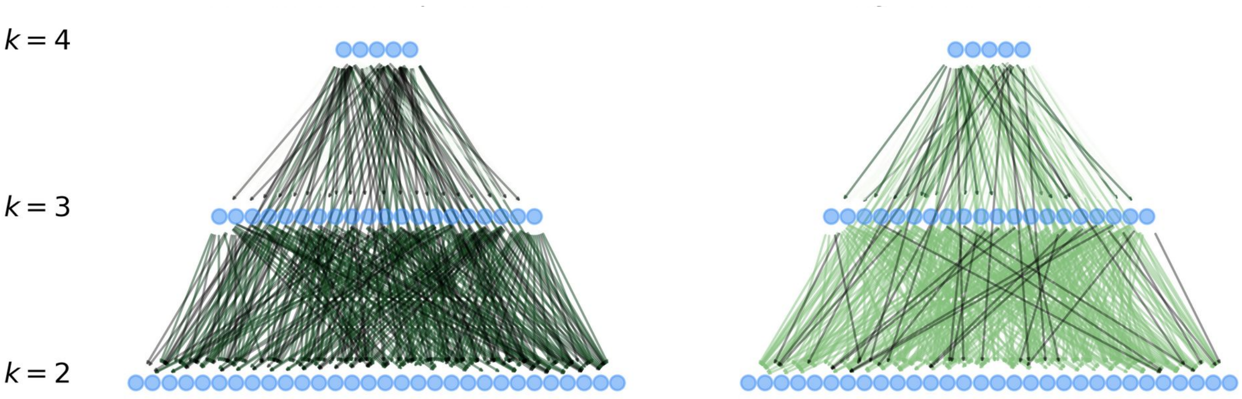

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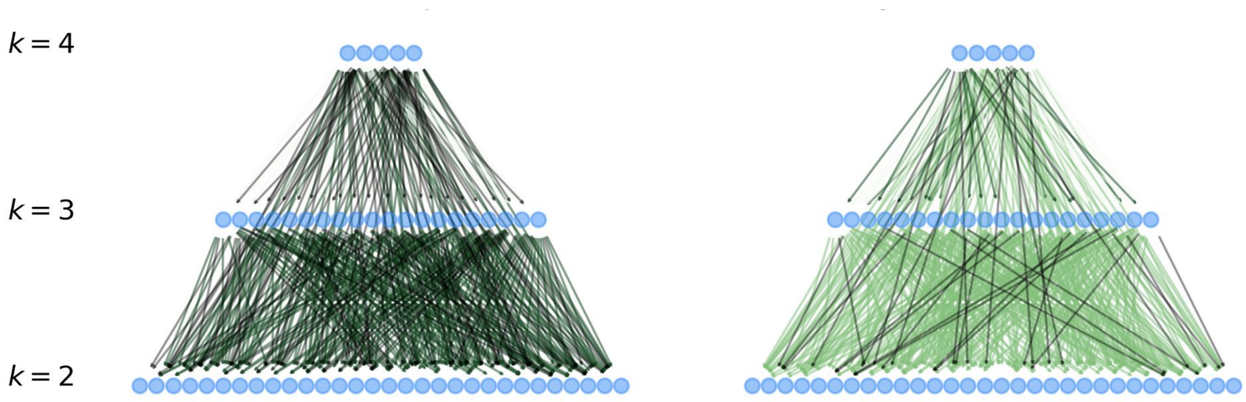

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- Proportion of the maximum DAG edges → “distance” from simplex assumption

Why encapsulation DAGs?

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The simpliciality of higher-order networks

Nicholas W. Landry, Jean-Gabriel Young & Nicole Eikmeier

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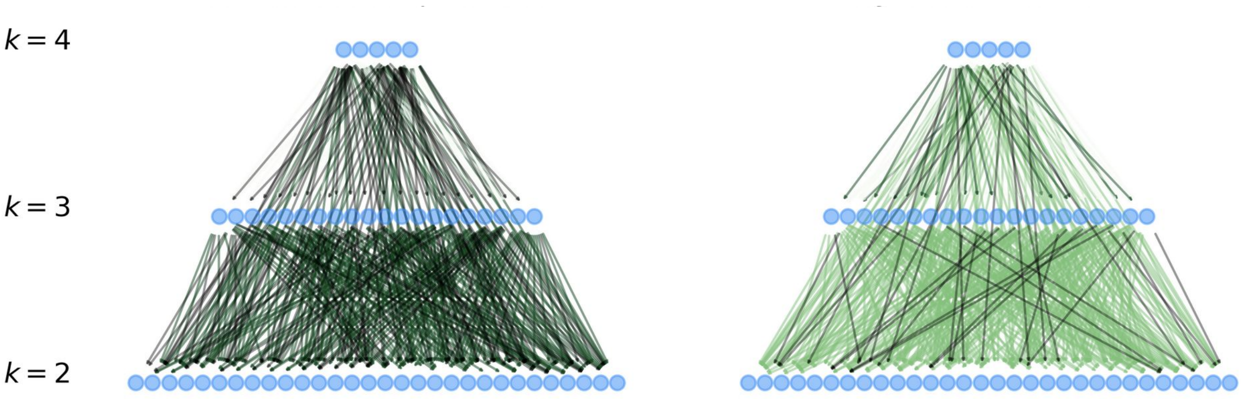

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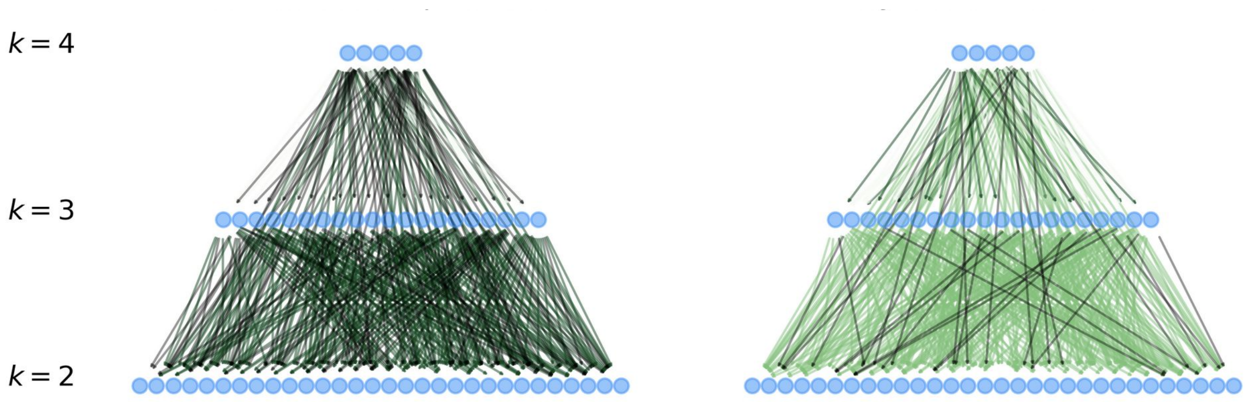

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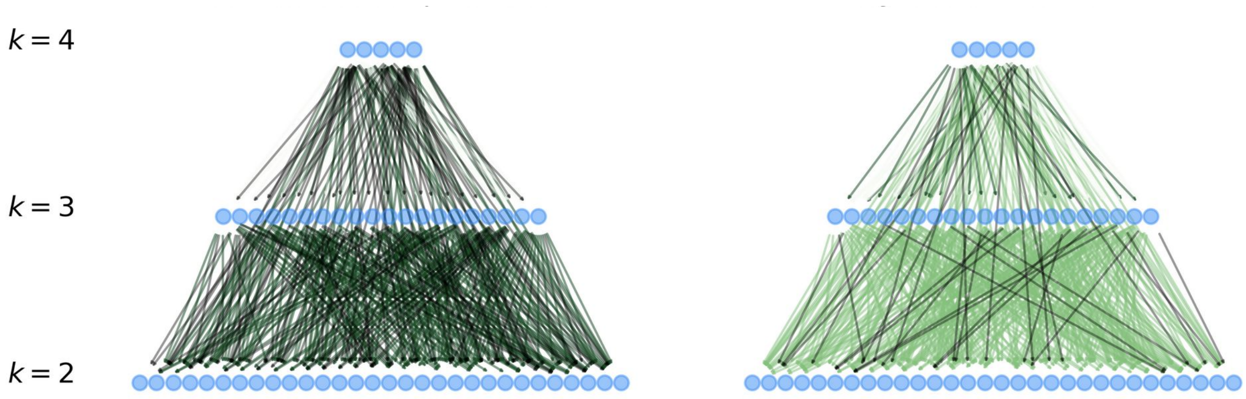

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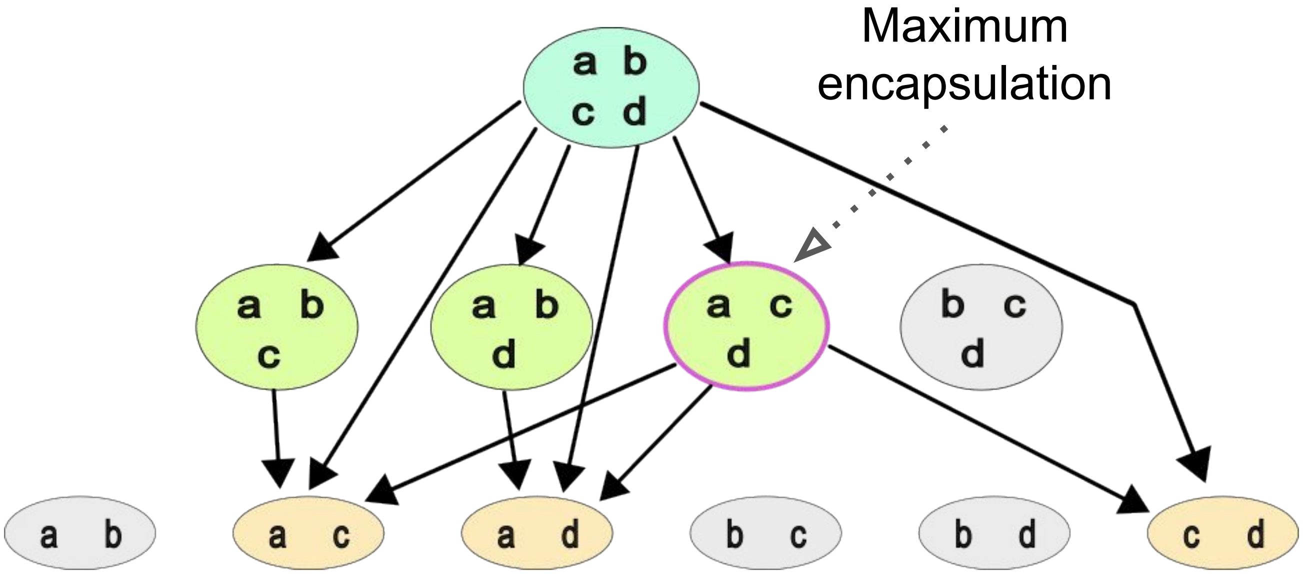
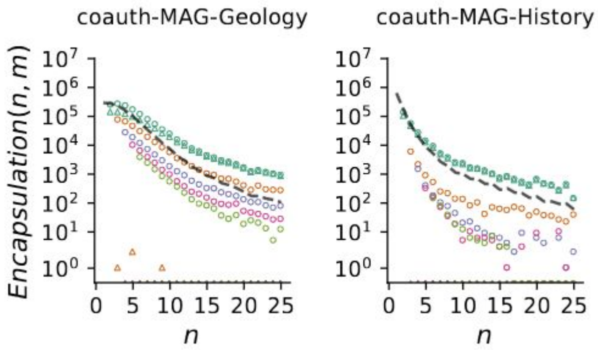
How can these quantities characterize hypergraph data?

DAG Out-degree

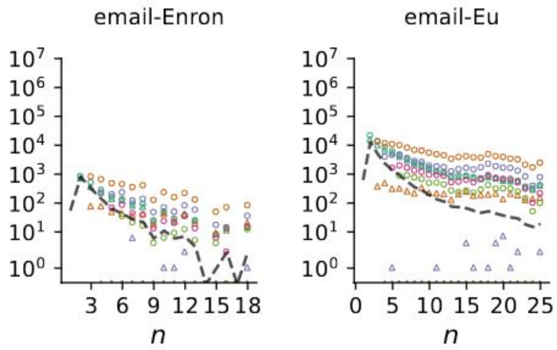


- Indicates how many hyperedges *e* encapsulates
- Maximum is the size of the powerset of *e*

DAG Out-degree

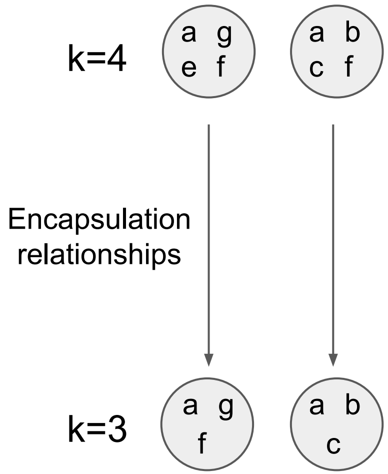


size-n hyperedges that encapsulate hyperedges of size m



Layer Randomization

Idea: Shuffle node labels within each size layer



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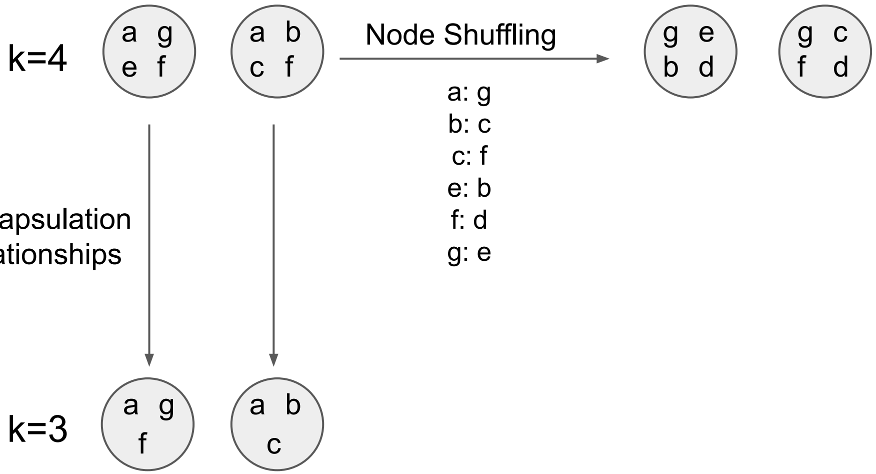

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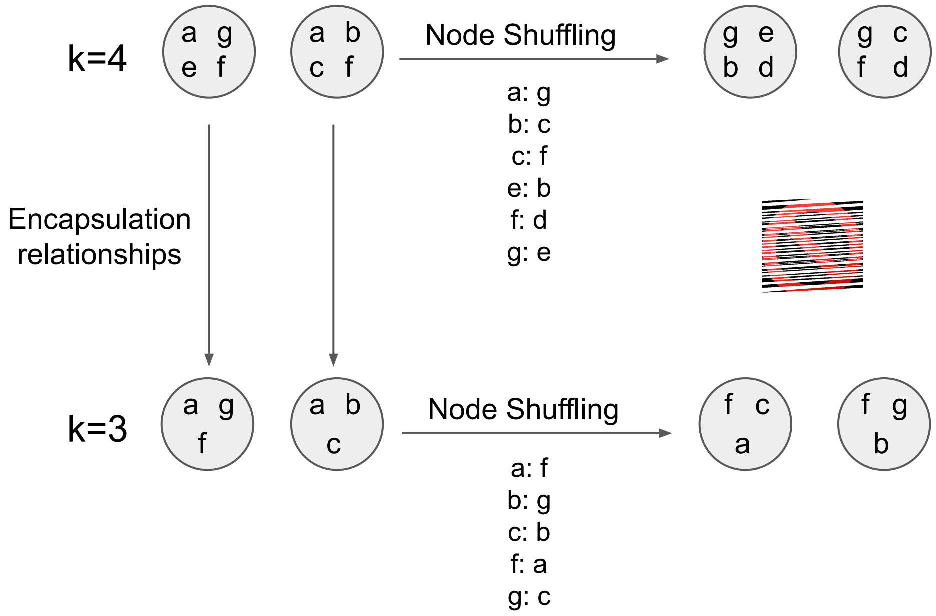

Layer Randomization

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Randomizes:

→ Encapsulation and overlap relationships

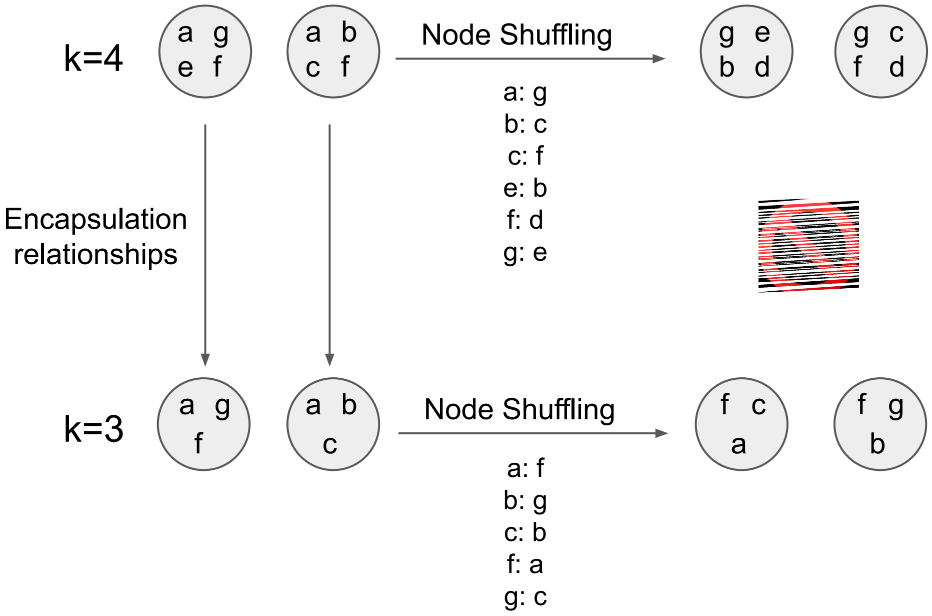
→ Labeled node-degree distributions within and across size layers

→ Unlabeled node-degree distribution across layers

Preserves:

→ Hyperedge size distribution

→ Unlabeled node-degree distribution within layers



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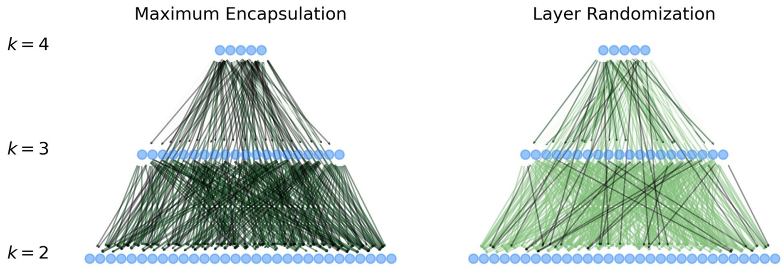

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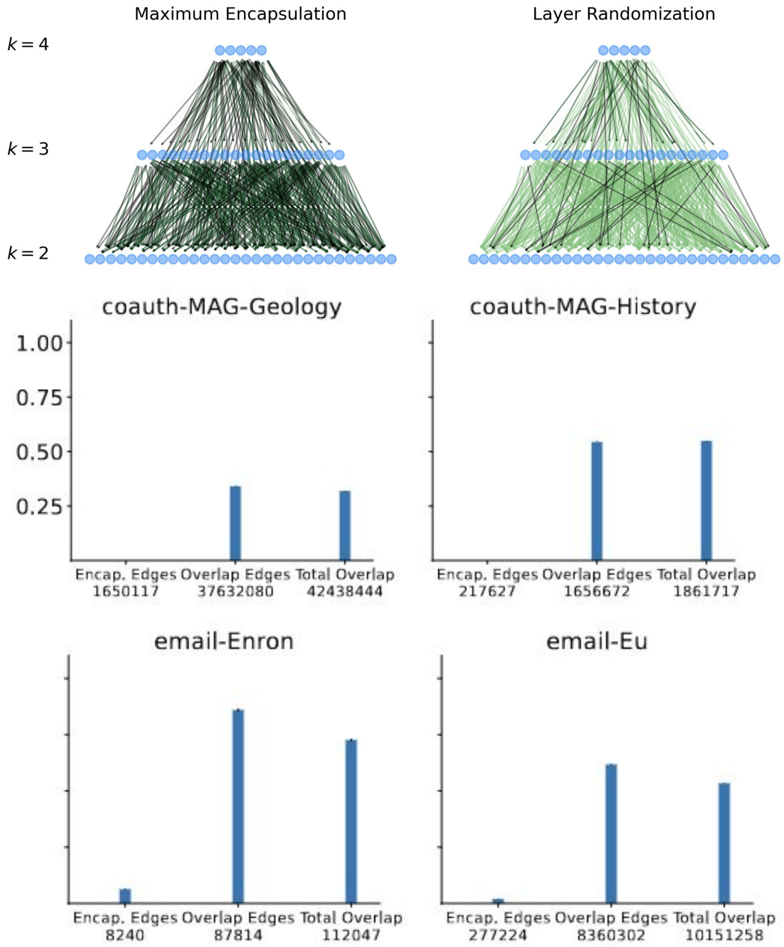
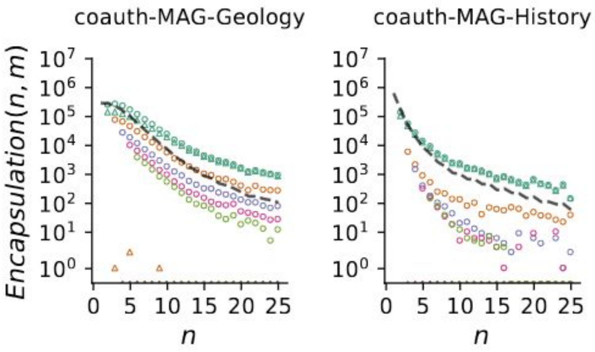
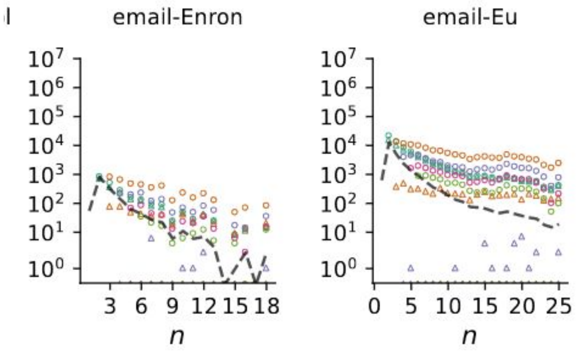


DAG Out-degree



After layer randomization, encapsulation relationships are substantially reduced



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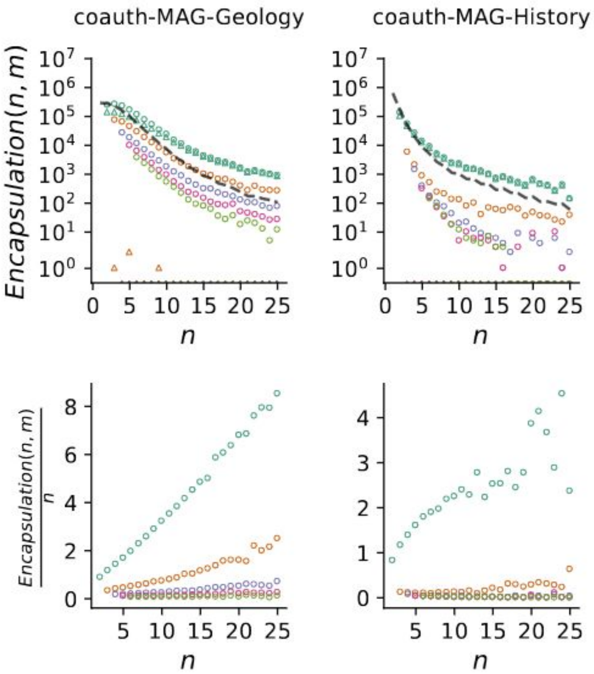
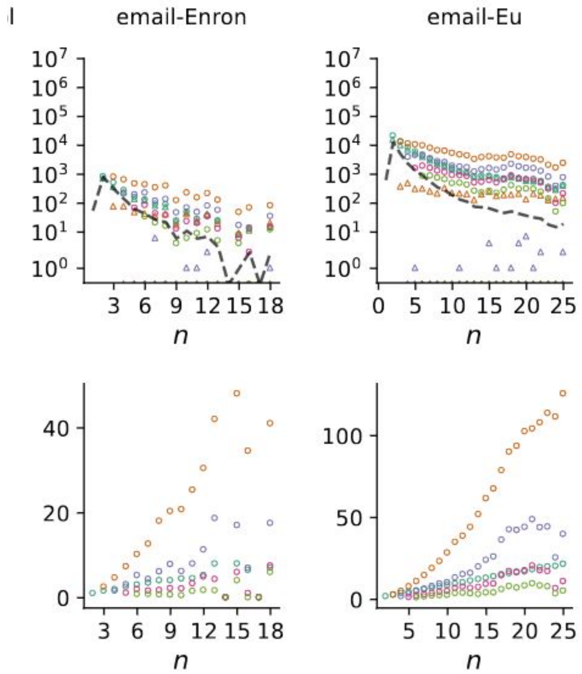


Encapsulations per hyperedge of size n

Simplex assumption would imply

$$y = \binom{n}{m}$$



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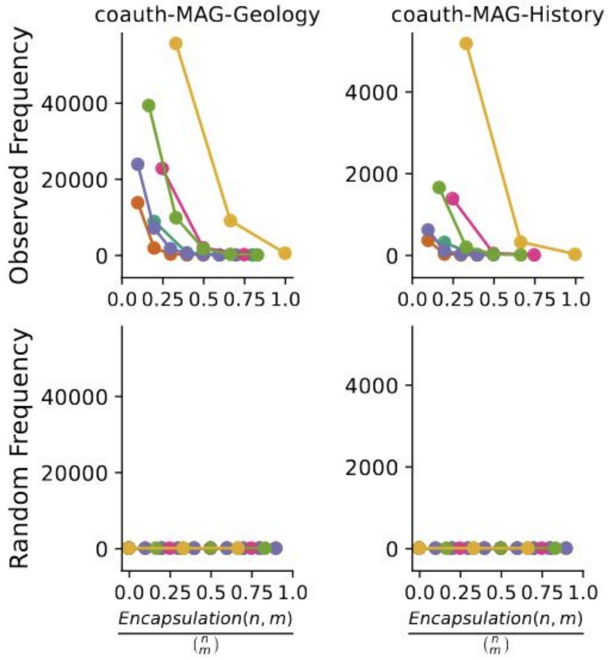
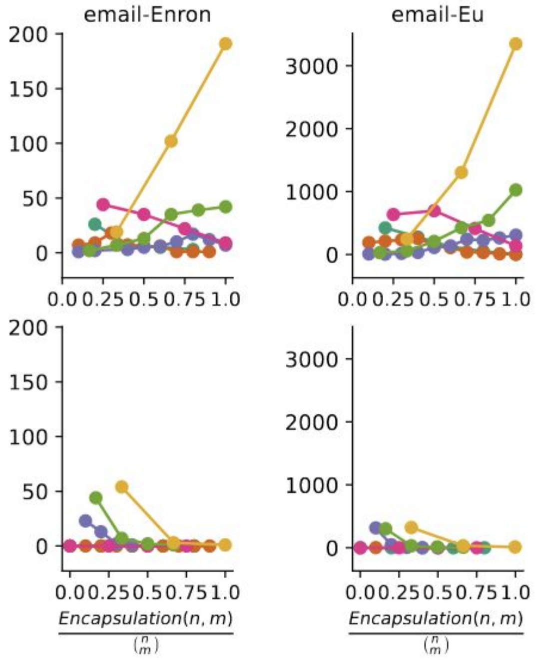
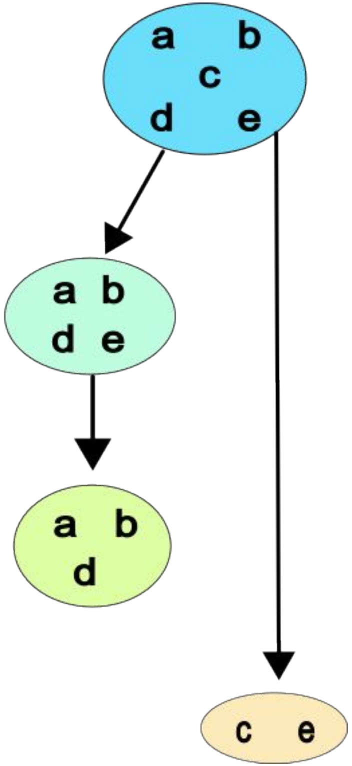


One histogram line for every point in the previous plot (up to n=5)

Concentration around 1 → full encapsulation/simplex assumption

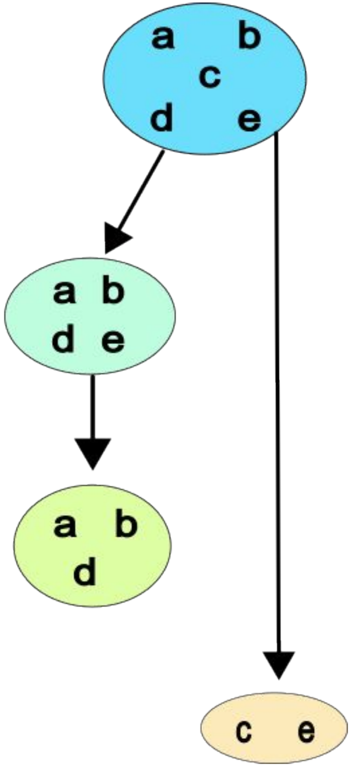


Paths through DAGs



Paths through DAGs

Rooted path length distribution in the **transitively reduced** DAG



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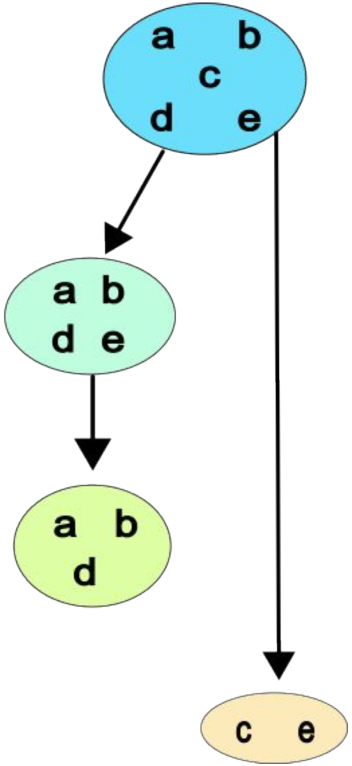
Roots → nodes (hyperedges) with no in-degree (facets)



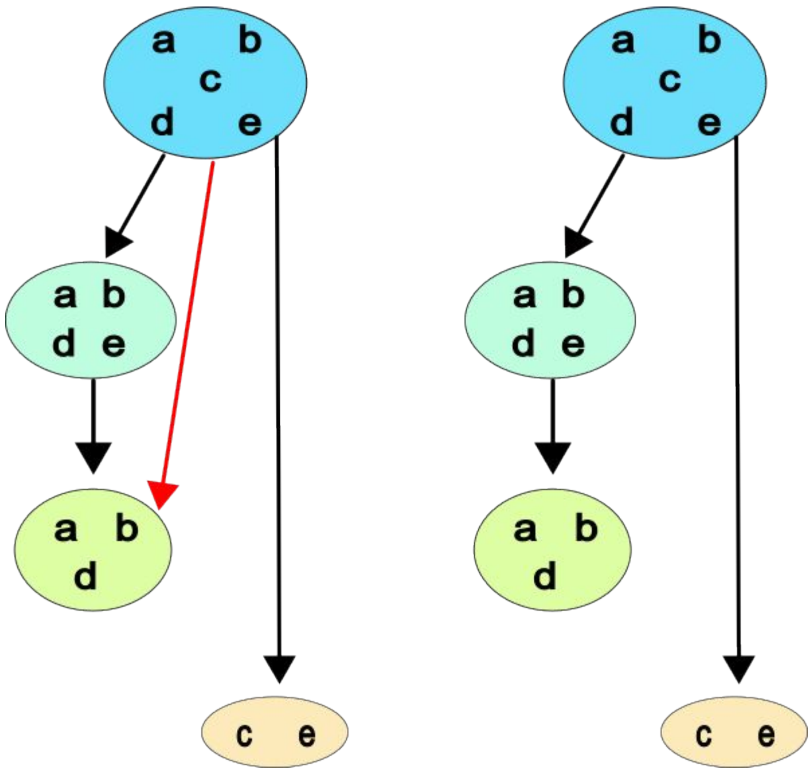
Paths through DAGs

Rooted path length distribution in the **transitively reduced** DAG

Roots → nodes (hyperedges) with no in-degree (facets)

Transitive reduction → remove “shortcut” edges (called a Hasse Diagram for simplicial complexes)

Transitive Reduction



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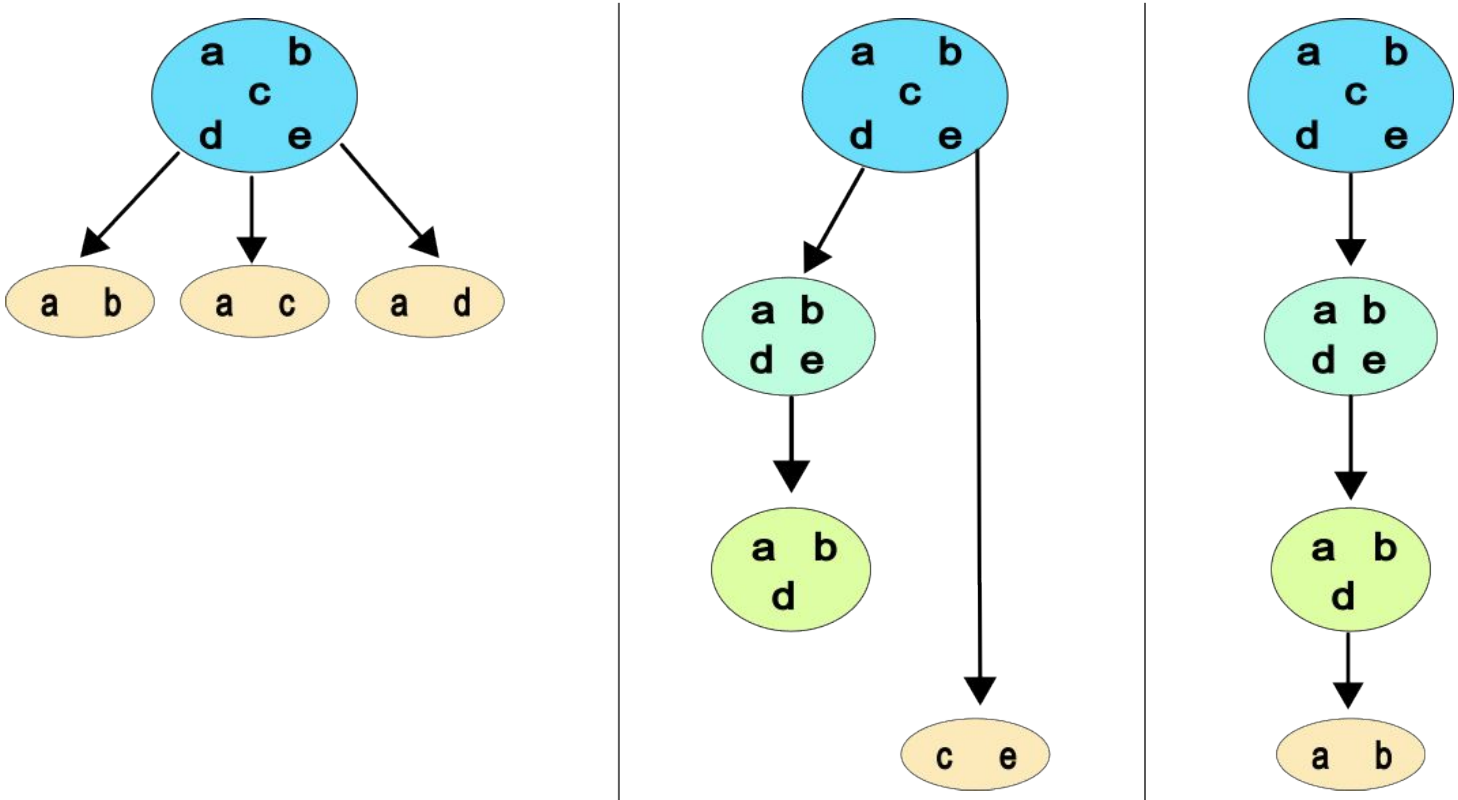
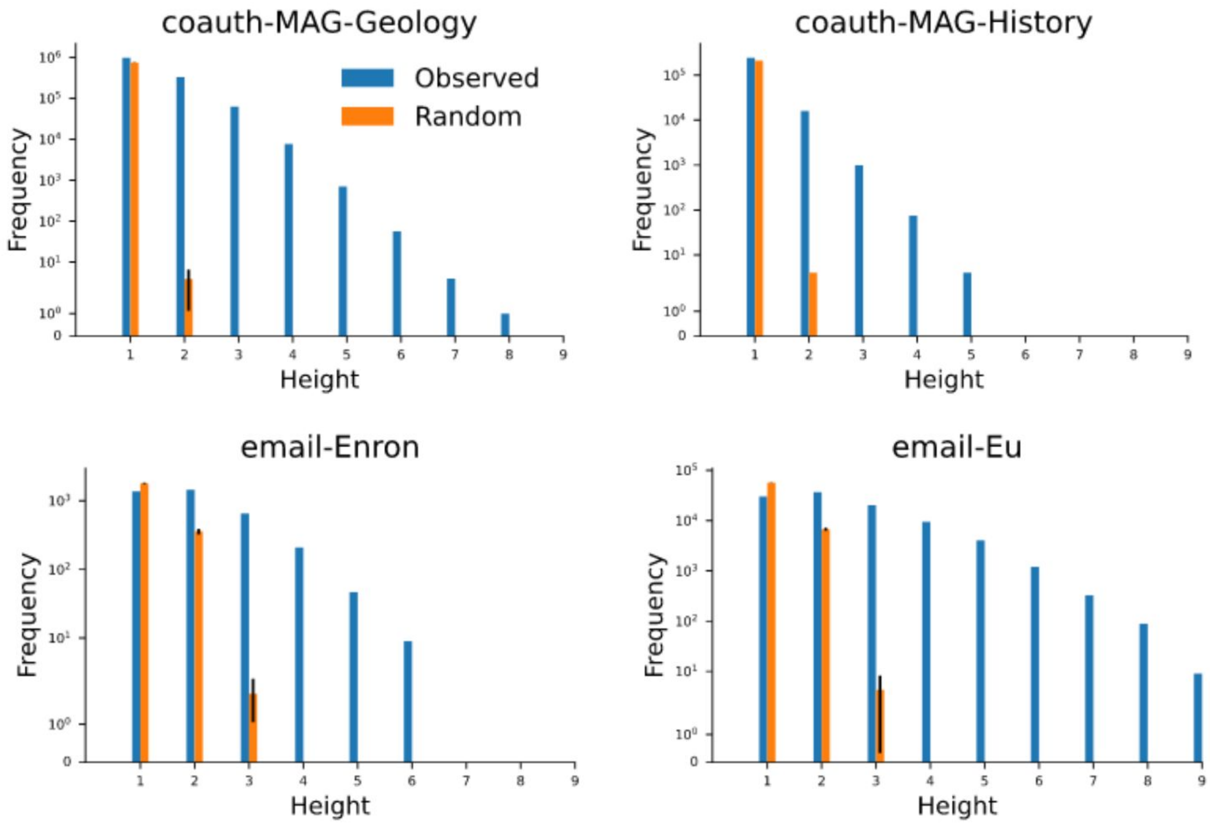


Shallow | Mixed | Deep

Paths through DAGs



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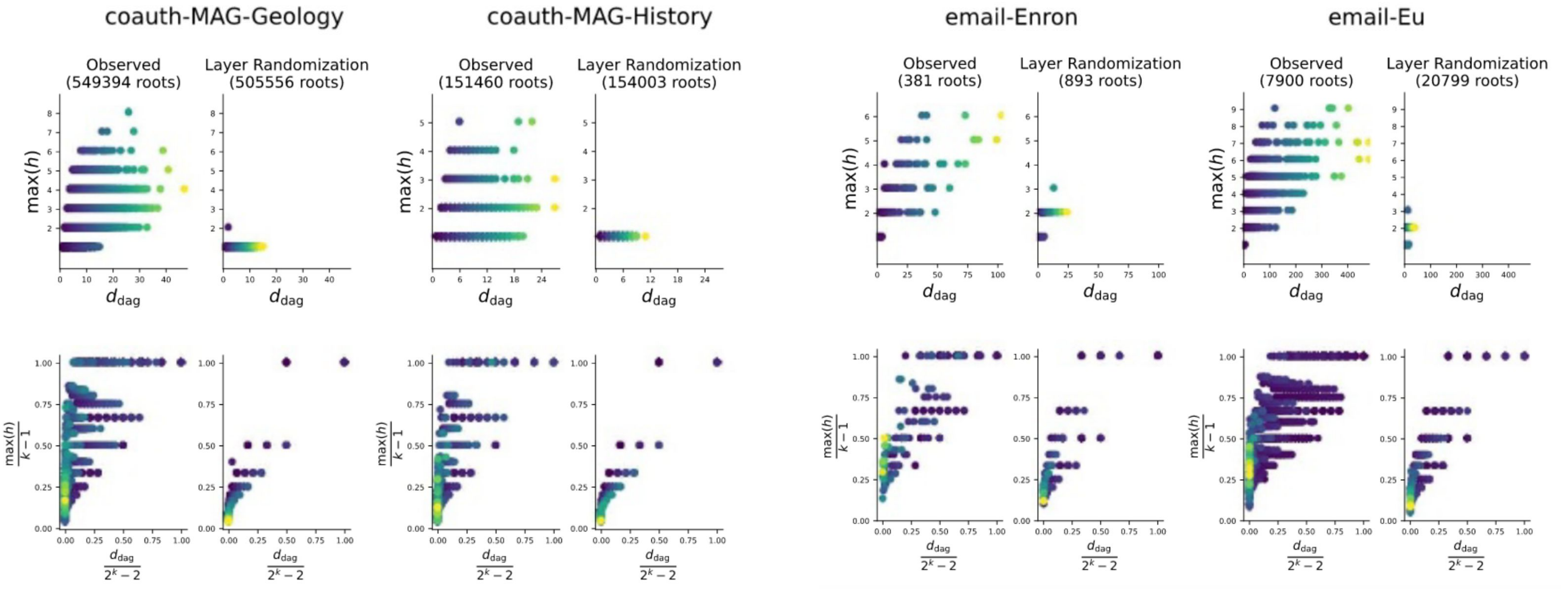

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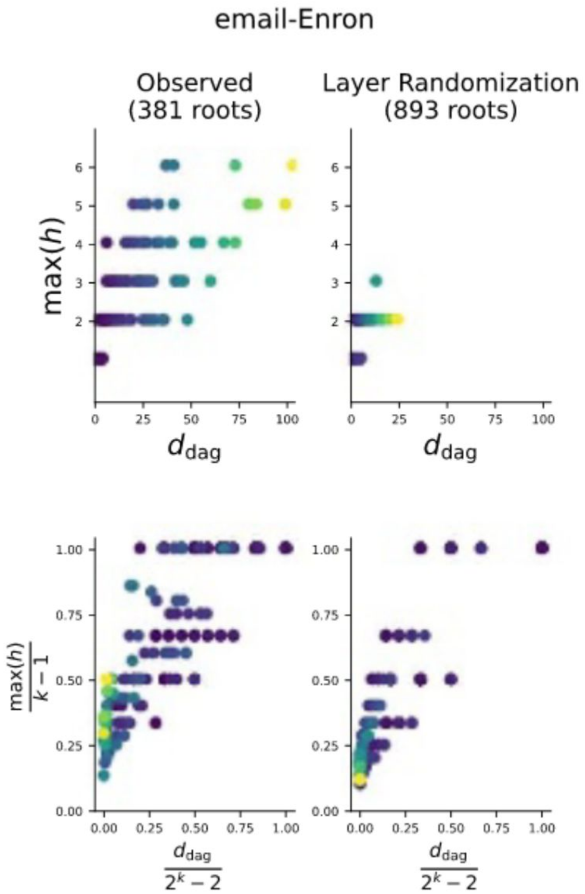

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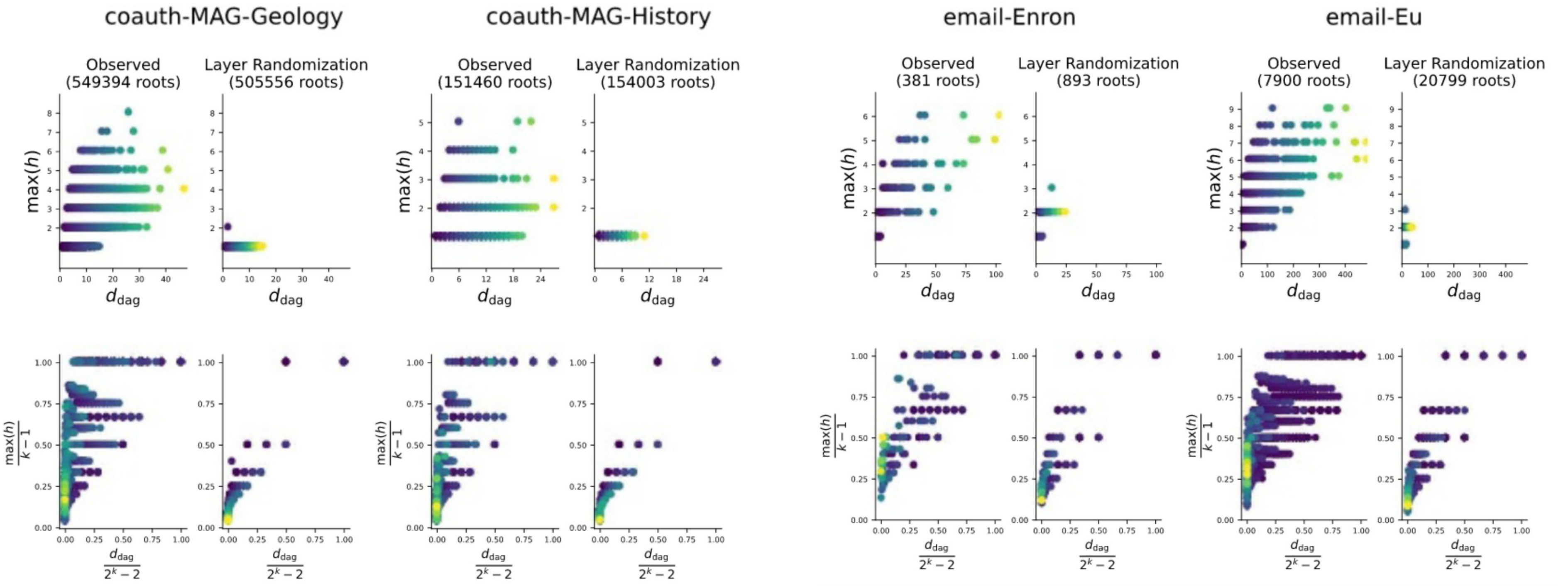

Real hypergraphs have encapsulation structure.

How can this structure affect dynamics?

Dynamics on hypergraphs

Node-based Threshold
Dynamics

A **node** becomes active if it participates in a hyperedge where more than a threshold of **nodes** become active.

Dynamics on hypergraphs

Node-based Threshold Dynamics

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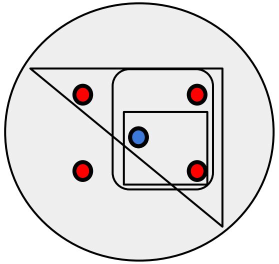

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Dynamics on hypergraphs

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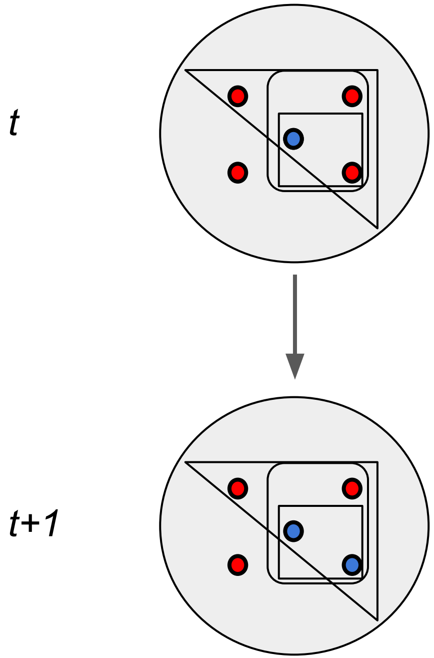

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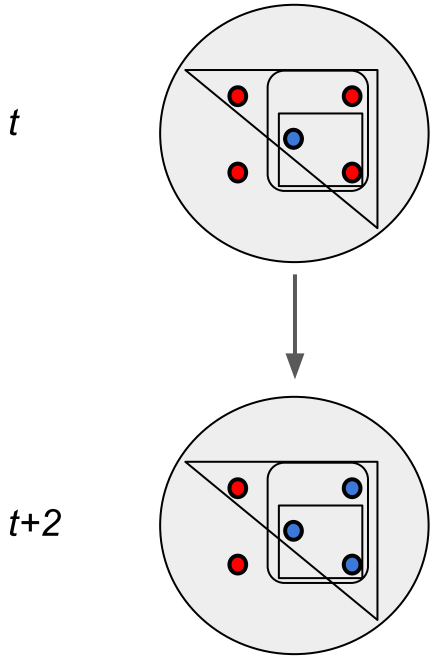
Dynamics on hypergraphs

Node-based Threshold Dynamics



A **node** becomes active if it participates in a hyperedge where more than a threshold of **nodes** become active.

Dynamics on hypergraphs

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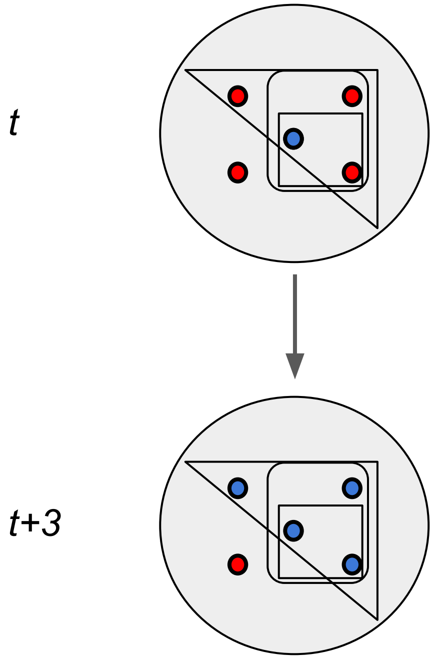

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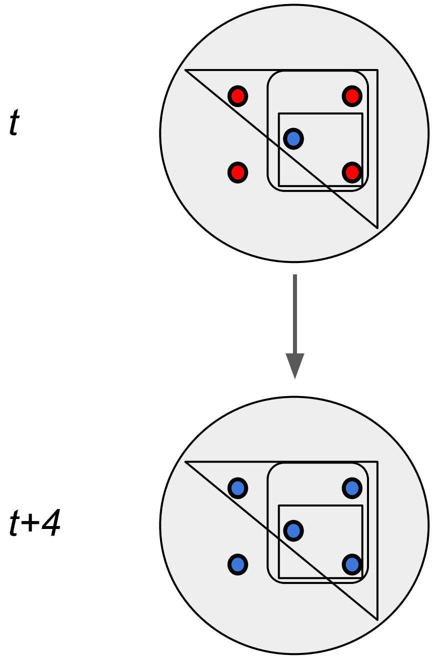

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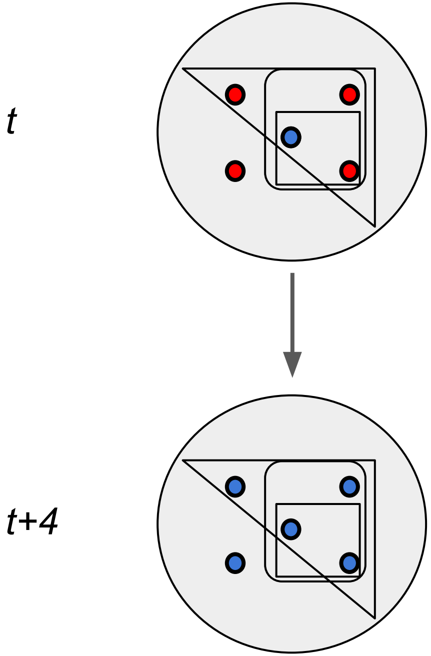

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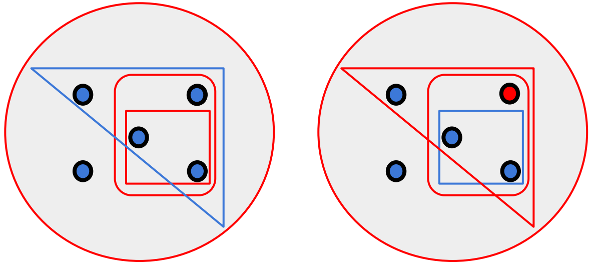
Encapsulation Dynamics

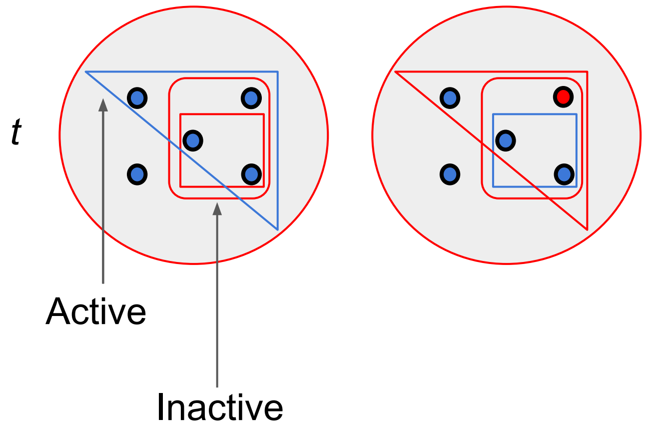


A **hyperedge** becomes active if more than a threshold of its **encapsulated hyperedges** becomes active.

Dynamics on hypergraphs

Nodes **and** edges in binary state, active or inactive

Encapsulation Dynamics



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Activation flows **upward from smallest to largest**

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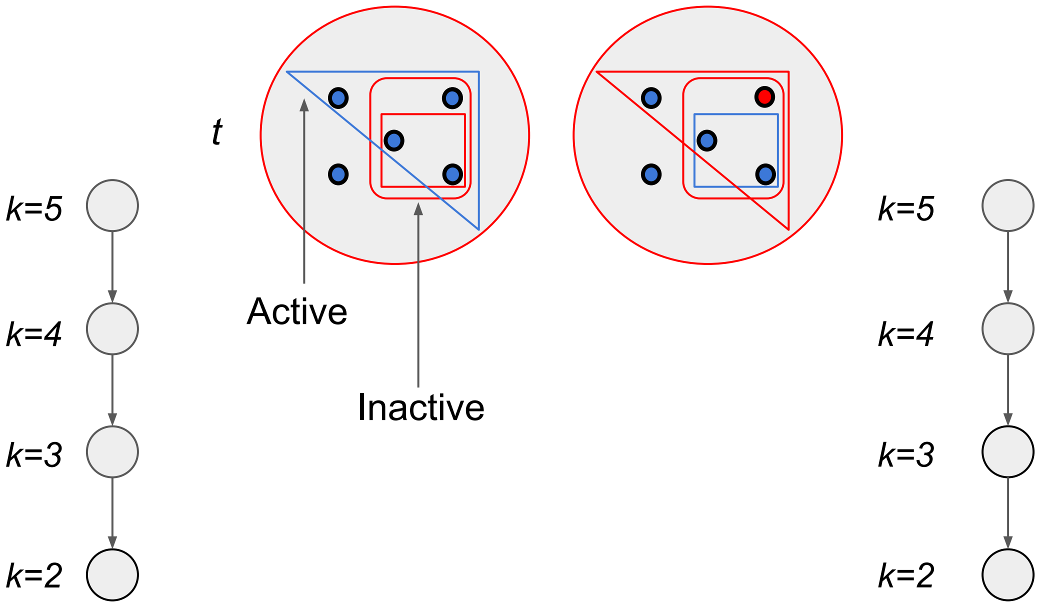

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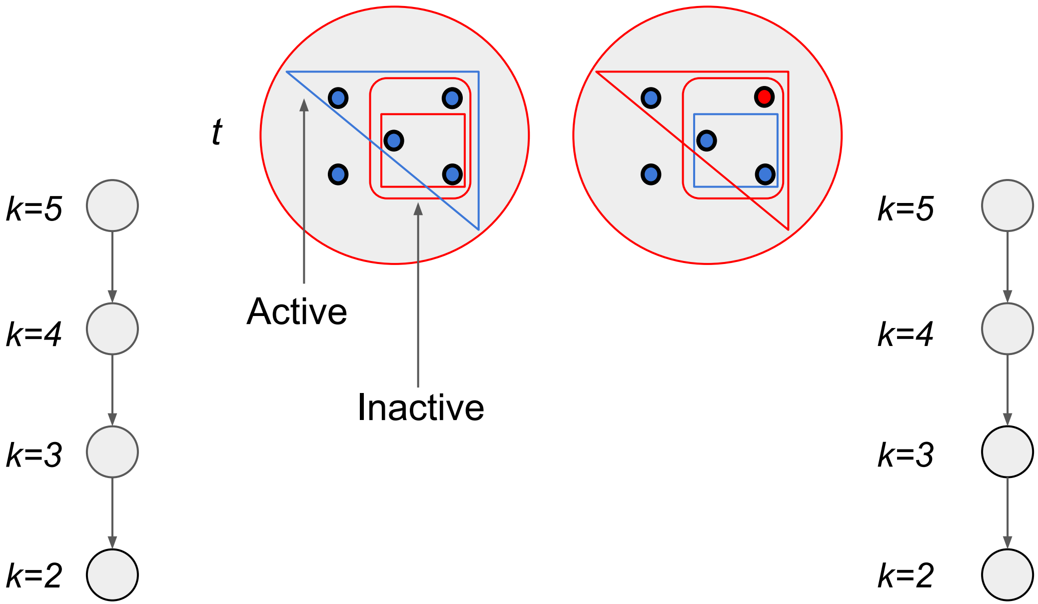

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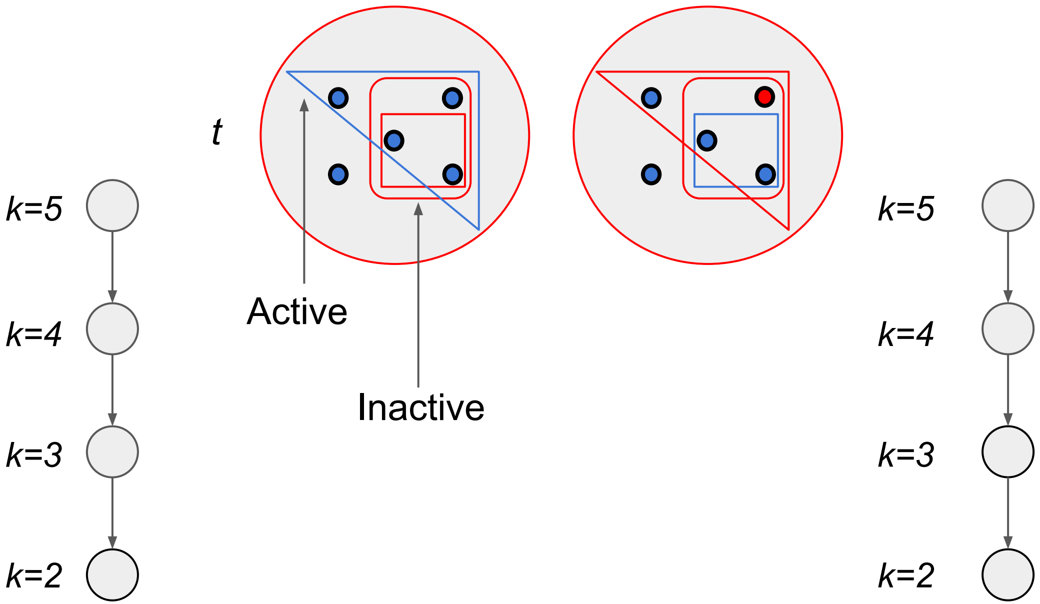

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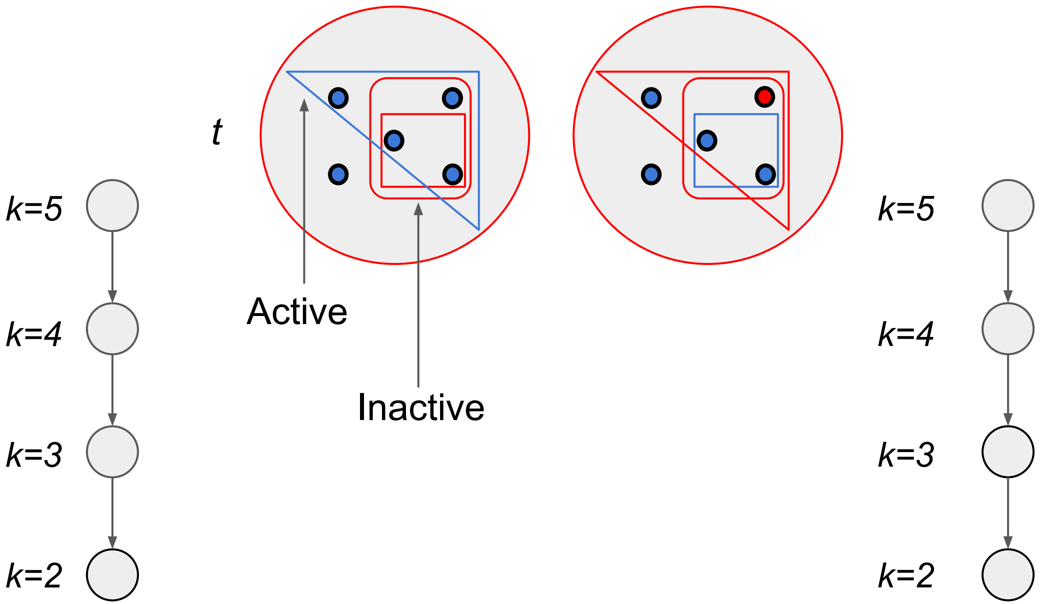
(Strict) Encapsulation Dynamics



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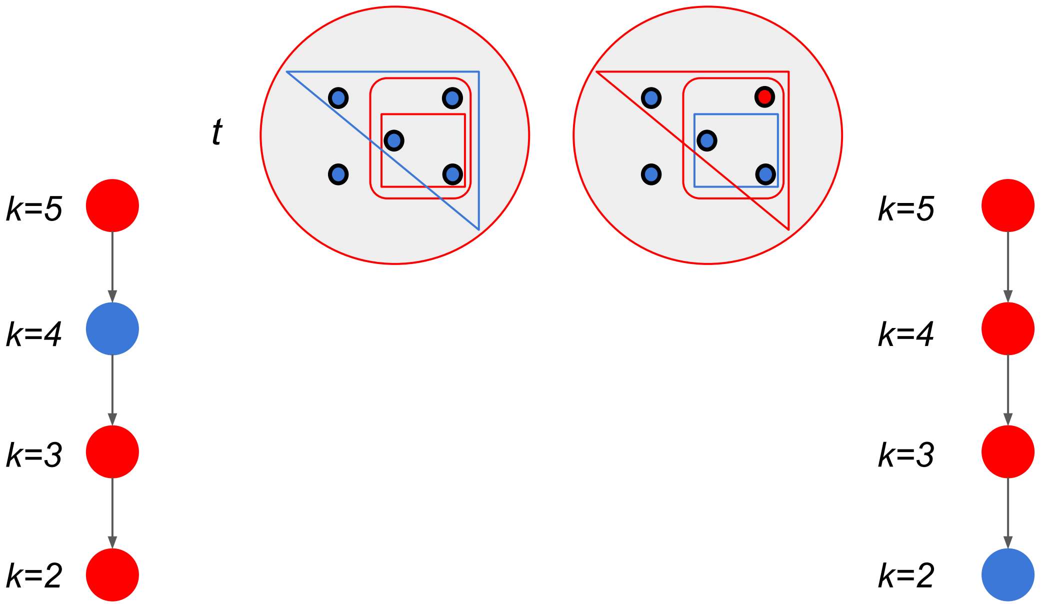

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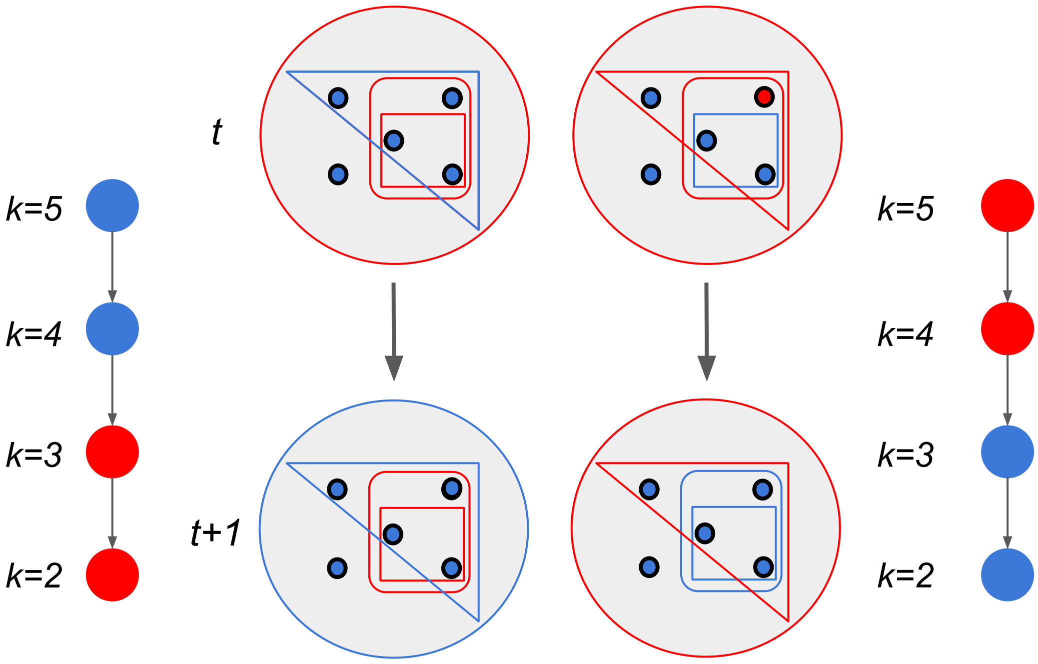

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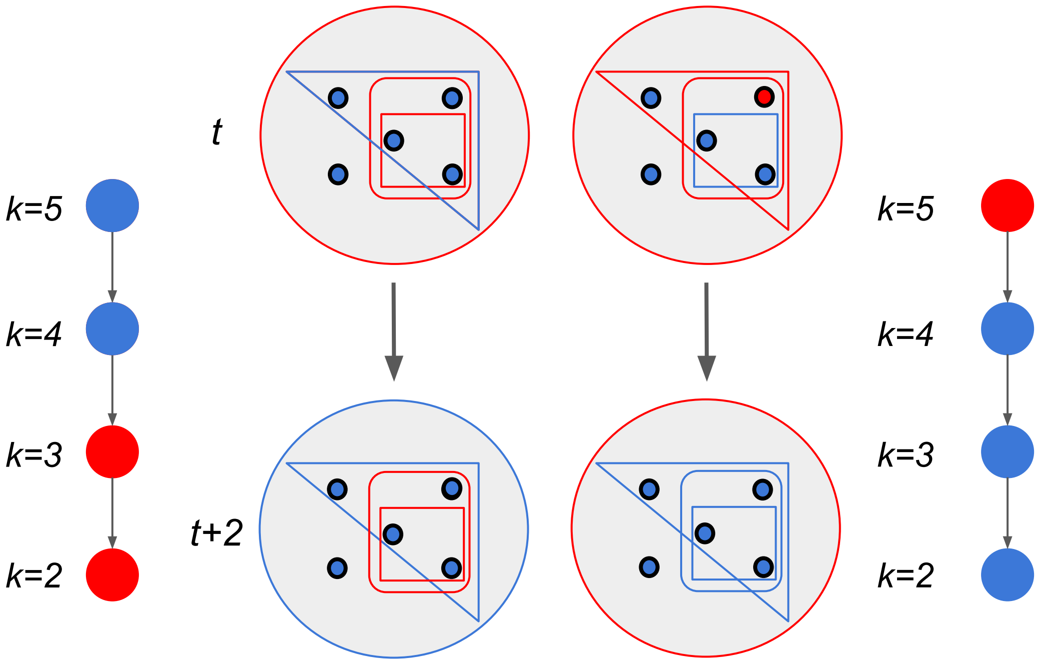

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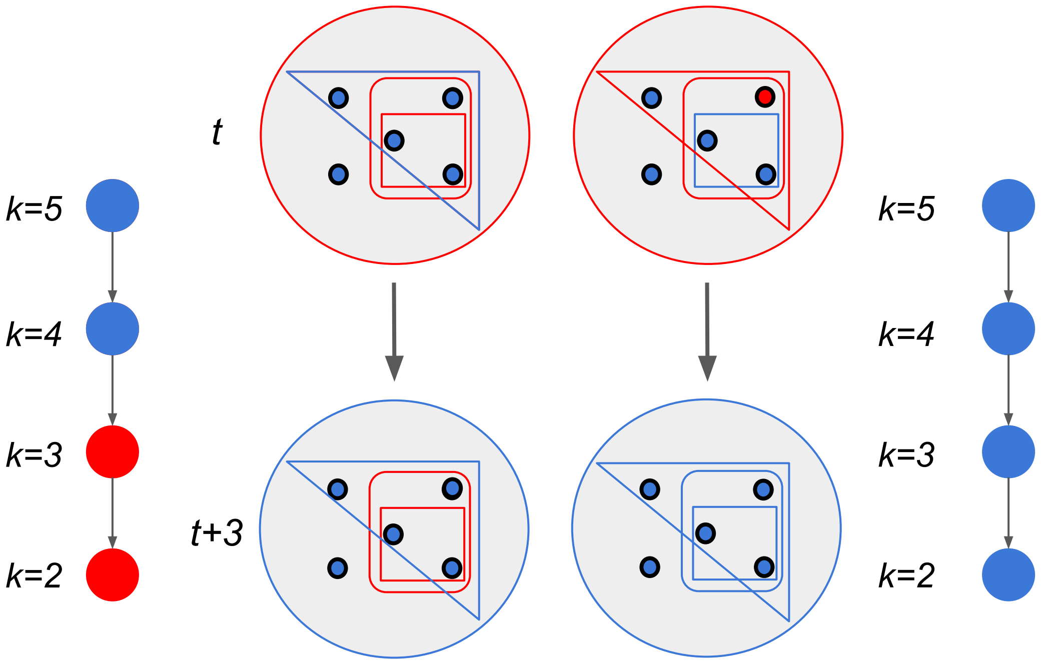


A **hyperedge** becomes active if more than a threshold of its **encapsulated hyperedges** becomes active.

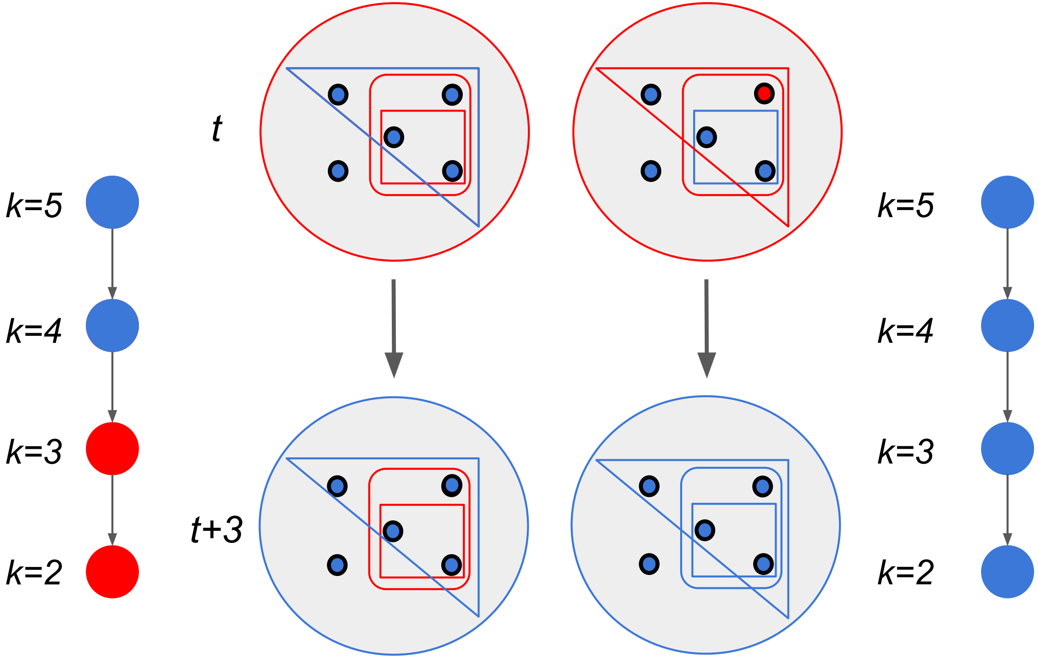
(Campfire) Dynamics on hypergraphs

Analogy to lighting a campfire: the smallest fuel must be lit before the logs can catch on fire!

Image: https://www.pelican.com/us/en/discover/pelican-flyer/post/how-to-start-a-campfire/

Correspond to a type of **coordinated behavior** where nodes not only share goals/opinion/information, but coordinate to pass to other groups they are embedded within.

(Strict) Encapsulation Dynamics



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Random Nested Hypergraph Model

Contagion dynamics on hypergraphs with nested hyperedges
Jihye Kim, Deok-Sun Lee, and K.-I. Goh
Phys. Rev. E **108**, 034313 – Published 28 September 2023

Idea: Start from a fully encapsulated hypergraph (simplicial complex), then selectively rewire hyperedges to destroy encapsulation relationships

Parameters:

N: Number of nodes

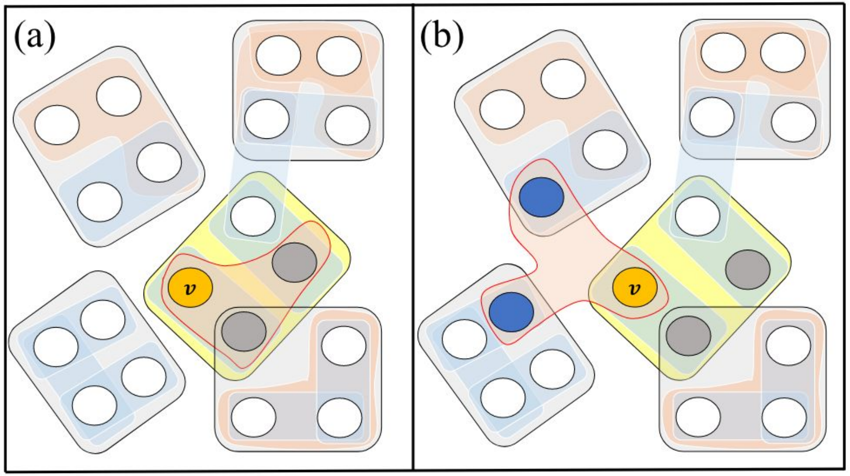
s_m: Maximum size hyperedge

H_s: Number of hyperedges of size s_m

ϵ_s: 1 minus probability of rewiring hyperedge of size s

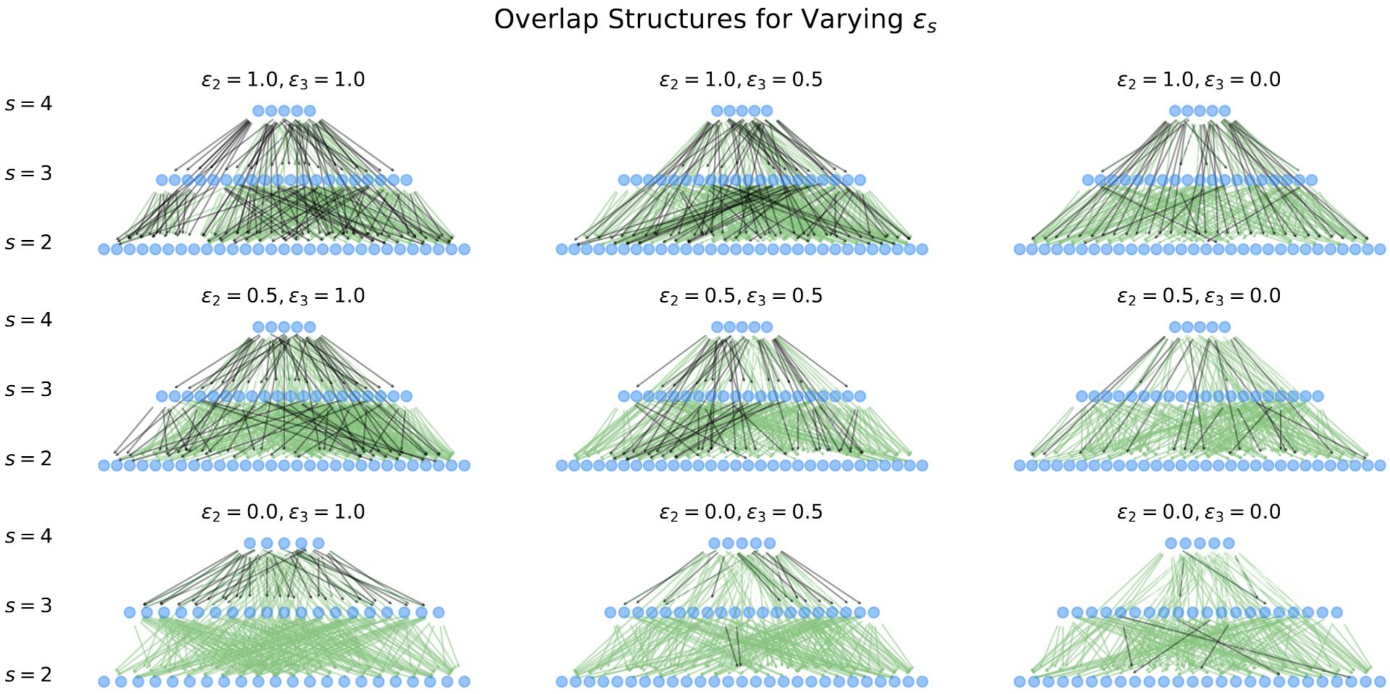
Procedure:

1. Generate random hyperedges of size s_m and all of their subhyperedges (power set)
2. For each hyperedge of size $s < s_m$, rewire with probability ϵ_s



Rewiring works by choosing a pivot node to keep, then randomizing other nodes by choosing nodes that are not in supersets of the original hyperedge.

Random Nested Hypergraph Model



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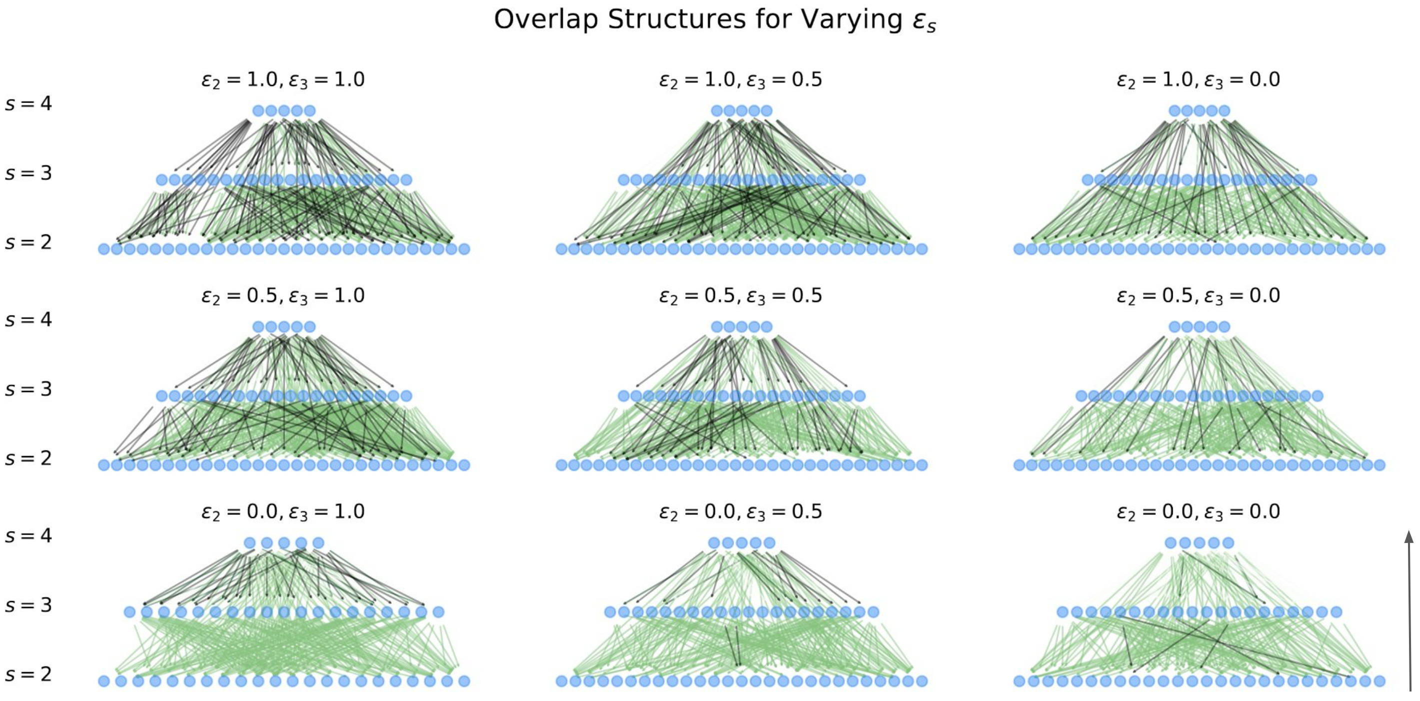
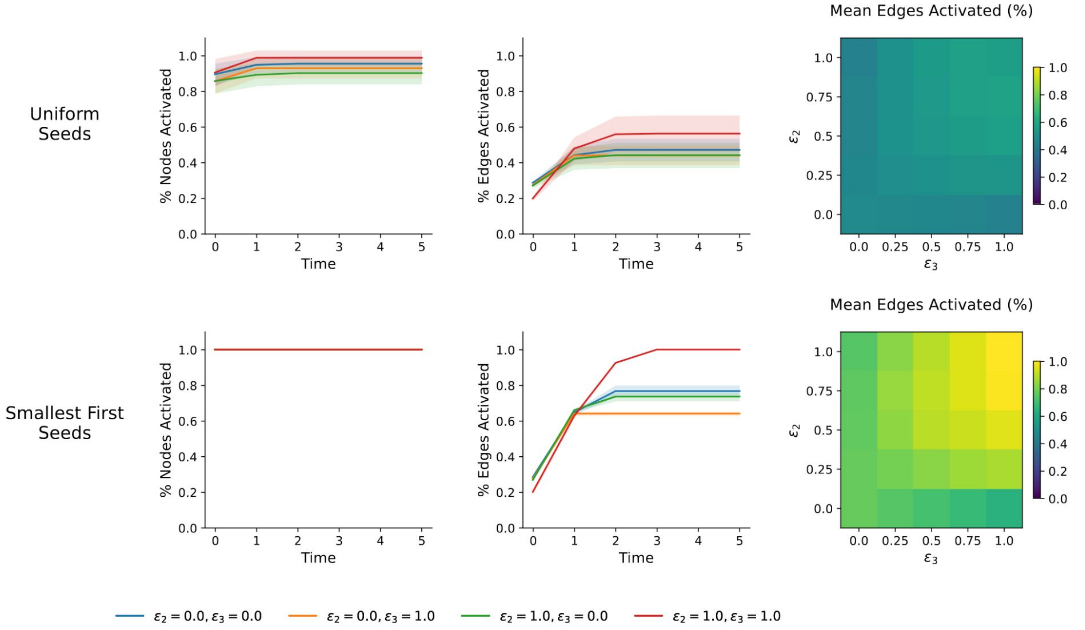


In strict encapsulation dynamics, **activation can only spread up the black edges!**

Simulation Results

Average results over 50 strict encapsulation dynamics simulations on 50 RNHM realizations



RNHM with N = 20, s_m = 4, H_s = 5, and varying ϵ_s, including individual nodes as hyperedges.

Number of seeds = N (number of nodes)

Simulation Results

Uniform seeding:

- Even with high proportion of nodes activated, only 50% of edges

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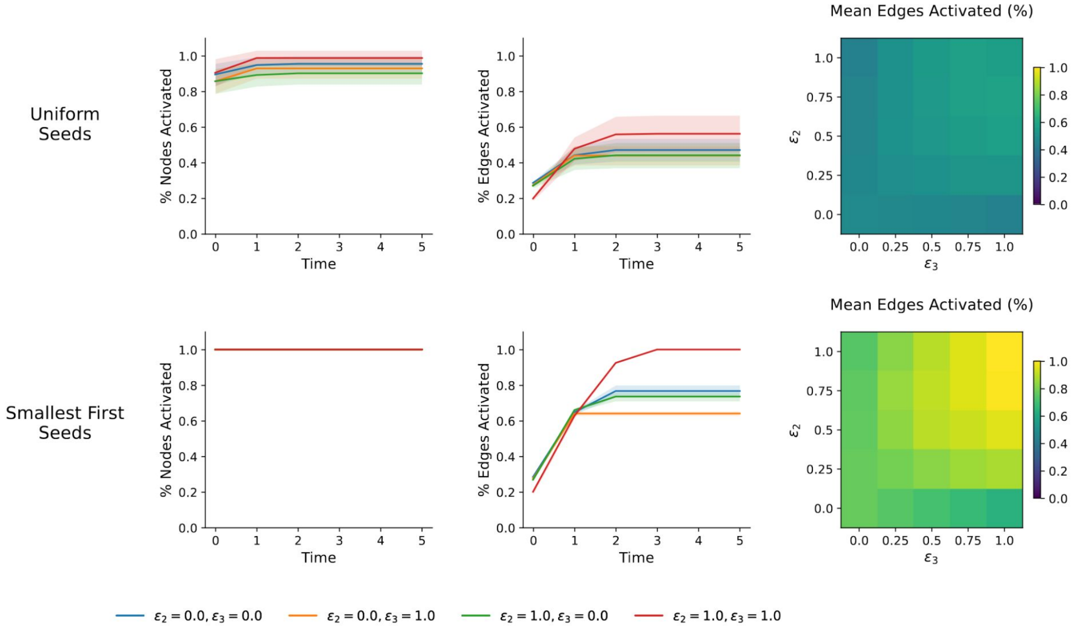

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- Even though nodes are activated by definition, all hyperedges do not become active. Key distinguishing feature from node-based threshold dynamics.

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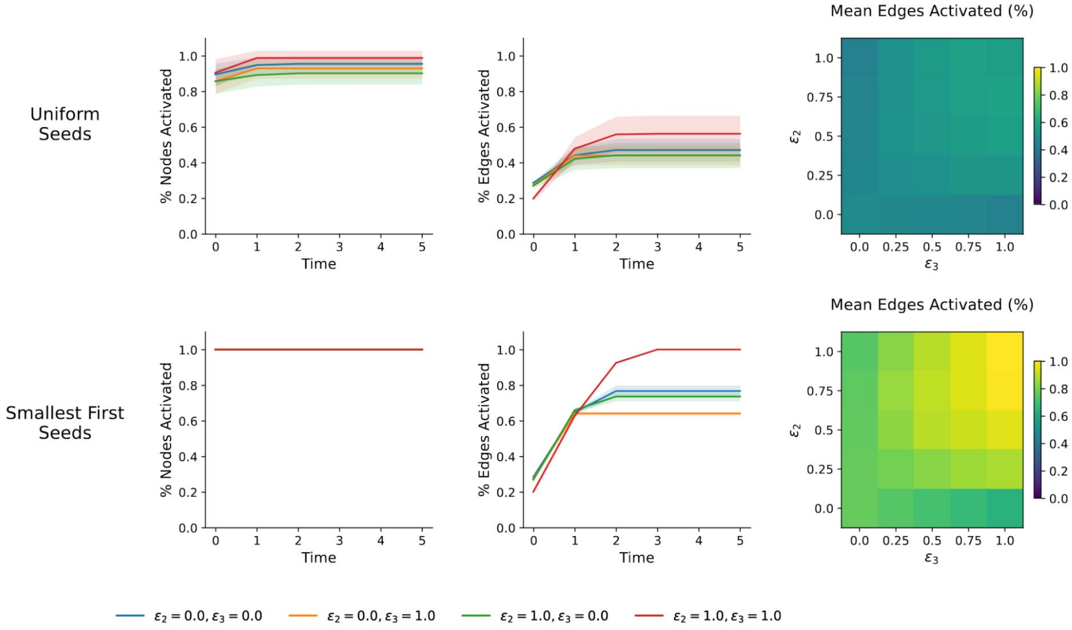

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These dynamics correspond not just to node influence, but to **coordinated behavior**!

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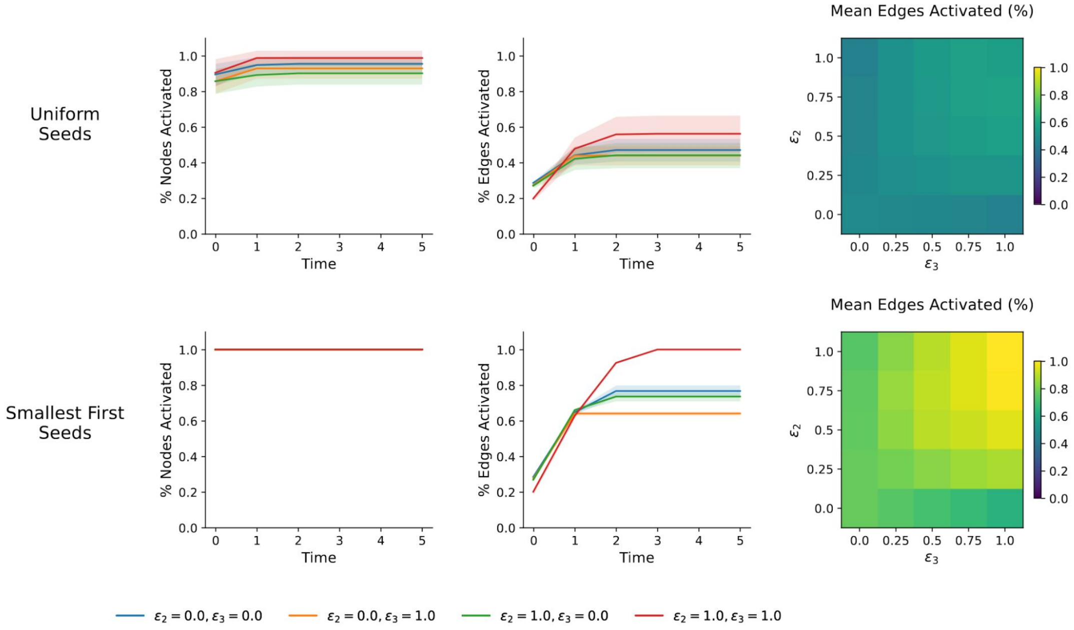

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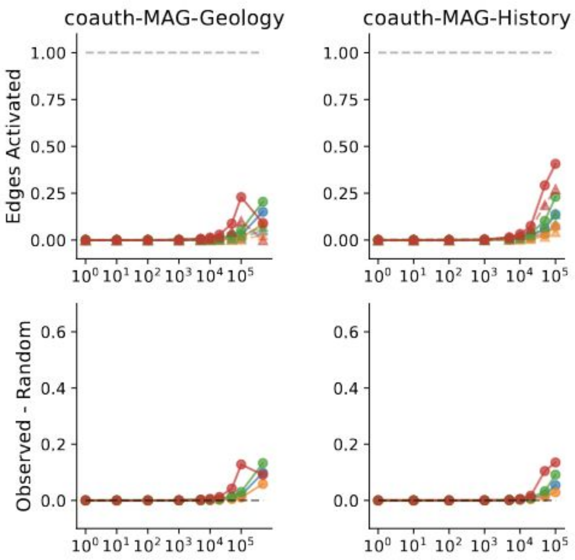
Simulation Results

- More encapsulation → more spread
- Smallest first seeding → more and faster spread

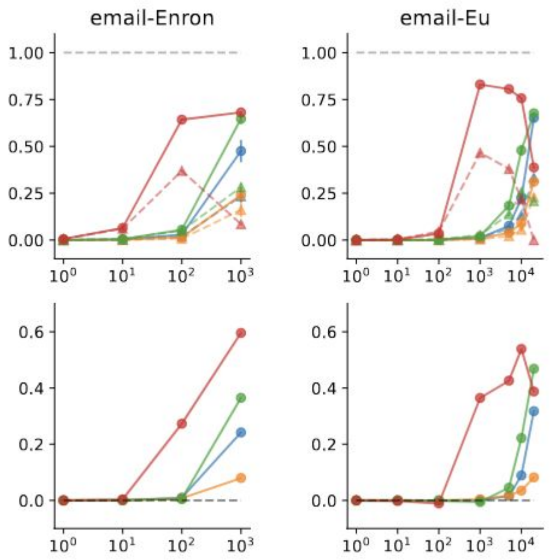
Empirical Simulation Results

Strict Encapsulation Dynamics, 25 steps, $\tau = \text{all}$



Large number of seeds required for spread in real data

Smallest first and inverse size seeding most effective



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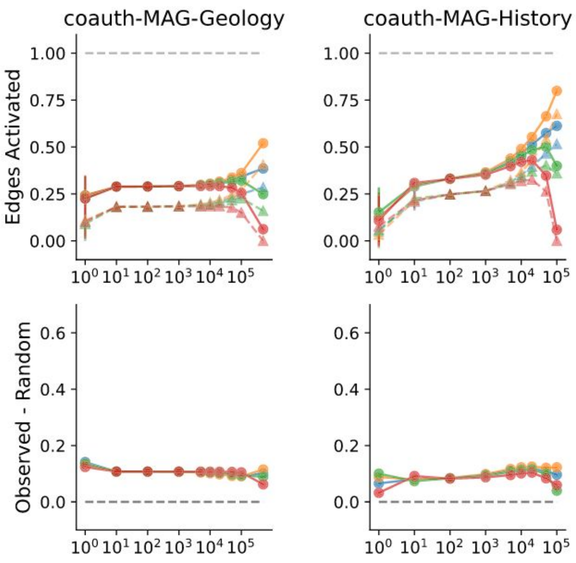
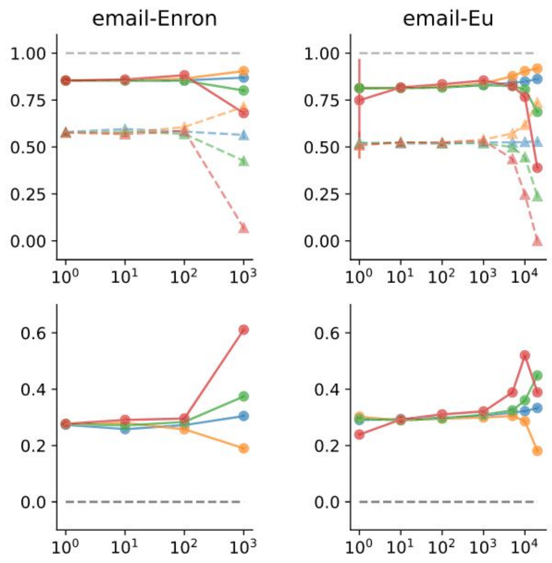


Non-strict dynamics: Individual node states influence 2-node hyperedges



Thank you!

Contact
larock@maths.ox.ac.uk
https://www.tlarock.github.io

Code: @tlarock on GitHub

https://www.github.com/tlarock/encapsulation-dynamics

PAPER • OPEN ACCESS

Encapsulation structure and dynamics in hypergraphs

Timothy LaRock[3,1] and Renaud Lambiotte[1,2]

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