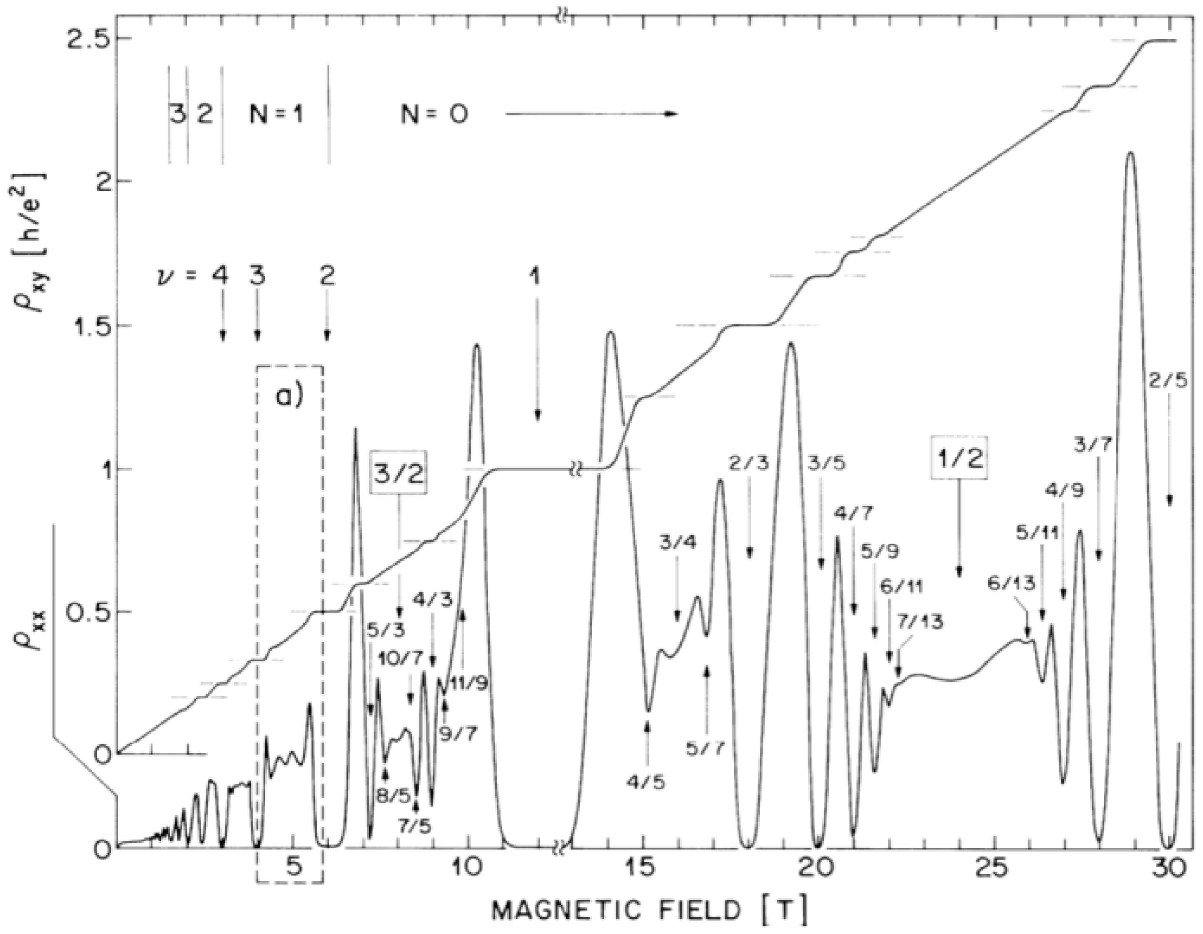
Splitting the Indivisible 2.0

Strongly correlated topological phases

Fractional Quantized Hall Effect

Existence proof for topological phases beyond band theory



Characterizing Many Body Topological Order in the Fractional Quantum Hall Effect

Many-Particle Wavefunction

- Laughlin wavefunction ($\nu = 1/m$)

$$\Psi_{\text{Laughlin}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

- Moore Read wavefunction

Wavefunctions in lowest Landau level are correlators in a **2D conformal field theory**

Effective Field Theory

- Bulk: 2+1D Topological Field Theory

Girvin, MacDonald '87
Zhang, Hansson,Kivelson '89
Read '89, ...

Chern Simons Theory

$$L = \sum_{IJ} \frac{K^{IJ}}{4\pi} \varepsilon_{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J$$

- Boundary: 1+1D Conformal Field Theory

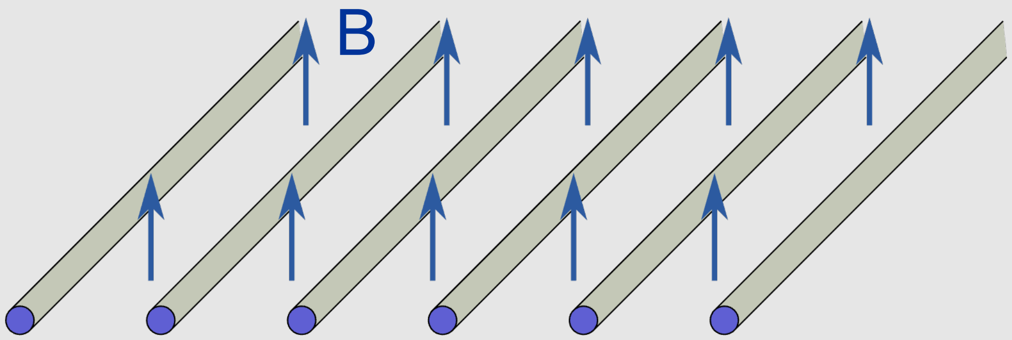
Wen '91

Chiral Luttinger liquid

$$L = \sum_{IJ} \frac{1}{4\pi} K^{IJ} \partial_x \phi^I \left(\partial_t \phi^J + v \partial_x \phi^J \right)$$

Coupled wire construction

- A bridge between microscopic electronic models and effective field theory.
- Explicit connection between electrons and field theory via 1D bosonization

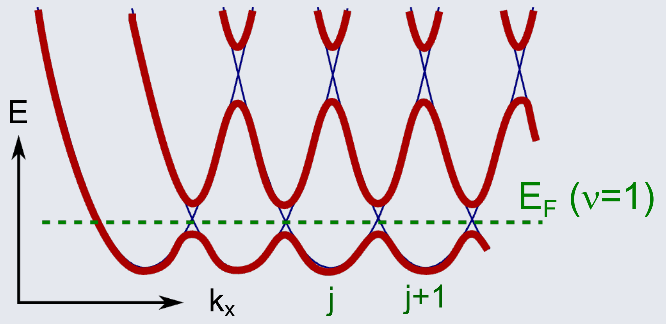


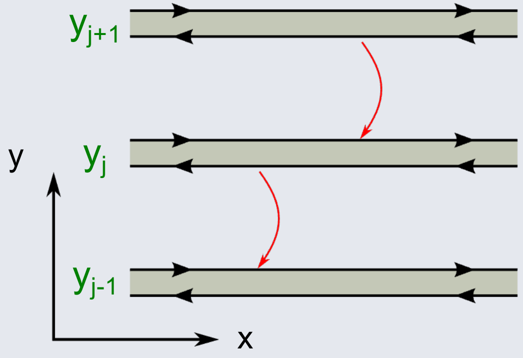
Coupled Wire Model

Integer Quantized Hall Effect

Sondhi, Yang 2001

$$E_j(k_x) = \frac{1}{2m}\left(\hbar k_x - eA_x(y_j)\right)^2 \qquad A_x(y) = By$$





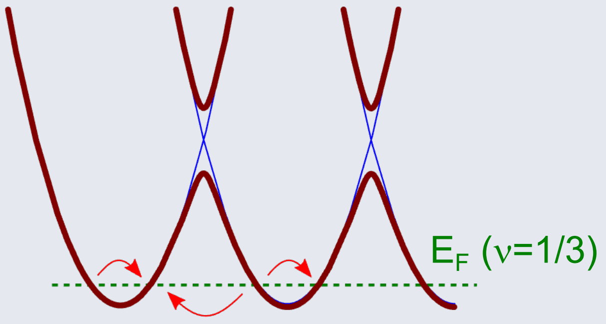
Low energy Hamiltonian

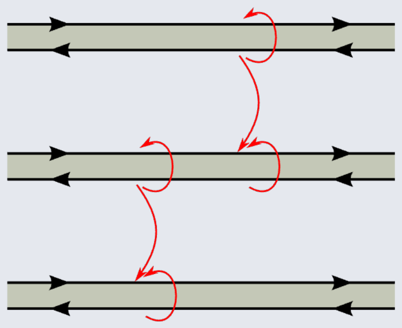
$$H_0 = -i v_F \sum_j (\psi^\dagger_{j,R}\partial_x\psi_{j,R} - \psi^\dagger_{j,L}\partial_x\psi_{j,L})$$

$$H = H_0 - t\sum_j\left(\psi^\dagger_{j,R}\psi_{j+1,L} + h.c.\right)$$

Laughlin State: ν = 1/m

Kane, Mukhopodhyay, Lubensky 2002

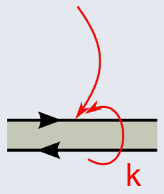




For filling 1/m (m odd) there is a momentum conserving m particle tunneling process

Composite electron operator

$$\Psi^\dagger_{j,R} = \psi^\dagger_{j,R}\left(\psi^\dagger_{j,R}\psi_{j,L}\right)^k$$



Charge e; momentum mk_F (m=1+2k)

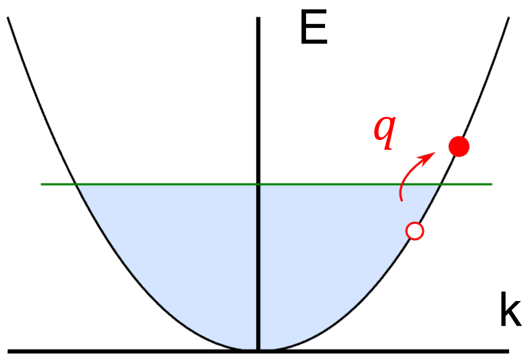
$$H = H_0 + t\sum_j\left(\Psi^\dagger_{j,R}\Psi_{j+1,L} + h.c.\right)$$

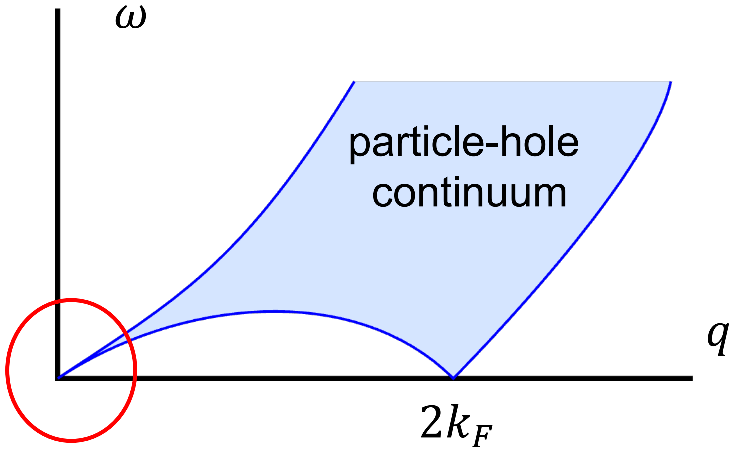
Digression: a primer on Bosonization and Luttinger Liquid

Classic reference: FDM Haldane, J. Phys. C 14, 2585 (1981).

Also: monograph by T Giamarchi, 'Quantum physics in one dimension'

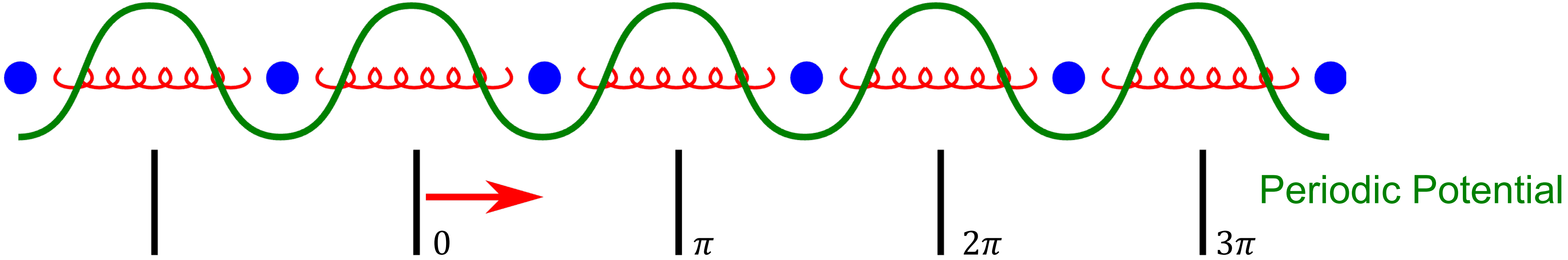
Intuitive picture : particle-hole spectrum of free Fermi gas





$\omega \sim v_F q$: resembles acoustic phonon

Strong Interaction Limit : Wigner Crystal



$\theta(x) = \pi\, u(x)\, / a$: phonon displacement

Lagrangian density for acoustic phonons

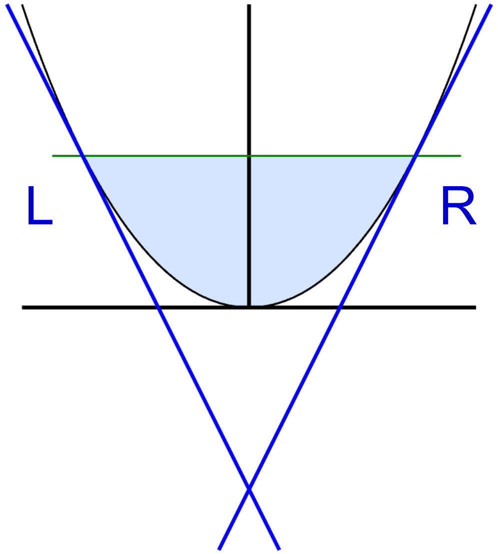
$$L = KE - PE = \frac{1}{2\pi K}\left[\frac{1}{\mathrm{v}}\left(\frac{\partial\theta}{\partial t}\right)^2 - \mathrm{v}\left(\frac{\partial\theta}{\partial x}\right)^2 + U\cos 2\theta\right]$$

Zero point fluctuations of phonon modes destroy long range crystal order : power law order at T=0.

Wigner crystal pinned by commensurate periodic potential

Bosonization of non-interacting Fermi gas

Luttinger Model : Chiral Dirac fermions



$$E_k = \pm\, v_F k$$

$$H = -i\, v_F (\,\psi_R^{+}\partial_x\psi_R \;-\; \psi_L^{+}\partial_x\psi_L\,) \;= H_R + H_L$$

Focus on right moving sector. Chiral density operator :

$$\rho_R(x) = \,:\psi_R^{+}(x)\psi_R(x):\; \equiv\; \psi_R^{+}(x)\psi_R(x) \;-\langle\psi_R^{+}(x)\psi_R(x)\rangle_0$$

$$\rho_R(q) = \sum_k :c_R^{+}(k+q)c_R(k):$$

Kac Moody commutation algebra

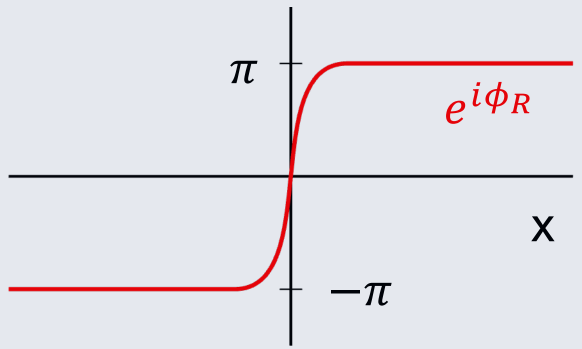
$$[\,\rho_R(q')\,,\rho_R(q)] \;=\; \frac{qL}{2\pi}\delta_{q+q'}$$

$$[\rho_R(-q),\rho_R(q)]|0\rangle = \int_{-q}^{0}\frac{Ldq}{2\pi}|0\rangle = \frac{qL}{2\pi}|0\rangle$$

Chiral boson operator : $b_{R,q}^{+} = \sqrt{\frac{2\pi}{qL}}\,\rho_R(q) \quad\rightarrow\quad \left[b_{R,,q}, b_{R,q}^{+}\right] = 1$

Exact mapping between free fermions and free bosons

Chiral Phase Field $\rho_R(x) = \dfrac{\partial_x \phi_R}{2\pi}$ $[\phi_R(x), \phi_R(x')] = i\,\pi\,\mathrm{sgn}(x - x')$



$e^{i\phi_R}$ introduces 2π kink in ϕ_R

Fermion creation operator : $\psi_R^+(x) \sim e^{i\phi_R}$

Chiral Hamiltonian : $H_R = -i\, v_F\, \psi_R^+ \partial_x \psi_R = \sum_q v_F q\, b_q^+ b_q = \dfrac{v_F}{4\pi} (\partial_x \phi_R)^2$

Chiral Lagrangian : $L_R = p\,\dot{q} - H = \dfrac{1}{4\pi} \partial_x \phi_R \,(\, \partial_t \phi_R - v_F\, \partial_x \phi_R \,)$

topological term : encodes Fermi statistics of $e^{i\phi_R}$

Combine right and left movers

$$L = L_R + L_L = \frac{1}{4\pi}[\partial_x \phi_R \, (\partial_t \phi_R - v_F \, \partial_x \phi_R \,) + \partial_x \phi_L \, (-\partial_t \phi_L - v_F \, \partial_x \phi_L \,)]$$

New variables : $\phi_R = \varphi + \theta \;\; ; \;\; \phi_L = \varphi - \theta$ $\qquad [\, \partial_x \theta(x'), \varphi(x) \,] = \pi \delta(x - x')$

$$L \; = \; \frac{1}{\pi} \partial_x \theta \partial_t \varphi \; - \frac{v_F}{2\pi} \, (\, (\partial_x \theta)^2 + (\partial_x \varphi)^2 \,) \;\; \Rightarrow \;\; \frac{1}{2\pi} \left(\frac{1}{v_F} (\partial_t \theta)^2 \; - v_F \, (\partial_x \theta)^2 \right)$$

Charge density $\qquad \rho =: \psi_R^+ \, \psi_R + \psi_L^+ \psi_L : \, = \rho_R + \rho_L = \frac{1}{\pi} \partial_x \theta$

Electron operators : $\qquad \psi_R^+ \sim e^{i(\varphi + \theta)} \;\; ; \;\; \psi_L^+ \sim e^{i(\varphi - \theta)}$

Periodic Potential : $\qquad \Delta L \; = \; V(2k_F) \, \psi_R^+ \, \psi_L + h.c. \;\; \sim \;\; V \cos 2\theta$

Interacting 1D Fermi gas : Luttinger Liquid

Forward scattering interactions : $H_{int} = \frac{1}{2} V_0 \int dx \, (\psi_R^+ \psi_R + \psi_L^+ \psi_L)^2 = \frac{V_0}{2\pi^2} (\partial_x \theta)^2$

$$L = \frac{1}{2\pi} \left(\frac{1}{v_F} (\partial_t \theta)^2 - v_F (\partial_x \theta)^2 \right) - \frac{V_0}{2\pi^2} (\partial_x \theta)^2$$

$$= \frac{1}{2\pi K} \left(\frac{1}{v} (\partial_t \theta)^2 - v (\partial_x \theta)^2 \right)$$

$$v = \frac{v_F}{K} \qquad K = \frac{1}{\sqrt{1 + \frac{V_0}{\pi v_F}}}$$

Luttinger parameter K

K < 1 : repulsive interactions
K = 1 : non-interacting
K > 1 : attractive interactions

Properties of a Luttinger Liquid

1. Almost a crystal

Density correlations at $q = 2k_F$: $\langle \psi_R^+ \psi_L(x)\, \psi_L^+ \psi_R(0) \rangle \sim \left\langle e^{2i\left(\theta(x)-\theta(0)\right)} \right\rangle \sim \dfrac{1}{x^{2K}}$

2. Almost a superfluid

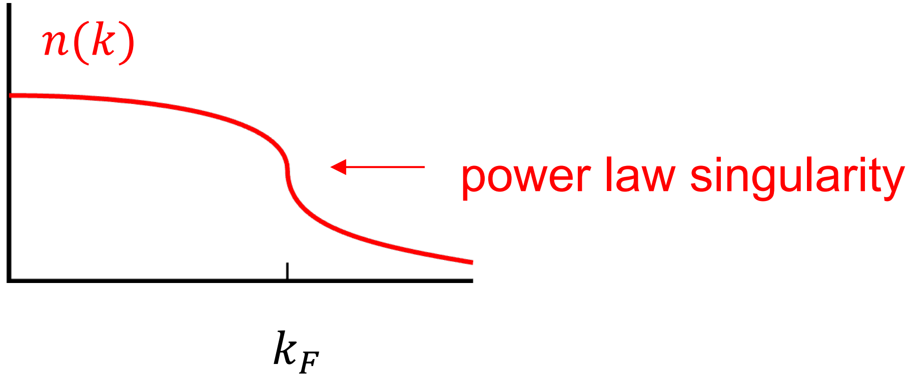
Pair correlations at $q = 2k_F$: $\langle \psi_R^+ \psi_L^+(x)\, \psi_L \psi_R(0) \rangle \sim \left\langle e^{2i\left(\varphi(x)-\varphi(0)\right)} \right\rangle \sim \dfrac{1}{x^{2/K}}$

3. Single particle correlations

Reduced density matrix : $G(x) = \langle \psi_R^+(x) \psi_R(0) \rangle \sim \left\langle e^{i(\varphi(x)+\theta(x)-\varphi(0)-\theta(0))} \right\rangle \sim \dfrac{1}{x^{\frac{1}{2}\left(K+\frac{1}{K}\right)}}$

4. Momentum distribution

$n(k) = \langle c_k^+ c_k \rangle = \int dk\, e^{ikx} G(x) \sim k^{\frac{1}{2}\left(K+\frac{1}{K}\right)-1}$

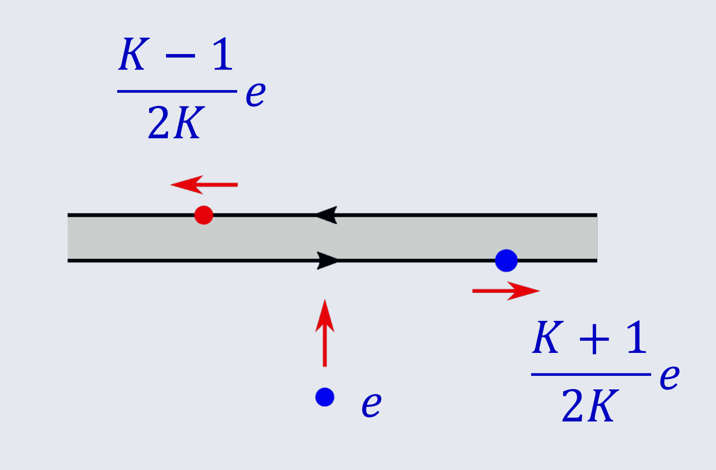


Factorization of a Luttinger Liquid

$$L = \frac{1}{2\pi K}\left(\frac{1}{v}(\partial_t\theta)^2 - v\,(\partial_x\theta)^2\right) = L_R + L_L$$

$$L_R = \frac{1}{4\pi K}\partial_x\tilde{\phi}_R\left(\partial_t\tilde{\phi}_R - v_F\,\partial_x\tilde{\phi}_R\right) \qquad L_L = \frac{1}{4\pi K}\partial_x\tilde{\phi}_L\left(-\partial_t\tilde{\phi}_L - v_F\,\partial_x\tilde{\phi}_L\right)$$

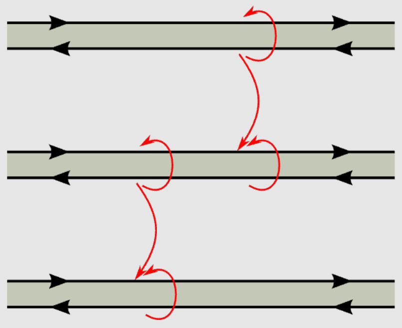
$$\tilde{\phi}_R = \frac{1+K}{2K}\phi_R + \frac{1-K}{2K}\phi_L \qquad \tilde{\phi}_L = \frac{1-K}{2K}\phi_R + \frac{1+K}{2K}\phi_L$$



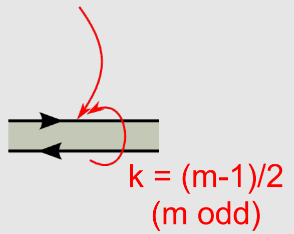
Electron operator : $\psi_R^+ \sim e^{i\phi_R} = e^{i\left(\frac{1+K}{2K}\tilde{\phi}_R+\frac{1-K}{2K}\tilde{\phi}_L\right)}$

Can the chiral modes of a Luttinger liquid be spatially separated ?

Wire construction revisited : Laughlin states at filling $\nu = 1/m$



Composite electron operator $\Psi^\dagger_{j,R} = \psi^\dagger_{j,R}\left(\psi^\dagger_{j,R}\psi_{j,L}\right)^{\frac{m-1}{2}}$



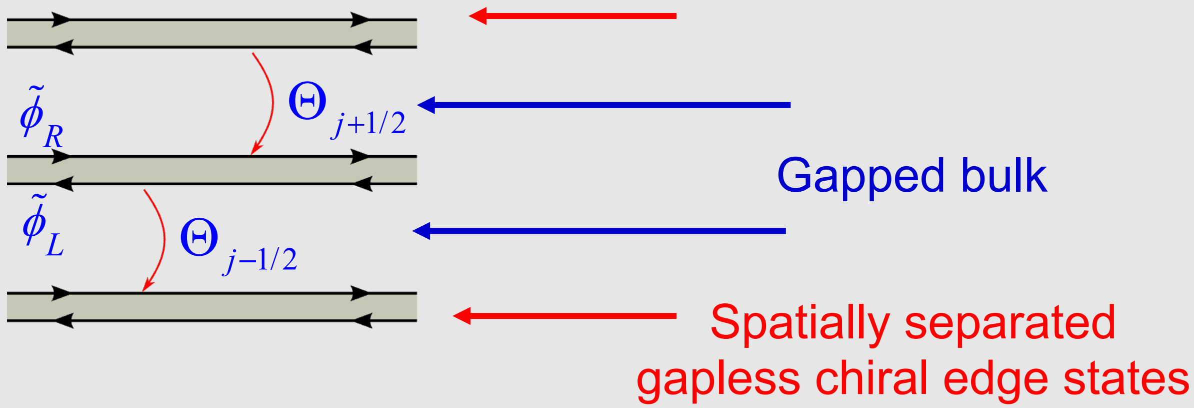
$$H = H_0 + t\sum_j \left(\Psi^\dagger_{j,R}\Psi_{j+1,L} + h.c.\right)$$

For $K = \frac{1}{m}$ the composite electron operator is purely chiral: $\Psi^\dagger_R \sim e^{i\left(\frac{1+m}{2}\phi_R + \frac{1-m}{2}\phi_L\right)} = e^{im\tilde{\phi}_R}$

Solvable wire model: $K = \frac{1}{m}$:

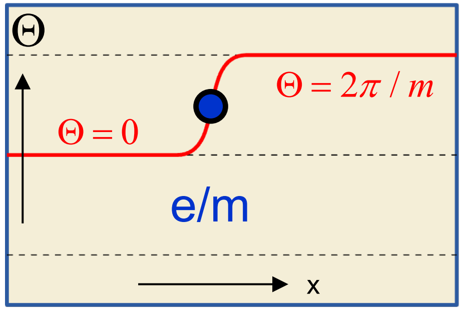
$$H = H_0 - t\sum_j \cos m\Theta_{j+1/2}$$

$$\left(\Theta_{j+1/2} = \tilde{\phi}_{j+1,R} - \tilde{\phi}_{j,L}\right)$$



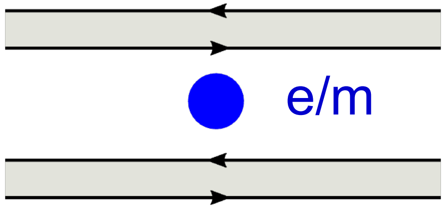
Fractional Charge

Fractionally charge quasiparticle excitations live on the links between wires



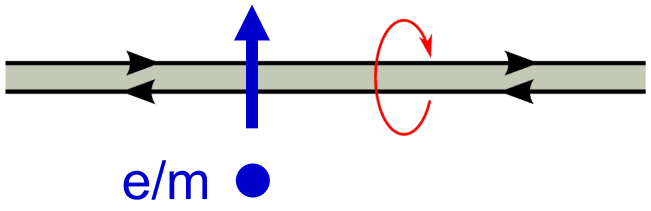
$t \cos m\Theta$ locks Θ, and supports solitons

$$Q = \frac{e}{2\pi}\int \partial_x \Theta = \frac{e}{m}$$

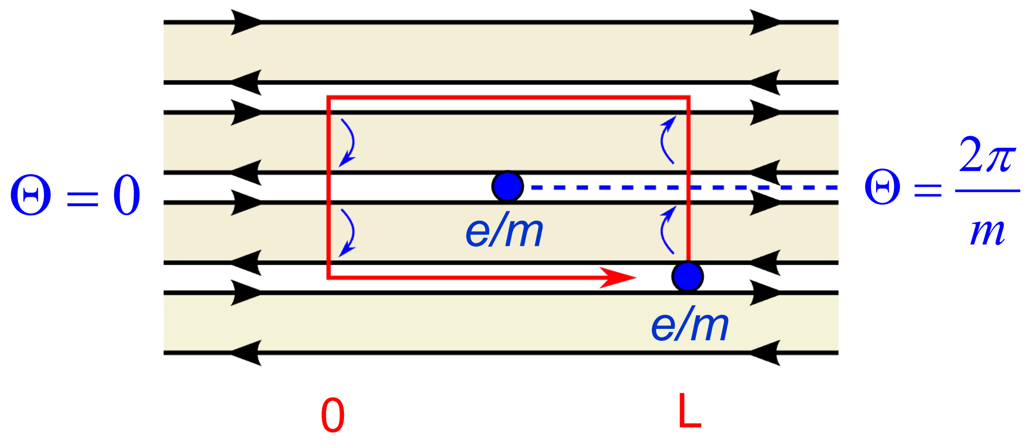


Quasiparticles can move along the links, and they can hop between links via the electron backscattering operator on a wire :

$$\psi_R^\dagger \psi_L \sim e^{i(\tilde{\phi}_R - \tilde{\phi}_L)}$$

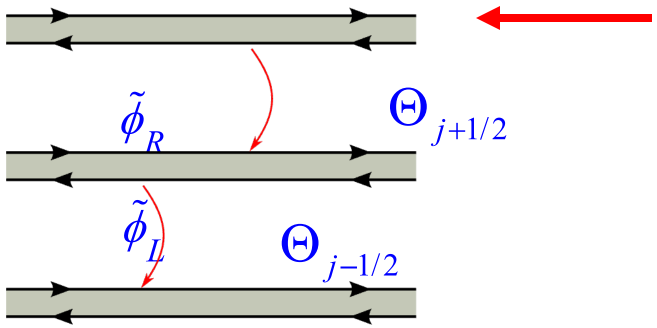


Fractional Statistics



Fractional braiding phase : $e^{i(\Theta(L)-\Theta(0))} = e^{2\pi i/m}$

Chiral Luttinger Liquid Edge States

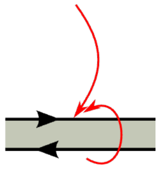


unpaired chiral mode

$$L_{edge} = \frac{m}{4\pi} \partial_x \tilde{\phi}_R \left(\partial_t \tilde{\phi}_R - v_F \, \partial_x \tilde{\phi}_R \right)$$

Bare electron operator : $\psi_R^{+} \sim e^{i\phi_R} = e^{i\left(\frac{1+K}{2K}\tilde{\phi}_R + \frac{1-K}{2K}\tilde{\phi}_L\right)}$: gapped

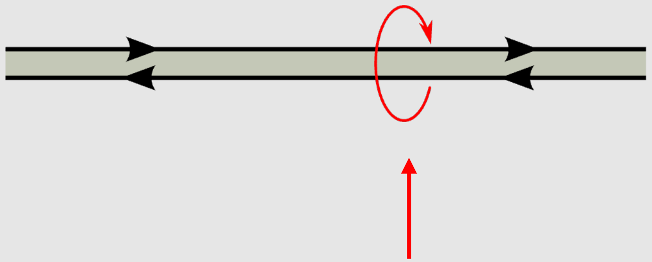
Composite electron operator : $\Psi_R^{\dagger} = \psi_R^{\dagger} (\psi_R^{\dagger} \psi_L)^{\frac{m-1}{2}} \sim e^{i\left(\frac{1+m}{2}\phi_R + \frac{1-m}{2}\phi_L\right)} = e^{im\tilde{\phi}_R}$



$\langle \Psi_R^{+}(t) \Psi_R(0) \rangle \sim \frac{1}{t^m}$ ⇒ Power law suppression of tunneling density of states : $\frac{dI}{dV} \sim V^{m-1}$

1+1D conformal field theory as building block for 2+1D topological phases

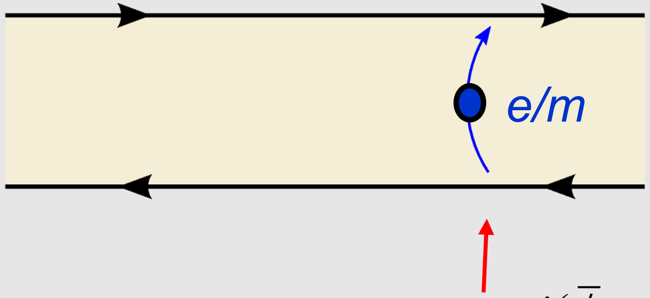
1D Electron Luttinger liquid
K=1/m



Electron back scattering $\psi_R^\dagger \psi_L$

=

Edge states of
ν=1/m strip



Quasiparticle tunneling: $e^{i(\bar{\phi}_R - \bar{\phi}_L)}$

=

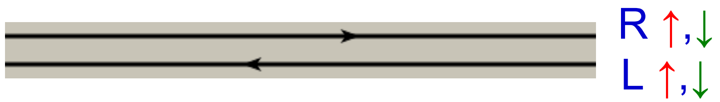
Rational Conformal
Field Theory

Free boson compactified
on circle with radius
$R^2 = K = 1/m$

Are there further factorizations ?

1. Spin ½ Fermions

1D



For chiral mode with spin (non-interacting) :

$$L_R = \frac{1}{4\pi}\left[\partial_x \phi_{R\uparrow}\left(\partial_t \phi_{R\uparrow} - v_F\, \partial_x \phi_{R\uparrow}\right) + \partial_x \phi_{R\downarrow}\left(\partial_t \phi_{R\downarrow} - v_F\, \partial_x \phi_{R\downarrow}\right)\right]$$

Charge – Spin variables : $\phi_{R\uparrow} = \phi_{R\rho} + \phi_{R\sigma}$; $\phi_{R\downarrow} = \phi_{R\rho} - \phi_{R\sigma}$ $\quad L = L_\rho + L_\sigma$

$$L_\rho = \frac{2}{4\pi}\partial_x \phi_{R\rho}\left(\partial_t \phi_{R\rho} - v_F\, \partial_x \phi_{R\rho}\right)$$

$K = \frac{1}{2}$ L.L. aka $U(1)_2$ Wess Zumino Witten (WZW) model

$$L_\sigma = \frac{2}{4\pi}\partial_x \phi_{R\sigma}\left(\partial_t \phi_{R\sigma} - v_F\, \partial_x \phi_{R\sigma}\right)$$

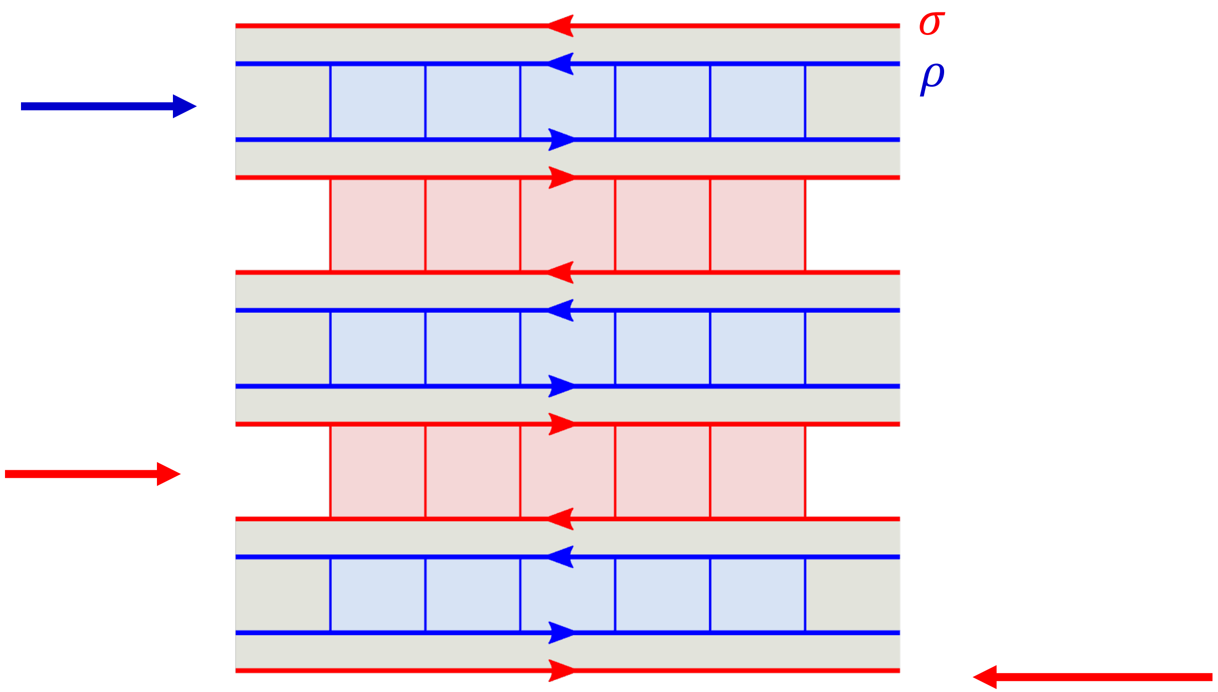
$K = \frac{1}{2}$ L.L. aka $SU(2)_1$ WZW theory

$S^z \sim \frac{\partial_x \phi_\sigma}{2\pi}$; $S^\pm \sim e^{\pm 2i\,\phi_\sigma}$ related by SU(2)

Chiral Spin Liquid State

Kalmeyer and Laughlin, 1988
Wen, 1989

Charge is gapped within each wire.



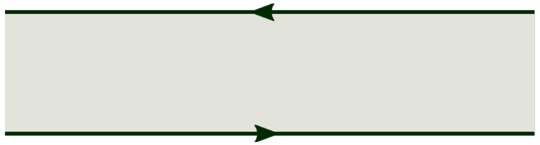
Spin is gapped between wires.

Unpaired $SU(2)_1$ chiral spin mode

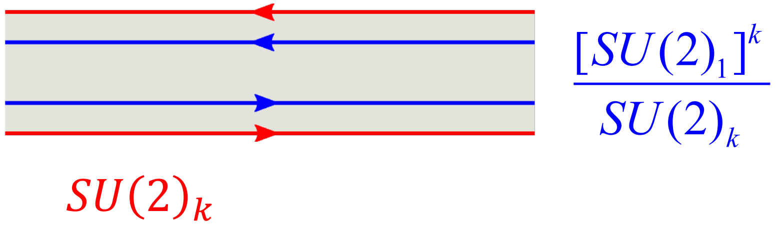
Chiral edge states of CSL resemble low energy excitations of 1D spin ½ Heisenberg model

A class of non-Abelian topological phases

A further factorization of k copies of $SU(2)_1$: $[SU(2)_1]^k \sim SU(2)_k \times \frac{[SU(2)_1]^k}{SU(2)_k}$ “coset construction” in conformal field theory



k copies of $SU(2)_1$



Allows construction of Read Rezayi states, based on $SU(2)_k$

k = 1 : Abelian

k = 2 : Non-Abelian : supports ‘Ising anyons’ (similar to topological superconductor)

k = 3 : Non-Abelian : supports ‘Fibonacci anyons’

Non – Abelian 2D topological superconducting phases

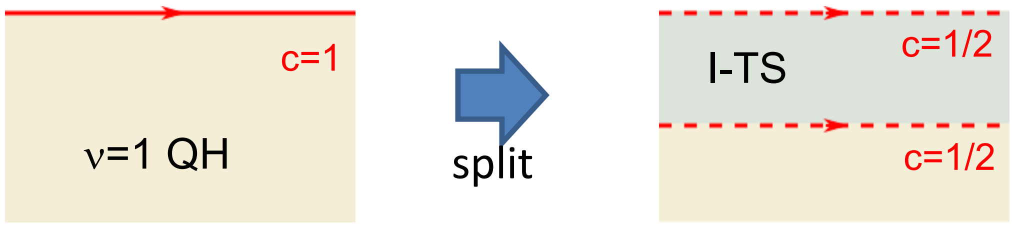
1. 2D Ising topological superconductor (I-TS): 1 = 1/2 + 1/2

Read, Green 2001
Qi, Hughes, Zhang, 2010

Split the chiral
Dirac fermion

$$c^\dagger = \gamma_1 + i\gamma_2$$

Majorana fermions



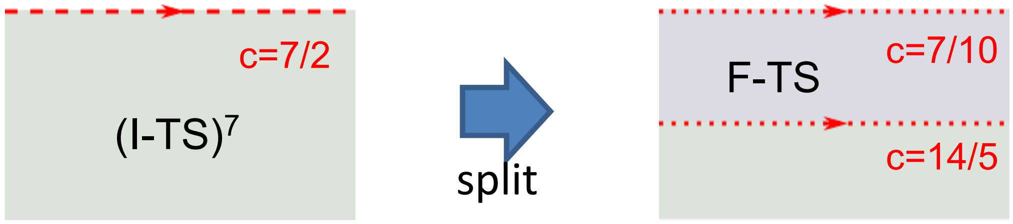
2. Fibonacci topological superconductor (F-TS): 7/2 = 7/10 + 14/5

Mong et al 2014,
Yi, Kane 2018

Split the chiral
Majorana fermion

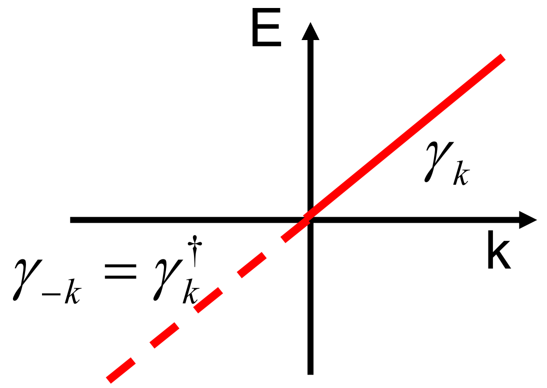
$$\gamma = \tau \ \varepsilon$$

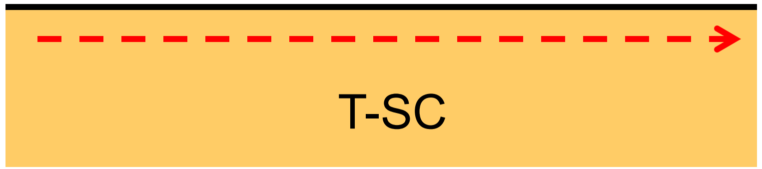
fibonacci anyons



2D □ topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





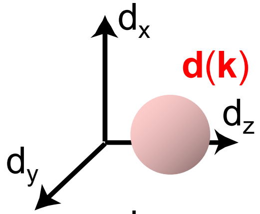
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

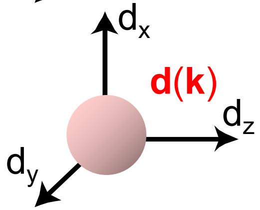
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \vec{\tau}$

$|\mu|>4t$: Strong pairing phase
trivial superconductor



Chern number 0

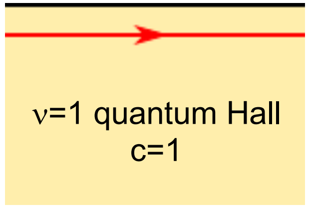
$|\mu|<4t$: Weak pairing phase
topological superconductor



Chern number 1

2D topological superconductivity near a quantum Hall transition

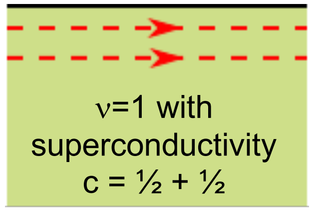
Factorization of Chiral Dirac Fermion Edge states



1 chiral Dirac mode = 2 chiral Majorana modes

$$\psi^\dagger = \gamma_1 + i\gamma_2$$

$$H = -iv\psi^\dagger\partial_x\psi = -iv\gamma_1\partial_x\gamma_1 - iv\gamma_2\partial_x\gamma_2$$



Chiral central charge: c = 1 = ½ + ½

Thermal Hall conductance: $\kappa = c\dfrac{\pi^2}{3}\dfrac{k_B^2}{h}T$

In presence of superconductivity the quantum Hall transition splits