Symmetry, Topology and Phases of Quantum Matter

or

"Splitting the Indivisible"

Charles Kane, University of Pennsylvania

Organizing Principles for Understanding Matter

Symmetry

- What operations leave a system invariant?
- Distinguish phases of matter by symmetries

wallpaper group p4

wallpaper group p31m

Topology

- What stays the same when a system is deformed?
- Distinguish topological phases of matter

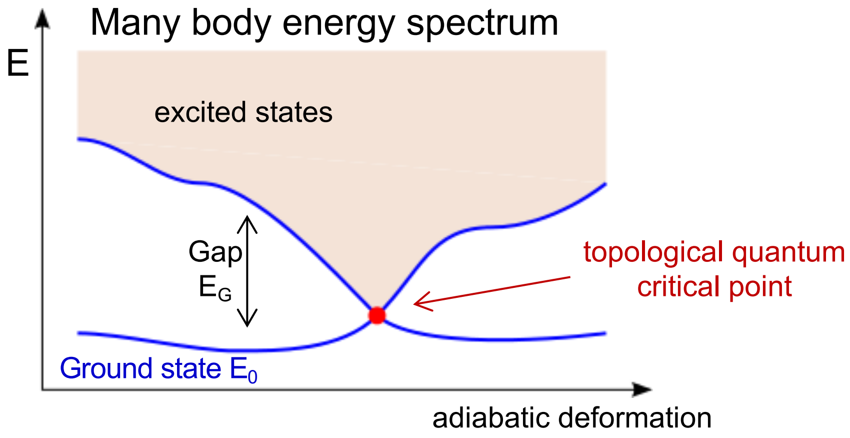
genus g = 1

Topology and Quantum Phases

Topological Equivalence : Principle of Adiabatic Continuity

Quantum phases with an energy gap are topologically equivalent if they can be smoothly deformed into one another without closing the gap.

Topologically distinct phases are separated by quantum phase transition.



Topological Electronic Phases

Free fermion topological phases : ‘topological band theory’

States adiabatically connected to independent electrons:

- Quantum Hall (Chern) insulators
- Topological insulators
- Weak topological insulators
- Topological crystalline insulators
- Topological (Fermi, Weyl and Dirac) semimetals …..

Many real materials and experiments

Topological Superconductivity

Proximity induced topological superconductivity

Majorana bound states, quantum information

Tantalizing experimental progress

Beyond Band Theory: Strongly correlated topological phases

State with intrinsic topological order

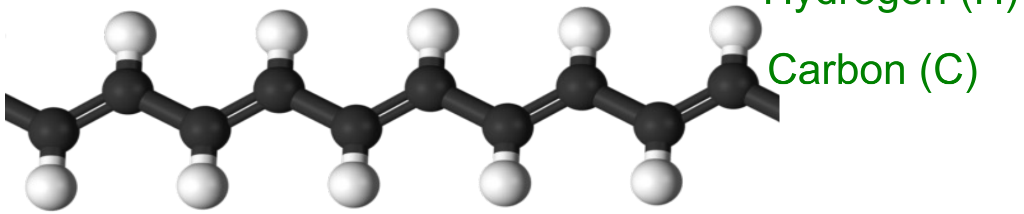
- fractional quantized Hall effect
- fractional quantum numbers, anyons
- topological ground state degeneracy

- Symmetry protected topological states
- Surface topological order ……

Much conceptual progress, but theory is still far from the real electrons

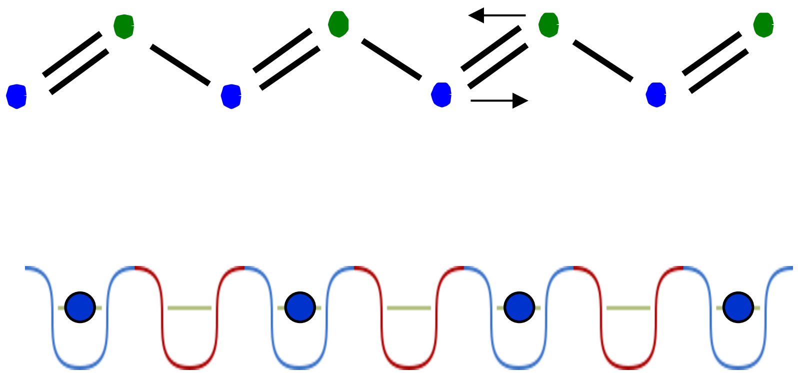
Cartoon Example

Polyacetylene: A 1-dimensional polymer $(CH)_n$

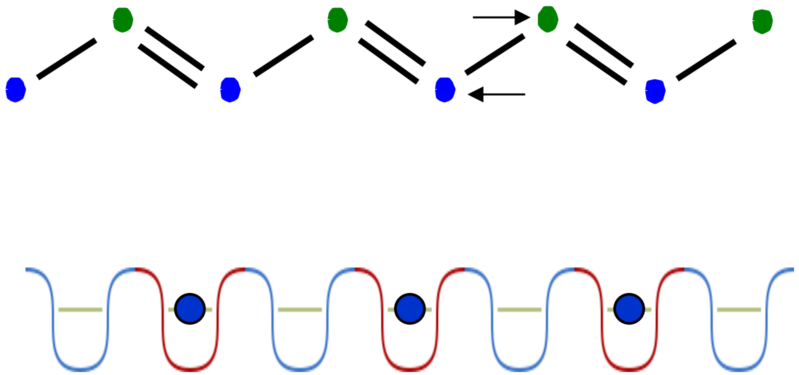


An Insulator with two 'topologically distinct' phases

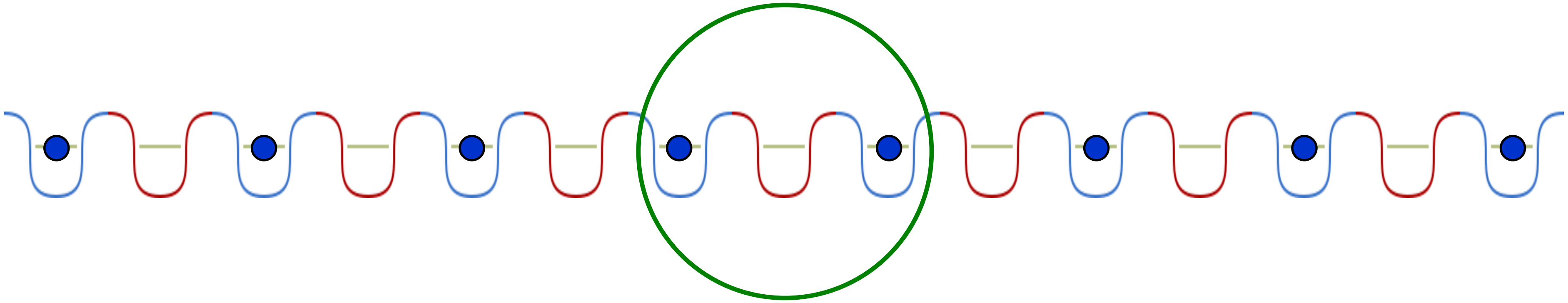
A phase



B phase

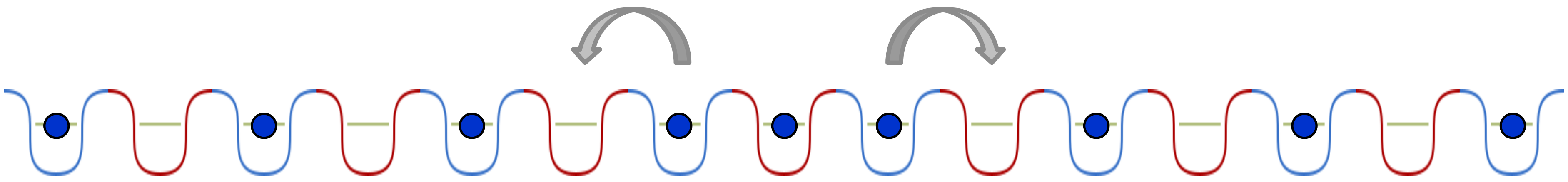


Magic Trick

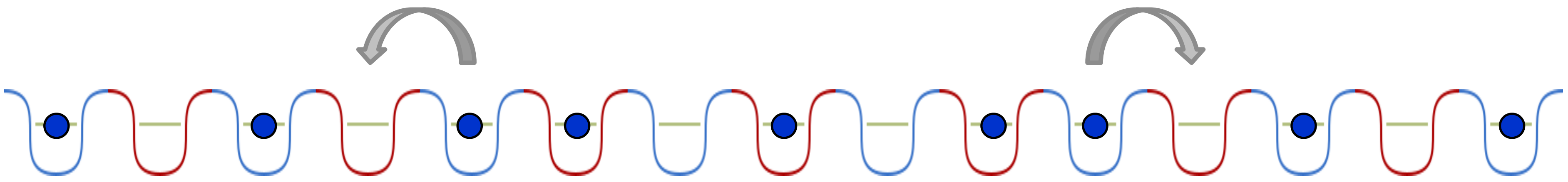


Extra electron:
charge - e

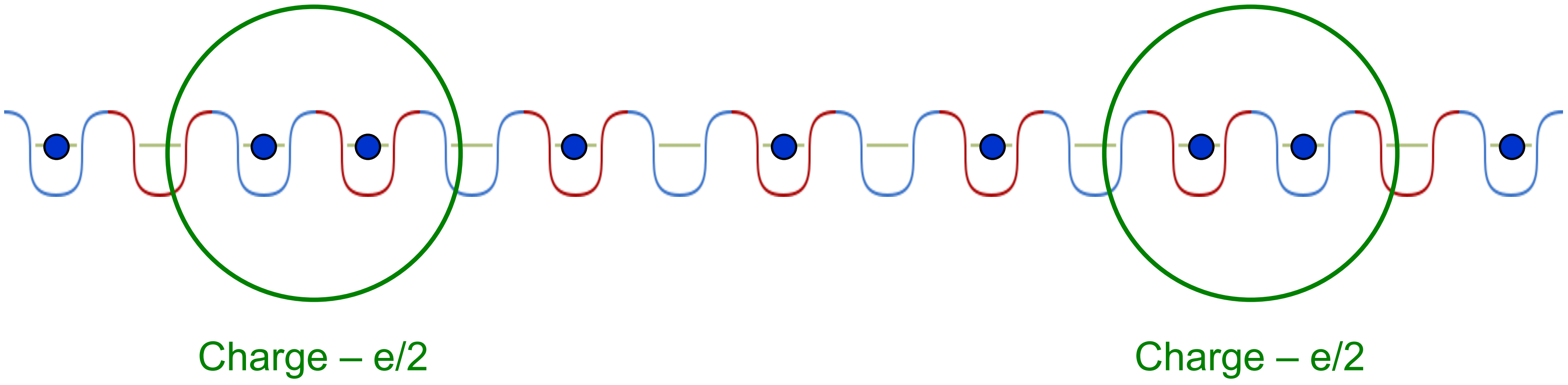
Magic Trick



Magic Trick

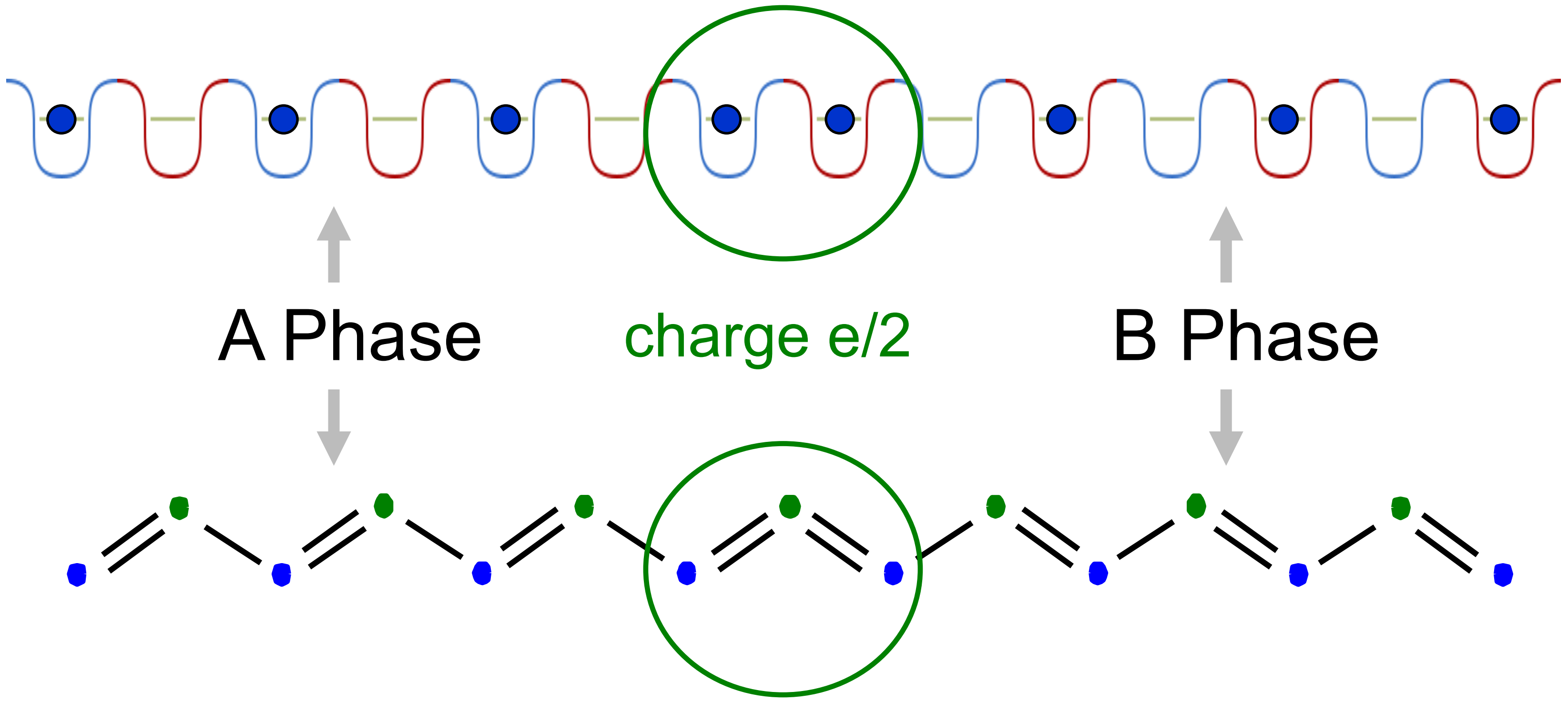


Magic Trick



The added electron has split in half!

The "impossible" occurs at the boundary between different topological phases



There are many more examples of this phenomenon

Variations on a theme of ‘splitting the indivisible’

Lecture I: Topological phases of non-interacting fermions

Topological Band Theory

1. Topology in D = 1
2. Topology in D = 2
3. Z_2 topological insulator
4. Topological superconductivity

Lecture II: Topological phases with strong correlation

Coupled wire construction

1. Bosonization, and Luttinger liquid
2. Wire construction for Laughlin state
3. Generalizations

Band Theory

Theory of electronic structure based on independent electron approximation

Many particle ground state : Slater determinant of occupied single particle energy eigenstates

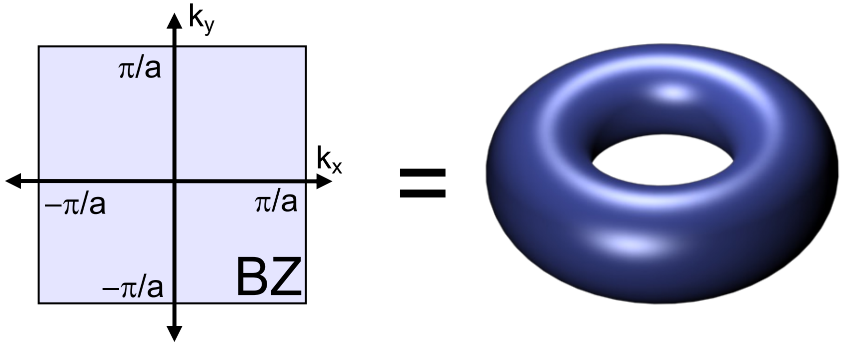
Bloch Theorem :

Lattice translation symmetry $T(\mathbf{R})|\psi\rangle = e^{i\mathbf{k}\cdot\mathbf{R}}|\psi\rangle$ $\qquad |\psi\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}|u(\mathbf{k})\rangle$

Bloch Hamiltonian $H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\mathrm{H}e^{i\mathbf{k}\cdot\mathbf{r}}$ $\qquad H(\mathbf{k})|u_n(\mathbf{k})\rangle = E_n(\mathbf{k})|u_n(\mathbf{k})\rangle$

$\mathbf{k} \in$ Brillouin Zone

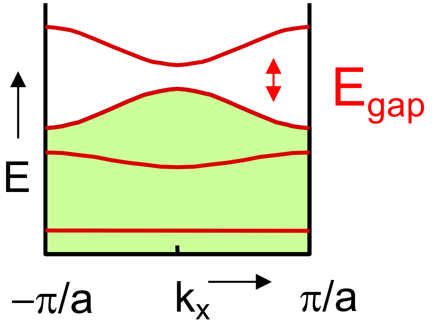
$=$ Torus, T^d



Band Structure :

A mapping $\mathbf{k} \mapsto H(\mathbf{k})$

(or equivalently to $E_n(\mathbf{k})$ and $|u_n(\mathbf{k})\rangle$)



Berry Phase

Phase ambiguity of quantum mechanical wave function

$$|u(\mathbf{k})\rangle \to e^{i\phi(\mathbf{k})}|u(\mathbf{k})\rangle$$

Berry connection : like a vector potential $\mathbf{A} = -i\langle u(\mathbf{k})|\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$

$$\mathbf{A} \to \mathbf{A} + \nabla_{\mathbf{k}}\phi(\mathbf{k})$$

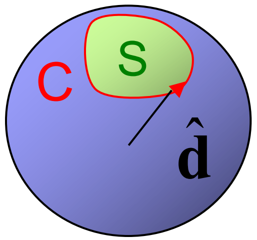
Berry phase : change in phase on a closed loop C $\gamma_C = \oint_C \mathbf{A}\cdot d\mathbf{k}$

Berry curvature : $\mathbf{F} = \nabla_{\mathbf{k}} \times \mathbf{A}$ $\gamma_C = \int_S \mathbf{F}\, d^2k$

Famous example : eigenstates of 2 level Hamiltonian

$$H(\mathbf{k}) = \mathbf{d}(\mathbf{k})\cdot\vec{\sigma} = \begin{pmatrix} d_z & d_x - id_y \\ d_x + id_y & -d_z \end{pmatrix}$$

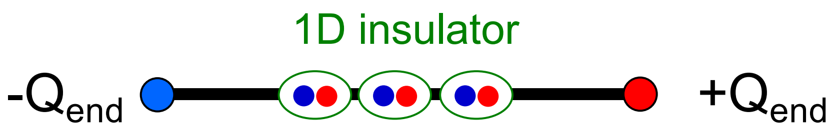
$$H(\mathbf{k})|u(\mathbf{k})\rangle = +|\mathbf{d}(\mathbf{k})||u(\mathbf{k})\rangle$$



$$\gamma_C = \frac{1}{2}\left(\text{Solid Angle swept out by } \hat{\mathbf{d}}(\mathbf{k})\right)$$

Topology in one dimension : Berry phase and electric polarization

Classical electric polarization :



$$P = \frac{\text{dipole moment}}{\text{length}} \qquad \rho_{bound} = \nabla \cdot P \qquad Q_{end} = P \cdot \hat{n}$$

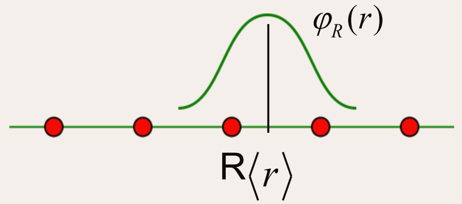
Quantum polarization : a Berry phase

Bloch states $\psi_k(r) = e^{ikr} u_k(r)$ are defined for periodic boundary conditions

Define localized Wannier States :

$$|\varphi(R)\rangle = \int_{BZ} \frac{dk}{2\pi} e^{-ikR} |\psi_k\rangle = \int_{BZ} \frac{dk}{2\pi} e^{-ik(R-r)} |u_k\rangle$$

$$P = e\langle \varphi(R) | r - R | \varphi(R) \rangle = \frac{ie}{2\pi} \int_{BZ} \langle u_k | \nabla_k | u_k \rangle$$

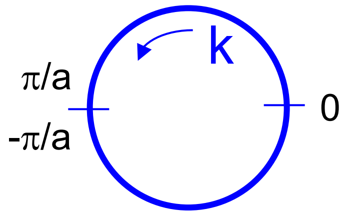


Wannier states associated with R are localized, but gauge dependent.

$$P = \frac{e}{2\pi} \int_{BZ} A(k)dk$$

$$\mathbf{A} = -i\langle u(\mathbf{k}) | \nabla_{\mathbf{k}} | u(\mathbf{k}) \rangle$$

BZ = 1D Brillouin Zone = S^1



Gauge invariance and intrinsic ambiguity of P

- The end charge is not completely determined by the bulk polarization P because integer charges can be added or removed from the ends :

$$Q_{\text{end}} = P \text{ mod } e$$

- The Berry phase is gauge invariant under continuous gauge transformations, but is not gauge invariant under "large" gauge transformations.

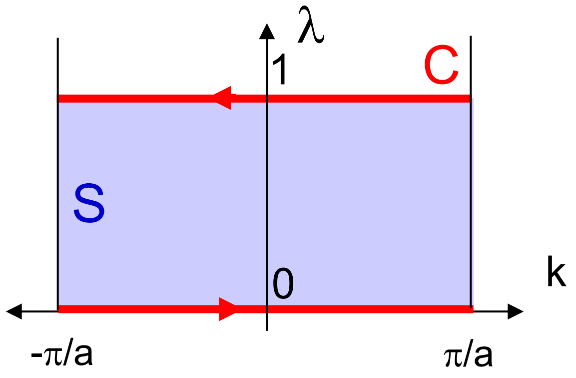
$$P \to P + en \quad \text{when} \quad |u(k)\rangle \to e^{i\phi(k)}|u(k)\rangle \quad \text{with} \quad \phi(\pi/a) - \phi(-\pi/a) = 2\pi n$$

Changes in P, due to adiabatic variation are well defined and gauge invariant

$$|u(k)\rangle \to |u(k,\lambda(t))\rangle$$

$$\Delta P = P_{\lambda=1} - P_{\lambda=0} = \frac{e}{2\pi}\oint_C \mathbf{A}dk = \frac{e}{2\pi}\int_S \mathbf{F}dk d\lambda$$

gauge invariant Berry curvature



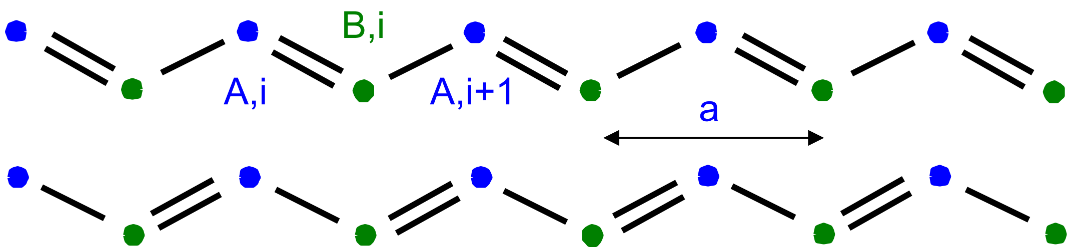
Su Schrieffer Heeger Model

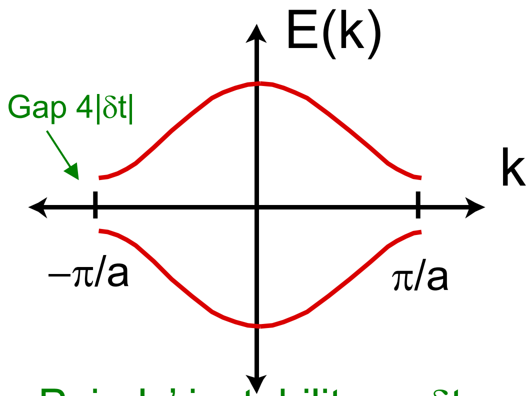
model for polyacetylene
simplest “two band” model

$$H=\sum_i (t+\delta t)c^\dagger_{Ai}c_{Bi}+(t-\delta t)c^\dagger_{Ai+1}c_{Bi}+h.c.$$

$\delta t>0$

$\delta t<0$





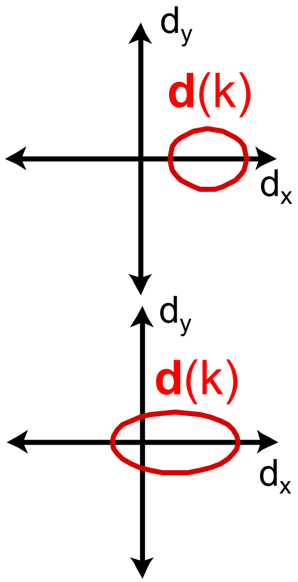
Peierls’ instability → δt

$$H(k)=\mathbf{d}(k)\cdot\vec{\sigma}$$

$$d_x(k)=(t+\delta t)+(t-\delta t)\cos ka$$

$$d_y(k)=(t-\delta t)\sin ka$$

$$d_z(k)=0$$



δt>0 : Berry phase 0
P = 0

δt<0 : Berry phase π
P = e/2

Provided symmetry requires $d_z(k)=0$, the states with δt>0 and δt<0 are distinguished by an integer winding number. Without extra symmetry, all 1D band structures are topologically equivalent.

Symmetries of the SSH model

"Chiral" Symmetry : $\{H(k),\sigma_z\}=0$ (or $\sigma_z H(k)\sigma_z = -H(k)$)

- Artificial symmetry of polyacetylene. Consequence of bipartite lattice with only A-B hopping: $c_{iA}\to c_{iA}$, $c_{iB}\to -c_{iB}$
- Requires $d_z(k)=0$: integer winding number
- Leads to particle-hole symmetric spectrum:

$$H\sigma_z|\psi_E\rangle = -E\sigma_z|\psi_E\rangle \quad \Rightarrow \quad \sigma_z|\psi_E\rangle = |\psi_{-E}\rangle$$

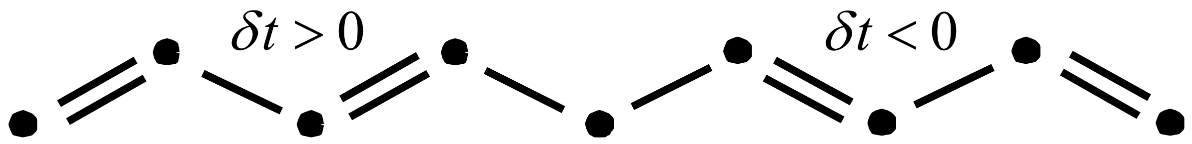
Reflection Symmetry : $H(-k)=\sigma_x H(k)\sigma_x$

- Real symmetry of polyacetylene.
- Allows $d_z(k)\neq 0$, but constrains $d_x(-k)= d_x(k)$, $d_{y,z}(-k)= -d_{y,z}(k)$
- No p-h symmetry, but polarization is quantized: Z_2 invariant

P = 0 or e/2 mod e

Domain Wall States

An interface between different topological states has topologically protected midgap states



Low energy continuum theory :

For small δt focus on low energy states with $k \sim \pi/a$

$$k \rightarrow \frac{\pi}{a} + q \quad ; \quad q \rightarrow -i\partial_x$$

$$H = -i\mathrm{v}_F \sigma_x \partial_x + m(x)\sigma_y \qquad \mathrm{v}_F = ta \;\; ; \;\; m = 2\delta t$$

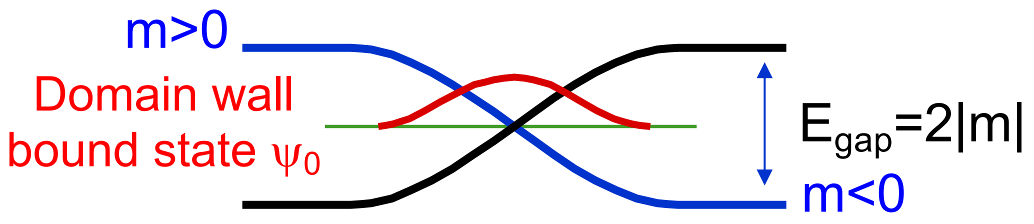
Massive 1+1 D Dirac Hamiltonian $\quad E(q) = \pm\sqrt{(\mathrm{v}_F q)^2 + m^2}$

"Chiral" Symmetry : $\quad \{\sigma_z, H\} = 0 \;\rightarrow\; \sigma_z |\psi_E\rangle = |\psi_{-E}\rangle$

Any eigenstate at +E has a partner at -E

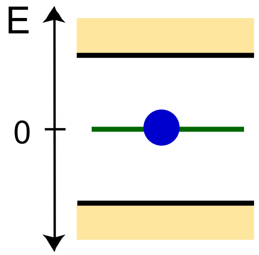
Zero mode : topologically protected eigenstate at E=0

(Jackiw and Rebbi 76, Su Schrieffer, Heeger 79)

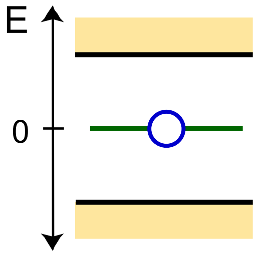


$$\psi_0(x) = e^{-\int_0^x m(x')dx'/\mathrm{v}_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Many body ground state: Charge fractionalization



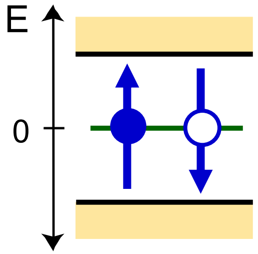
charge + e/2



charge - e/2

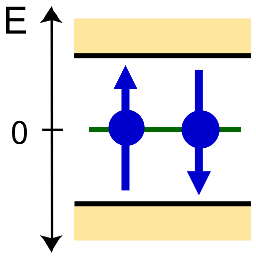
Splitting the indivisible

Real polyacetylene has spin : spin – charge separation



spinon :

charge 0
spin 1/2



holon :

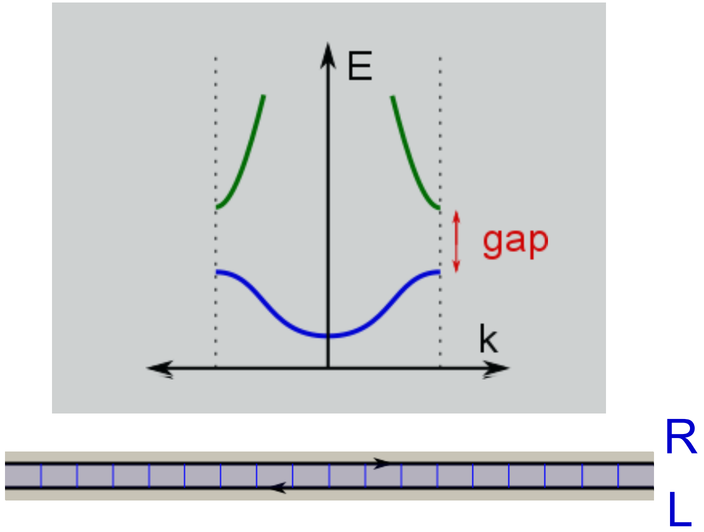
charge e
spin 0

From 1 Dimension to 2 Dimensions

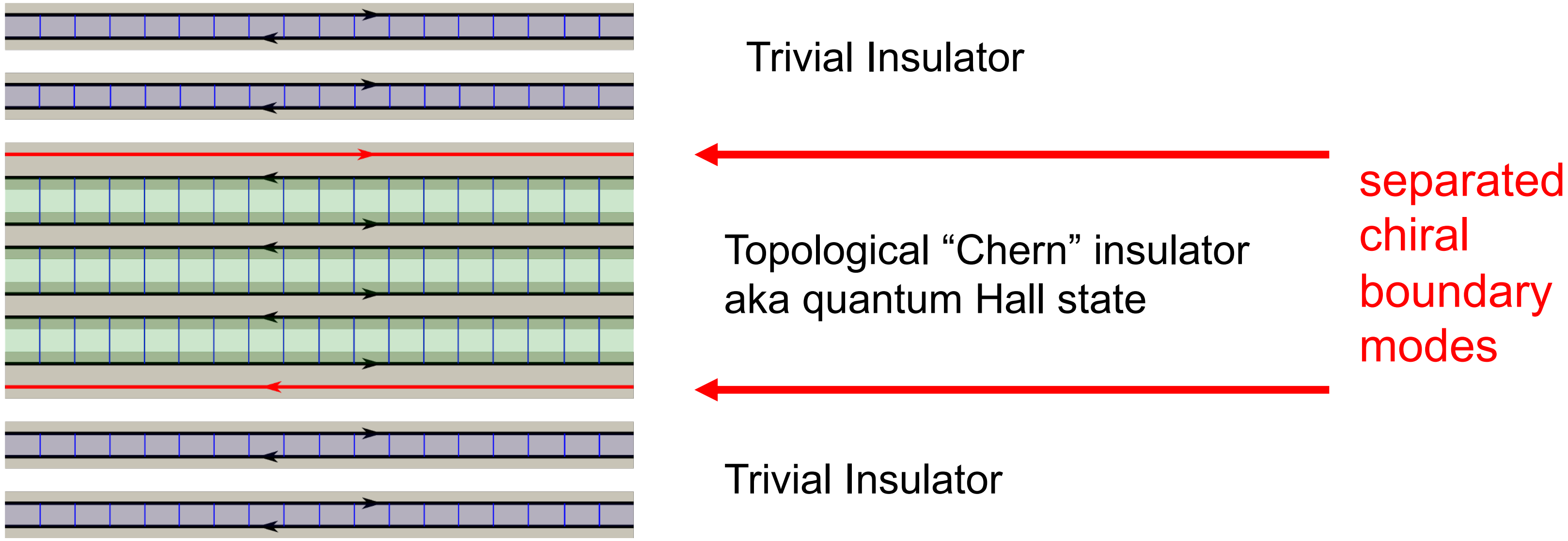
1D electrical conductor

Low energy excitations:

- Right/Left moving "chiral fermions"
- Localized by commensurate periodic potential or disorder.

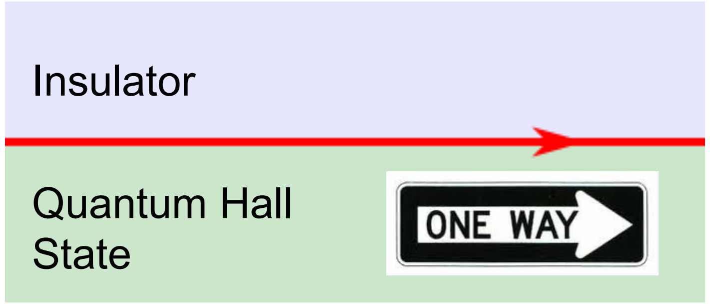


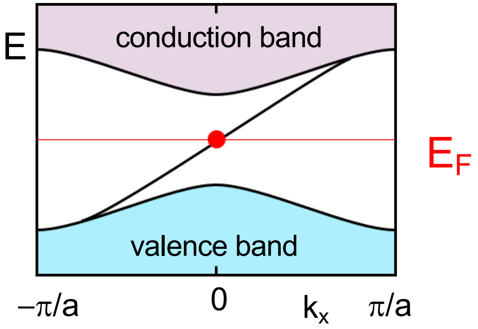
Split the 1D chiral modes in a 2D insulator:



Chiral Edge States

Single-particle edge spectrum : one-way propagating single particle states





Many-body edge spectrum : "chiral Fermi liquid"

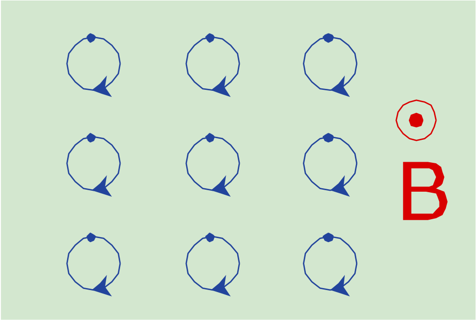
- Free Dirac fermion conformal field theory $H = -i v \psi^\dagger \partial_x \psi$
- Quantized electrical conductance $G = \nu \frac{e^2}{h}$ $\nu = 1$
- Quantized thermal conductance: $\kappa = c \frac{\pi^2}{3} \frac{k_B^2}{h} T$ $c = 1$ chiral central charge

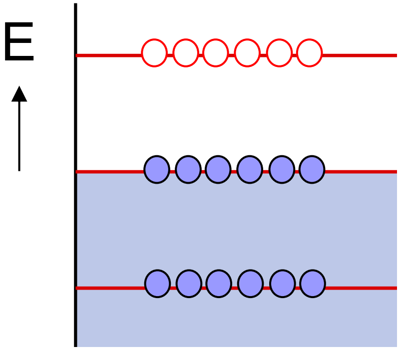
Chiral Anomaly : In presence of electric field the edge charge density not conserved

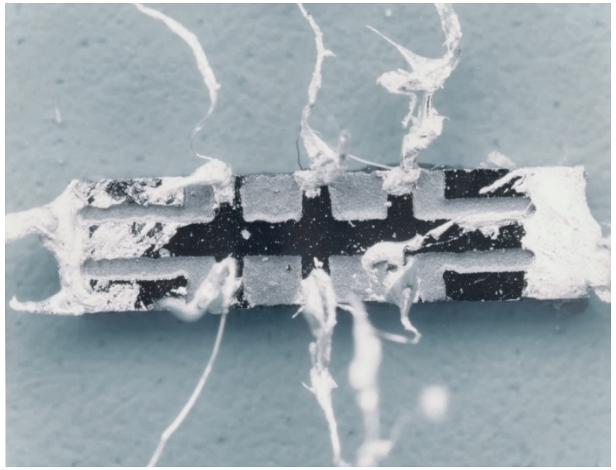
$$\frac{d\rho_+}{dt} = \frac{e}{2\pi}\frac{dk}{dt} = \frac{e}{2\pi}\frac{eE}{\hbar} = \sigma_{xy} E$$

Quantized Hall Effect

2D electrons in magnetic field







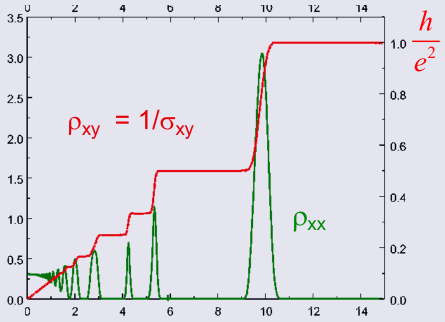
Stormer, Tsui, Gossard 1981

Quantized Hall conductance

von Klitzing et al. ‘1980

$$\sigma_{xy} = n\ e^2 / h$$

n = Integer to 10^{-9}

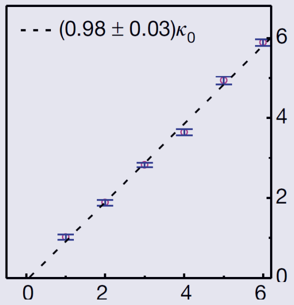


Quantized thermal Hall conductance

Banerjee, Heiblum, et al. 2017

$$\kappa_{xy} = c\ (\pi^2/3)(k_B^2/h)T$$

$$c\ =\ 0.98\ \pm\ .03\ \sim\ 1$$

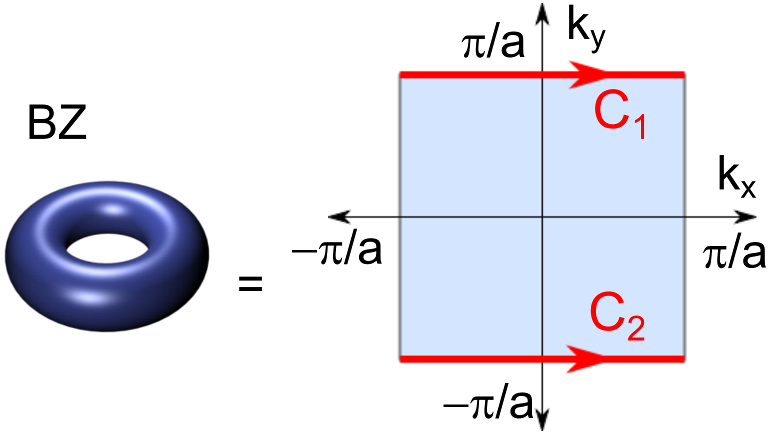


Topology in 2 Dimensions: Chern number (aka TKNN Invariant)

Thouless, Kohmoto, Nightingale and den Nijs 82

For 2D band structure, define $\mathbf{A}(\mathbf{k}) = -i\langle u(\mathbf{k})|\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$

BZ

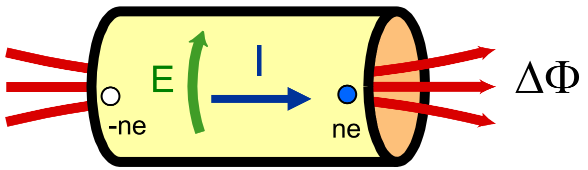


$$n = \frac{1}{2\pi}\int_{C_1}\mathbf{A}\cdot d\mathbf{k} - \frac{1}{2\pi}\int_{C_2}\mathbf{A}\cdot d\mathbf{k} \in \mathbb{Z}$$

$$= \frac{1}{2\pi}\int_{BZ} d^2k\,\mathbf{F}(\mathbf{k})$$

Physical meaning: Quantized Hall conductivity $\sigma_{xy} = n\frac{e^2}{h}$

Laughlin argument: Thread flux $\Delta\Phi = h/e$



$Ea = d\Phi / dt$ (Faraday's law)

$I = \sigma_{xy} Ea = \sigma_{xy} d\Phi / dt$

$\Delta P = \int I dt = \sigma_{xy}\Delta\Phi = \sigma_{xy} h / e$

Thouless pump: Cylinder with circumference 1 lattice constant (a)

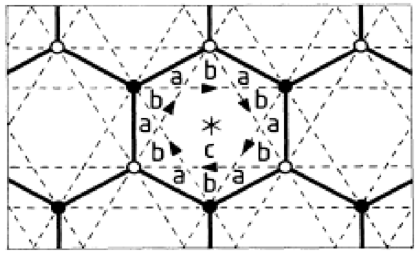
Φ plays role of k_y $\quad \Delta\Phi = h/e \;\Rightarrow\; \Delta k_y = 2\pi / a \qquad \Delta P = ne$

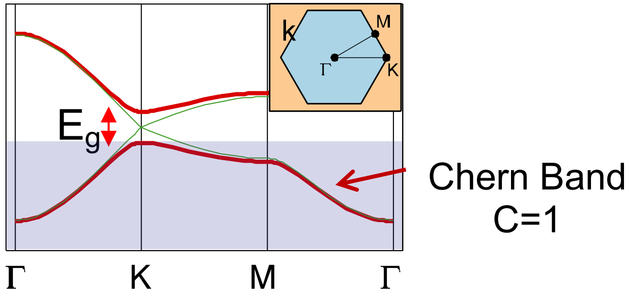
Alternative calculation: compute σ_{xy} via Kubo formula

Realizing a Chern Insulator

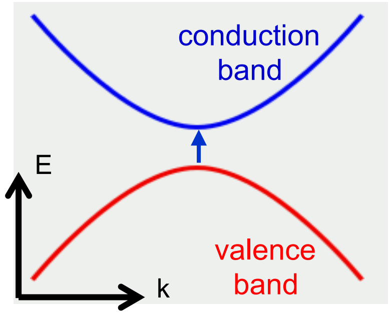
Quantized Hall effect without Landau levels

Haldane model

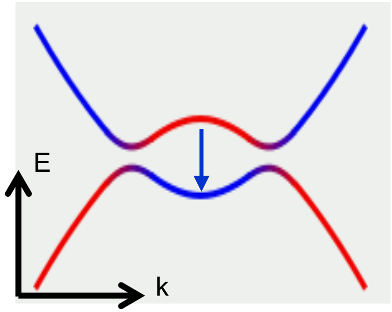




Band Inversion Paradigm



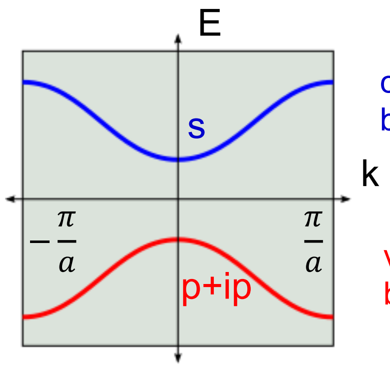
trivial insulator



inverted
topological insulator

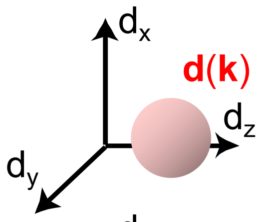
Lattice model for Chern insulator

$$H(\mathbf{k}) = \tau_z\left(2t_0\left[\cos k_x a + \cos k_y a\right] + \Delta E_{sp}\right)$$
$$+ 2t_{sp}\left(\tau_x \sin k_x a + \tau_y \sin k_y a\right) = \mathbf{d}(k)\cdot\vec{\tau}$$



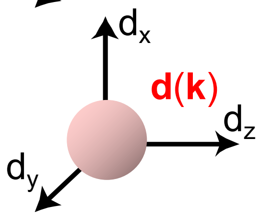
Square lattice model with inversion of bands with s and p_x+ip_y symmetry near Γ

$|\Delta E_{sp}| > 4t$: Uninverted Trivial Insulator



Chern number 0

$|\Delta E_{sp}| < 4t$: Inverted Chern Insulator



Chern number 1

Regularized continuum model for Chern insulator

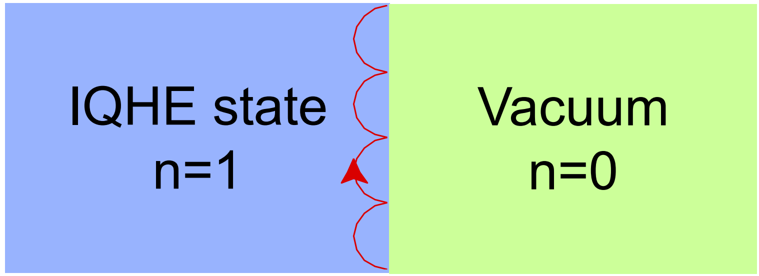
$$H(\mathbf{k}) = \tau_z\left(m + ak^2\right) + \mathrm{v}\left(k_x\tau_x + k_y\tau_y\right) = \mathbf{d}(k)\cdot\vec{\tau}$$

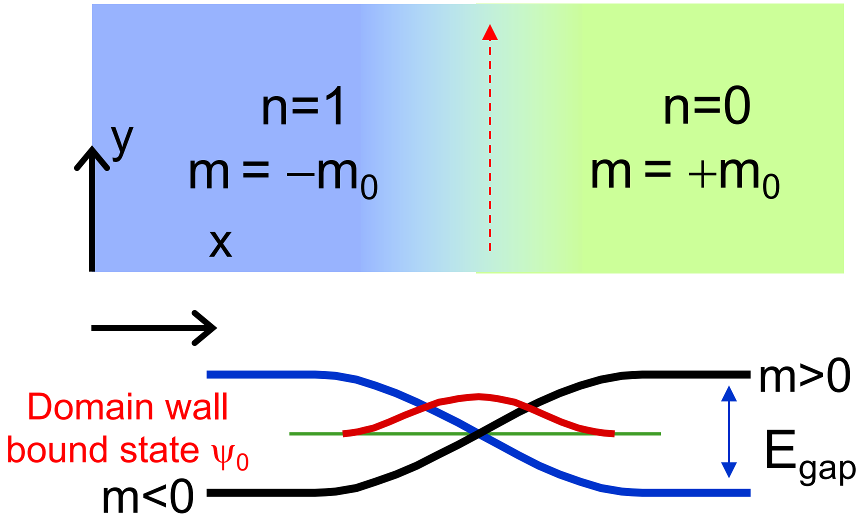
$m = 4t_0 - \Delta E_{sp}$
$a = t_0\, a$
$v = 2t_{sp}\, a$

Inverted near k=0 for m<0. Uninverted for $k \to \infty$

Edge States

Gapless states at the interface between topologically distinct phases



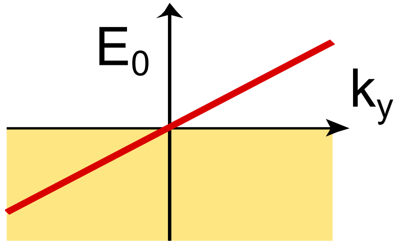


Edge states ~ skipping orbits
Lead to quantized transport

Band inversion transition : Dirac Equation

$$H = \mathrm{v_F}(-i\sigma_x\partial_x + \sigma_y k_y) + m(x)\sigma_z$$

$$\psi_0(x) \sim e^{ik_y y} e^{-\int_0^x m(x')dx'/\mathrm{v_F}} \qquad E_0(k_y) = \mathrm{v_F} k_y$$



Chiral Dirac Fermions

Generalizations

d=4 : 4 dimensional generalization of IQHE   Zhang, Hu '01

$$\mathbf{A}_{ij} = \langle u_i(\mathbf{k}) | \nabla_{\mathbf{k}} | u_j(\mathbf{k}) \rangle \cdot d\mathbf{k}$$ Non-Abelian Berry connection 1-form

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$ Non-Abelian Berry curvature 2-form

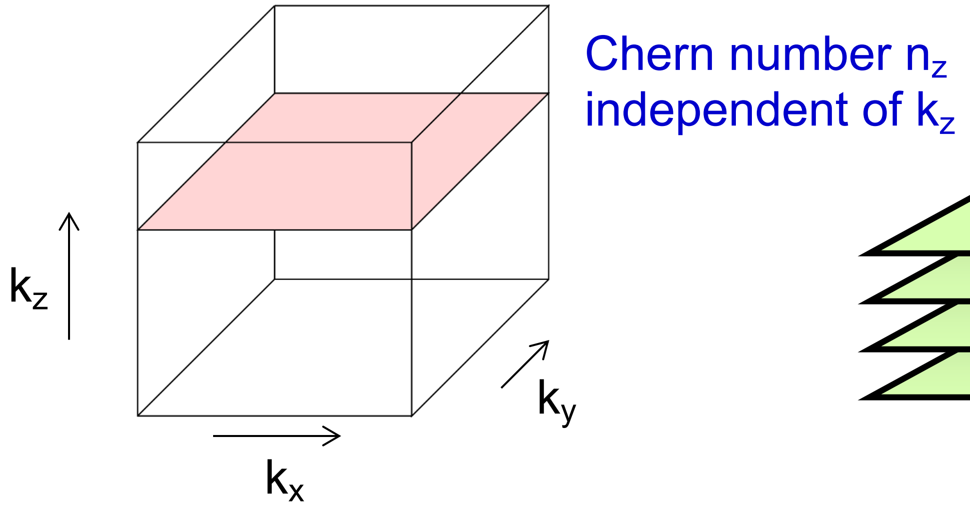
$$n = \frac{1}{8\pi^2} \int_{T^4} \mathrm{Tr}[\mathbf{F} \wedge \mathbf{F}] \in \mathbb{Z}$$ 2nd Chern number = integral of 4-form over 4D BZ

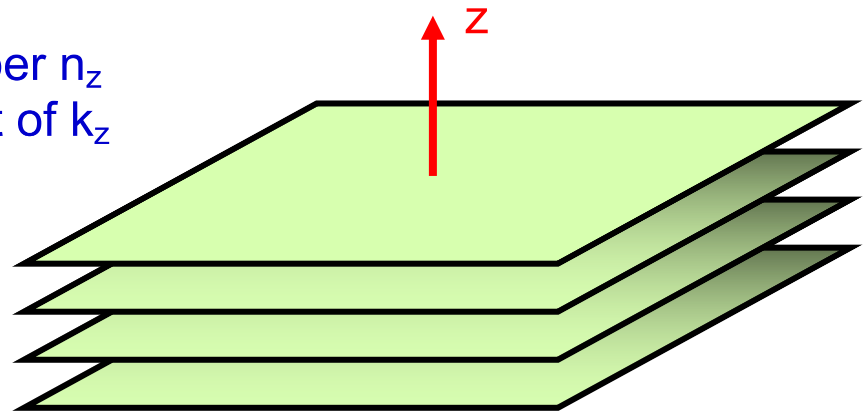
Boundary states : 3+1D Chiral Dirac fermions

Higher Dimensions : "Bott periodicity" d → d+2

	d							
	1	2	3	4	5	6	7	8
no symmetry	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
chiral symmetry	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

3D Quantum Hall Effect





More generally, the 3 independent Chern numbers (n_x,n_y,n_z) define a reciprocal lattice vector **G** that characterizes a family of lattice planes.

$$\sigma_{ij} = \frac{e^2}{2\pi h}\varepsilon_{ijk}G_k$$

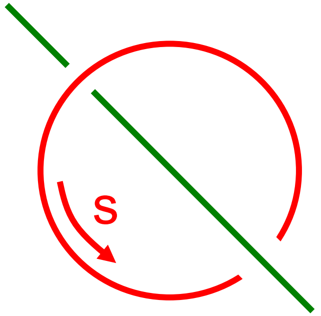
Topological Defects

Consider insulating Bloch Hamiltonians that vary slowly in real space

Teo, Kane '10

$$H = H(\mathbf{k}, s)$$

1 parameter family of 3D Bloch Hamiltonians



2nd Chern number : $n = \frac{1}{8\pi^2}\int_{T^3\times S^1} \mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}]$

Generalized bulk-boundary correspondence :

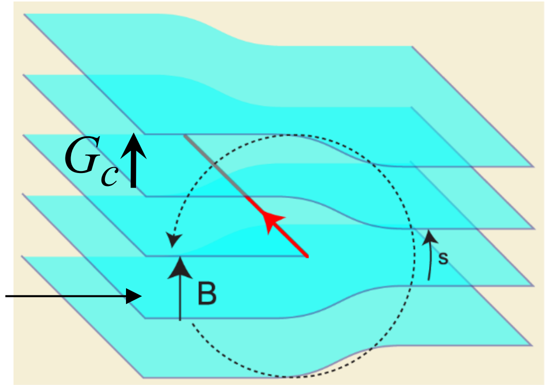
n specifies the number of chiral Dirac fermion modes bound to defect line

Example : dislocation in 3D layered IQHE

$$n = \frac{1}{2\pi}\mathbf{G}_c \cdot \mathbf{B}$$

3D Chern number
(vector ⊥ layers)

Burgers' vector



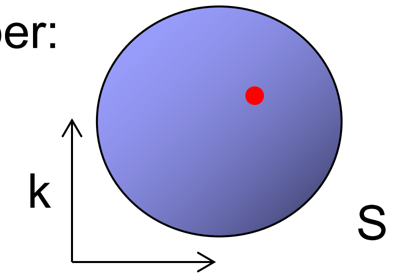
Are there other ways to engineer
1D chiral dirac fermions?

Weyl Semimetal

Gapless "Weyl points" in momentum space are topologically protected in 3D

A sphere in momentum space can have a Chern number:

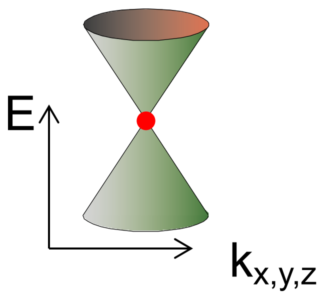
$$n_S = \int_S d^2k \mathbf{F}(\mathbf{k}) \in \mathbb{Z}$$



n_S=+1: S must enclose a degenerate Weyl point:
Magnetic monopole for Berry flux

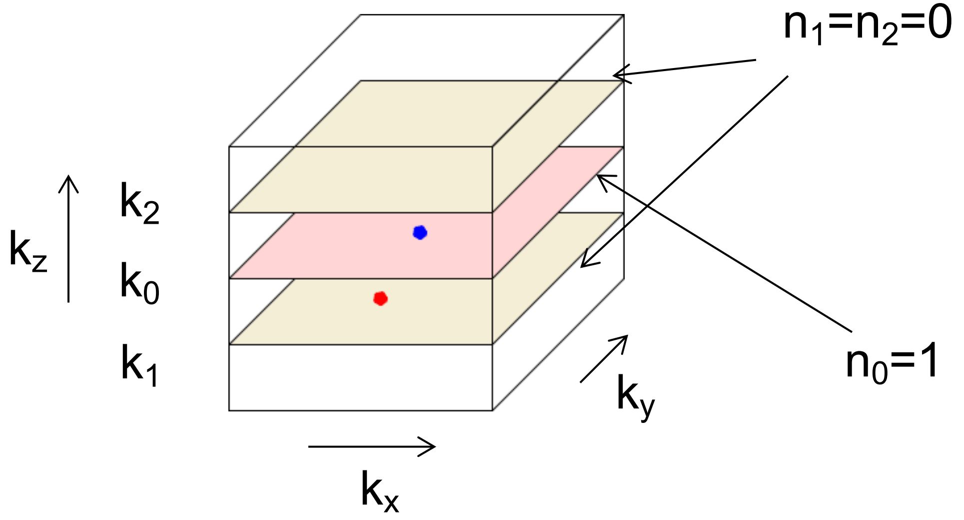
$$H(k_0 + q) = \mathrm{v}(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z)$$

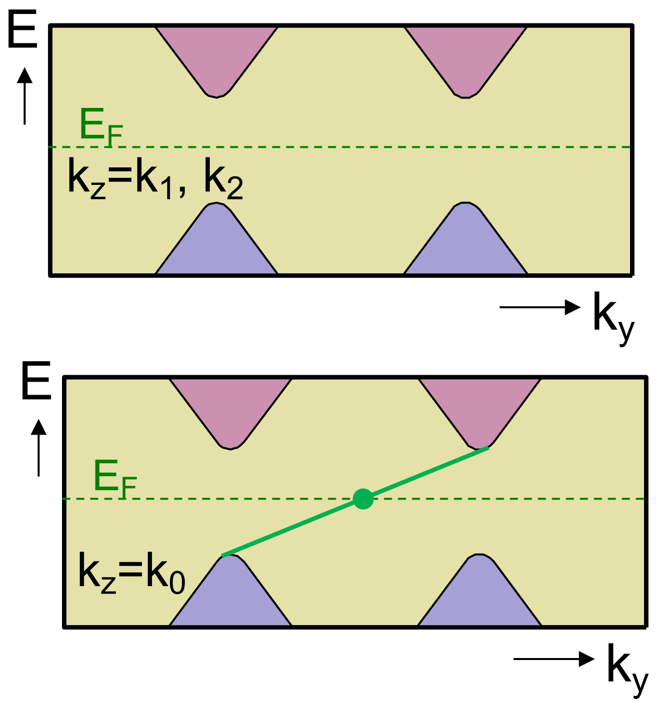
(or $\mathrm{v}_{ia} q_i \sigma_a$ with $\det[\mathrm{v}_{ia}] > 0$)



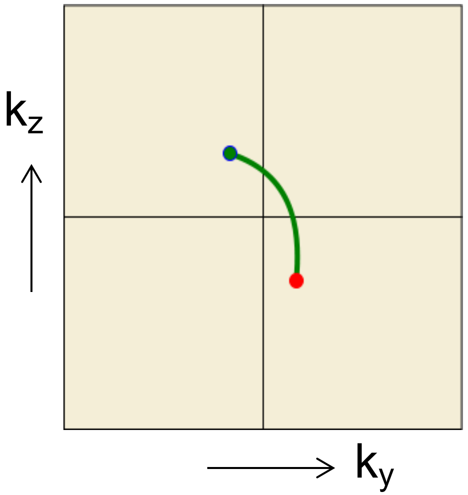
Total magnetic charge in Brillouin zone must be zero: Weyl points must come in +/- pairs.

Surface Fermi Arc





Surface BZ



Chiral Anomaly

In the presence of E and B, the charge at one (or the other) Weyl point is not conserved:

$$\frac{dn_+}{dt} = -\frac{dn_-}{dt} = \frac{e^2}{h^2}\mathbf{E}\cdot\mathbf{B}$$

I. Anti Unitary Symmetries

- Time Reversal Symmetry $[H,\Theta]=0$

  Chern number n=0
  Z_2 topological insulator in d=2,3

- Particle-hole symmetry of single particle BdG Hamiltonian $\{H,\Xi\}=0$

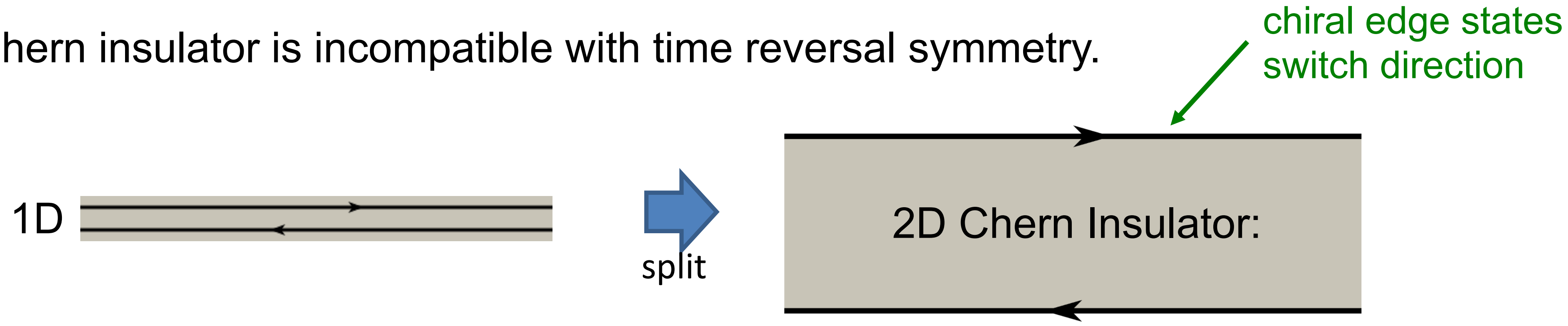
  Z_2 topological superconductor d=1
  Z topological superconductor d=2

II. Topological Superconductivity

- Majorana modes, Kitaev model
- Quantum Information, Topological quantum computation

Time Reversal Symmetry

A Chern insulator is incompatible with time reversal symmetry.



Are there time reversal invariant topological phases?

Anti Unitary time reversal operator : $\Theta\psi = e^{i\pi S^y/\hbar}\psi^*$ $\quad [H,\Theta]=0$

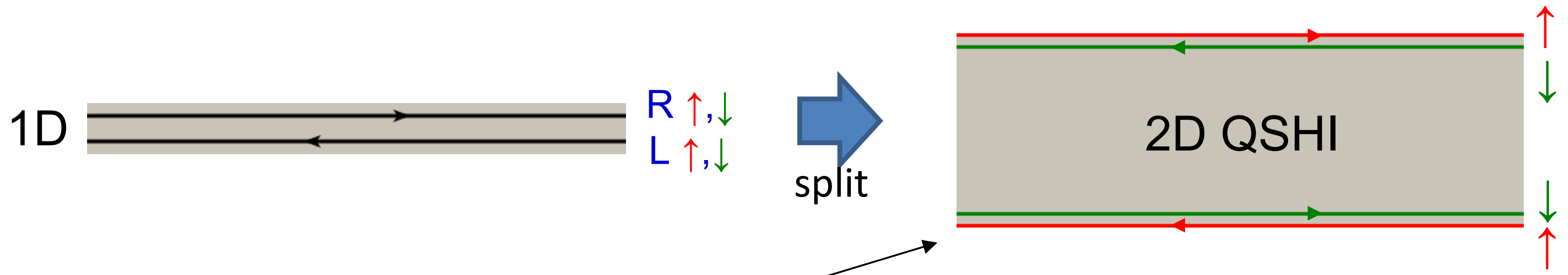
Spin ½ : $\Theta\begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix} = \begin{pmatrix}\psi^*_\downarrow \\ -\psi^*_\uparrow\end{pmatrix}$ $\quad \Theta^2 = -1$

Kramers' Theorem: for spin ½ all eigenstates are at least 2-fold degenerate

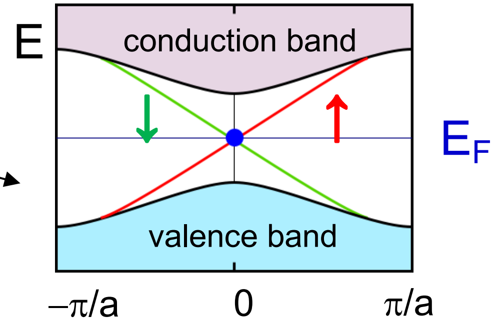
Proof : for a non degenerate eigenstate $\Theta|\chi\rangle = c|\chi\rangle$, $\Theta^2|\chi\rangle = |c|^2|\chi\rangle$ $\quad \Theta^2 = |c|^2 \neq -1$

Time Reversal Invariant Topological Insulator

Quantum spin Hall insulator : Split 1D conductor preserving T symmetry



Helical Edge States :



- Crossing at k=0 is Kramers' degeneracy: protected by time reversal symmetry
- Requires spin – orbit interaction
- Simplest model: two copies of Chern insulator

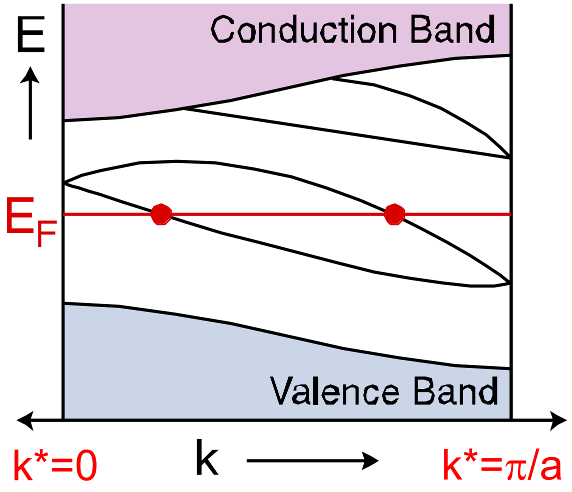
Z_2 Topological Insulator

There are two (and only two) classes of time reversal invariant insulators

Distinguished by Z_2 topological invariant $\nu = 0, 1$

Two patterns of edge states:

ν=0 : Conventional Insulator



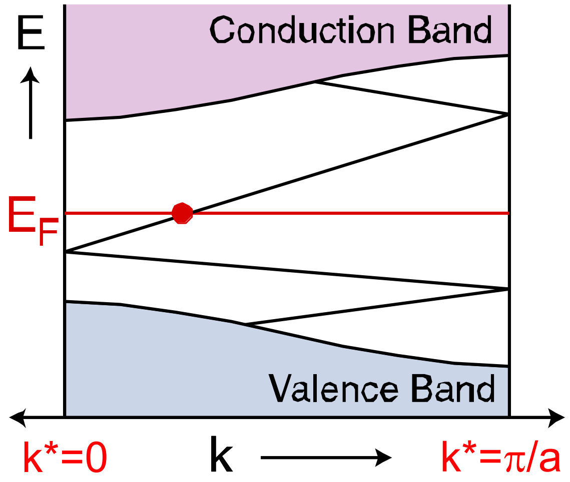
Even number of bands cross Fermi energy

Kramers degenerate at

time reversal invariant momenta

$k^* = -k^* + G$

ν=1 : Topological Insulator



Odd number of bands cross Fermi energy
“impossible” in 1D T-invariant system

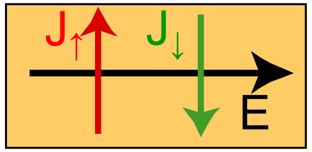
Determining the Z_2 invariant

Simplest for systems with extra symmetry:

1. S_z conserved : independent spin Chern integers :

$$n_\uparrow = -\, n_\downarrow \quad \text{(due to time reversal)}$$

Quantum spin Hall Effect :



$$\boxed{\nu = n_{\uparrow,\downarrow} \ \mathrm{mod}\ 2}$$

2. Inversion (P) Symmetry : determined by Parity of occupied 2D Bloch states

$$P|\psi_n(\Lambda_a)\rangle = \xi_n(\Lambda_a)|\psi_n(\Lambda_a)\rangle$$

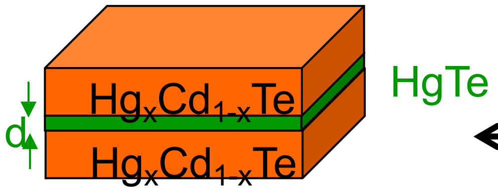
$$\xi_n(\Lambda_a) = \pm 1$$

$$\boxed{(-1)^\upsilon = \prod_{a=1}^{4}\prod_n \xi_{2n}(\Lambda_a)}$$

Numerical methods have been developed to compute the Z_2 invariant in systems without extra symmetry.

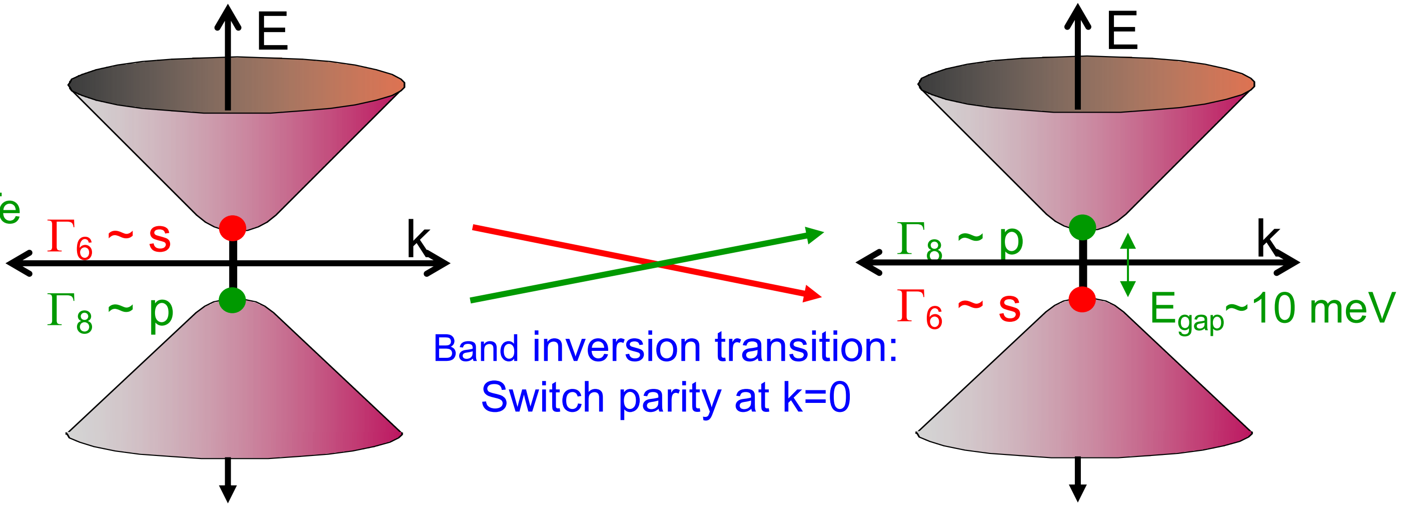
Quantum Spin Hall Effect in HgTe quantum wells

Theory: Bernevig, Hughes and Zhang, Science '06



d < 6.3 nm : Normal band order

d > 6.3 nm : Inverted band order



Conventional Insulator

$$\prod \xi_{2n}(\Lambda_a) = +1$$

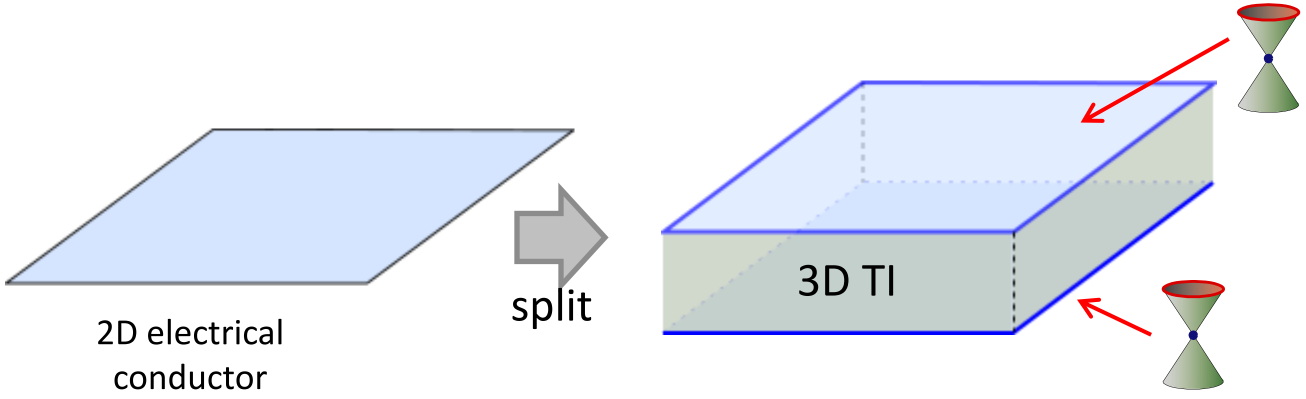
Quantum spin Hall Insulator
with topological edge states

$$\prod \xi_{2n}(\Lambda_a) = -1$$

BHZ Model : 4 band T-invariant band inversion model

$$H(\mathbf{k}) = \tau_z\left(m + ak^2\right) + \mathrm{v}\left(k_x \tau_x \sigma_x + k_y \tau_x \sigma_y\right)$$

3D Z_2 Topological Insulator

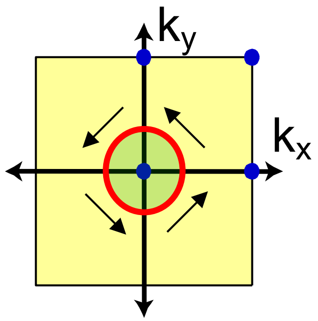


“impossible” 2D T-invariant conductor on surface :

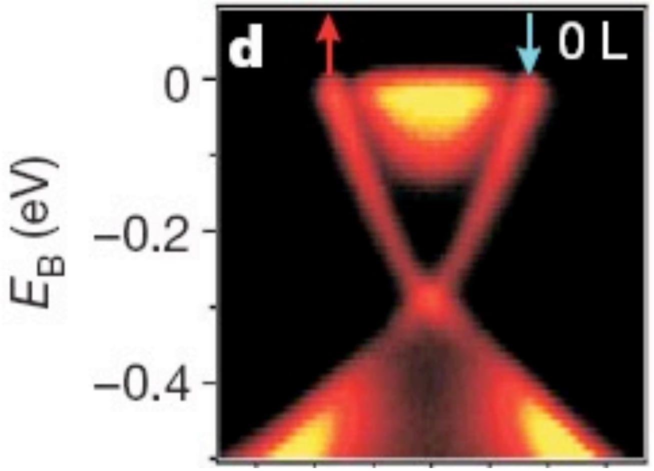
“single Dirac cone” massless quasi-particles

‘Helical metal’ :

- Half the degrees of freedom of ordinary 2DEG
- Berry’s phase π
- Quantized magnetoelectric effect
  - ‘$\theta=\pi$’
  - half quantized surface quantized Hall effect : $\sigma_{xy} = \pm e^2/2h$
- Impossible to localize
  - robust to disorder
  - similar to quantum Hall transition, but tuned by time reversal symmetry



Angle resolved photoemission spectroscopy on Bi_2Se_3



(Xia, …, Hasan et al ‘09)

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \quad \Rightarrow \quad \langle \Psi^\dagger\Psi^\dagger \rangle \Psi\Psi = \Delta^*\Psi\Psi$

$$H = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} \Psi^\dagger & \Psi \end{pmatrix} H_{BdG} \begin{pmatrix} \Psi \\ \Psi^\dagger \end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG} \qquad \Xi\varphi = \tau_x \varphi^* \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

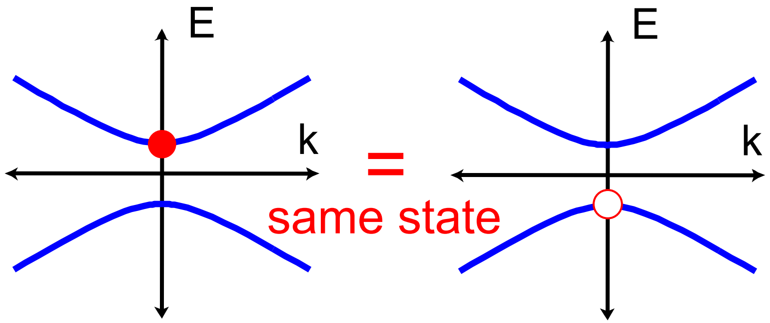
$$\Xi^2 = +1$$

Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

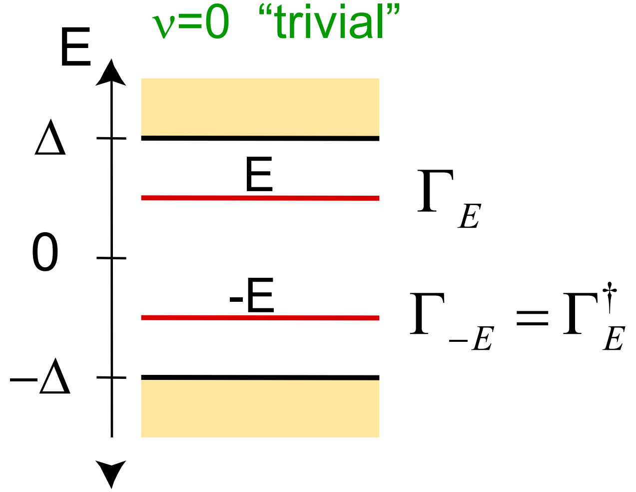
Particle – hole redundancy

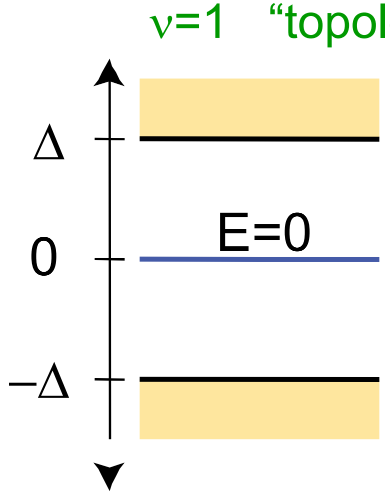
$$\varphi_{-E} = \Xi\varphi_E \quad \Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



1D $\mathbb{Z}_2$ Topological Superconductor : $\nu = 0,1$ (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum





Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion operator: Particle = Antiparticle $\qquad \gamma = \gamma^\dagger$

γ is the *real* part of a Dirac fermion :

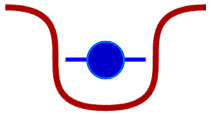
$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases} \qquad \begin{array}{c} \gamma_i^2 = 1 \\ \{\gamma_i, \gamma_j\} = 2\delta_{ij} \end{array}$$

Split the Qubit

Two state system defined fermion occupation of a single state :

$|0\rangle$ = empty
$|1\rangle$ = occupied

$|1\rangle = \Psi^+|0\rangle$



split

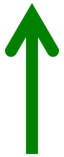
γ_1 1D topological superconductor γ_2

$\Psi^+ = \gamma_1 + i\gamma_2$
$\Psi^+\Psi = 2i\gamma_1\gamma_2$

Two state system with fixed fermion parity

e.g. spin qubit

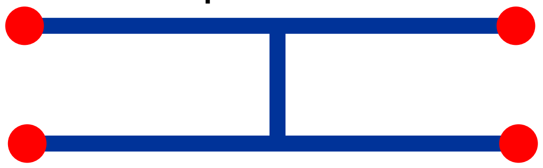
$\vec{S} = \frac{\hbar}{2}\,\vec{\sigma}$



split

2 coupled 1D TSC

γ_1 γ_2



γ_3 γ_4

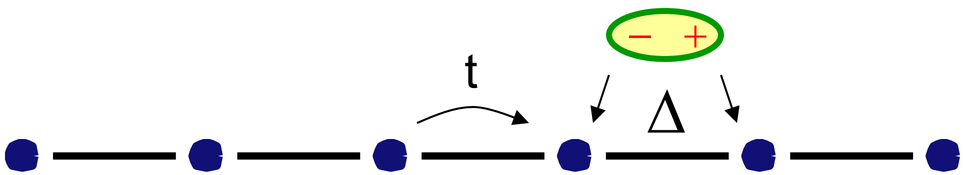
Fixed fermion parity : $(-1)^N = \gamma_1\gamma_2\gamma_3\gamma_4$

$S_z = i\gamma_1\gamma_2$, $S_x = i\gamma_1\gamma_3$, $S_y = i\gamma_2\gamma_3$

Kitaev Model for 1D p wave superconductor

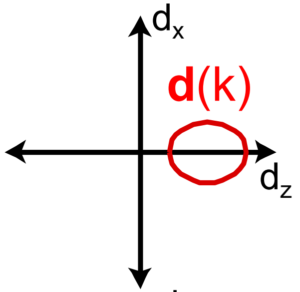
$$H-\mu N=\sum_i t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)-\mu c_i^\dagger c_i+\Delta(c_i c_{i+1}+c_{i+1}^\dagger c_i^\dagger)$$

$$=\sum_k \begin{pmatrix} c_k^\dagger & c_{-k}\end{pmatrix} H_{BdG}(k)\begin{pmatrix} c_k \\ c_{-k}^\dagger\end{pmatrix}$$

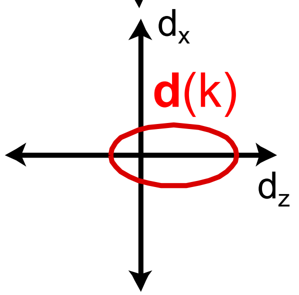


$$H_{BdG}(k)=\tau_z(2t\cos k-\mu)+\tau_x\Delta\sin k=\mathbf{d}(k)\cdot\vec{\tau}$$

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x,d_y,d_z)\big|_k=(-d_x,-d_y,d_z)\big|_{-k}$

Majorana Chain

$$c_i \rightarrow \gamma_{1i} + i\gamma_{2i}$$

$$H = 2i\sum_i t_1\gamma_{1i}\gamma_{2i} + t_2\gamma_{2i}\gamma_{1i+1}$$

$$\mu c_i^\dagger c_i \rightarrow 2i\mu\gamma_{1i}\gamma_{2i}$$

$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \rightarrow 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \rightarrow 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

For Δ=t : nearest neighbor Majorana chain

$$t_1 = \mu, \;\; t_2 = 2t$$