Quantum Connections in Sweden-15
Summer School 2025
Quantum Geometry in Superconductivity, background material

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The purpose of these notes

- To understand my lectures, you should be familiar with the concepts in these notes, at least at some level. The contents are basics of mean-field BCS theory of superconductivity, and the basics of quantum geometry. The latter will be taught in many other lectures of the school, so you will probably learn them during the school, if not already familiar with them.
- I will go through this background information only very briefly in the beginning of my lectures. Therefore, if you're not familiar with it, especially the superconductivity part, and want to understand my lectures well, you may study these notes on your own before the lectures.

The BCS theory of superconductivity

Literature for the BCS theory: A.L. Fetter and J.D. Walecka, Quantum theory of many-particle systems, Dover, Chapters 36-37; P.G. De Gennes, Superconductivity of metals and alloys, Westview Press, Chapters 4-5

1 Fermi sea and the possibility of condensation for fermions

We will learn the basic description of superconductivity/superfluidity of Fermionic interacting particles. Due to the Pauli exclusion principle, identical fermions occupy the energy levels of the system until the Fermi level. We assume you know the concepts of Fermi level μ, Fermi energy E_F, Fermi wave vector k_F (E_F and k_F are given by the non-interacting, $T = 0$ case), and Fermi sphere. Note also the relation between the density $n = N/V$ and Fermi wave vector k_F for spin 1/2 particles, where N is the number of particles and V is the volume:

$$n = \frac{N}{V} = 2\frac{4\pi k_F^3}{3}\frac{1}{(2\pi)^3} = \frac{k_F^3}{3\pi^2}. \tag{1.1}$$

Fermi sea refers to the particles (for instance electrons) that are below the Fermi level. The concept is used often when we wish to remind that the presense of many other particles in different energy states affects the behaviour of what happens to one particle, or to two particles in a scattering process. The existence of the Fermi sea turned out to be essential in explaining the phenomenon of superconductivity, as we will learn in this lecture.

Condensation is by definition a phenomenon where one single quantum state (e.g. the ground state) becomes occupied by a macroscopic number of particles: it cannot happen for noninteracting fermions due to the Pauli exclusion principle.

However, if there is a system with two different types of (non-identical, distinguishable) Fermions, they can via interactions form effective bosons which may

then form a Bose-Einstein condensate (BEC). This is essentially what happens, e.g., in superconductivity in metals and in superfluidity in ultracold Fermi gases.

In metals, the electrons have repulsive interactions due to the Coulomb force. However, the electrons interact also with the lattice and this can cause effective attractive interactions between the electrons. Or, spin fluctuations may cause effetive interactions. Consequently, a spin up and a spin down electron can form a so-called Cooper pair which is effectively a boson. Superconductivity is the condensation of these Cooper pairs.

2 Cooper instability

L.N. Cooper presented the following calculation:
Consider
- the scattering of two particles
- which have an attractive interaction
- in the presence of a Fermi sea (restricting the possible momenta where the particles can scatter).

Based on this, he showed that even for **arbitrarily small interactions**, pairs (later named as Cooper pairs) will be formed in the system. This is referred to as **instability of the Fermi sea**: pair formation happens for any non-zero, attractive interaction. Thus it is enough to consider only two-particle scattering (i.e. not many-body physics), added with the constraint of the Fermi sea (which is a many-body effect), to predict that there will be pairing. It is interesting to note that also here the simple existence of the Fermi sea and its ability to restrict the phase space for scattering is crucial; remember above where the restriction of the phase space for scattering was essential to the existence of well-defined (long-lived) quasiparticles in the Fermi liquid.

Condensation and superfluidity are essentially many-body effects, and to predict and describe them one needs a many-body description. We will learn below the BCS (Bardeen-Cooper-Schrieffer) mean-field description of superconductivity/superfluidity.

3 The BCS theory

Let us consider a system of two types of spin up and spin down electrons. The Hamiltonian, using field operators, is

$$H = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m_\sigma} - \mu_\sigma \right) \psi_\sigma(\mathbf{r}) +$$

$$+\frac{1}{2} \sum_{\alpha,\beta=\uparrow,\downarrow,\alpha\neq\beta} \int d\mathbf{r} \int d\mathbf{r}' V_{\uparrow\downarrow}(\mathbf{r},\mathbf{r}') \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}') \psi_\beta(\mathbf{r}') \psi_\alpha(\mathbf{r}) . \qquad (3.1)$$

We set here

$$\mu_\uparrow = \mu_\downarrow, \quad m_\uparrow = m_\downarrow. \qquad (3.2)$$

Note that here the only degree of freedom, in addition to the spatial one, is the spin. This could describe spin up and spin down fermions in a continuum system,

or in a lattice with only one (relevant) orbital and thus only one band. In the rest of the lectures, however, we will introduce orbital degrees of freedom, because the quantum geometric effects of superconductivity arise from them. However, in order to learn the basics, we consider here this simplest possible case.

For pedagogical purposes, let us assume that the interaction is a contact interaction; even when interctions in real materials are more complicated, the description using this simplification gives some valuable qualitative insight to superconductivity:

$$V_{\uparrow\downarrow}(\mathbf{r},\mathbf{r}') = V_0\delta(\mathbf{r}-\mathbf{r}'). \tag{3.3}$$

The Hamiltonian becomes

$$H = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} - \mu\right) \psi_\sigma(\mathbf{r}) + \\ + V_0 \int d\mathbf{r} \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}). \tag{3.4}$$

3.1 The mean-field approximation

It is usually extremely difficult or practically impossible to solve exactly a quantum many-body system with a large number of particles. One of the most used approximations to make the problems tractable is the **mean-field approximation**. This means basically that we replace some of the operators in the Hamiltonian by their mean values (which are just complex numbers, not operators), assuming that the deviations from the mean values are small.

Let us consider the example of operators A and B and their product AB. Let us write the operators A and B as their mean values and the deviation from the mean value (no approximation made at this point, just rewriting), and then calculate AB:

$$A = \langle A\rangle + \delta A \tag{3.5}$$

$$B = \langle B\rangle + \delta B \tag{3.6}$$

$$\Rightarrow$$

$$AB = \langle A\rangle\langle B\rangle + \langle A\rangle\delta B + \langle B\rangle\delta A + \delta A\delta B. \tag{3.7}$$

Assuming that the fluctuations around the mean values are small, one can neglect the last term which is quadratic in the fluctuations, i.e.

$$\delta A\delta B \approx 0 \tag{3.8}$$

Then insert $\delta A = A - \langle A\rangle$, $\delta B = B - \langle B\rangle$ in (3.7)

$$\Rightarrow$$

$$AB = \langle A\rangle B + \langle B\rangle A - \langle A\rangle\langle B\rangle. \tag{3.9}$$

One can generalize this kind of consideration for products of more than two operators (Wick's theorem).

Now we do the BCS mean-field approximation to the interaction term of the Hamiltonian, leading to Hartree-fields and pairing-fields (Fock-fields are zero in the BCS theory). Let us denote $\psi_\sigma(\mathbf{r}) \equiv \psi_\sigma$. Terms of the form $\langle\rangle\langle\rangle$ are neglected here (they are just numbers, not operators, and thus will cause only a constant shift in the energy; note, however, that they can be important sometimes, for instance if one is interested in the absolute energy of the state e.g. compared to some other state).

Note that here we organize the operators in *pairs* and then take the expectation values, unlike in the example above where we had expectation values of single operators A and B. This is done because in a Fermi system which is expected to show pairing correlations, this is a good choice: expectation values of single fermion operators are zero in this case. In general, when doing a mean-field approximation, some pre-knowledge or an educated guess/argument about the relevant non-zero expectation values and correlations are needed. When organizing the fermion operators in pairs, one has to sometimes move them with respect to each other. This may give minus signs due to the Fermi statistics, and one has to keep track on them. To understand these things more deeply, you may search more information on the topic "Wick's theorem".

$$\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow = \langle\psi_\downarrow\psi_\uparrow\rangle\,\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \left\langle\psi_\uparrow^\dagger\psi_\downarrow^\dagger\right\rangle\psi_\downarrow\psi_\uparrow + \left\langle\psi_\uparrow^\dagger\psi_\uparrow\right\rangle\psi_\downarrow^\dagger\psi_\downarrow + \left\langle\psi_\downarrow^\dagger\psi_\downarrow\right\rangle\psi_\uparrow^\dagger\psi_\uparrow$$

$$- \left(\left\langle\psi_\uparrow^\dagger\psi_\downarrow\right\rangle\psi_\downarrow^\dagger\psi_\uparrow + \left\langle\psi_\downarrow^\dagger\psi_\uparrow\right\rangle\psi_\uparrow^\dagger\psi_\downarrow\right) \tag{3.10}$$

Pairing fields $\langle\psi_\downarrow\psi_\uparrow\rangle\,\psi_\uparrow^\dagger\psi_\downarrow^\dagger$, $\left\langle\psi_\uparrow^\dagger\psi_\downarrow^\dagger\right\rangle\psi_\downarrow\psi_\uparrow$

Hartree fields $\left\langle\psi_\uparrow^\dagger\psi_\uparrow\right\rangle\psi_\downarrow^\dagger\psi_\downarrow$, $\left\langle\psi_\downarrow^\dagger\psi_\downarrow\right\rangle\psi_\uparrow^\dagger\psi_\uparrow$

The **Fock fields** $\left\langle\psi_\uparrow^\dagger\psi_\downarrow\right\rangle = \left\langle\psi_\downarrow^\dagger\psi_\uparrow\right\rangle = 0$ here, so the corresponding terms in (3.10) are zero.

Note that $\langle\psi_\downarrow\psi_\uparrow\rangle \neq 0$, c.f. $\langle a\rangle \neq 0$ for BEC.

The **order parameter** of the BCS theory turns out to be given by the pairing fields in the following way:

$$\Delta(\mathbf{r}) = V_0\langle\psi_\downarrow(\mathbf{r})\,\psi_\uparrow(\mathbf{r})\rangle\,. \tag{3.11}$$

This quantity will also be the energy gap in the excitation spectrum of the BCS theory, as we will see later. One can also assume that it does not depend on the spatial coordinate, this is a good assumption for a homogeneous system within the usual BCS theory:

$$\Delta(\mathbf{r}) = \Delta = \Delta^*. \tag{3.12}$$

We also insert the Hartree fields within redefined chemical potentials. Denoting $\left\langle\psi_\sigma^\dagger\psi_\sigma\right\rangle = n_\sigma$, we get

$$-\mu\psi_\uparrow^\dagger\psi_\uparrow + V_0 n_\downarrow\psi_\uparrow^\dagger\psi_\uparrow = -\tilde{\mu}\psi_\uparrow^\dagger\psi_\uparrow \tag{3.13}$$

$$\tilde{\mu} = \mu - V_0 n_\downarrow. \tag{3.14}$$

The Hamiltonian becomes now (terms of the form $\langle\rangle\,\langle\rangle$ are neglected here)

$$H = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} - \tilde{\mu} \right) \psi_\sigma(\mathbf{r}) + \\ + \int d\mathbf{r} \left(\Delta \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) + \Delta \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}) \right). \tag{3.15}$$

Now, let us expand the field operators using the annihilation operators for momentum states

$$\psi_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} c_{\mathbf{k}\sigma}. \tag{3.16}$$

The Hamiltonian is then (it is a nice exercise to calculate this):

$$H = \sum_{\mathbf{k}} \xi_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} + \xi_{\mathbf{k}} c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} + \Delta c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger + \Delta c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \tag{3.17}$$

where

$$\xi_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m} - \tilde{\mu}. \tag{3.18}$$

3.2 Diagonalizing the Hamiltonian: the Bogoliubov transformation

When describing a quantum system, the first thing to do is find the eigenvalues and eigenfunctions of the Hamiltonian. Since we have done the mean-field approximation, the Hamiltonian has transformed into the above simple quadratic form which in fact can be written in matrix form and diagonalized, leading to results that give a lot of insight into the system. The Hamiltonian in the matrix form is (the coefficient A is straightforward to calculate):

$$H = A + \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}. \tag{3.19}$$

The Hamiltonian is hermitian, so there exists a unitary transformation that diagonalizes it, i.e. we can write

$$H = A + \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} UU^\dagger \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} UU^\dagger \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} \tag{3.20}$$

$$= A + \sum_{\mathbf{k}} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow}^\dagger & \gamma_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} E_{\uparrow\mathbf{k}} & 0 \\ 0 & -E_{\downarrow\mathbf{k}} \end{pmatrix} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}, \tag{3.21}$$

where we have defined

$$\begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} = U^\dagger \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} \tag{3.22}$$

and

$$\begin{pmatrix} E_{\uparrow\mathbf{k}} & 0 \\ 0 & -E_{\downarrow\mathbf{k}} \end{pmatrix} = U^\dagger \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} U \tag{3.23}$$

is the diagonalized Hamiltonian, with the eigenvalues $E_{\sigma\mathbf{k}}$. The minus sign in the second eigenvalue is chosen for convenience. Writing

$$U = \begin{pmatrix} u_{\uparrow\mathbf{k}} & u_{\downarrow\mathbf{k}} \\ v_{\uparrow\mathbf{k}} & v_{\downarrow\mathbf{k}} \end{pmatrix}, \tag{3.24}$$

and multiplying both sides of Eq. (3.23) by U from the left leads to the eigenvalue equations

$$\begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} u_{\sigma\mathbf{k}} \\ v_{\sigma\mathbf{k}} \end{pmatrix} = \pm E_{\sigma\mathbf{k}} \begin{pmatrix} u_{\sigma\mathbf{k}} \\ v_{\sigma\mathbf{k}} \end{pmatrix}, \quad \sigma = \uparrow, \downarrow, \tag{3.25}$$

where the plus corresponds to $\sigma = \uparrow$ and minus to $\sigma = \downarrow$. Solving these, one obtains the eigenvalues

$$E_{\uparrow\mathbf{k}} = E_{\downarrow\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}, \tag{3.26}$$

and the eigenvectors are given by

$$\xi_{\mathbf{k}} u_{\sigma\mathbf{k}}^2 + \Delta u_{\sigma\mathbf{k}} v_{\sigma\mathbf{k}} = E_{\sigma\mathbf{k}} u_{\sigma\mathbf{k}}^2, \tag{3.27}$$

$$-\xi_{\mathbf{k}} v_{\sigma\mathbf{k}}^2 + \Delta u_{\sigma\mathbf{k}} v_{\sigma\mathbf{k}} = E_{\sigma\mathbf{k}} v_{\sigma\mathbf{k}}^2. \tag{3.28}$$

Note that since $\xi_{\mathbf{k}}$ and Δ are real, U is real as well. The unitarity $U^\dagger U = I$ implies $u_{\sigma\mathbf{k}}^2 + v_{\sigma\mathbf{k}}^2 = 1$, from which we obtain

$$u_{\mathbf{k}} = u_{\uparrow\mathbf{k}} = v_{\downarrow\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)} \tag{3.29}$$

$$v_{\mathbf{k}} = v_{\uparrow\mathbf{k}} = -u_{\downarrow\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)} \tag{3.30}$$

$$\Rightarrow$$

$$u_{\mathbf{k}} v_{\mathbf{k}} = \frac{1}{2} \frac{\Delta}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}. \tag{3.31}$$

The unitary transformation above is called the Bogoliubov transformation. It defines the relations between the original annihilation operators and the operators corresponding to the diagonalized Hamiltonian:

$$U \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}. \tag{3.32}$$

$$\begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{k}\uparrow}^\dagger & \gamma_{-\mathbf{k}\downarrow} \end{pmatrix} U^\dagger \tag{3.33}$$

Thus

$$c_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} - v_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow}^\dagger \tag{3.34}$$

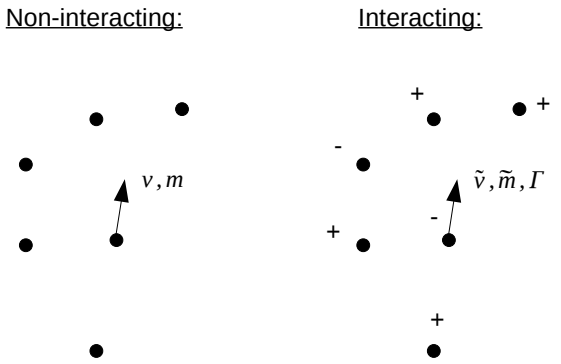
$$c_{-\mathbf{k}\downarrow}^\dagger = v_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + u_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow}^\dagger \tag{3.35}$$

and

$$\gamma_{\mathbf{k}\uparrow} = u_{\mathbf{k}} c_{\mathbf{k}\uparrow} + v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \tag{3.36}$$

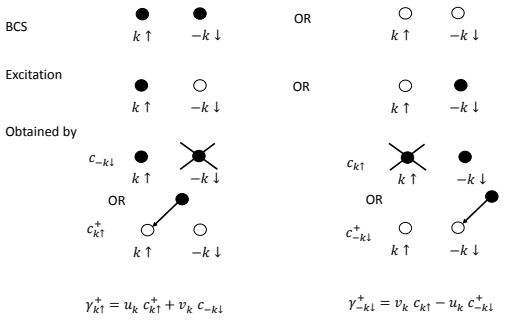
$$\gamma^{\dagger}_{-\mathbf{k}\downarrow} = v_{\mathbf{k}} c_{\mathbf{k}\uparrow} - u_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow}. \tag{3.37}$$

The γ's are called **quasiparticle** annihilation (creation) operators. They fulfill the same fermionic anticommutation relations than the original operators, thus they are well-defined fermionic quasiparticles. A quasiparticle is, in general, an excitation of an interacting system, characterized e.g. by its energy, effective mass, life time, etc. Compare to the usual concept of a quasiparticle in a Fermi liquid:



In the BCS theory, the quasiparticles are linear combinations of annihilation and creation operators of opposite spins, so it is not easy to make an intuitive picture about them. They are superpositions of a particle, and a hole of opposite spin and momentum. In the Fermi liquid, the quasiparticles could be understood, roughly, as the original particles "dressed" by the presence of and interactions with the other particles. Here the quasiparticles are somewhat different, they are superpositions of particles and holes and thus describe the pairing correlations present in the BCS state.

The nature of the quasiparticles is described in the below picture. In the BCS state, one either has, with some probability, both $\mathbf{k}$ and $-\mathbf{k}$ states occupied, or both empty. An excitation means that one of these states can be occupied while the other one is empty. As the picture shows, there are two routes for forming such an excitation: to destroy one particle, or to create another. The quasiparticle is a superposition of these possibilities.



At zero temperature, there are no quasiparticles in the BCS description. This is easy to see: the Hamiltonian is diagonal when expressed with the quasiparticle operators. This means that the quasiparticles are non-interacting (remember that in the Fermi liquid theory, the residual interactions between the quasiparticles were essential), and we can directly apply the statistical physics of non-interacting fermionic particles. For instance, the occupation number of the quasiparticles is simply given by the Fermi distribution:

$$\left\langle \gamma^\dagger_{\mathbf{k}\sigma}\gamma_{\mathbf{k}\sigma}\right\rangle = \frac{1}{e^{\frac{E_\sigma}{kT}}+1} \equiv f\left(E_\sigma, T\right). \qquad \sigma = \uparrow,\downarrow \tag{3.38}$$

Note also that

$$\left\langle \gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\downarrow}\right\rangle = \left\langle \gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{\mathbf{k}\downarrow}\right\rangle = 0. \tag{3.39}$$

The eigenvalues are always positive, thus in the limit $T = 0$ the occupation numbers go to zero. Therefore, in the ground state there are no quasiparticles. By giving energy to the system, one may create excitations, that is, quasiparticles. Breaking a Cooper pair is equivalent to creating two quasiparticles. To create two quasiparticles, the energy of $2E_{\mathbf{k}}$ is needed:

$$E_{\uparrow\mathbf{k}} + E_{\downarrow\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + \Delta^2} + \sqrt{\xi^2_{\mathbf{k}} + \Delta^2}. \tag{3.40}$$

The minimum of $\xi^2_{\mathbf{k}} = \left(\frac{\hbar^2\mathbf{k}^2}{2m} - \tilde{\mu}\right)^2$ is zero, thus

$$\left[E_{\uparrow\mathbf{k}} + E_{\downarrow\mathbf{k}}\right]_{\min} = 2\Delta. \tag{3.41}$$

This means that the order parameter defines the energy gap for creating excitations. The existence of an energy gap is behind many important properties: one cannot make single particle excitations to the system if one tries to give or take an amount of energy that is less than the value of the energy gap. In other words, dissipation is restricted. This is basically why supercurrents can flow without resistance. However, to really prove the existence of supercurrents and superflows one has to describe the dynamics of the system in a way that goes beyond this lecture. To describe collective modes of the superfluid, one has to introduce interactions between the quasiparticles (this can be conveniently done, e.g. with the so-called generalized random phase approximation (GRPA)), just like in the case of the Fermi liquid. However, in case of the BCS theory, one can predict many important phenomena,

such as the existence of an energy gap, even without considering quasiparticle interactions.

3.3 The gap equation

Now, let us see how does one actually calculate the value of the order parameter. Using the Bogoliubov transformation, the order parameter becomes:

$$\Delta = -\frac{V_0}{V}\sum_{\mathbf{k}}\langle c_{\mathbf{k}\uparrow}c_{-\mathbf{k}\downarrow}\rangle = -\frac{V_0}{V}\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}\left(1-\left\langle\gamma^{\dagger}_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow}\right\rangle-\left\langle\gamma^{\dagger}_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow}\right\rangle\right) \tag{3.42}$$

$$\Rightarrow$$

$$1 = -\frac{V_0}{V}\sum_{\mathbf{k}}\frac{1-f\left(E_{\uparrow\mathbf{k}},T\right)-f\left(E_{\downarrow\mathbf{k}},T\right)}{2\sqrt{\xi^2_{\mathbf{k}}+\Delta^2}}. \tag{3.43}$$

This is the so-called **gap equation**. Because we assumed a contact interaction, this equation is actually divergent. This unphysical divergence can be avoided by renormalization, for instance, by simply removing the diverging part

$$\frac{1}{V_0} \to \frac{1}{T^{2B}} - \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2\epsilon_{\mathbf{k}}}, \tag{3.44}$$

where $\epsilon_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$, and the two-body T-matrix $T^{2B} = \frac{4\pi a_s\hbar^2}{m}$ describes the scattering for two particles.

Usually the gap equation is written in the continuum

$$\frac{1}{V}\sum_{\mathbf{k}} \to \left(\frac{1}{2\pi}\right)^3\int d^3\mathbf{k} \tag{3.45}$$

$$\Rightarrow$$

$$1 = -\frac{T^{2B}}{2\pi^2}\int_0^{\infty} k^2 dk\left(\frac{1-f\left(E_{\uparrow\mathbf{k}},T\right)-f\left(E_{\downarrow\mathbf{k}},T\right)}{2\sqrt{\xi^2_{\mathbf{k}}+\Delta^2}}-\frac{1}{2\epsilon_{\mathbf{k}}}\right). \tag{3.46}$$

From this form, the order parameter Δ can be solved numerically, at a given temperature T and for a given interaction strength and a chemical potential. If the order parameter Δ has a finite value, one generally has a superconducting/superfluid ground state (i.e. a condensate of Cooper pairs). If it is zero, the ground state is simply a normal state. Thus one can obtain the critical temperature where the condensation happens by setting $\Delta = 0$ in the gap equation and then solving the temperature T from the equation. This can be done analytically (with some approximations) and leads to

$$T_c = \frac{8E_F}{k_B\pi}e^{\gamma-2}\exp\left(-\frac{\pi}{2k_F\left|a_s\right|}\right), \tag{3.47}$$

where γ is Euler's constant.

One can see from the above that the critical temperature cannot be developed into a Taylor series with respect to the scattering length (interaction strength) a_S. This means that one cannot predict superconductivity from the ideal gas ($a_S = 0$) by perturbation theory!

3.4 The BCS wave function

Apart from the diagonalization by the Bogoliubov transformation, an alternative route leading to the same end-results is a **variational calculation** where one makes a parametrized guess (ansatz) for the ground state of the system, calculates the expectation value of the Hamiltonian using the guess wave function, and then minimizes this expectation value, leading to certain values for the parameters. If the BCS ansatz gives a lower energy than the normal state wave function, then one knows that the normal state is not the ground state.

The BCS ansatz (the BCS wavefunction) is of the following form:

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} \right) |0\rangle \,. \tag{3.48}$$

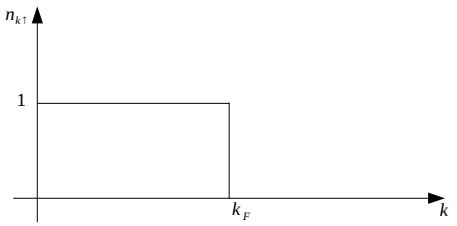
One can then calculate

$$\langle \mathrm{BCS}| \, H \, |\mathrm{BCS}\rangle \tag{3.49}$$

and minimize it. The minimization gives values for the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, and they are the same as found above by the Bogoliubov transformation.

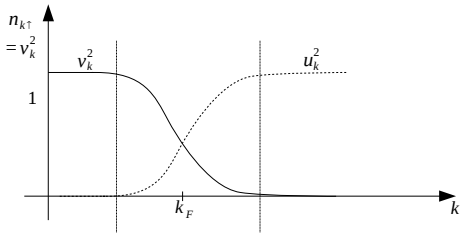
The BCS wave function is quite intuitive. For the noninteracting case, it becomes trivial: all the $v_{\mathbf{k}}$'s are unity below the Fermi level and zero above it ($v_{\mathbf{k}} = 1$ for $|\mathbf{k}| \leq |\mathbf{k}_F|$), and for $u_{\mathbf{k}}$ vice versa ($u_{\mathbf{k}} = 0$ for $|\mathbf{k}| \leq |\mathbf{k}_F|$). The momentum distributions show a sharp edge. A sharp feature at the edge tells about the existence of a well-defined **Fermi surface**:

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k} \leq \mathbf{k_F}} c^{\dagger}_{-\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} |0\rangle \,. \tag{3.50}$$



$n_{k\uparrow} = \langle c^{+}_{k\uparrow} c_{k\uparrow} \rangle = v^2_k$ *(exercise)*

When there are interactions and a finite order parameter Δ, for certain momenta both the $u_{\mathbf{k}}$'s and $v_{\mathbf{k}}$'s are nonzero. The momentum distributions become smoothened and there is no more a well defined Fermi surface (sharp edge/drop in the momentum distribution like for the normal state or the Fermi liquid).



$\Delta \propto u_k v_k \rightarrow$ the particles around the Fermi level form Cooper pairs

Basics of quantum geometry and topology

Literature: R. Resta, The insulating state of matter: a geometrical theory, The European Physical Journal B 79, 121 (2011); B.A. Bernevig and T.L. Hughes, Topological Insulators and Topological Superconductors, Princeton University Press (2013).

A key question in quantum physics is how to classify, in a conceptually elegant and efficient way, the various states of matter that have been observed in nature, and which nowadays can be created by artificial quantum systems and simulators. The band structure theory for electrons in solids has been quite powerful in this: we know that electrons in periodic potentials (i.e., lattices, formed for example by the crystal of nuclei) may have a band structure. If the Fermi level of the system is within the band gap, we have an *insulator*. If the band-gap is small, the system is *semiconducting*. If the Fermi level is in the conduction band, we have a *metal*. However, there are also concepts like the *Mott insulator*: there the insulating behaviour is not explained by the simple band theory, but requires taking into account the strong interactions (correlations) in the system. Another example of an insulator not explained by band theory is the *Anderson insulator*, where disorder felt by the electrons is the underlying cause of non-conductance of current. In recent years, the concept of a *topological insulator* has become extremely important. There the insulating behaviour results from quantum geometric and topological properties of the system. An interesting feature of topological insulators is that, even when the bulk material is insulating, there can be transport (current) on the surface of the material. This current is robust against perturbations (such as scattering from a material defect) due to the topological properties. In practice, for instance, backscattering of electrons from a defect can be forbidden due to topological properties.

In recent times, it has become more and more clear that a key underlying concept for classifying different states of matter are the localization properties of wavefunctions that describe the system. The eigenfunctions of a periodic system are the Bloch functions, parametrized by the lattice wavevector (also called crystal quasimomentum) $\mathbf{k}$ which is a good quantum number in a perfectly periodic system:

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}), \tag{3.51}$$

where $u_{\mathbf{k}}(\mathbf{r})$ is periodic with the lattice period. One can transform the Bloch functions to define so-called Wannier functions

$$w_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}\in BZ} e^{-i\mathbf{k}\cdot\mathbf{R}} \psi_{\mathbf{k}}(\mathbf{r}), \tag{3.52}$$

where N is the number of lattice sites, the $\mathbf{k}$ summation is over the first Brillouin zone (BZ), and $\mathbf{R}$ a vector of the Bravais lattice (that is, the position of one lattice site; since in an infinite lattice they are all equal, it does not matter which one we choose). The Wannier function characterizes how the particle (e.g., electron) is spread around the position $\mathbf{R}$ (a useful discussion on Wannier functions can be found for instance in Marzari *et al.*, Rev. Mod. Phys. 84, 1419 (2012)). Whether the Wannier functions are delocalized, or localized over some finite range of lattice sites, whether they overlap, etc. has turned out to be a crucial feature determining the state of the system. It is easy to understand that totally localized and disconnected Wannier functions lead to an insulating state while completely delocalized functions help carry current. Localized but overlapping functions are an interesting intermediate case. The story of course becomes more complicated when the particles interact with each other.

The properties of Wannier functions naturally depend on the Bloch function. The key properties turn out to be those related to the quantum geometry and topology of the system. The basic concepts of quantum geometry and topology, such as Berry phase, Berry connection, Berry curvature, Chern number and quantum metric have become important basic building blocks of modern quantum physics.

3.5 Geometry in quantum mechanics: phase and distance

Let us consider quantum states that are eigenstates of the Schrödinger equation

$$H(\mathbf{k})|\Psi(\mathbf{k})\rangle = E_{\mathbf{k}}|\Psi(\mathbf{k})\rangle. \tag{3.53}$$

Here $\mathbf{k}$ is a parameter (real number); for instance in a lattice system it could be the lattice wavevector, but the discussion here is completely general and it can be something else. The set of wavevectors $|\Psi(\mathbf{k})\rangle$ that fulfil the equation (3.53) form a sub-manifold of the Hilbert space. In the lattice case they would be the Bloch functions of the different energy bands of the lattices. Interesting quantum geometrical or topological effects usually require a multi-band (multi-component) system, where the bands (components) can come for instance from lattice geometry, effective finite lattice unit cell size (caused, e.g., by an effective or real magnetic field), or existence of two spin components and spin-orbit coupling.

In classical physics, the definition of distance between two points is quite straightforward: draw the shortest possible line between them and measure it. Of course, this is more tricky if the geometry is non-trivial: for instance, on the surface of a sphere the distance between two points is defined differently, as function of the coordinates, than on a flat surface. Curved space-time geometries appear also in the context of general relativity. The quantity that takes into account the geometry of the system in defining the distance is called **metric**.

Now, we may ask whether it is possible to define the distance between quantum states on a given manifold (for instance the sub-manifold defined by Equation (3.53)). We use the Bures distance (there are other definitions too, for instance the standard definition of distance in a Hilbert space has a factor of two difference to the Bures distance):

$$D_{12} = \sqrt{1 - |\langle\Psi(\mathbf{k}_1)|\Psi(\mathbf{k}_2)\rangle|^2}. \tag{3.54}$$

As one can easily see, the distance is zero for equal states, and one for orthogonal ones. Since the definition contains square of the inner product, we obviously loose any information related to phase of the inner product. Indeed one can define also the concept of phase difference:

$$e^{-i\Delta\varphi_{12}} = \frac{\langle\Psi(\mathbf{k}_1)|\Psi(\mathbf{k}_2)\rangle}{|\langle\Psi(\mathbf{k}_1)|\Psi(\mathbf{k}_2)\rangle|} \tag{3.55}$$

$$\Delta\varphi_{12} = -Im\log\langle\Psi(\mathbf{k}_1)|\Psi(\mathbf{k}_2)\rangle. \tag{3.56}$$

You have probably learned in earlier quantum mechanics courses that one can always add an arbitrary phase factor to a quantum state and it does not change any observable quantity. In another language, multiplying a quantum state by an arbitrary phase factor is a so-called gauge transformation, and the overall phase of the wavefunction is a gauge-dependent quantity. Gauge-dependent quantities are not physical properties of the system, in the sense that they cannot be observed. The phase difference defined above is gauge-dependent, and therefore does not as such have a physical meaning. This relation holds also the other way round: if one

can show that some quantity is gauge independent, it is physically meaningful and can (at least in principle) be measured. The Bures distance is an example of such gauge-independent, measurable quantity.

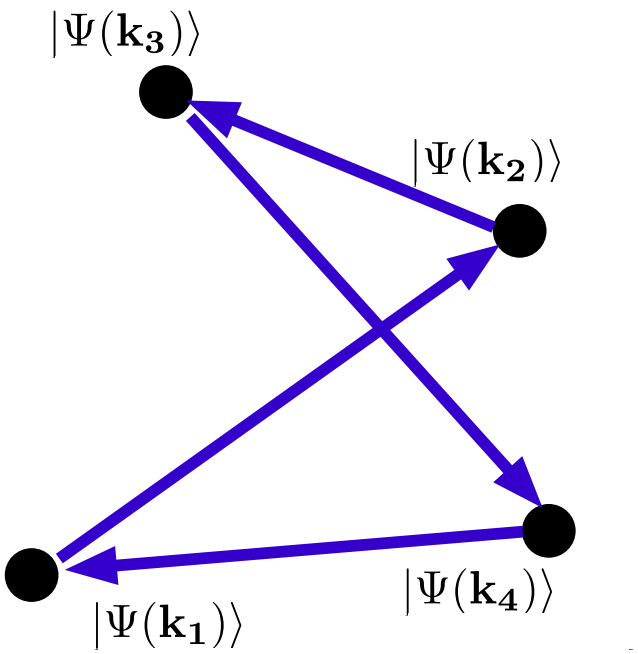


Figure 1: A closed path that connects four states in the $\mathbf{k}$ space.

So why are we still interested in the phase difference? Well, let us see what we get if we try to measure the total phase difference when we make a closed loop between a set of states, see Figure 1, that is, calculate the phase difference between states 1 and 2, then 2 and 3, 3 and 4, and finally between 4 and 1, and sum them. The total phase difference becomes

$$\begin{aligned} \gamma &= \Delta\varphi_{12} + \Delta\varphi_{23} + \Delta\varphi_{34} + \Delta\varphi_{41} \\ &= -Im \log\langle\Psi(\mathbf{k}_1)|\Psi(\mathbf{k}_2)\rangle\langle\Psi(\mathbf{k}_2)|\Psi(\mathbf{k}_3)\rangle\langle\Psi(\mathbf{k}_3)|\Psi(\mathbf{k}_4)\rangle\langle\Psi(\mathbf{k}_4)|\Psi(\mathbf{k}_1)\rangle. \end{aligned} \tag{3.57}$$

Now all gauge-arbitrary phases cancel. Example: if for instance $|\Psi(\mathbf{k}_1)\rangle = |1\rangle + e^{i\phi}|2\rangle$ where $|1\rangle$, $|2\rangle$ are some orthogonal basis states, then the state $e^{i\theta}|\Psi(\mathbf{k}_1)\rangle = e^{i\theta}(|1\rangle + e^{i\phi}|2\rangle)$ is a physically equivalent state and the gauge-arbitrary phase θ cancels away. However, the phase ϕ is physically meaningful and affects the value of the inner products with $|\Psi(\mathbf{k}_2)\rangle$ and $|\Psi(\mathbf{k}_4)\rangle$, if those states contain $|2\rangle$.

As a gauge invariant quantity, the total phase difference (3.57) is potentially an observable quantity! Let us now consider, instead of four states, a smooth curve in the parameter space $\mathbf{k}$, and discretize it. The phase difference between two points in the curve separated by a small distance $\Delta\mathbf{k}$ is

$$e^{-i\Delta\varphi} = \frac{\langle\Psi(\mathbf{k})|\Psi(\mathbf{k}+\mathbf{\Delta k})\rangle}{|\langle\Psi(\mathbf{k})|\Psi(\mathbf{k}+\mathbf{\Delta k})\rangle|} \tag{3.58}$$

$$\Delta\varphi = -Im \log\langle\Psi(\mathbf{k})|\Psi(\mathbf{k}+\mathbf{\Delta k})\rangle. \tag{3.59}$$

If the phase varies in a differentiable way, then (take a Taylor series of the above and keep terms up to the order Δk)

$$i\Delta\varphi \simeq \langle\Psi(\mathbf{k})|\nabla_{\mathbf{k}}\Psi(\mathbf{k})\rangle \cdot \Delta\mathbf{k}. \tag{3.60}$$

If the set of points in the path becomes dense, that is, $\Delta\mathbf{k}$ is infinitesimally small, we can write the discrete sum of distances from point 1 to point M as an integral over the corresponding smooth curve C from 1 to M:

$$\gamma = \sum_{s=1}^{M} \Delta\varphi_{s,s+1} \longrightarrow \gamma = \int_C \mathcal{A} \cdot d\mathbf{k}. \tag{3.61}$$

Here the vector $\mathcal{A}$ is called the **Berry connection**:

$$\mathcal{A} = i\langle\Psi(\mathbf{k})|\nabla_{\mathbf{k}}\Psi(\mathbf{k})\rangle. \tag{3.62}$$

The state vectors are assumed to be normalized at any $\mathbf{k}$ which means that the Berry connection is real (just take the derivative, using the chain rule, of both sides of $\langle\Psi|\Psi\rangle^2 = 1$). Therefore one can also write

$$\mathcal{A} = -Im\langle\Psi(\mathbf{k})|\nabla_{\mathbf{k}}\Psi(\mathbf{k})\rangle. \tag{3.63}$$

The integral γ is called the **Berry phase**. It has already been observed in numerous physical systems and has become an important concept of modern quantum (and classical optical) physics.

3.5.1 Berry curvature

If the curl of the Berry connection is well-defined on a surface Σ whose boundary is the curve C (notation $C = \partial\Sigma$), then one can use Stokes theorem to write the Berry phase in an alternative way

$$\gamma = \int_{\partial\Sigma} \mathcal{A} \cdot d\mathbf{k} = \int_{\Sigma} \mathbf{\Omega} \cdot \mathbf{n} d\sigma. \tag{3.64}$$

Stokes theorem transforms a line integral to an integral over an area, with the integrand replaced by its curl. Here $\mathbf{n}$ is a vector normal to the surface that is integrated over. The quantity $\mathbf{\Omega}$ is the **Berry curvature**

$$\begin{aligned} \mathbf{\Omega} &= \nabla_{\mathbf{k}} \times \mathcal{A} && (3.65)\\ &= -Im\langle\nabla_{\mathbf{k}}\Psi(\mathbf{k})| \times |\nabla_{\mathbf{k}}\Psi(\mathbf{k})\rangle && (3.66)\\ &= i\langle\nabla_{\mathbf{k}}\Psi(\mathbf{k})| \times |\nabla_{\mathbf{k}}\Psi(\mathbf{k})\rangle. && (3.67) \end{aligned}$$

In dimensions other than three the Berry curvature can be written component-wise ($(i,\, j)$ denote Cartesian coordinates and $\partial_i \equiv \partial/\partial k_i$) as

$$\Omega_{i,j}(\mathbf{k}) = -2Im\langle\partial_i\Psi(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle. \tag{3.68}$$

The Berry curvature is a gauge-invariant quantity, and naturally then also the Berry phase calculated from it. Indeed, both can (and have been) observed. The Berry connection and Berry curvature play a similar role as the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, respectively, in elementary magnetostatics. The vector potential is gauge-dependent and one can choose the gauge (for instance the Coulomb gauge by $\nabla \cdot \mathbf{A} = 0$), while the magnetic field is a gauge-independent, physical quantity that can be measured.

3.5.2 Chern number

We discussed above the Berry phase defined for a certain path C, and via Stokes theorem, for a surface Σ that the path C is a boundary for. But one can define such a phase also for a closed surface Σ. Naturally, the path C is then vanishing. Think about a part of a sphere being the surface: that can be circulated by a finite path. But if the whole sphere is the surface, then the "path" is vanishing. It can be shown that the integral of the Berry curvature over such a closed surface is quantized:

$$\int_{\Sigma} \mathbf{\Omega} \cdot \mathbf{n} d\sigma = 2\pi C_1. \tag{3.69}$$

Here C_1 is an integer, called Chern number of the first class (or often just Chern number). The closed surface can be, for example, the first Brillouin zone in a lattice system. We will not go through the proof for Equation (3.69), but if you are interested, you can find it from Section 2.4 of the article by Resta mentioned in *Literature*, or Chapter 3.6 of Bernevig's book. The system is said to be topologically trivial if $C_1 = 0$, and topologically non-trivial if C_1 is some finite integer. It can be shown that in order to have non-zero Chern number, the Berry connection must have singularities somewhere on the surface (i.e. somewhere in the Brillouin zone in a lattice system).

The Chern number is a **topological invariant**. It stays invariant between two topological spaces that are connected by a homeomorphism. Sounds abstract, but just go to https://en.wikipedia.org/wiki/Homeomorphism to see a movie about how a coffee cup transforms continuously to a donut which shows that they are homeomorphic. The topological invariant in that case is the number of holes in the object. Of course in quantum physics things are more abstract: we are now talking about topological properties and invariants of the eigenstates of a certain system. Analogously to the number of holes in a cup and a donut, the topological invariant of a quantum system does not necessarily change due to small changes in the system Hamiltonian. The fact that the topological invariant is insensitive to small changes and perturbations can potentially be used, for instance for robust quantum computing, or protected uni-directional currents. See Figure 2 for illustration of this point. In the previous section we discussed the analogue between Berry curvature and the magnetic field. Within this analogy, a non-zero Chern number corresponds to having a magnetic monopole. This gives some intuition to why the Chern number must be quantized and it why it is a topological invariant.

A famous example of the significance of the Chern number is the quantum Hall effect. There the conductance of the system is quantized, that is, it changes in steps when a magnetic field that penetrates the sample is changed, see Figure 3. It was shown in 1982 by Thouless, Kohmoto, Nightingale and den Nijs that the conductance is given by the Chern number and thereby topological properties of the system are the origin of the quantum Hall effect. Thouless got the Nobel prize 2016 for this and his other theoretical work on topological physics. There are also other types of topological invariants than the Chern number. Topological quantum physics has become an extremely important and fast growing field of research, inspired by the experimental observation of topological insulators and by the concept of a topological superconductor.

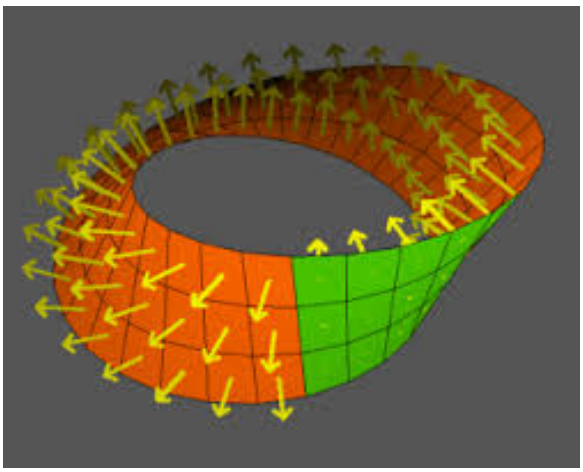


Figure 2: Arrows on a Möbius stripe. The existence of the twist in the Möbius stripe means that the arrows change direction when you go around the loop. The change of the direction cannot be eliminated just by deformations of the stripe, without cutting it. Analogously, certain properties of quantum states are robust to small perturbations, when those properties are caused by the quantum geometry and topology related to the system's eigenstates. Image from plus.maths.org.

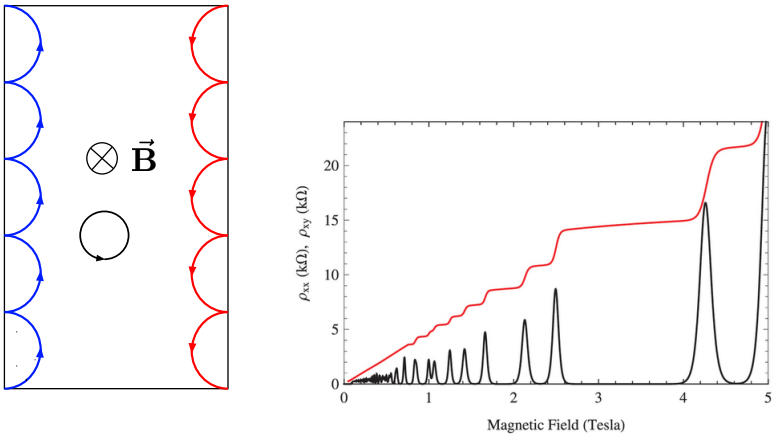


Figure 3: Quantum Hall effect. Magnetic field penetrates the sample and quantized conductance as function of the magnetic field emerges. Image on the right from Research Gate.

3.5.3 Quantum metric

Based on the definition of the Bures distance, Equation (3.54), one can calculate the infinitesimal distance. Start from

$$D_{12}^2 = 1 - |\langle\Psi(\mathbf{k})|\Psi(\mathbf{k}+d\mathbf{k})\rangle|^2 \tag{3.70}$$

and use the Taylor expansion to second order

$$|\Psi(\mathbf{k}+d\mathbf{k})\rangle \simeq |\Psi(\mathbf{k})\rangle + \sum_i |\partial_i\Psi(\mathbf{k})\rangle dk_\alpha + \frac{1}{2}\sum_{i,j} |\partial_j\partial_i\Psi(\mathbf{k})\rangle dk_i dk_j. \tag{3.71}$$

In the calculation (a nice exercise) it is good to remember which quantity is imaginary, and separate the real and imaginary parts. Higher than second order terms are neglected. The result becomes

$$D^2_{\mathbf{k},\mathbf{k}+d\mathbf{k}} = \sum_{i,j=1}^{d} g_{ij}(\mathbf{k}) dk_i dk_j, \tag{3.72}$$

where we have the **quantum metric** tensor

$$g_{ij}(\mathbf{k}) = Re\left(\langle\partial_i\Psi(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle - \langle\partial_i\Psi(\mathbf{k})|\Psi(\mathbf{k})\rangle\langle\Psi(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle\right). \tag{3.73}$$

One can express it also using the projector $Q(\mathbf{k}) = 1 - |\Psi(\mathbf{k})\rangle\langle\Psi(\mathbf{k})|$

$$g_{ij}(\mathbf{k}) = Re\langle\partial_i\Psi(\mathbf{k})|Q(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle. \tag{3.74}$$

Now let us recall the definition of the Berry curvature:

$$\Omega_{ij}(\mathbf{k}) = -2Im\langle\partial_i\Psi(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle. \tag{3.75}$$

We can insert the projector $Q(\mathbf{k})$ into this definition, because as said above, $\langle\partial_i\Psi(\mathbf{k})|\Psi(\mathbf{k})\rangle$ is imaginary, and thus the product $\langle\partial_i\Psi(\mathbf{k})|\Psi(\mathbf{k})\rangle\langle\Psi(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle$ real. We have

$$\Omega_{ij}(\mathbf{k}) = -2Im\langle\partial_i\Psi(\mathbf{k})|Q(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle. \tag{3.76}$$

We see that (apart from the factor of two difference which is trivial) the quantum metric and the Berry curvature are the real and imaginary parts of the same quantity. This quantity is called **quantum geometric tensor** (this tensor is also called the Fubini-Study metric):

$$\mathcal{B}_{ij} = \langle\partial_i\Psi(\mathbf{k})|Q(\mathbf{k})|\partial_j\Psi(\mathbf{k})\rangle. \tag{3.77}$$

A simple example that illustrates how the quantum metric is related to distances between the states is presented in Figure 4. The two-level system forms a simple manifold where the quantum metric can be easily calculated for the angles that parametrize the possible states (it is left as an exercise for you to calculate it). When the angle φ is changed, at the north or south poles the state itself doesn't change, because φ gives just an overall phase factor. Correspondingly, the quantum metric is zero there. While in the equator, change in φ can make the new state (somewhat) orthogonal to the initial one, i.e., there is a finite distance; this is indicated by the quantum metric which is non-zero.

In summary, the quantum geometric tensor has a real part, the symmetric tensor called quantum metric describing the amplitude distance between quantum

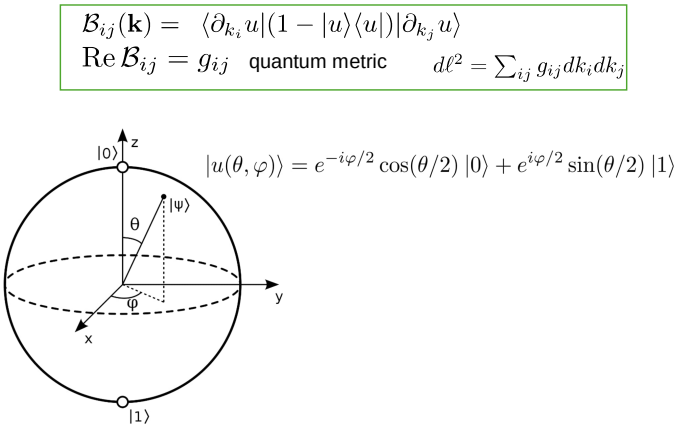


Figure 4: Illustration of the quantum metric for a generic two-state system.

states, and an imaginary part, the antisymmetric tensor called Berry curvature which is related to the phase distance between two states. The quantum metric, Berry curvature and the quantum geometric tensor are all gauge-invariant, measurable quantities. However, they are basis, or lattice geometry dependent (Simon and Rudner, Phys. Rev. B 102, 165148 (2020)), i.e. they are influenced by the spatial positions of the orbitals within the unit cell. This is important to keep in mind when relating physical quantities to quantum geometric ones; we will discuss subtleties related to this in the context of superconductivity.

The Berry phase has been observed in a multitude of systems, and also Berry curvature has been measured. The first direct measurements of the quantum metric were published in 2019, and it is a concept whose significance in physics is emerging right now. It has been predicted to appear in a few contexts, one of them is superconductivity; it has been predicted that the quantum metric of the system affects superfluidity and superconductivity. In particular, its non-zero value guarantees that supercurrent exists even in a situation where the group velocities of non-interacting electrons are zero (so called flat energy bands). The rest of the lectures will focus on this topic.

Finally, let us come back to where this discussion started. It was mentioned that localization properties of the Wannier functions are crucial in describing properties of quantum phases of matter. As you have learned above, the quantum metric and Berry curvature depend on derivatives of the eigenfunctions, in a lattice system this would mean derivatives of the Bloch functions. Via this, there is a connection between the localization properties of Wannier functions (which are combinations of Bloch functions) and quantum geometry.

The understanding of all the consequences of quantum geometry, especially in interacting many-body systems, is only in the beginning. The summer school lectures will tell you more about this.

A recent perspective article about the potential significance of quantum geometry can be found here: P. Törmä, Phys. Rev. Lett. 131, 240001 (2023)

In the lectures of this summer school, we will utilize the review: Quantum geometry in superfluidity and superconductivity S. Peotta, K.-E. Huhtinen, P. Törmä, Proceedings of the International School of Physics "Enrico Fermi", Ebook Volume 211: Quantum Mixtures with Ultra-cold Atoms, Pages 373-404, DOI 10.3254/ENFI250023 (2025). This review is easiest to obtain from arXiv, arXiv:2308.08248

The relevance of quantum geometric superconductivity and superfluidity in the context of moiré materials is reviewed here: Superconductivity, superfluidity and quantum geometry in twisted multilayer systems, P. Törmä, S. Peotta, B.A. Bernevig, Nature Reviews Physics 4, 528 (2022)

There is also a recent review on the role of quantum geometry in quantum materials in general (not restricted to superconductivity): Quantum Geometry in Quantum Materials J. Yu, B.A. Bernevig, R. Queiroz, E. Rossi, P. Törmä, B.J. Yang, arXiv:2501.00098 (2025)