Quantum Metrology meets Quantum Matter

* Quantum Connections Summer School, June 2025

refs: Degen et al RMP 89, 035002 (2017)

Pezze et al RMP 90, 035005 (2018)

Ma et al Physics Reports 509, 89-165 (2011)

Block, Ye et al Nature Physics 20, 1575-1581 (2024)

Wu, Davis et al arXiv: 2503.14585 (2025)

To begin: why quantum?

(i) b/c the thing you're measuring is small

$\Longrightarrow$ spatial resolution

(ii) b/c the thing is quantum $\Longrightarrow$ certain fundamental quantities, α

use sensors that are governed by quantum mechanics.

(iii) b/c entanglement can enhance the precision/sensitivity of a measurement

$\Longrightarrow$ Heisenberg limit.

are few & far between

use quantum mech to make better measurements than can be done in a classical world

[focus of our lectures]

remarks:

① {accuracy, precision}

assume that we have designed proper expt to accurately measure signal

② how many measurements does one need to perform in order to distinguish a small variation in signal / parameter one is sensing?

$\Longrightarrow$ variance + scaling

Goal: general principle for classifying / making those q. entangled states when the precision scaling better than classical?

A bit of cold water ... [Huelga PRL (1997)]
[Zhou et al Nat Comm 2018]

$\Rightarrow$ in the presence of generic noise / decoherence, one cannot achieve such an q-enhanced precision scaling ...

"wins" must be more nuanced ..

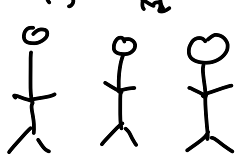
1| Classical Estimation Theory

problem: construct an estimator that takes in a sample from $p(\underline{x} \mid \underline{\theta})$ & yields an estimate $\tilde{\theta}$ for our parameter of interest θ

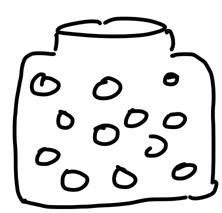
exe: [F. Galton, Nature 1907, Vox Populi]

jelly beans in the jar?

$x_3 = \dots$ $x_2 = \dots$ $x_1 = 1235$



$\tilde{\theta}$ = estimated value



θ = true value = 1198

w/ N "measurements", better estimator $\tilde{\theta} = \frac{1}{N} \sum_n x_n$

central result of classical estimation theory: variance of any estimator is bounded by Cramer-Rao bound [Rao 1945, Cramer 1946]

$$\Rightarrow \quad \Delta\theta^2 = \overline{(\theta - \tilde{\theta})^2} \geq \Delta\theta_{CR}^2 = \frac{C}{N}$$

∴ intuition: sampling from distribution always leads to noisy estimates!

Content of CR bound:

(i) familiar scaling w/ N

(ii) coefficient C is actually specified & can be calculated

$$\Rightarrow \quad \frac{1}{C} = \underset{F(\theta)}{\text{Fisher Information}} = \sum_x \frac{1}{p(x|\theta)} \left[\partial_\theta \, p(x|\theta)\right]^2$$

1.2 | Quantum Sensing

N.B. "classical" cramer-rao bound applies to unentangled q. sensors

$\Longrightarrow$ "standard quantum limit"

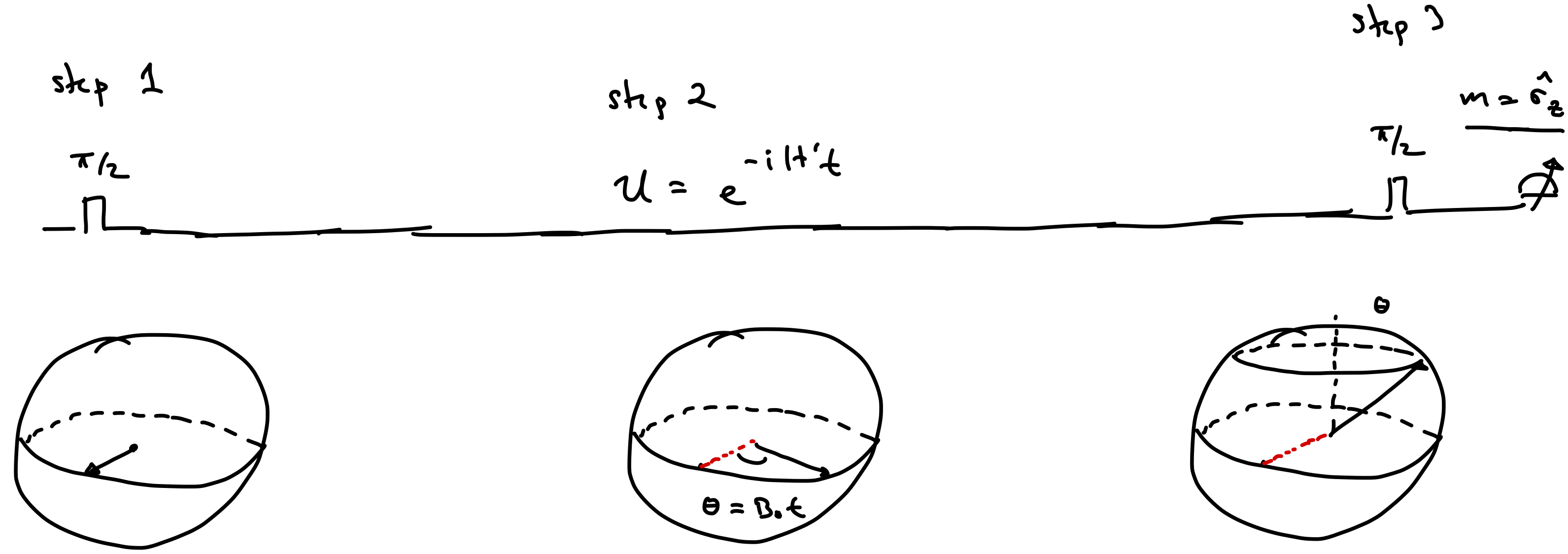
problem: calculate an estimate $\tilde{\theta}$ for θ given quantum state $\rho(\theta)$

workflow.

step 1: prepare a probe state ρ $\longrightarrow$ step 2: interact w/ signal via $\mathcal{U}_\theta$ $\Rightarrow \rho_\theta = \mathcal{U}_\theta \, \rho \, \mathcal{U}_\theta^\dagger$ $\longrightarrow$ step 3: pick observable $\hat{M}$ & measure the probe state $\Big\{ \Rightarrow$ sample from $p(n|\theta)$ (q. measurement sampling) $\longrightarrow$ step 4: post processing $\Rightarrow \tilde{\theta}$

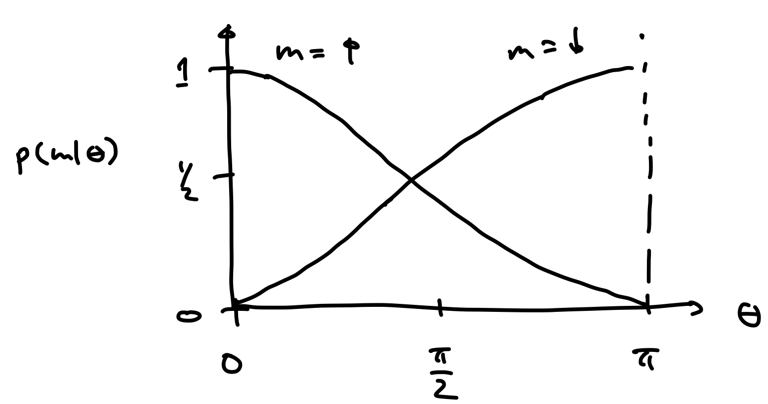
exe: a single q. spin d.o.f used to estimate the strength of B-field via Ramsey spectroscopy

$$\Longrightarrow \quad H' = B_0 \hat{\sigma}_z$$



$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle + |\uparrow\rangle\right) \qquad |\psi_\theta\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta/2}|\uparrow\rangle + e^{-i\theta/2}|\downarrow\rangle\right) \qquad |\psi_\theta\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + i\sin\frac{\theta}{2}|\downarrow\rangle$$

what does $p(m|\theta)$ look like?



$\therefore$ can see that a single measurement of $\sigma^z = \uparrow$ doesn't really tell you all that much about θ

absent of entanglement ... uncorrelated sensors.

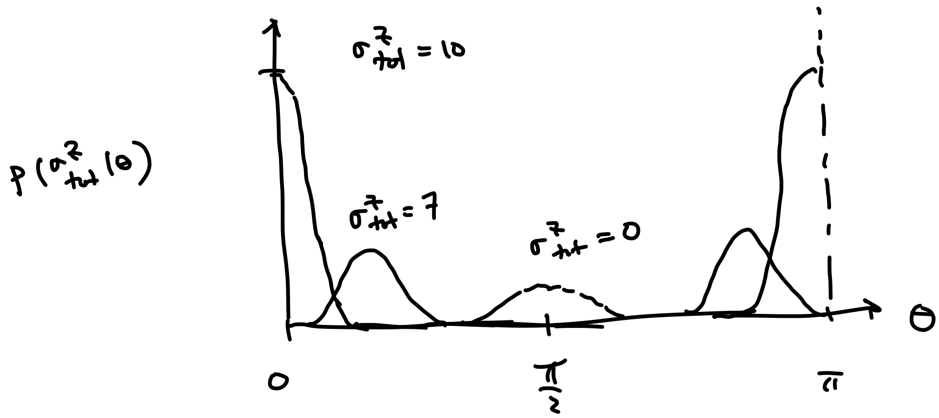
How does this change for N spins? purely classical intuition ... $\sim \frac{1}{\sqrt{N}}$

(i) if one gets a big value of σ^z_{tot}, then distribution should be sharply peaked at $\theta = 0, \pi$

(ii) if one gets a small value of $\sigma^z_{tot} \sim 0$, distribution should be peaked around $\theta = \frac{\pi}{2}$

$$p(\sigma^z_{tot}|\theta) = \left[\binom{N}{\frac{N}{2}+\sigma^z_{tot}}^{1/2} \left(\cos\frac{\theta}{2}\right)^{\frac{N}{2}-\sigma^z_{tot}} \left(\sin\frac{\theta}{2}\right)^{\frac{N}{2}+\sigma^z_{tot}} \right]^2$$

$N = 20$



$\therefore$ what is the Fisher information:

$$F(\theta) = \sum_{\sigma^z_{tot} = -\frac{N}{2}}^{N/2} \frac{1}{p(\sigma^z_{tot}|\theta)} \left[\partial_\theta \, p(\sigma^z_{tot}|\theta)\right]^2 = \underline{\underline{N}}$$

(independent of θ)!

Dual perspective on scaling of such N spin ramsey sensing expt:

analysis above: one "big spin" σ^z_{tot}	complementary analysis: a spin 1/2
measured once ($N = 1$)	measured N times
	$\therefore F(\theta) = 1$
$\Delta\theta_{CR}^2 = \frac{C}{1} = \frac{1}{N}$	$\Delta\theta_{CR}^2 = \frac{C}{N} = \frac{1}{N}$

1.3 | Quantum $\underbrace{\text{Cramer-Rao bound}}$

depends on particular set of measurements

$\therefore$ considering all possible measurements, what is the optimal Fisher information for some given state ρ?

$$\underset{\sim\sim}{\Delta\theta^2_{CR}} \geq \Delta\theta^2_{QCR} = \frac{1}{N} \times \frac{1}{\underset{\hat{M}}{\text{opt}}\, F_q[\rho_\theta, \hat{M}]}$$

for $\underline{\text{pure states}}$: $\Rightarrow$ $F_q[\rho_\theta, \hat{M}] = 4\,\Delta M^2$

variance of $\underline{\hat{M}}$ in state $\underline{\rho_\theta}$

two helpful limits:

case 1: $\Delta M^2 \longrightarrow 0 \Rightarrow F_q(\rho_\theta, \hat{M}) = 0$

implies ρ_θ is an eigenstate of $\hat{M}$

thus evolution under rotations that need to commute w/ $\hat{M}$

case 2: how does one get the maximum Fisher information?

eigenbasis of $\hat{M}$: $M|m_i\rangle = m_i|m_i\rangle$

$$\underset{\sim\sim\sim}{4(\Delta M^2)} \leq (m_{max} - m_{min})^2$$

is there any pure quantum state that can saturate this bound?

Yes! the cat state $|\psi\rangle_{cat} = \frac{1}{\sqrt{2}}(|m_{max}\rangle + |m_{min}\rangle)$

∴ going back very quickly to example of N spins & $\hat{O}^z_{tot}$

GHZ state

$\frac{|\downarrow\cdots\downarrow\rangle + |\uparrow\cdots\uparrow\rangle}{\sqrt{2}}$

$$m_{max} = \frac{N}{2}, \quad m_{min} = -\frac{N}{2} \implies 4(\Delta M^2) \leq N^2$$

→ shining exe # 1

thus, surprisingly, w/ a cat state "big spin"

$$(N=1) \implies \Delta\theta^2 \sim \frac{c}{1} \sim \frac{1}{N^2} \implies \text{Heisenberg limit of sensing}$$

the above is really the essence of the quantum cramer-rao bound:

$$\Delta\theta^2 = \overline{(\theta - \tilde{\theta})^2} \geq \Delta\theta^2_{QCR} = \frac{c'}{N^2}$$

1.4 | Metrology & Parameter Estimation: combining classical & quantum Cramer-Rao

Combine our cramer-rao bounds together: integrated variance associated w/ any quantum sensing experiment

$$\Delta\theta^2 = \Delta\theta^2_{cl} + \Delta\theta^2_{QSN}$$

quantum shot noise: an additional source of error b/c q. measurements are inherently stochastic e.g. another source of randomness in its own right

classical sources of fluctuations / errors.

(cramer - rao)

unentangled

(SQL ⟷ cramer - rao)

entanglement

(Heisenberg limit)

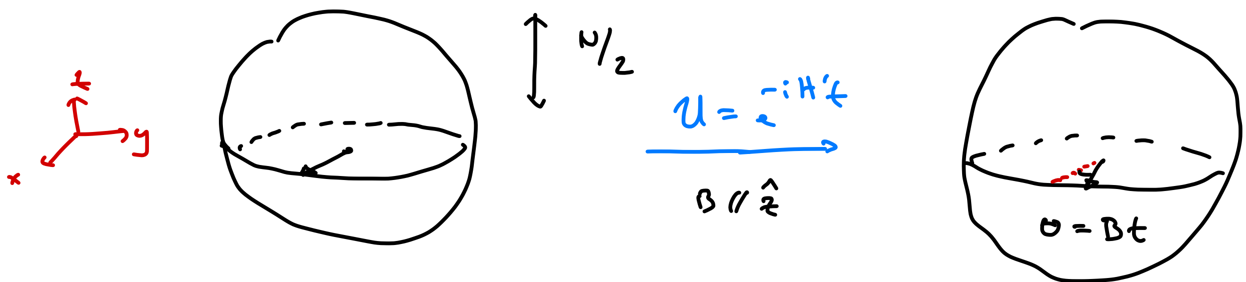
quantum cramer - rao

2 | Quantum Shot Noise & Spin-Squeezed Entanglement

Let us unpack this shot noise:

recall our previous Ramsey expt, but lets redraw it for N spins instead

N spins are in product state, aka coherent spin state (CSS)



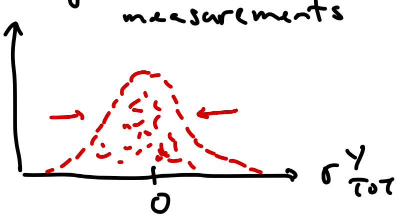
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle + |\uparrow\rangle\right)^{\otimes N}$$

$$= |\rightarrow \cdots \rightarrow\rangle_x$$

∴ to measure the angle θ need to know how much spin has deflected, $\hat{\sigma}^Y_{Tot}$ ⇒ how much has CSS precessed?

What is the intrinsic limitation to how well measure this?

⇒ intuition: compare size of signal in σ^Y_{Tot} to inherent variance associated w/ measuring σ^Y_{Tot}

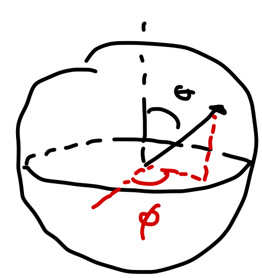


⇒ variance of binomial distribution $\sigma^2 = Np(1-p)$

N spin 1/2 in $|\psi_0\rangle$, $\Delta\sigma^{Y\,2}_T = \frac{N}{4}$

One nice way to visualize this "quantum shot noise":

Husimi-Q representation of state

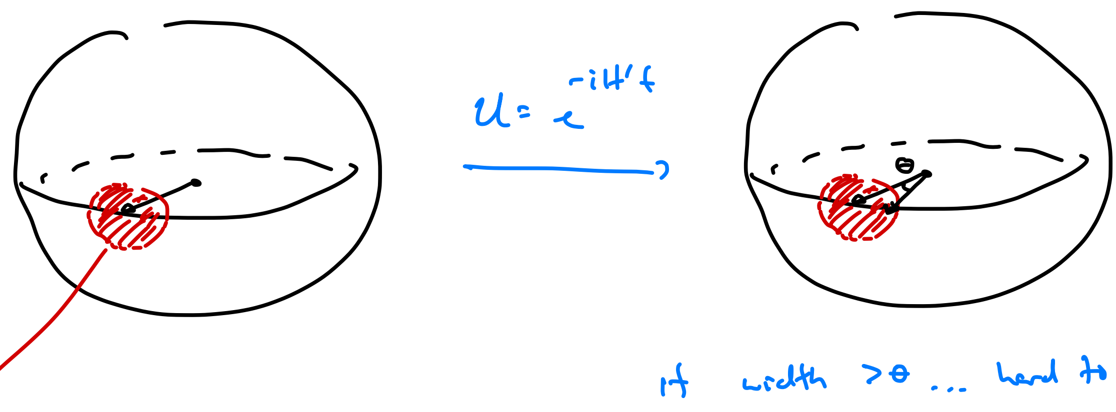


define: $|\theta, \phi, N\rangle$ as CSS state on Bloch sphere

$$= \left(\cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle\right)^{\otimes N}$$

for any state $\hat{\rho}$: $\quad Q(\theta,\phi) = \frac{N+1}{4\pi} \langle \theta,\phi,N | \hat{\rho} | \theta,\phi,N \rangle$

for $|\psi_0\rangle$



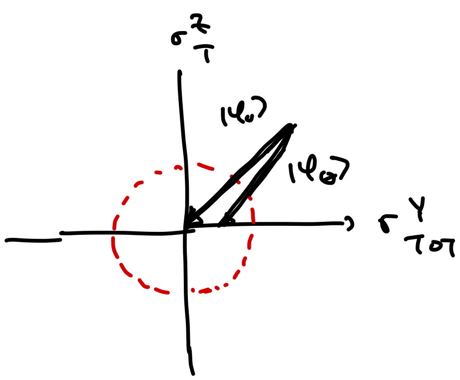
quantum projection noise: think of it as a little "blob" of uncertainty of tip of spin vector

if width $> \theta$ ... hard to resolve the signal!

<u>2.11</u> Spin squeezing as a way to overcome the QPN-limit

↳ classic exe #2 of type of metrologically-useful entanglement

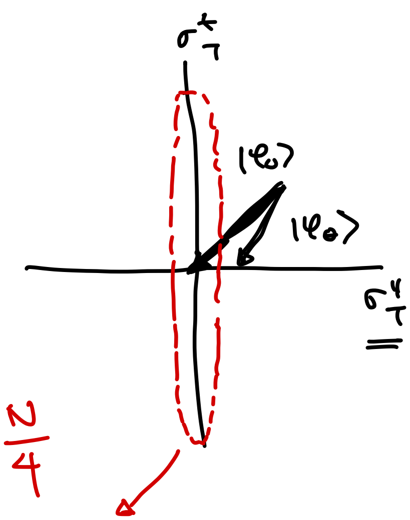
again considering: $|\psi_0\rangle = |\rightarrow \cdots \rightarrow\rangle_x$, look at the QPN in yz-plane



1) make the QPN smaller?

2) reshape the QPN up to limits imposed by Heisenberg uncertainty

$$\Delta\sigma^y_T \, \Delta\sigma^z_T \geq \frac{1}{2} |\langle \sigma^x_T \rangle| = \frac{N}{4}$$



spin-squeezed state

Carlton Caves, 1991

is this realizable? Yes! [Kitagawa, Ueda 1993]

<u>Exe:</u> model system / Hamiltonian known as One-Axis-Twisting (OAT)

N spin 1/2

w/ $\sigma^z_T = \sum_i \sigma^z_i$

$$\mathcal{H}_{OAT} = \frac{1}{N} \sum_{i,j} \sigma^z_i \sigma^z_j = (\sigma^z_T)^2 / N$$

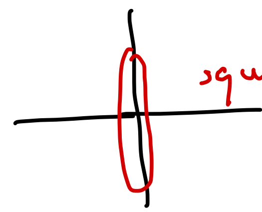
∴ all-to-all coupled Ising model

N.B. many expts demonstrating OAT model cavity QED

[Leroux et al PRL 2010]
Riedel et al Nature 2010

claim: starting w/

$$|\psi_0\rangle = |\rightarrow \cdots \rightarrow\rangle_x \xrightarrow{e^{-iH_{OAT}t}}$$

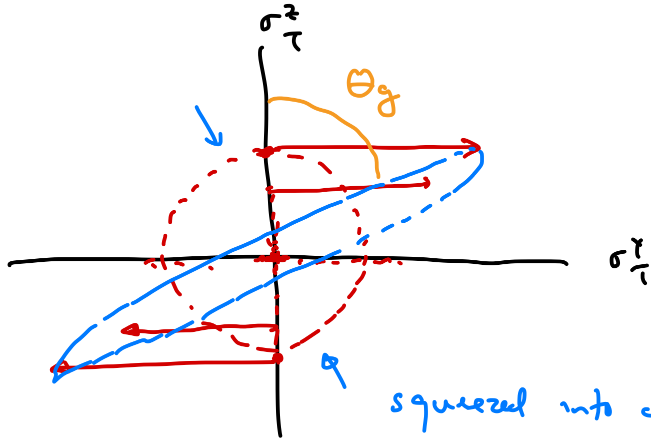


pf by pictures: recall generate rotations around $\hat\sigma^z_T$ as $e^{-i\theta \hat\sigma^z_T}$

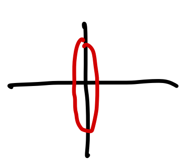
$$e^{-iH_{OAT}t} = e^{-\frac{i}{N}\sigma^z_T \underbrace{\sigma^z_T}\, t}$$

implements rotation around z by angle proportional to the value of σ^z_T

How does the QPN evolve under $\mathcal{U} = e^{-iH_{OAT}t}$??



$\Rightarrow$ evolution under H_{OAT} literally shears original QPN into an ellipse!

* to get back to , global rotate around $\hat x$ by θ_g

Properties of the squeezed state generated by OAT. (Wineland PRA 94) to initial CSS

figure of merit: $\xi^2 = \dfrac{\min_{\perp} \mathrm{var}(\sigma_T)}{\langle \sigma_T^x \rangle^2}$ → smallest possible var in yz-plane

length of CSS → length of collective spin two start count as Ramsey

e.g. how much one wins in signal-to-noise at Ramsey spectroscopy

ξ^2 small is good!

it turns out that for any system size N, there is an optimal stopping time t_{opt} which yields ξ^2_{opt}, b/c of curvature of Bloch sphere at some point the denominator gets smaller:

$$t_{opt} \sim N^{1/3} \Rightarrow \xi^2_{opt} \sim \frac{1}{N^{2/3}} \Rightarrow \Delta\theta \sim \frac{1}{N^{5/6}}$$

recall: classical cramer rao (SQL) $\Delta\theta \sim \frac{1}{\sqrt{N}}$

quantum cramer rao (Heisenberg limit) $\Delta\theta \sim \frac{1}{N}$

does better than classical but not as good optimal QCR allows for

N.B. there are squeezed states that saturate $\sim \frac{1}{N}$ scaling
see e.g. two-axis-twisting (TAT)

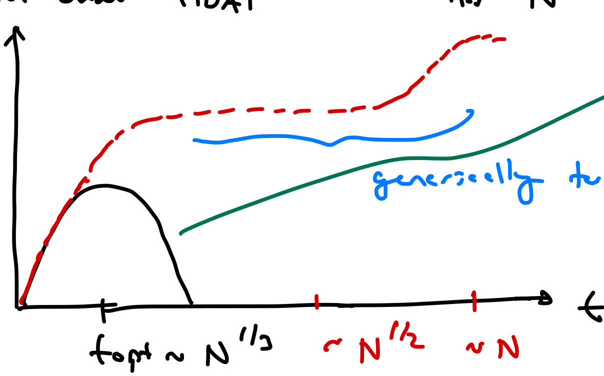
Note: that the squeezing parameter is NOT a proxy for the Q. Fisher information!

* different types of metrological gain, & the QFI characterize the metrological utility assuming all possible measurements

$|\psi_0\rangle \to$ evolve under H_{OAT} for N

$\frac{1}{\xi^2}$, QFI

"normalized properly"



metrological gain can be used/readout easily

generically to states realized in this time window after

but very hard to use that metrological utility

3 | "Universal" principles for "predicting" metrologically useful states

↳ spontaneous symmetry breaking

thus far, we have seen two "shining examples" of entangled states that beat classical cramer rao bound: GHZ state, spin squeezed

seems like these states are very "finely" constructed: (i) $\frac{1}{\sqrt{2}}(|m_{ax}\rangle + |m_{in}\rangle)$ (ii) shear QPN of CSS state..

Goal: more general framework for predicting / classifying metrologically useful entanglement

aside (if time permits)

not done at QC 2025

Story about what interacting systems of spins want to do at low temperatures?

Imagine: $\mathcal{H}$, write down the symmetries of $\mathcal{H}$

$$\mathcal{U}\mathcal{H}\mathcal{U}^{\dagger} = \mathcal{H}$$

SSB: possible for the system to exhibit a state that has a smaller symmetry than its parent Hamiltonian

transverse field Ising model $d=2$

$$\mathcal{H} = -\sum_{\langle ij\rangle} J\sigma_i^z\sigma_j^z - h\sigma_i^x$$

symmetry: rot by 180° around z

$|J|$ larger $|h|$ $\Longrightarrow$ ferromagnetic regime / phase

$$|\downarrow\cdots\downarrow\rangle \text{ or } |\uparrow\cdots\uparrow\rangle$$

$\Longrightarrow$ order can be diagnosed by order parameter simplest order parameter: magnetization $M_z = \sum_i \sigma_i^z$

turns out: one distinguishing feature of Long-range order that is useful for connecting to metrology / QFI: connected correlation function

N.B. perhaps not as common an object in cond. mat. as disconnected CF

connected correlation $= \sum_{i,j} \langle \sigma_i^z \sigma_j^z \rangle - \langle \sigma_i^z \rangle \langle \sigma_j^z \rangle$

$\Longrightarrow$ just by looking at it, already see natural connection to QFI

in fact, for pure quantum states there is a 1-to-1 relation between

long-range order (SSB) $\longleftrightarrow$ scaling of the QFI

order	scaling of the QFI
Long range order: connected c.f. $\xrightarrow[\lvert i-j\rvert \to \infty]{}$ plateau	QFI $\sim N^2$ (q. scaling rao)
Algebraic long range order ...	$N \le$ QFI $< N^2$
no order: connected c.f. $\to e^{-\lvert i-j\rvert/\xi}$	QFI $\sim N$ (classical rao)

take home message from the table? if one can generate a pure state w/ long range order, then such a state must be metrologically-useful in the sense that it beats classical CR-bound, saturates QCR bound!

caveats:
(i) it may not be easy to generate such a state

(ii) even if one does generate such a state, may be difficult to use it / i.e. readout a sensing signal from it.

Is there any universal strategy based on "connection" between order $\leftrightarrow$ QFI to leverage SB to generate metrologically useful q. states: YES!
$\hookrightarrow$ based on ETH!

Strategy:

step 1: find a H that exhibits finite T order below T_c

step 2: pick initial state $|\psi_0\rangle$ s.t. the temperature of state is $T_{|\psi_0\rangle} < T_c$ $\therefore$ energy density of state is lower critical energy density

preparable

step 3: time evolve $|\psi_0\rangle$ under H

$\Rightarrow$ $|\psi(t)\rangle \longrightarrow$ evolve into one w/ entanglement allows one to saturate QCR bound!

Let us explore this strategy in two distinct cases via examples:

case 1: spontaneous symmetry breaking of discrete symmetry

exe: $d=2$ Transverse field Ising model

$$H_{TFIM} = -\sum_{\langle i,j\rangle} J\,\sigma_i^z\sigma_j^z - \sum_i h\,\sigma_i^x \qquad \therefore \mathbb{Z}_2 \text{ symmetry}$$

take H to be in the ferromagnetic Ising phase: $|J| > |h|$

$\Rightarrow$ there exists finite T_c for FM order in $d \geq 2$

step 1 ✓

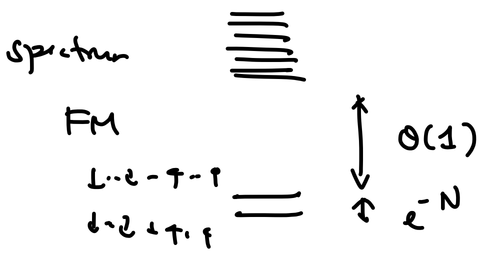
take initial state to be $|\varphi_0\rangle = |\downarrow \cdots \downarrow\rangle$, which is clearly very cold! i.e. $T_{|\varphi_0\rangle} < T_c$ step 2 ✓

step 3: what happens when one time evolves GHZ-like state!

$$|\varphi_0\rangle = |\downarrow \cdots \downarrow\rangle \xrightarrow{\; e^{-iH_{TFIM}t} \;} \sim \frac{|\downarrow \cdots \downarrow\rangle + |\uparrow \cdots \uparrow\rangle}{\sqrt{2}}$$

why? for any finite size N

the many-body spectrum:



key issue: to get to a GHZ-like state, one has to wait a time $t_{opt} \sim e^{N}$!

∴ conclusion: for any discrete symmetry breaking model, the ETH-based metrology strategy is not very useful b/c time to prepare a metrologically-useful state diverges exponentially in system size

case 2: spontaneous symmetry breaking of continuous symmetry

exe: $d = 3$ Ferromagnetic XY model

$$\mathcal{H}_{XY} = -\sum_{\langle i,j\rangle} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \quad ∴ \text{ easy plane ferromagnet}$$

$U(1)$ symmetry

$\Rightarrow \exists$ a finite T_c for FM XY order for dimn $d \geq 3$

step 1 ✓

take initial state to be $|\varphi_0\rangle = |\rightarrow \cdots \rightarrow\rangle$, which is clearly very cold! i.e. $T_{|\varphi_0\rangle} < T_c$ step 2 ✓

step 3: what happens when one time evolves

$$|\varphi_0\rangle = |\rightarrow \cdots \rightarrow\rangle \xrightarrow{e^{-iH_{XY}t}}$$

at late times $t \sim \mathrm{poly}(N)$ get a metrologically-useful thermalized state...

interestingly, even before full "thermalization", one gets a metrologically-useful entangled state ...... a spin squeezed state!

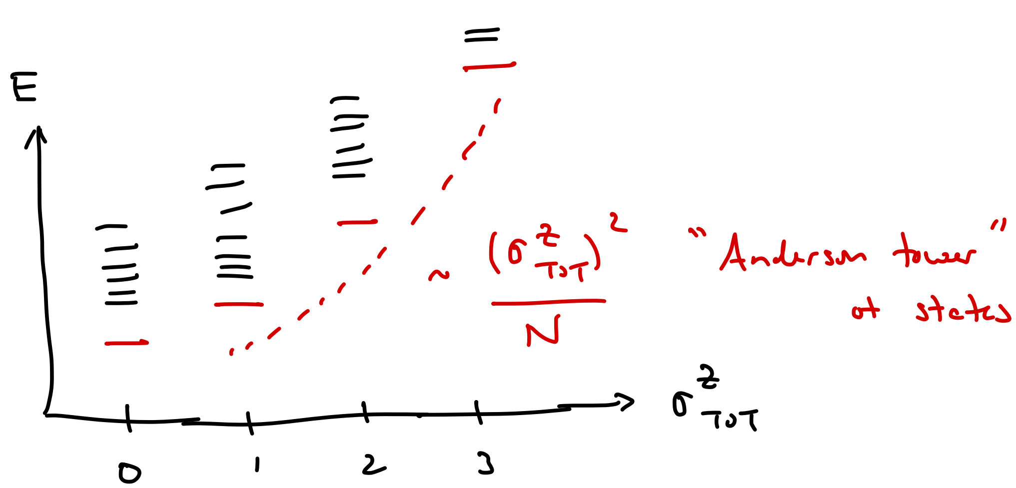
* Nature Physics 20, 1575-1581 (2024)

in particular: at $t_{opt} \sim N^{2/5}$ get a spin-squeezed state

$$\text{w/ } \xi^2_{opt} \sim \frac{1}{N^{2/5}} \Rightarrow \Delta\theta \sim \frac{1}{N^{7/10}}$$

intuition? why does one get a spin squeezed state?

carefully look at low-energy spectrum of H_{XY}



$\therefore$ look at spectrum resolved by σ^z_{TOT} b/c of $U(1)$ symmetry concept: each low energy state in σ^z_{TOT} sector has a natural counterpart in other sectors that is an energy $\sim \frac{(\sigma^z_{TOT})^2}{N}$ away!

$\Rightarrow$ if $|\psi_0\rangle$ is sufficiently low temperature, then one can expand $|\psi_0\rangle$ over the low energy spectrum & evolution is governed by effective OAT-like Hamiltonian!

Q: if this is true, why is ξ^2_{opt} scaling different & worse than OAT?

A: because of thermalization! b/c $\mathcal{H}_{XY}$ is not integrable like H_{OAT}, the area of the QPN is not conserved as a function of time, so the shearing dynamics must compete w/ growth of area of QPN associated w/ thermalization

Impact on expt: connection between spin-squeezing & continuous symmetry breaking has dramatically increased the # of models $\Rightarrow$ scalable spin-squeezing

dipolar interactions in $d=2$!

for example: NN FM XY in $d=3$, etc...

3d spin-exchange w/ Cs atoms arXiv: 2409.17398 (2024)

experiments:

dipolar Rydberg atom array Nature 621, 728 (2023)

dipolar magnetic atom in optical lattice arXiv: 2411.07219 (2024)

dipolar NV centers in solid-state arXiv: 2503.14585 (2025)