



Aalto University
School of Science

Quantum geometry in Superconductivity

Slides 1

Päivi Törmä
Aalto University

Quantum Connections in Sweden-15 Summer School 2025, Stockholm, Sweden

12.-13.6.2025

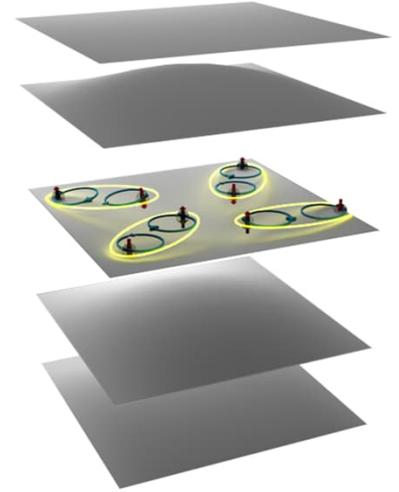


Institute





Overview of the lectures



First and second lectures (roughly)

- Brief recap of BCS theory and basics of quantum Geometry, from background material sent to you
- Basics of the role of quantum geometry in superconductivity (superfluid weight/stiffness, flat bands), from arxiv:2308.08248

Third and fourth lectures (roughly)

- Brief recap of the basics of quantum geometry in superconductivity
- Recent research results on superconductivity and quantum geometry, especially in flat bands (specific materials, beyond-mean-field, non-equilibrium, etc.)
- Other contexts where quantum geometry matters (Bose-Einstein Condensation, light-matter interactions, conductivity)
- The many-body quantum metric and physical responses: example

From slides

Contents

Factor of two problem

Quantum geometry and superconductivity

**Quantum geometric superconductivity
in twisted bilayer graphene**

**Flat band ratio and quantum metric: DMFT studies of
non-isolated flat bands**

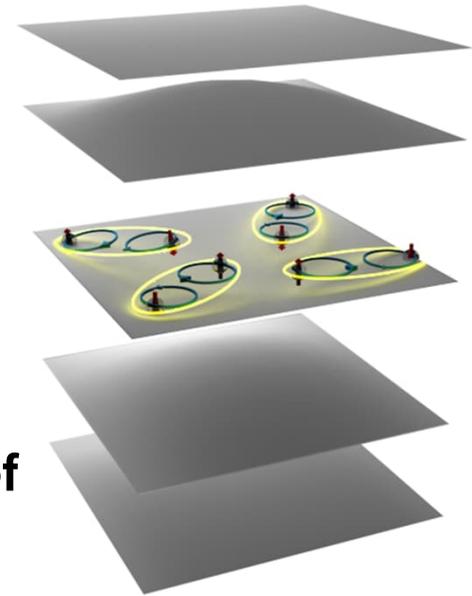
Non-equilibrium transport in a flat band

Conductivity in a flat band

Many-body quantum metric and the Drude weight

Quantum geometry and Bose-Einstein condensation

Quantum geometry and light-matter interactions



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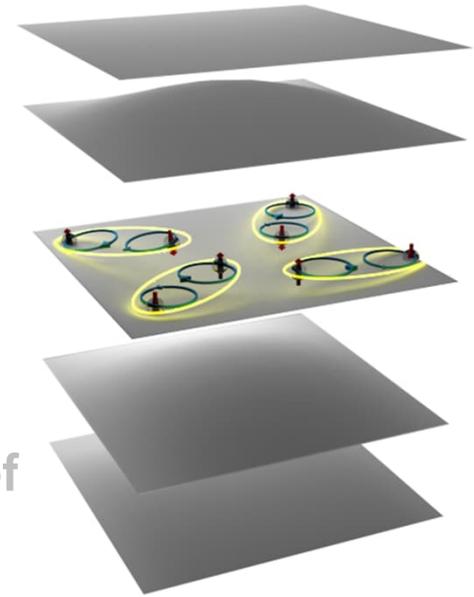
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Superconductivity

WHY NOT AT ROOM TEMPERATURE?

Highest T_c (ambient pressure)
~150 K – just a factor of two!

LHC: The Large Hadron Collider

The protons have not yet been accelerated to their full energy.

accelerator handle...

SPS now at 306.0 GeV (68)%...

Lift handle to accelerate the stream



Superconductivity: Cooper pair formation competes with kinetic energy



Weak interaction U

Large kinetic energy (Fermi level)

Low critical temperature

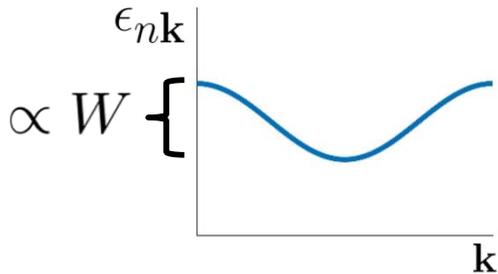
$$T_c \propto e^{-1/(U n_0(E_f))}$$

Constituents: interactions, density of states (DOS)

**Remove the kinetic energy/maximize DOS:
interaction effects dominate!**

Flat bands: interactions dominate

Dispersive band $U \ll W$:



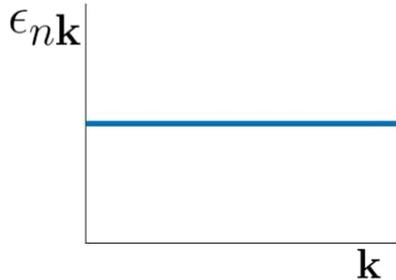
$$\psi_n(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$$

(periodic part of) the Bloch function

T_c for Cooper pairing

$$T_c \propto e^{-1/(U n_0(E_f))}$$

Flat band $U \gg W$:



$$\epsilon_{n\mathbf{k}} = \text{constant}$$

$$\text{Group velocity: } \frac{\partial \epsilon_{n\mathbf{k}}}{\partial k} = 0$$

No interactions: insulator at any filling

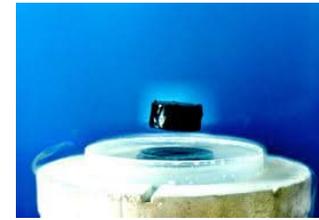
$$T_c \propto UV_{\text{flat band}}$$

High T_c for pairing
(Khodel, Shaginyan, Volovik,
Kopnin, Heikkilä)

This is the critical temperature for Cooper pairing

$$\Delta(\mathbf{r}) = \langle \psi_\sigma(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) \rangle \quad \Delta(\mathbf{r}) = |\Delta(\mathbf{r})|$$

Superfluid weight: supercurrent and Meissner Effect



Supercurrent

$$\mathbf{j} = -D_s \mathbf{A}$$

Current

$$\mathbf{j} = \sigma \mathbf{E} \quad \mathbf{E} = -\partial \mathbf{A} / \partial t$$

Order parameter phase gradient $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})| e^{2i\phi(\mathbf{r})}$

$\nabla \phi - e\mathbf{A}/\hbar$ Invariant under gauge transformations

Free energy change associated with phase gradient

$$\Delta F = \frac{\hbar^2}{2e^2} \int d^3\mathbf{r} \sum_{ij} [D_s]_{ij} \partial_i \phi(\mathbf{r}) \partial_j \phi(\mathbf{r})$$

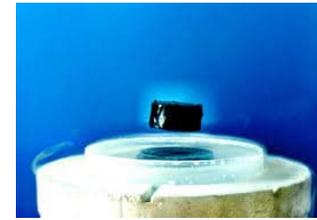
London equation and penetration depth

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\nabla^2 \mathbf{B} = \mu_0 D_s \mathbf{B}$$

$$\lambda_L = (\mu_0 D_s)^{-1/2}$$

Superfluid weight: supercurrent and Meissner Effect



Supercurrent

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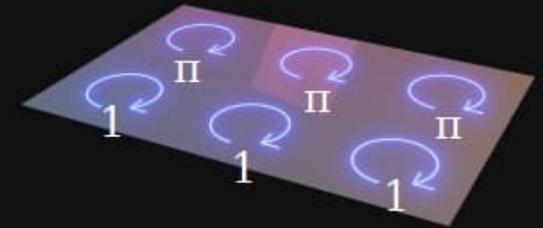
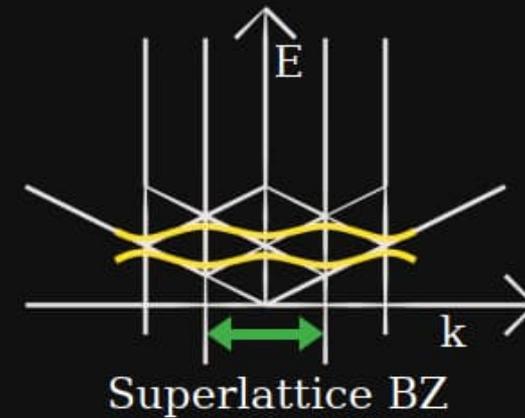
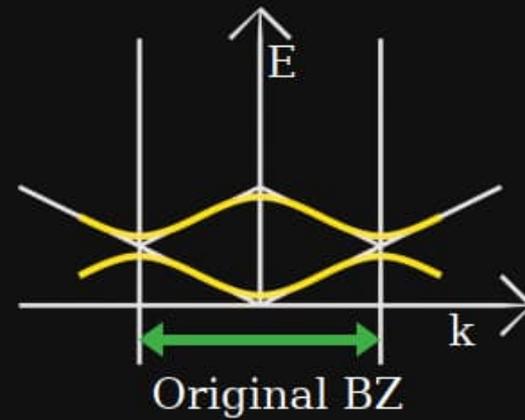
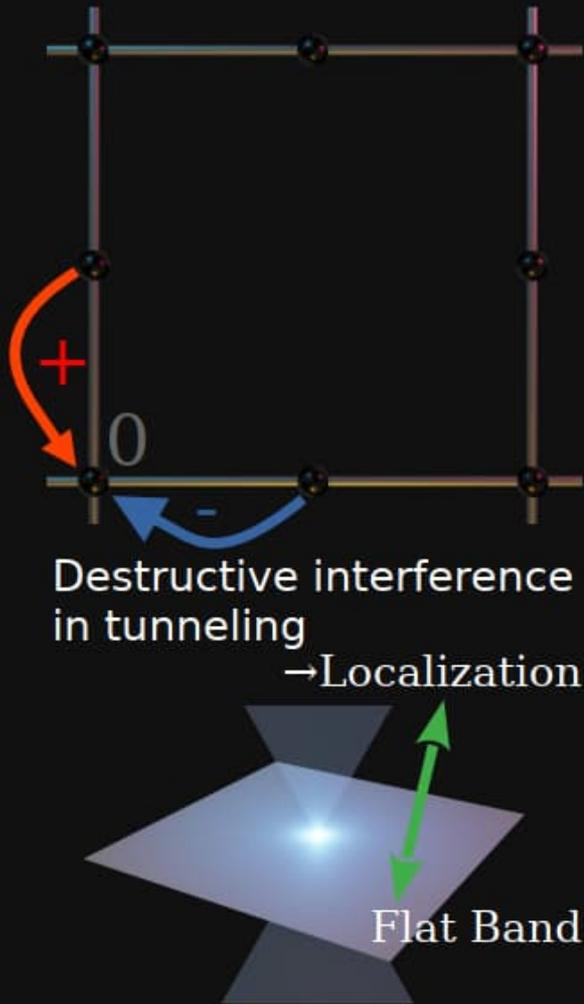
Conventional BCS: $D_s = \frac{e^2 n_p}{m_{\text{eff}}} \left(1 - \left(\frac{2\pi \Delta}{k_B T} \right)^{1/2} e^{-\Delta/(k_B T)} \right)$ **Zero at a flat band!!!**

n_p Particle density

$$\frac{1}{m_{\text{eff}}} \propto J \propto \partial_{k_i} \partial_{k_j} \epsilon_{\mathbf{k}}$$

Bandwidth $i, j = x, y, z$

Formation of flat bands



Landau levels

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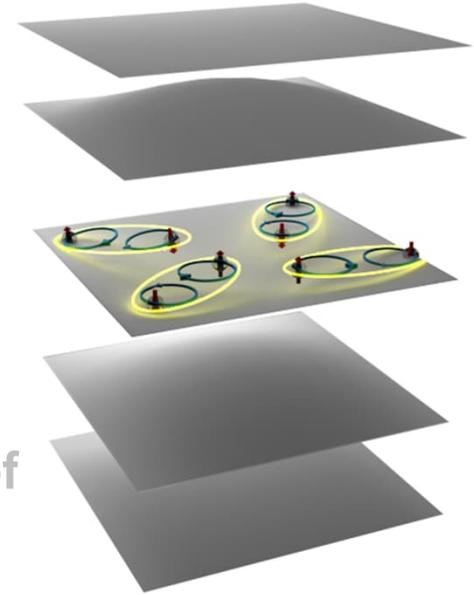
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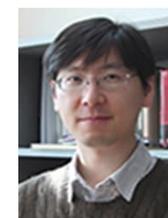
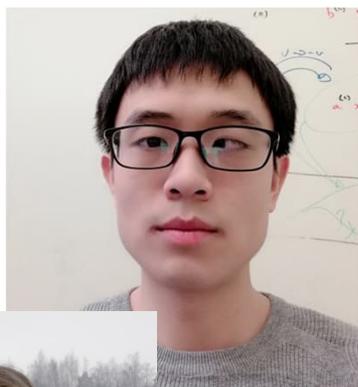
Many-body quantum metric and the Drude weight

Quantum geometry and Bose-Einstein condensation

Quantum geometry and light-matter interactions



Superfluidity and quantum geometry



Sebastiano Peotta

Long Liang

Andrei Bernevig

Sebastian Huber

Murad Tovmasyan

Kukka-Emilia Huhtinen

Jonah Herzog-Arbeitman

Aaron Chew

Peotta, PT, Nat Comm 2015

Julku, Peotta, Vanhala, Kim, PT, PRL 2016

Tovmasyan, Peotta, PT, Huber, PRB 2016

Liang, Vanhala, Peotta, Siro, Harju, PT, PRB 2017

Liang, Peotta, Harju, PT, PRB 2017

Tovmasyan, Peotta, Liang, PT, Huber, PRB 2018

PT, Liang, Peotta, PRB(R) 2018

Huhtinen, Herzog-Arbeitman, Chew, Bernevig, PT, PRB 2022

Herzog-Arbeitman, Chew, Huhtinen, PT, Bernevig, arXiv 2022

Aleksis Julku

Dong-Hee Kim

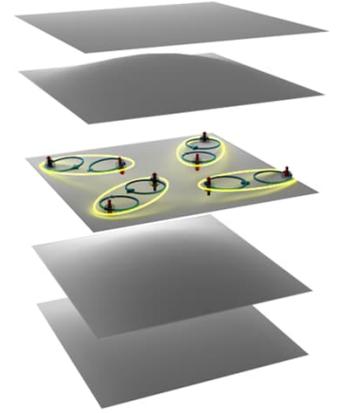
Tuomas Vanhala

Ari Harju

Topi Siro

Our multiband approach

MULTIBAND BCS MEAN-FIELD THEORY
 multiband two-component attractive
 Fermi-Hubbard model $-U < 0$



$$H = - \sum_{ij\alpha\beta\sigma} t_{i\alpha j\beta}^{\sigma} c_{i\alpha\sigma}^{\dagger} c_{j\beta\sigma} - U \sum_{i\alpha} n_{i\alpha\uparrow} n_{i\alpha\downarrow} - \mu \sum_{i\alpha\sigma} n_{i\alpha\sigma}$$

Introduce supercurrent

$$\Delta(\mathbf{r}) \rightarrow \Delta(\mathbf{r}) e^{2i\mathbf{q}\cdot\mathbf{r}}$$

$$\nabla\phi - e\mathbf{A}/\hbar$$

$2\mathbf{q}$ Cooper pair momentum

$$\langle j_i(\omega, \mathbf{q}) \rangle = - \sum_j \chi_{ij}(\omega, \mathbf{q}) A_j(\omega, \mathbf{q})$$

$$[D_s]_{ij} = \frac{e^2}{W} \frac{d^2 \Omega}{d\omega_{ij} d\omega_{ij}} \Big|_{\mathbf{q}=\mathbf{0}}$$

$$D_s = \lim_{\mathbf{q} \rightarrow 0} \chi(\omega = 0, \mathbf{q})$$

$i, j = x, y, z$

Superfluid weight in a multiband system

$$D_s = D_{s,\text{conventional}} + D_{s,\text{geometric}}$$

$$\propto \partial_{k_i} \partial_{k_j} \epsilon_{\mathbf{k}}$$

$i, j = x, y, z$

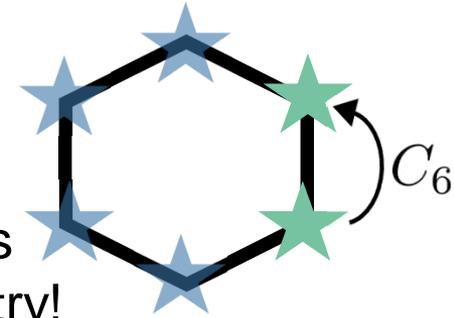
Can be nonzero also in a flat band
Present only in a multiband case
Proportional to the quantum metric

$$[D_{s,\text{geometric}}]_{ij} \propto U g_{ij}$$

Superfluid weight and quantum metric

Isolated flat band: $W \ll U \ll E_{\text{band gap}}$

Uniform pairing: $\Delta_{\text{orbital } \#} = \Delta$ Valid when orbitals related by symmetry!



$$\rightarrow [D_s]_{ij} = \frac{4e^2 U \nu (1 - \nu)}{(2\pi)^{d-1} N_{\text{orb}}} \mathcal{M}_{ij}^{\text{R}, \text{min}}$$

$$\mathcal{M}_{ij}^{\text{R}} = \frac{1}{2\pi} \int_{\text{B.Z.}} d^d \mathbf{k} \underbrace{\text{Re } \mathcal{B}_{ij}(\mathbf{k})}_{\text{quantum metric } g_{ij}}$$

$$\Delta = \frac{U}{N_{\text{orb}}} \sqrt{\nu(1 - \nu)}$$

$$[D_s]_{ij} = \frac{2e^2}{\pi \hbar^2} \frac{\Delta^2}{U N_{\text{orb}}} \mathcal{M}_{ij}^{\text{R}, \text{min}}$$

Mean-field

Exact many-body

- Peotta, PT, Nat Comm 2015
- Huhtinen, Herzog-Arbeitman, Chew, Bernevig, PT, PRB 2022
- Herzog-Arbeitman, Chew, Huhtinen, PT, Bernevig, arXiv 2022

Lower bound for flat band superfluidity

Peotta, PT, Nat Comm 2015

The quantum geometric tensor \mathcal{B}_{ij}
is complex positive semidefinite

$$\rightarrow D_s \geq \int_{B.Z.} d^d \mathbf{k} |\Omega_{\text{Berry}}(\mathbf{k})| \geq C$$

Time reversal symmetry assumed; C is a spin Chern number

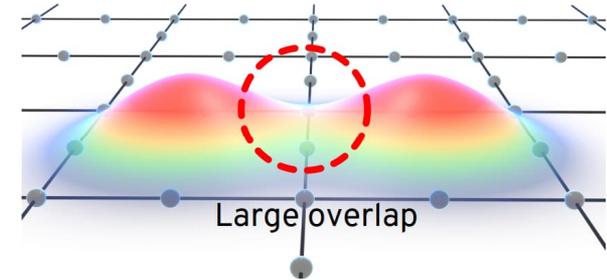
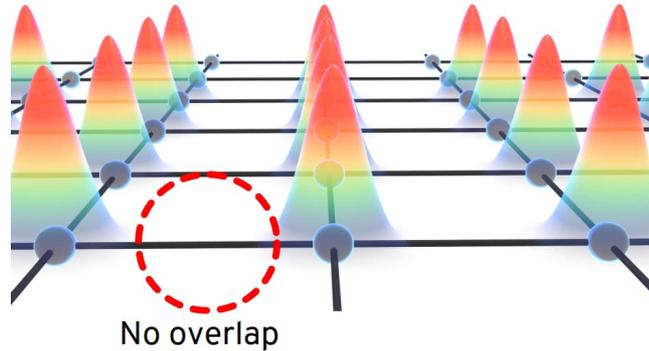
**Constituents: interactions, density of states (DOS)
and Bloch functions = quantum geometry and topology**

Why can there be transport in a flat band?

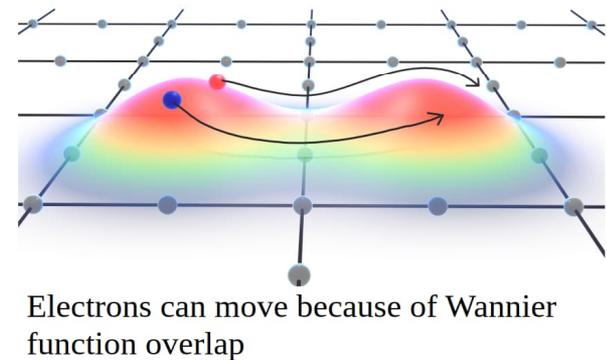
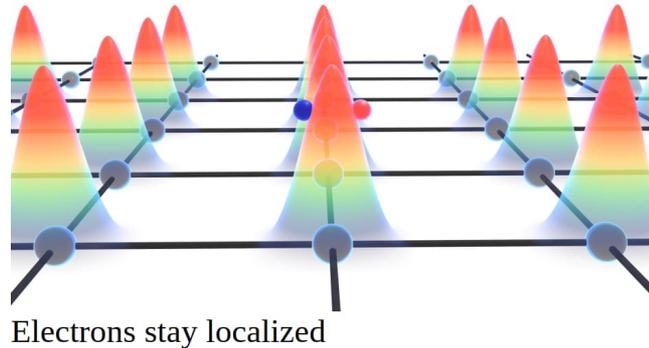
Localization and flat band due to vanishing overlap

Localization and flat band due to interference

Non-interacting



Interacting



$$C \neq 0 \Leftrightarrow \text{non-localized } w(\mathbf{r}) = \mathcal{F}[u(\mathbf{k})]$$

Brouder, Panati, Calandra, Marzari, PRL 2007

$$D_s \propto g_{ij} \geq C$$

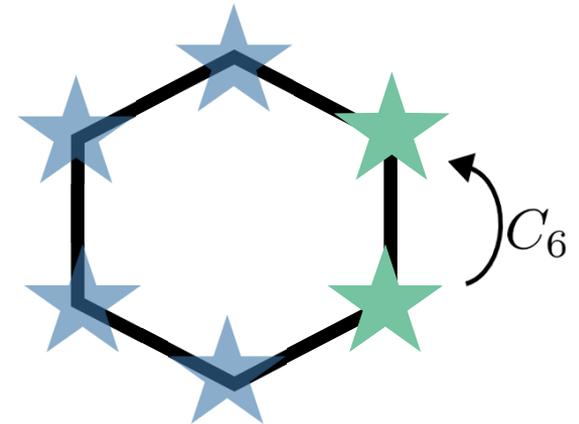
Uniform Pairing Condition from Symmetry

Are uniform pairing flat bands just fine tuning?

No! Uniform pairing is guaranteed by space group symmetry and the orbitals

Intuition: orbitals related by symmetry have uniform pairing

Precise statement: Orbitals forming an irrep of the site-symmetry group of a single Wyckoff position have uniform pairing



Complete equation for the superfluid weight

Huhtinen, Herzog-Arbeitsman, Chew, Bernevig, PT, PRB (2022)

$$\left. \frac{d^2 \Omega}{dq_i dq_j} \right|_{q=0} = \left. \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \right|_{q=0} - \left. (d_i \text{Im}(\Delta)) \mathbf{A} (d_j \text{Im}(\Delta)) \right|_{q=0}$$

conserved
not conserved
not conserved

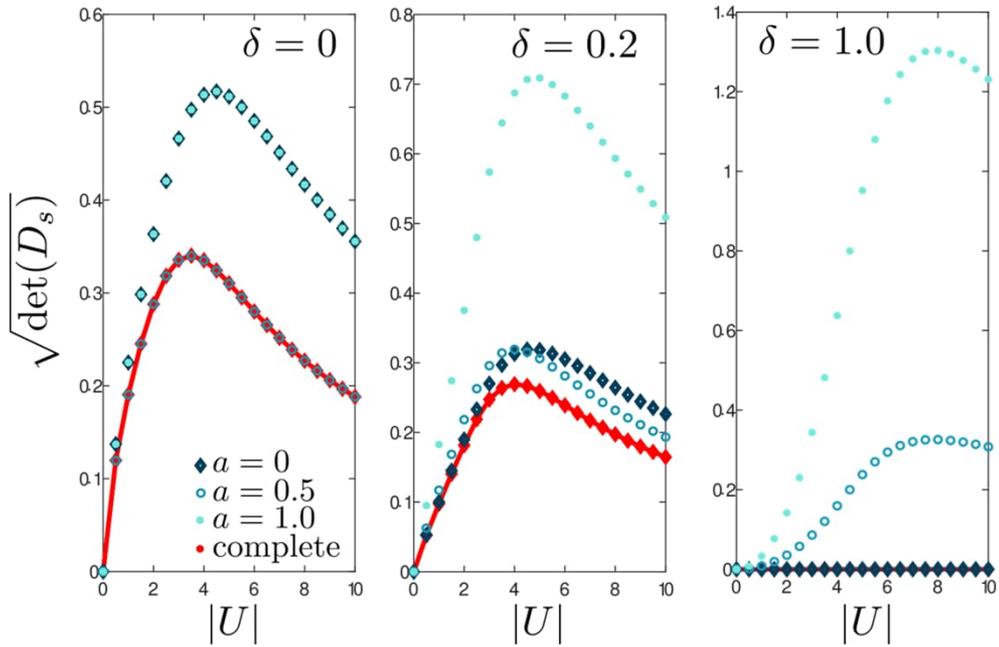
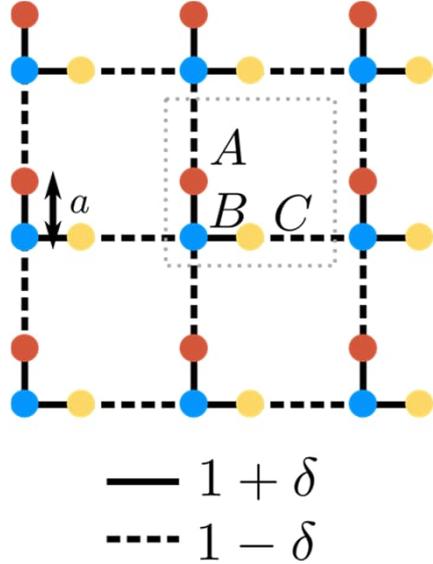
$$\mathbf{A} = \begin{pmatrix} \frac{\partial^2 \Omega}{\partial \text{Im} \Delta_2 \partial \text{Im} \Delta_2} & \cdots & \frac{\partial^2 \Omega}{\partial \text{Im} \Delta_2 \partial \text{Im} \Delta_{N_{\text{orb}}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \Omega}{\partial \text{Im} \Delta_{N_{\text{orb}}} \partial \text{Im} \Delta_2} & \cdots & \frac{\partial^2 \Omega}{\partial \text{Im} \Delta_{N_{\text{orb}}} \partial \text{Im} \Delta_{N_{\text{orb}}}} \end{pmatrix}$$

$$d_i \text{Im}(\Delta) = \left(\frac{d \text{Im} \Delta_2}{dq_i}, \dots, \frac{d \text{Im} \Delta_{N_{\text{orb}}}}{dq_i} \right)$$

- The minimal quantum metric, i.e. the one with the smallest possible trace, is related to the superfluid weight in isolated flat bands with TRS and uniform pairing.

When the orbitals are at high-symmetry positions, the quantum metric is guaranteed to be minimal

Example of the orbital dependence: the Lieb lattice



At worst, $\frac{1}{V} \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \Big|_{q=0}$ can give an incorrectly nonzero superfluid weight.

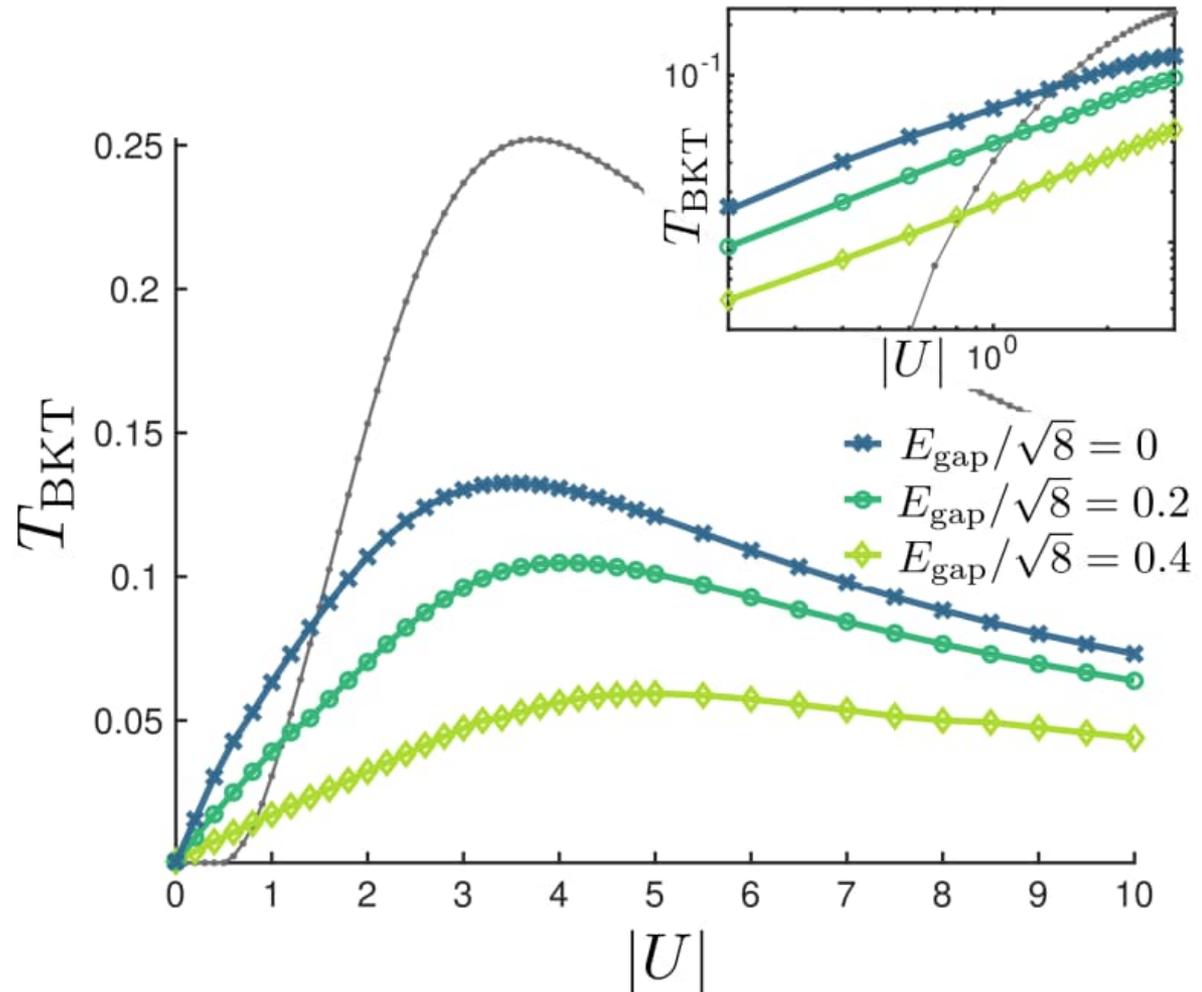
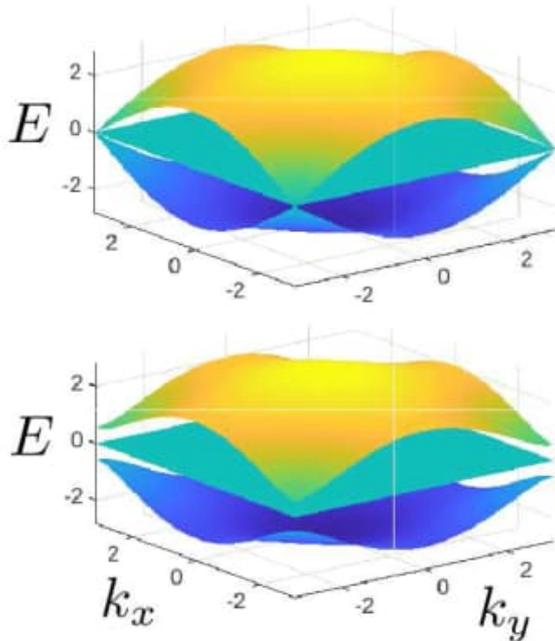
Orbital dependence of quantum geometric quantities: Simon and Rudner, PRB (2020)

When the orbitals are at high-symmetry positions, the quantum metric is guaranteed to be minimal

Non-isolated bands:

Band touchings **increase** the critical temperature

$$T_{\text{BKT}} = \frac{\pi}{8} \sqrt{\det D^s(T_{\text{BKT}})}$$



Quantum geometric superconductivity: confirmed beyond mean-field

BCS-state is the exact ground state at T=0

Julku, Peotta, Vanhala, Kim, PT, PRL 2016

Tovmasyan, Peotta, PT, Huber, PRB 2016

Exact diagonalization, DMFT, QMC, DMRG

Julku, Peotta, Vanhala, Kim, PT, PRL 2016

Liang, Vanhala, Peotta, Siro, Harju, PT, PRB 2017

Mondaini, Batrouni, Grémaud, PRB 2018

Hofmann, Berg, Chowdhury, PRB 2020, PRL 2023

Peri, Song, Bernevig, Huber, PRL 2021

Chan, Grémaud, Batrouni, PRB 2022 (x 2)

Herzog-Arbeitman, Peri, Schindler, Huber, Bernevig, PRL 2022

Prefomed pairs

Tovmasyan, Peotta, Liang, PT, Huber, PRB 2018

Perturbation theory with a Hamiltonian projected to a flat band

Tovmasyan, Peotta, Törmä, Huber, PRB 2016

$$H = \frac{|U|}{2} \sum_{i\alpha} (\bar{n}_{i\alpha\uparrow} - \bar{n}_{i\alpha\downarrow})^2$$

Exact results on the excitations possible! next

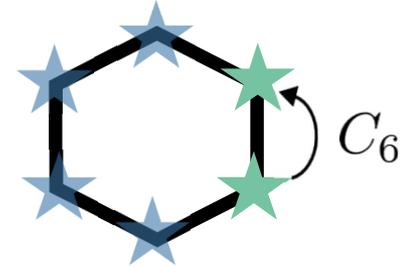
Quantum geometric superconductivity: exact results on Cooper pair mass and excitations

Project to the flat band and assume the uniform pairing condition

$$\bar{c}_{\mathbf{k}\alpha\sigma}^\dagger = \sum_{\beta} c_{\mathbf{k}\beta\sigma}^\dagger P_{\beta\alpha}^\sigma(\mathbf{k})$$

$$P^\sigma(\mathbf{k}) = \sum_{n \in \mathcal{B}} |n_{\mathbf{k}\sigma}\rangle \langle n_{\mathbf{k}\sigma}|$$

$$\frac{1}{N_c} \sum_{\mathbf{k}} P_{\alpha\alpha}(\mathbf{k}) = \frac{N_f}{N_{\text{orb}}}$$



$$\longrightarrow H = \frac{|U|}{2} \sum_{i\alpha} (\bar{n}_{i\alpha\uparrow} - \bar{n}_{i\alpha\downarrow})^2$$

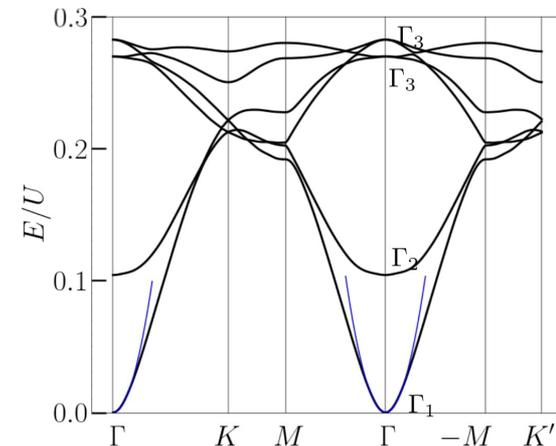
Ground state $|n\rangle \propto \eta^{\dagger n} |0\rangle$ $\eta^\dagger = \sum_{\mathbf{k}\alpha} \bar{c}_{\mathbf{k}\alpha\uparrow}^\dagger \bar{c}_{-\mathbf{k}\alpha\downarrow}^\dagger$

Cooper pair excitations governed by an effective single particle Hamiltonian

$$h_{\alpha\beta}(\mathbf{q}) = -\frac{|U|}{N_c} \sum_{\mathbf{k}} P_{\alpha\beta}(\mathbf{k} + \mathbf{q}) P_{\beta\alpha}(\mathbf{k})$$

- Single particles immobile
- Cooper pair mass from quantum geometry
- Leggett and Goldstone modes

$$\left[\frac{1}{m^*} \right]_{ij} = \frac{|U|}{N_{\text{orb}}} \mathcal{M}_{ij}^{\text{R}, \text{min}}$$



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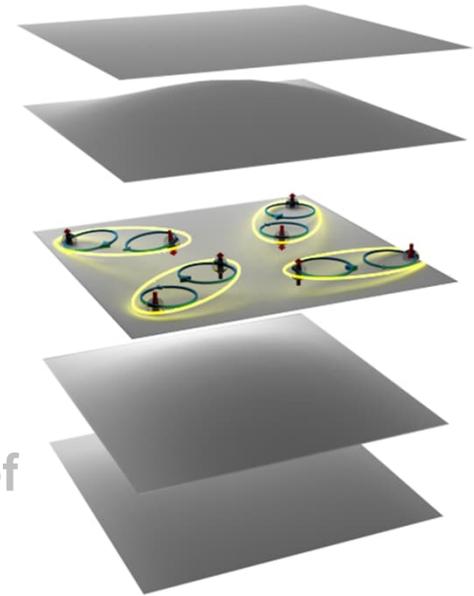
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Twisted Bilayer Graphene (TBG) superconductivity since 2018

Reviews: Balents, Dean, Efetov, Young, Nat Phys 2020

Andrei, Efetov, Jarillo-Herrero, MacDonald, Mak, Senthil, Tutuc, Yazdani, Young, Nat Rev Mater 2021

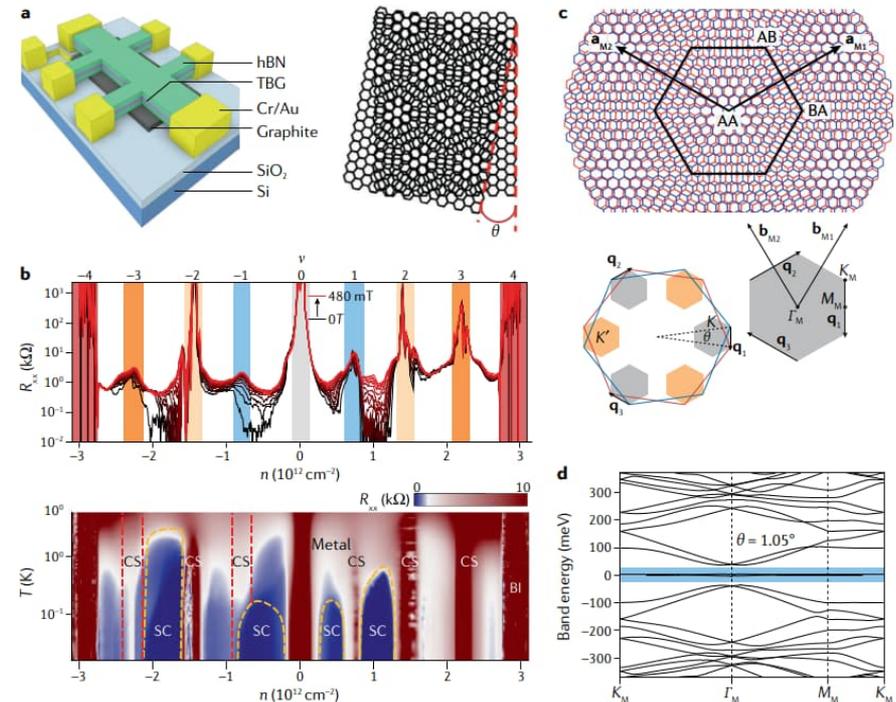
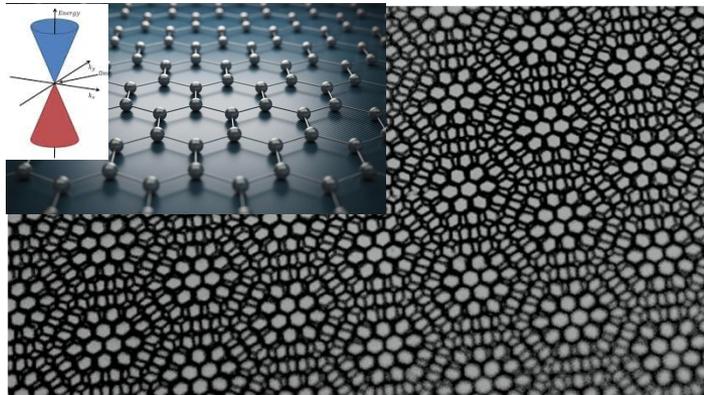
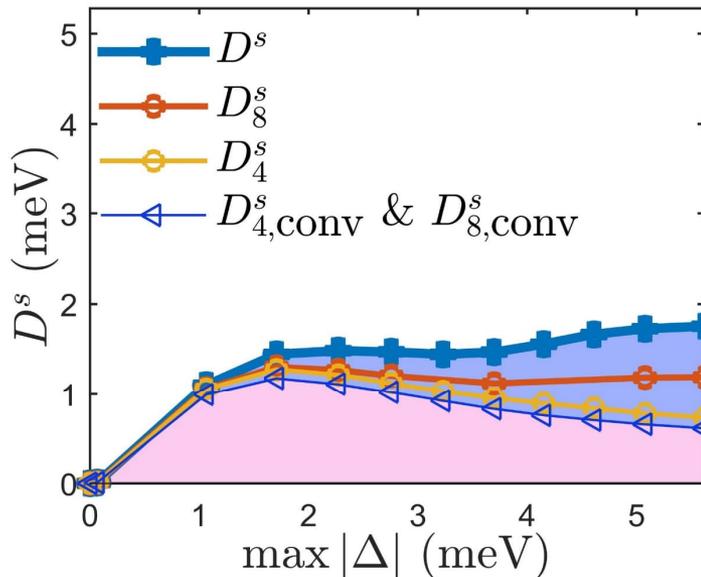


Figure credits see Fig.1 in
PT, Peotta, Bernevig,
Nat Rev Phys 2022

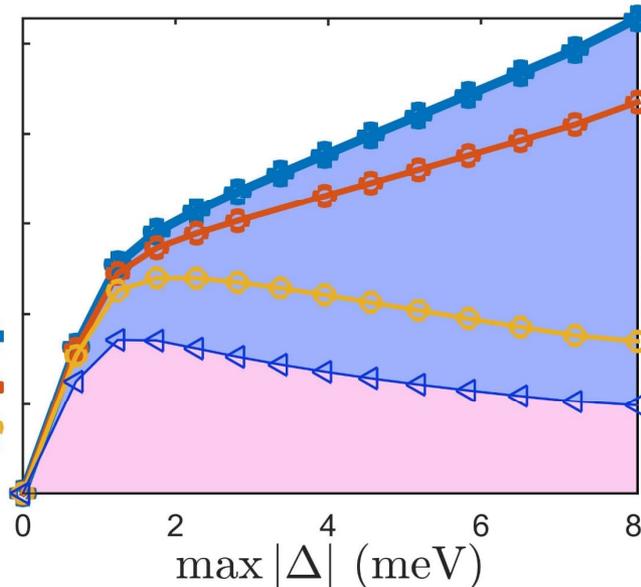
Geometric contribution in TBG

$$D^s = D_{\text{conv}}^s + D_{\text{geom}}^s$$

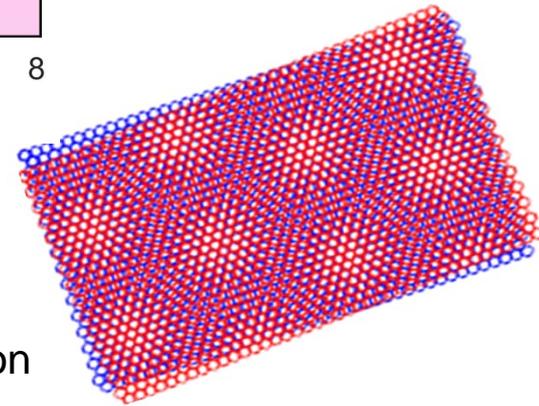
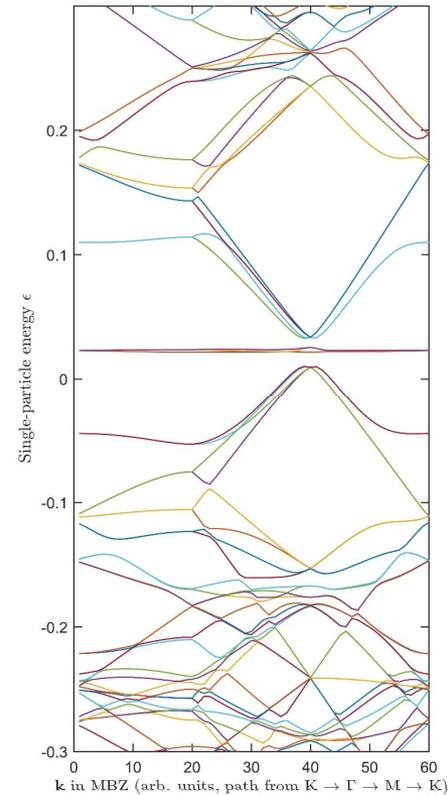
$$T_{\text{BKT}} = \frac{\pi}{8} \sqrt{\det D^s(T_{\text{BKT}})}$$



Non-local (RVB) interaction



Local (s-wave) interaction



Julku, Peltonen, Liang, Heikkilä, PT, PRB(R) (2020); Editors' Suggestion

Confirmed by (only s-wave): Hu, Hyart, Pikulin, Rossi, PRL (2019)

Euler class bound of TBG superconductivity: Xie, Song, Lian, Bernevig, PRL (2020)

The calculation should be revisited with modern understanding of TBG

Relevant for TBG superconductivity?

Theoretically suggested:

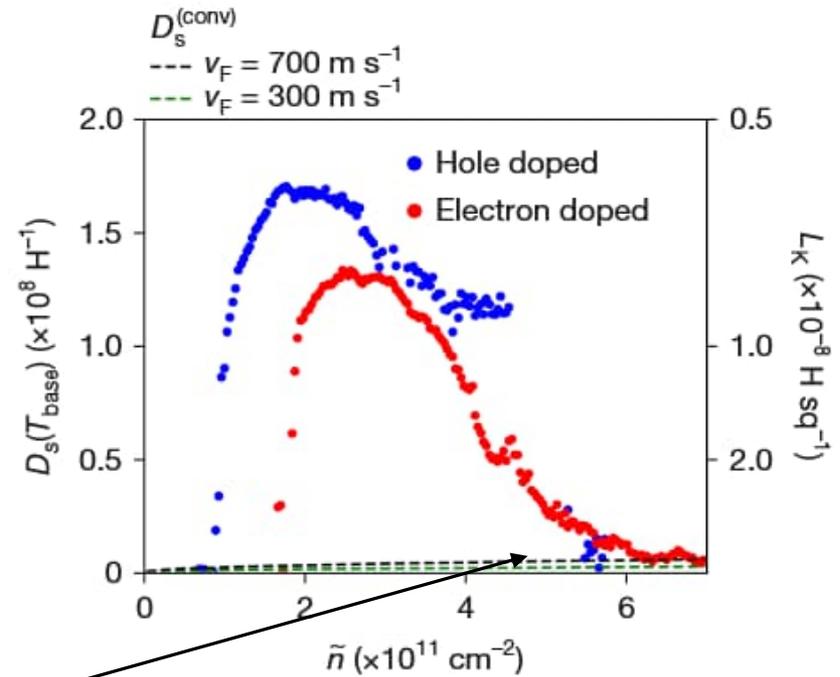
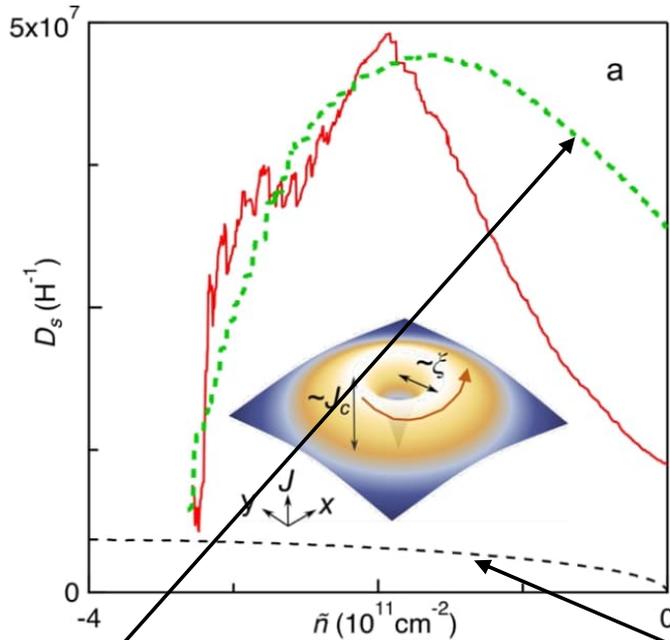
Julku, Peltonen, Liang, Heikkilä, PT, PRB 2020

Hu, Hyart, Pikulin, Rossi, PRL 2020

Xie, Song, Lian, Bernevig PRL2020

Tian, ... , Lau, Bockrath, Nature 2023

Tanaka, ... , Jarillo-Herrero, Oliver, Nature 2025



$$D_s(0, \tilde{n}) \approx b \frac{e^2}{\hbar^2} \Delta(0, \tilde{n})$$

$$D_s(T) = \frac{e^2 n_s(T)}{m}$$

Isolated flat band
Peotta, PT, Nat Comm 2015

$$[D_s]_{ij} = \frac{2}{\pi \hbar^2} \frac{\Delta^2}{UN_{\text{orb}}} \mathcal{M}_{ij}^R$$

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**Flat band ratio and quantum metric: DMFT studies of
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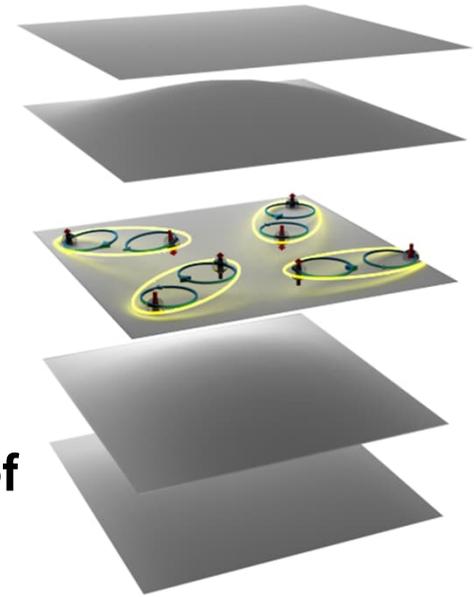
Non-equilibrium transport in a flat band

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Flat-band ratio and quantum metric in the superconductivity of modified Lieb lattices - how general are the isolated flat band results?



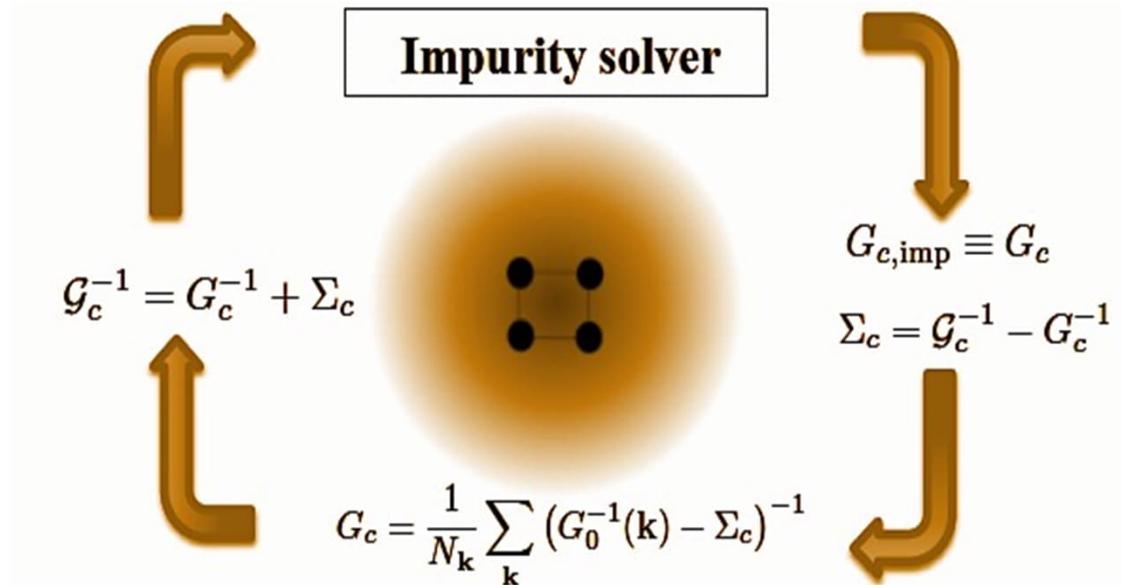
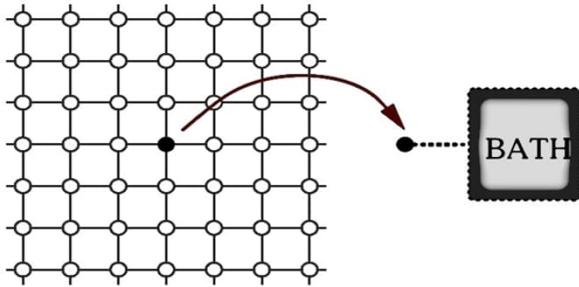
Reko Penttilä



Kukka-Emilia Huhtinen

Penttilä, Huhtinen, PT, Communications Physics (2025)
Focus Issue “Flat band engineering: from quantum materials
to quantum simulation”

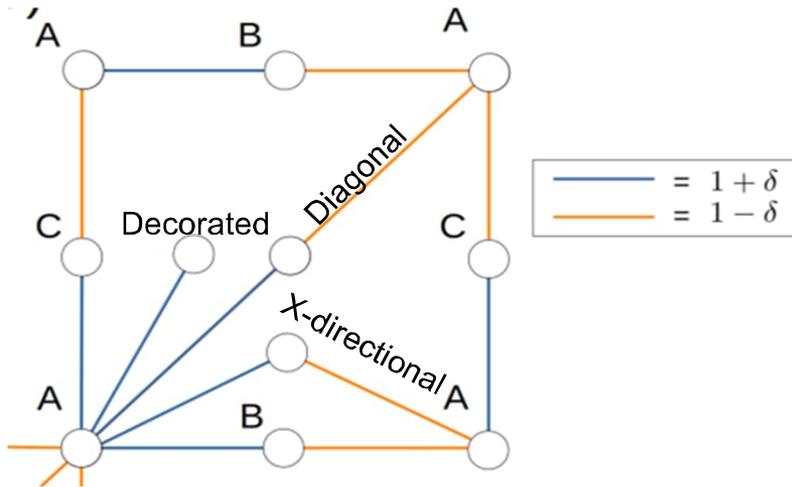
Dynamical Mean Field Theory (DMFT) to capture quantum effects *beyond mean-field*



Single site DMFT

Cellular/cluster DMFT; Non-local correlations

Different extended Lieb lattices



Isolated flat band, mean-field:

$$[D_s]_{ij} = \frac{4U\nu(1-\nu)N_f}{(2\pi)^{d-1}N_{\text{orb}}} \mathcal{M}_{ij}^{\text{R},\text{min}}$$

1/2

2/3

2/3

2/3

Lieb

diagonal

x-directional

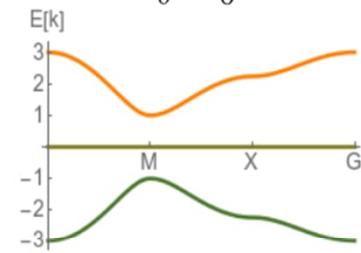
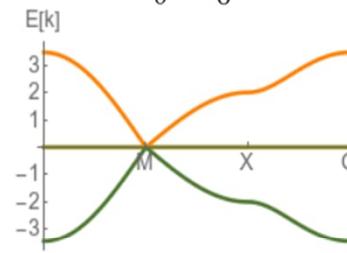
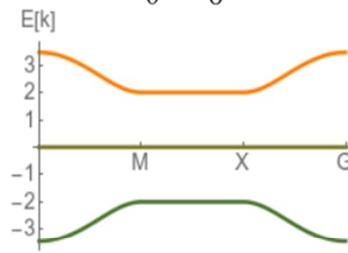
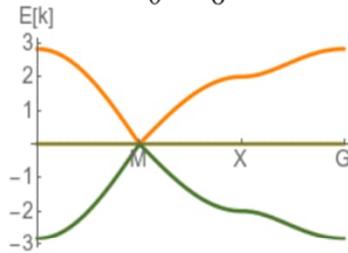
decorated

$\delta = 0$

$\delta = 0$

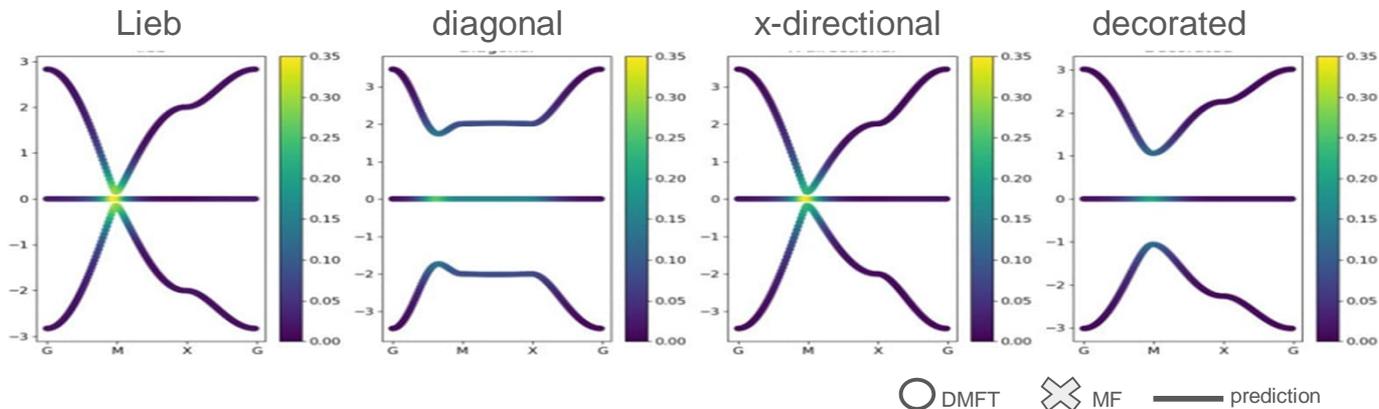
$\delta = 0$

$\delta = 0$

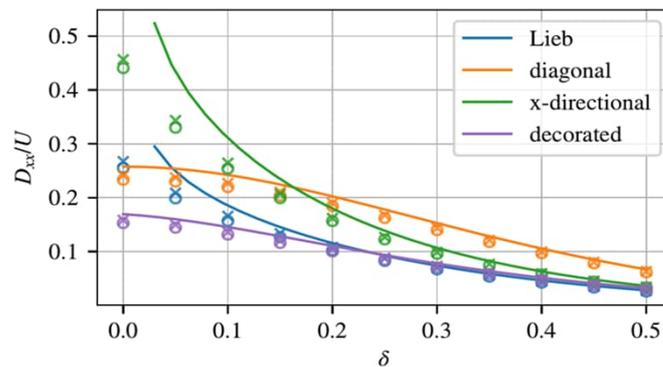


Quantum metric determines the superfluid weight also in DMFT results, T=0

$\delta = 0.05$

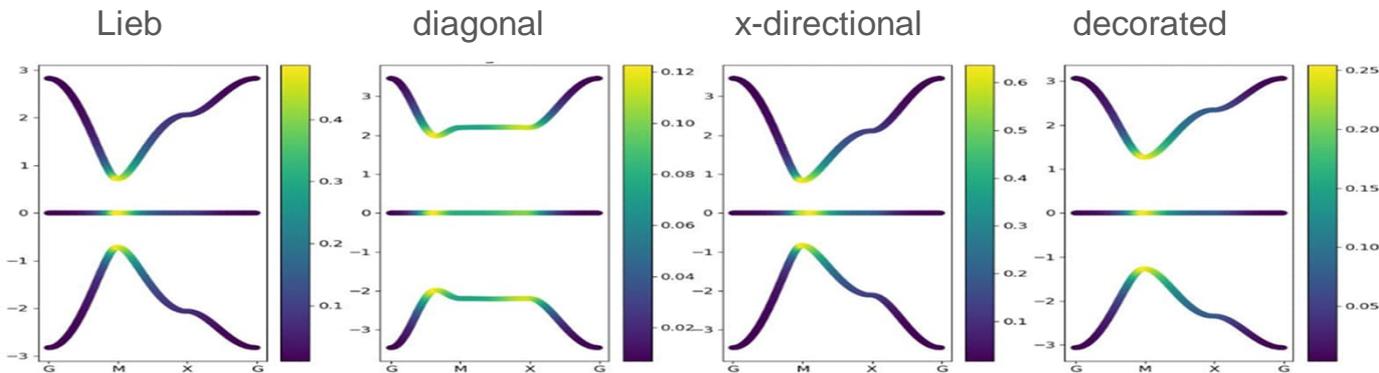


$$[D_s]_{ij} = \frac{4U\nu(1-\nu)N_f}{(2\pi)^{d-1}N_{\text{orb}}} \mathcal{M}_{ij}^{\text{R},\text{min}}$$



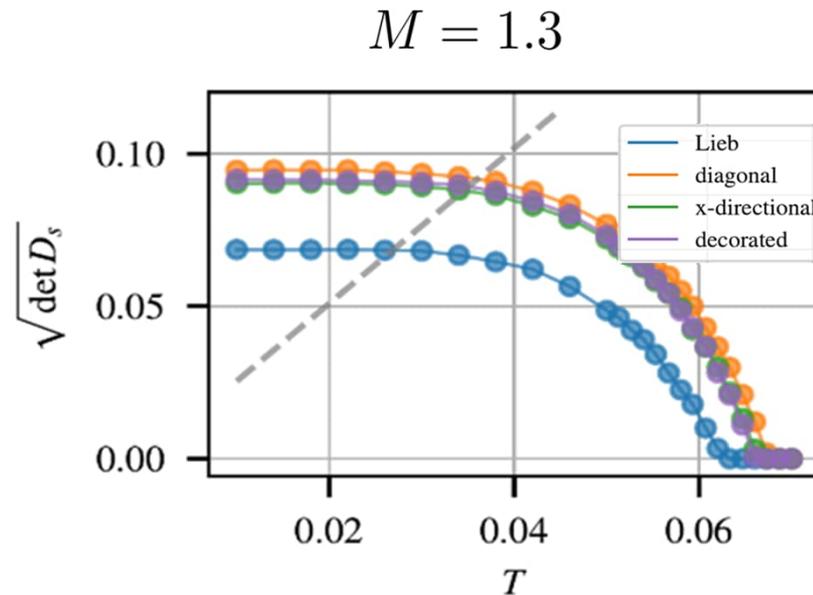
$|U| = 0.3$

$\delta = 0.3$



BKT temperature

The flat-band ratio and minimal quantum metric are able to qualitatively predict the BKT temperatures.

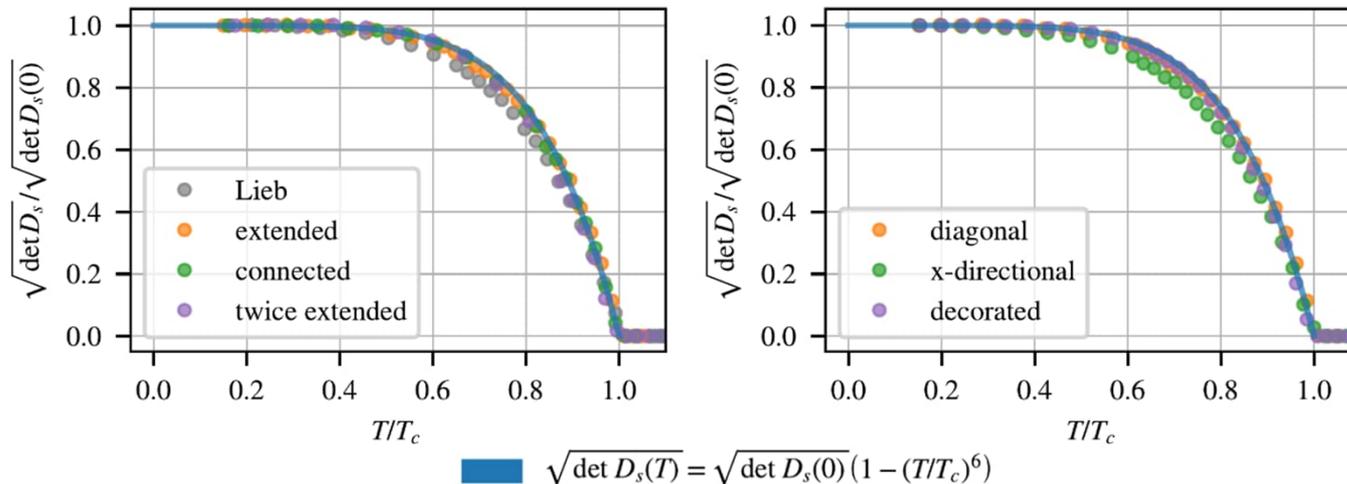


All lattices have the same minimal integrated quantum metric but the Lieb lattice has a smaller flat-band ratio.

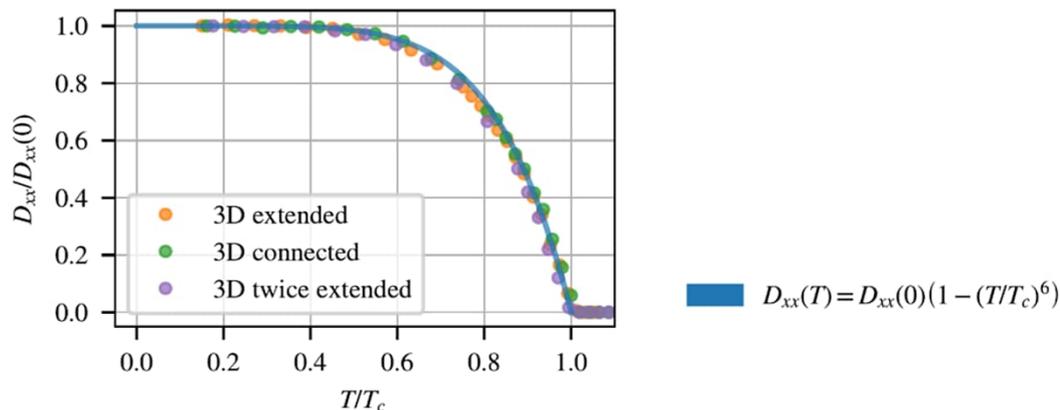
Temperature dependence of the superfluid weight

All lattices have similar behavior of the superfluid weight as a function of temperature

2D extensions



3D extensions



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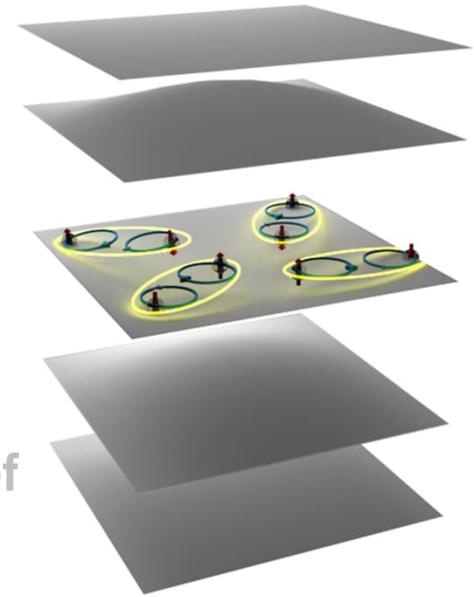
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SYNOPSIS

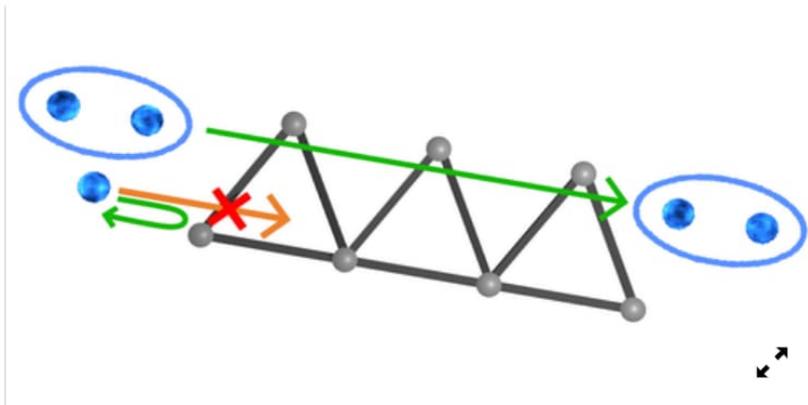
PDF Version



Static Electrons in Flat-Band Nonequilibrium Superconductors

May 25, 2023 • *Physics* 16, s76

Single electrons stay stationary in superconductors with “flat-band” electronic structures, which could lead to low-energy-consumption devices made from such materials.



A. Paraoanu/Aalto University

Suppression of Nonequilibrium Quasiparticle Transport in Flat-Band Superconductors

Ville A. J. Pyykkönen, Sebastiano Peotta, and Päivi Törmä

Phys. Rev. Lett. **130**, 216003 (2023)

Published May 25, 2023

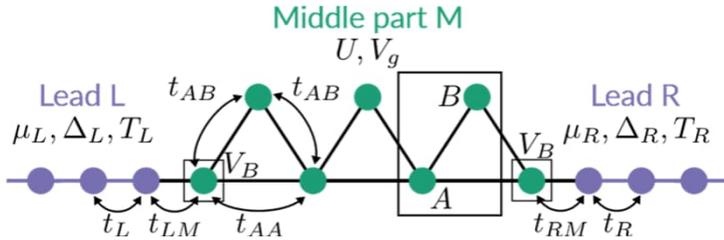


Ville Pyykkönen

Sebastiano Peotta

Pyykkönen, Peotta, PT, PRL 2023
Editors' Suggestion

Flat band transport in Keldysh formalism



Fermi-Hubbard Hamiltonian

$$\hat{H} = \sum_{\alpha i, \beta j, \sigma} T_{\alpha i, \beta j} \hat{c}_{\alpha i \sigma}^\dagger \hat{c}_{\beta j \sigma} + \sum_{\alpha i} U_{\alpha i} \hat{c}_{\alpha i \uparrow}^\dagger \hat{c}_{\alpha i \downarrow}^\dagger \hat{c}_{\alpha i \downarrow} \hat{c}_{\alpha i \uparrow}$$

$$\alpha, \beta \in \{L, R, M\}$$

Mean-field approximation

$$\hat{H}_{\text{MF}}(t) = \sum_{\alpha i, \beta j} \hat{d}_{\alpha i}^\dagger \begin{pmatrix} T_{\alpha i, \beta j} + V_{H, \alpha i}(t) \delta_{\alpha i, \beta j} & \Delta_{\alpha i} \delta_{\alpha i, \beta j} \\ \Delta_{\alpha i}^* \delta_{\alpha i, \beta j} & -T_{\alpha i, \beta j}^* - V_{H, \alpha i}(t) \delta_{\alpha i, \beta j} \end{pmatrix} \hat{d}_{\beta j}$$

$$\hat{d}_{\alpha i} = \left(\hat{c}_{\alpha i \uparrow}, \hat{c}_{\alpha i \downarrow} \right)^T$$

Superconducting order parameter

$$\Delta_{\alpha i}(t) = U_{\alpha i} \langle \hat{c}_{\alpha i \downarrow}(t) \hat{c}_{\alpha i \uparrow}(t) \rangle$$

Hartree potential

$$V_{H, \alpha i}(t) = U_{\alpha i} \langle \hat{c}_{\alpha i \uparrow}^\dagger(t) \hat{c}_{\alpha i \uparrow}(t) \rangle$$

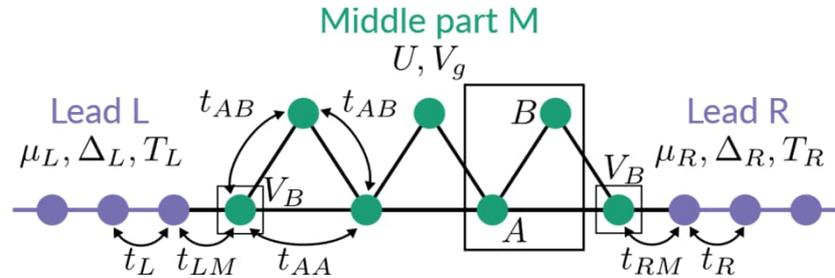
Keldysh formalism, non-equilibrium Green's functions

Dyson equation $G^{R/A}(\omega) = g^{R/A}(\omega) + g^{R/A}(\omega) \Sigma^{R/A}(\omega) G^{R/A}(\omega)$

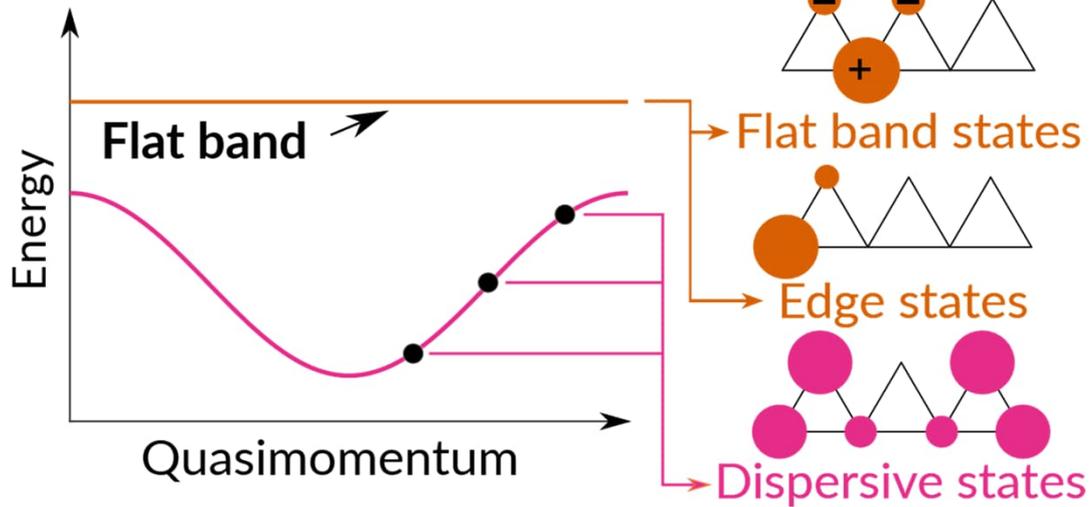
Kadanoff-Baym kinetic equation

$$G^<(\omega) = [I + G^R(\omega) \Sigma^R(\omega)] g^<(\omega) [I + \Sigma^A(\omega) G^A(\omega)] + G^R(\omega) \Sigma^<(\omega) G^A(\omega)$$

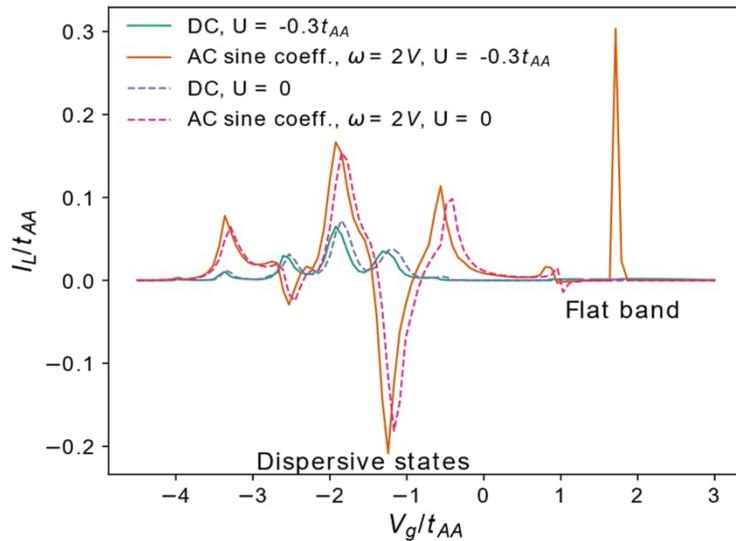
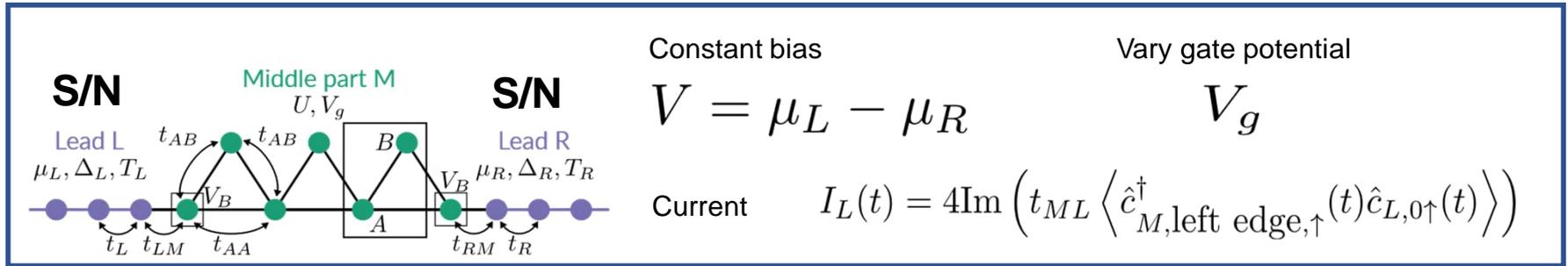
Flat, edge and dispersive states in the sawtooth ladder



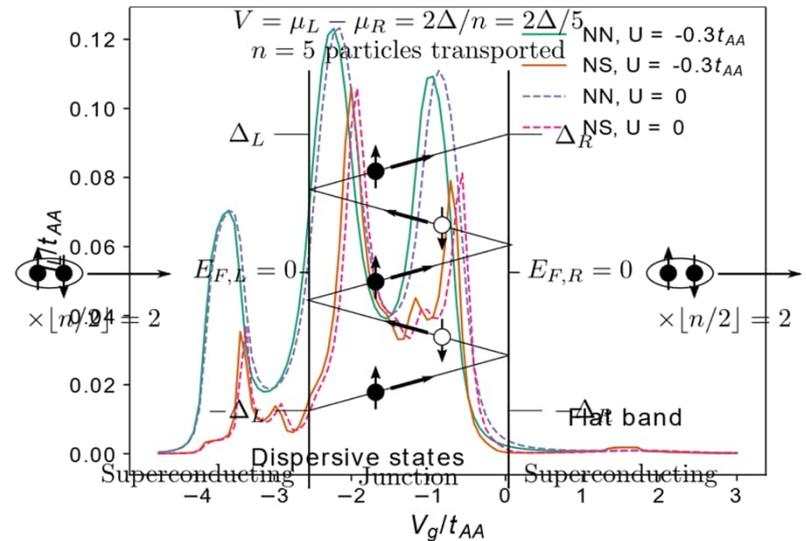
Select a state by gate potential V_g



Transport



Superconducting junction: at finite interaction **flat band AC Josephson current is finite but DC current (multiple Andreev reflections) quenched**



Normal-normal and normal-superconducting junction: **flat band current is quenched**

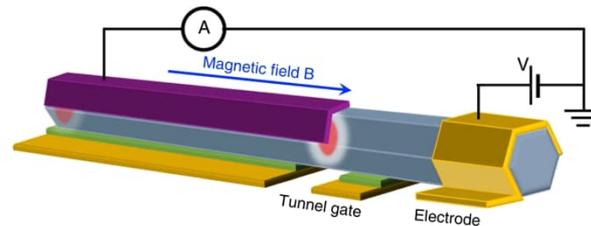
Quasiparticle transport quenched at flat band! Pure supercurrent!

Quasiparticle transport quenched at flat band! Pure supercurrent!

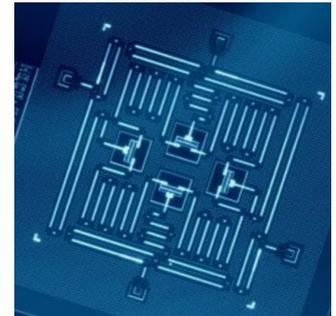
Quasiparticle poisoning

Nonequilibrium Quasiparticles and $2e$ Periodicity in Single-Cooper-Pair Transistors

J. Aumentado, Mark W. Keller, John M. Martinis, and M. H. Devoret
Phys. Rev. Lett. **92**, 066802 – Published 13 February 2004



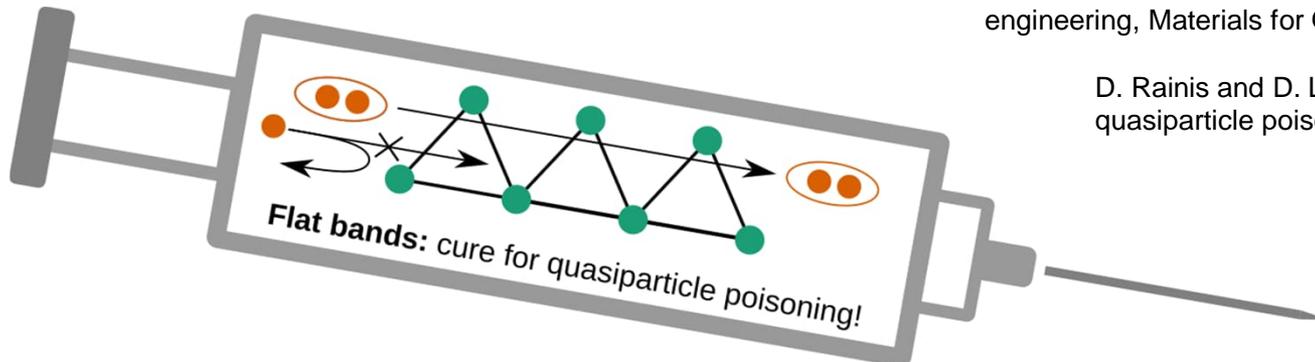
Majorana nanowire. H. Zhang, D.E. Liu, M. Wimmer, L.P. Kouwenhoven (Nat Commun 10, 5128, 2019) by CC BY 4.0 license



Four transmons. F.J.M. Gambaetta, J.M. Chow, and M. Steffen (npj QuantumInformation 3:2, 2017) by CC BY 4.0 license

G. Catelani and J. P. Pekola, Using materials for quasiparticle engineering, Materials for Quantum Technology 2, 013001 (2022)

D. Rainis and D. Loss, Majorana qubit decoherence by quasiparticle poisoning, Phys. Rev. B 85, 174533 (2012)



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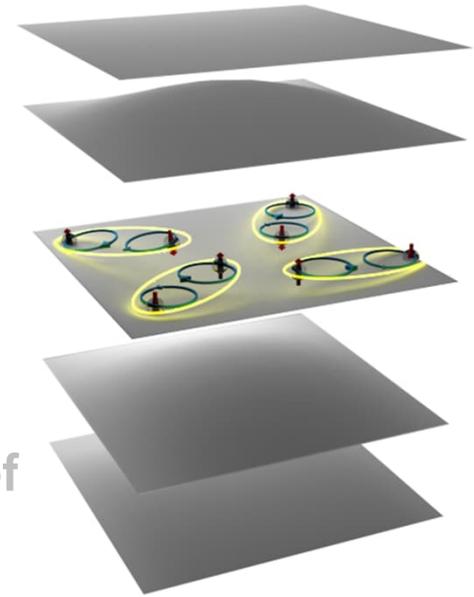
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Conductivity in a flat band



Kukka-Emilia Huhtinen

KE Huhtinen, PT, PRB (2023)

Conductivity in a flat band

Semiclassical Boltzmann theory of transport:

$$\sigma_{\mu\nu}(\omega) = -\frac{e^2}{\hbar} \sum_n \int_{\text{B.Z.}} \frac{d^D \mathbf{k}}{(2\pi)^D} \frac{\partial n_F(E)}{\partial E} \Big|_{E=\epsilon_n(\mathbf{k})} \partial_\mu \epsilon_n(\mathbf{k}) \partial_\nu \epsilon_n(\mathbf{k}) \frac{\eta}{(\hbar\omega)^2 + \eta^2} \quad \partial_\mu = \partial/\partial k_\mu$$

Full Kubo-Greenwood formula:

$$\sigma_{\mu\nu}(\omega) = \frac{e^2}{i\hbar V} \sum_{\mathbf{k}} \sum_{mn} \frac{n_F(\epsilon_n(\mathbf{k})) - n_F(\epsilon_m(\mathbf{k}))}{\epsilon_n(\mathbf{k}) - \epsilon_m(\mathbf{k})} \frac{[j_\mu(\mathbf{k})]_{nm} [j_\nu(\mathbf{k})]_{mn}}{\epsilon_n(\mathbf{k}) - \epsilon_m(\mathbf{k}) + \hbar\omega + i\eta}$$
$$[j_\mu(\mathbf{k})]_{mn} = \partial_\mu \epsilon_m(\mathbf{k}) \delta_{mn} + (\epsilon_m(\mathbf{k}) - \epsilon_n(\mathbf{k})) \langle \partial_\mu m_{\mathbf{k}} | n_{\mathbf{k}} \rangle$$

At low temperatures and finite scattering rate η , the interband geometric part is dominant on a flat band.

Inspired by

G. Bouzerar and D. Mayou, Phys. Rev. B 103, 075415 (2021)

J. Mitscherling and T. Holder, Phys. Rev. B 105, 08515 (2022)

B. Mera and J. Mitscherling, Phys. Rev. B 106, 165133 (2022)

G. Bouzerar, Phys. Rev. B 106, 125125 (2022)

Conductivity in a flat band

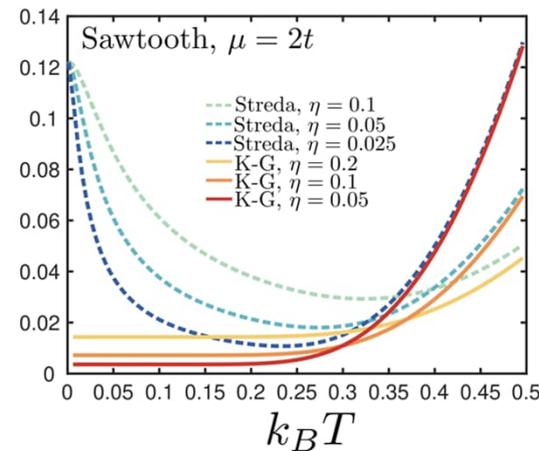
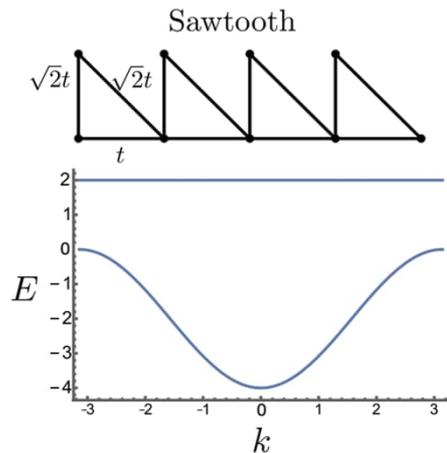
Streda formula:

$$\sigma_{\mu\nu}^{\text{sym}}(\omega = 0) = -\frac{e^2}{\hbar\pi} \int_{-\infty}^{\infty} d\epsilon \frac{\partial n_F(\epsilon)}{\partial \epsilon} \text{Tr}[\text{Im}[G_{\mathbf{k}}(\epsilon + i\eta)]j_{\mu}(\mathbf{k})\text{Im}[G_{\mathbf{k}}(\epsilon + i\eta)]j_{\nu}(\mathbf{k})]$$

$$G_{\mathbf{k}}(E) = (E - H_{\mathbf{k}})^{-1}$$

This gives a result proportional to the integrated quantum metric in the limit $\eta \rightarrow 0^+$ when $T \rightarrow 0$ is taken *first*.

This occurs only in *perfectly* (partially) flat bands due to ill-defined terms for states at the Fermi energy. **The Kubo-Greenwood and Streda formulas do not give the same conductivity when a flat band is in the vicinity of the Fermi energy.**



Lack of Fermi surface requires extra care in transport calculations.

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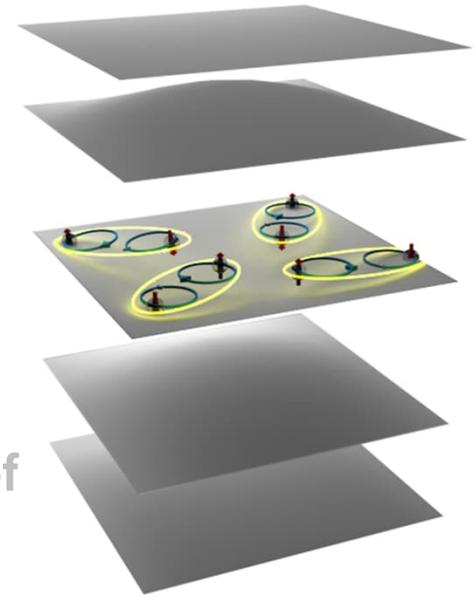
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Drude weight and the many-body quantum metric



Grazia Salerno



Tomoki Ozawa

Salerno, Ozawa, PT, PRB Letter (2023)

The many-body quantum metric (MBQM)

Defined on many-body states with respect to the twisted boundary condition phase

$$\mathbf{g}(\phi) = \text{Re} [\langle \partial_\phi \Psi_0 | (1 - |\Psi_0\rangle\langle\Psi_0|) | \partial_\phi \Psi_0 \rangle]$$

determines the “quantum distance” along a given path in ϕ space.

➔ Many-body generalization of the quantum metric

$$\mathbf{g}(0) = \text{Re} \left[\sum_{m \neq 0} \frac{|\langle \Psi_m | \partial_\phi \hat{H}(\phi) | \Psi_0 \rangle|^2}{(E_m(0) - E_0(0))^2} \right]$$

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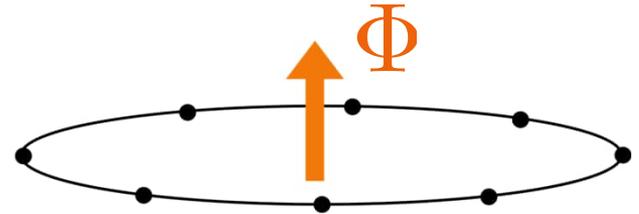
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➔ Many-body generalization of the quantum metric

$$\mathbf{g}(0) = \text{Re} \left[\sum_{m \neq 0} \frac{|\langle \Psi_m | \partial_\phi \hat{H}(\phi) | \Psi_0 \rangle|^2}{(E_m(0) - E_0(0))^2} \right]$$

$$\partial_\phi \hat{H}(\phi) = L \partial_\Phi \hat{H}(\Phi) = \hat{J} + \mathcal{O}(\Phi)$$

Drude weight and twisted boundary conditions



$$\hat{H}_0 = \hat{H}_{\text{kin}} + \hat{H}_V + \hat{H}_U = (\hat{K} + \hat{K}^\dagger) + \hat{H}_V + \hat{H}_U$$

Superfluid response of the system to a small external flux Φ introduced by the twisted boundary conditions:

$$D_w = \pi L \left. \frac{\partial^2 E(\Phi)}{\partial \Phi^2} \right|_{\Phi=0}$$

$$\hat{H}(\Phi) = \hat{K} e^{i\Phi/L} + \hat{K}^\dagger e^{-i\Phi/L} + \hat{H}_V + \hat{H}_U$$

Drude weight within perturbation theory

$$\hat{H}(\Phi) = \hat{H}_0 + \hat{H}_{\text{pert}} \quad \text{with} \quad \hat{H}_{\text{pert}} = \frac{\Phi}{L} \hat{J} - \frac{1}{2} \left(\frac{\Phi}{L} \right)^2 \hat{H}_{\text{kin}}$$

$$\text{Current operator } \hat{J} = i(\hat{K} - \hat{K}^\dagger)$$

$$D_w = 2\pi L \frac{E(\Phi) - E(0)}{\Phi^2} = -\frac{\pi}{L} \langle \Psi_0 | \hat{H}_{\text{kin}} | \Psi_0 \rangle - \frac{2\pi}{L} \underbrace{\sum_{m \neq 0} \frac{|\langle \Psi_m | \hat{J} | \Psi_0 \rangle|^2}{E_m(0) - E_0(0)}}_{> \mathfrak{g}(0) \cdot \varepsilon}$$

Can be bounded by the **many-body quantum metric**
if the system has a gap ε

Independent of particle statistics and spatial dimensions!

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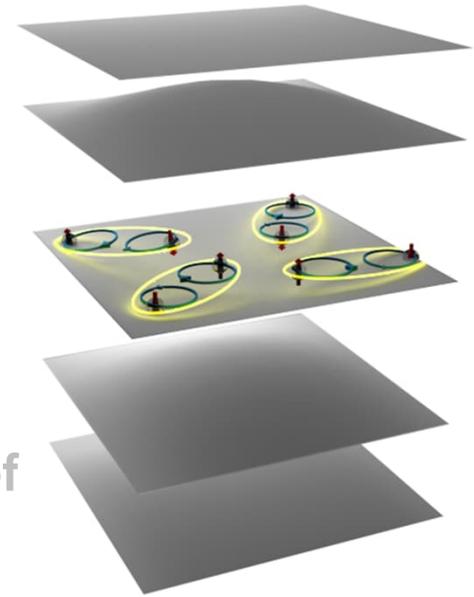
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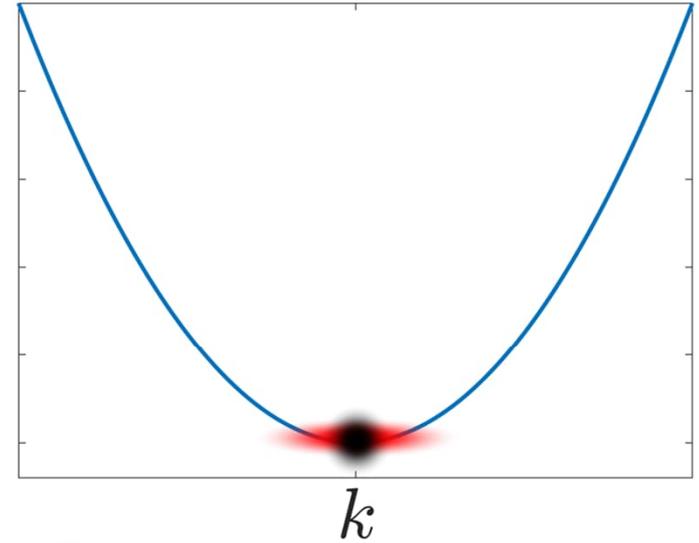
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BEC in continuum

$$\epsilon(\mathbf{k}) = \frac{k^2}{2m_{eff}}$$



$$n_{ex}(\mathbf{k}) = \frac{\epsilon(\mathbf{k}) + Un_0}{\sqrt{\epsilon(\mathbf{k})[\epsilon(\mathbf{k}) + 2Un_0]}} - 1$$

n_0 condensate density U repulsive contact interaction strength

n_{ex} = density of non-condensed bosons (quantum depletion)

BEC in continuum

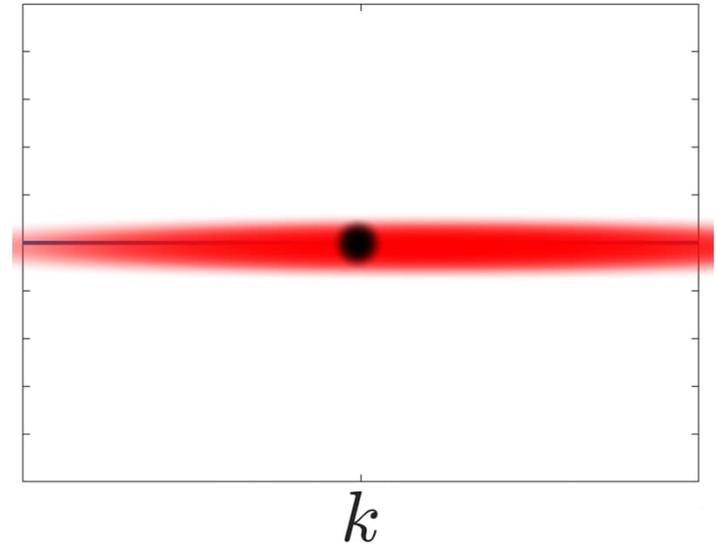
$$m_{eff} \rightarrow \infty$$

$$\epsilon(\mathbf{k}) = \frac{k^2}{2m_{eff}} \rightarrow 0$$

For any finite interaction:

$$n_{ex} \rightarrow \frac{Un_0}{\sqrt{\epsilon(\mathbf{k})2Un_0}} - 1 \rightarrow \infty$$

Obviously, flat band BEC in a single band model not possible.



BEC in a flat band?

- BEC in a flat band should not be possible???
- True for a single band system, **not for multiband models** (as shown in earlier works)

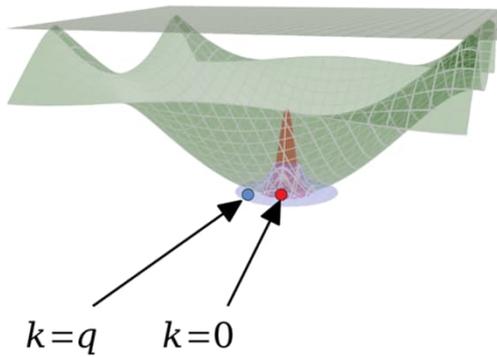
Huber & Altman, PRB 82, 184502 (2010)

You et al., PRL 109, 265302 (2012) (H. Zhai group)

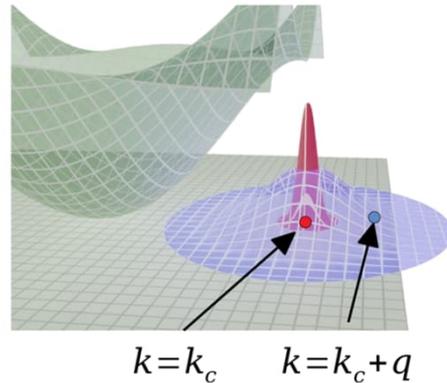
- **What determines the stability?**

Flat band BEC & quantum geometry

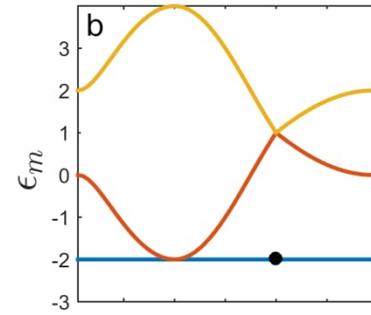
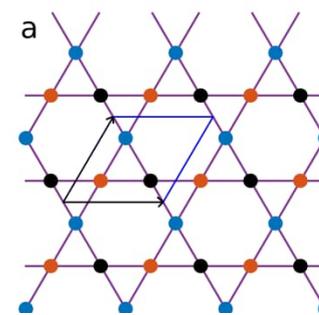
DISPERSIVE BAND



FLAT BAND



Kagome lattice:

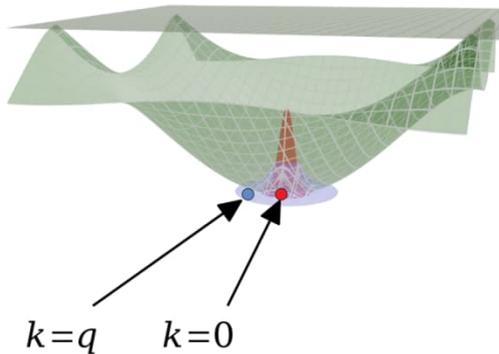


Alekski Julku Georg Bruun Grazia Salerno

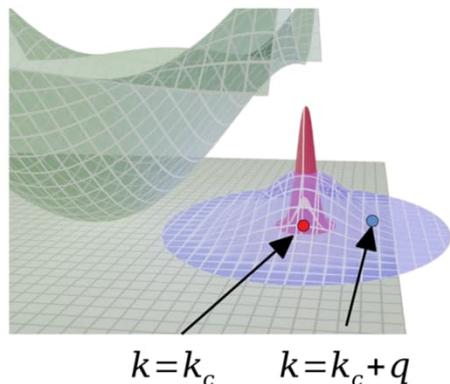
Julku, Bruun, PT, PRL 2021, PRB 2021
Julku, Salerno, PT, Fizika Nizkikh Temperatur (journal of ILTPE, Kharkiv, Ukraine) 49, 770 (2023); special issue coordinated by Andrei Bernevig

Flat band BEC & quantum geometry

DISPERSIVE BAND



FLAT BAND



n_0 Condensate density n_e Excitation density U Interaction $u(k)$ Bloch function

SPEED OF SOUND

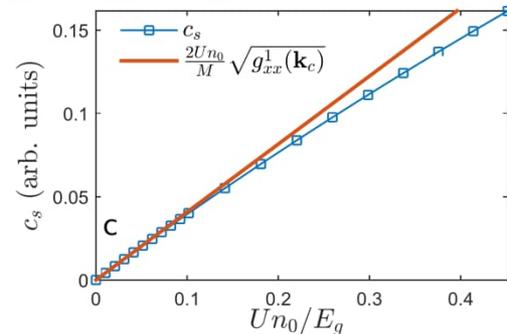
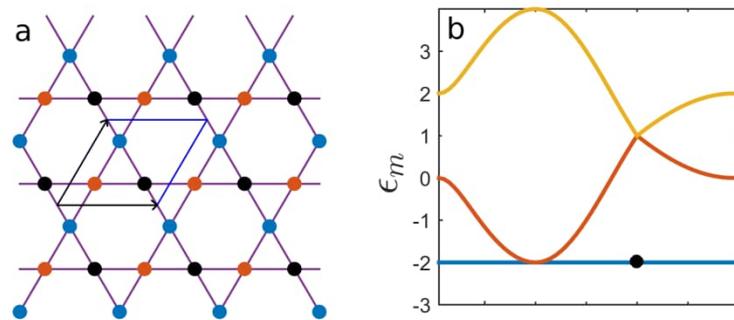
$$c_s \propto \sqrt{U} n_0$$

$$c_s \propto U n_0 \sqrt{g_{\alpha\beta}(k_c)}$$

Quantum metric

$$g_{\alpha\beta} = \Re[\langle \partial_\alpha u | \partial_\beta u \rangle - \langle \partial_\alpha u | u \rangle \langle u | \partial_\beta u \rangle]$$

Kagome lattice:



Quantum metric dictates the speed of sound



Aleksii Julku Georg Bruun Grazia Salerno

Julku, Bruun, PT, PRL 2021, PRB 2021

Julku, Salerno, PT, Fizika Nizkikh Temperatur (journal of ILTPE, Kharkiv, Ukraine) 49, 770 (2023); special issue coordinated by Andrei Bernevig, Princeton

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Flat band BEC & quantum geometry

Excitations do not cost energy; can BEC be stable?

Yes: finite **quantum distance** between Bloch states sets the limit for excitation density \rightarrow stable BEC

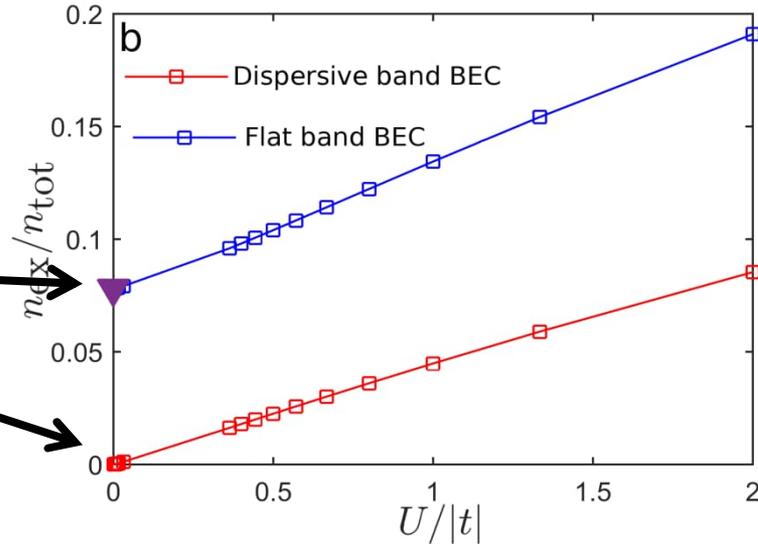
$$n_e(k) \xrightarrow{U \rightarrow 0} \frac{1-D}{2D}$$

Quantum distance

$$D = \sqrt{1 - |\langle u(k_c + q) | u(k_c - q) \rangle|^2}$$

Excitation density can be finite in the non-interacting limit...

...in contrast to dispersive band BEC



Interaction effects prominent even in the limit of vanishing interactions

Julku, Bruun, PT, PRL 2021, PRB 2021

Julku, Salerno, PT, Fizika Nizkikh Temperatur (journal of ILTPE, Kharkiv, Ukraine) 49, 770 (2023); special issue coordinated by Andrei Bernevig

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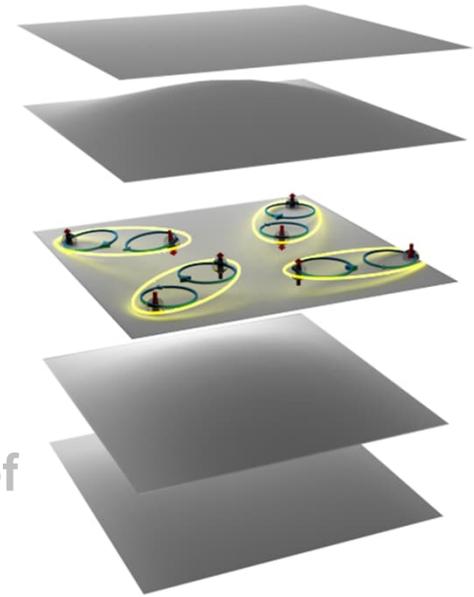
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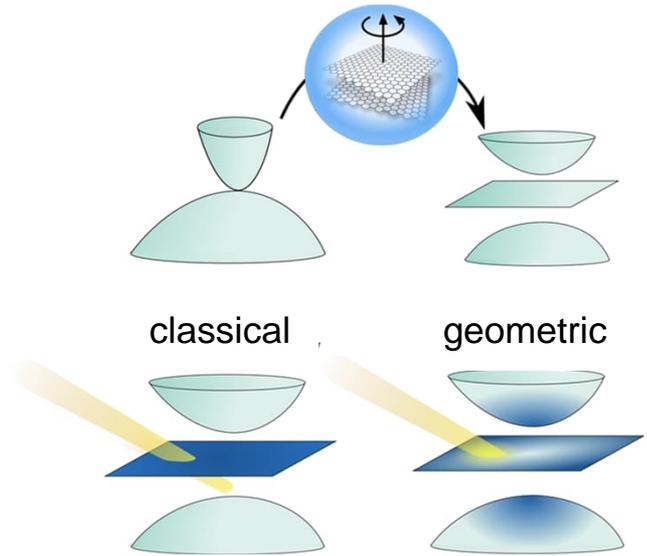
Quantum geometry and light-matter interactions



Light-matter coupling (LMC) in multi-band systems



G. E. Topp, C. J. Eckhardt, D. M. Kennes, M. A. Sentef, and PT, PRB 2021



Reminder: Single-band LMC

$$H_{\text{LMC}}^{\text{single}} = \sum_{\mu} \partial_{k\mu} \epsilon(k) \cdot A_{\mu} + \frac{1}{2} \sum_{\mu\nu} \partial_{k\mu} \partial_{k\nu} \epsilon(k) \cdot A_{\mu} A_{\nu}$$

paramagnetic
diamagnetic

Linear (A_{μ})

Quadratic ($A_{\mu} A_{\nu}$)

Intra-band (n)

$$\partial_{\mu} \epsilon_n$$

$$\partial_{\mu} \partial_{\nu} \epsilon_n$$

$$- \sum_{n' \neq n} (\epsilon_n - \epsilon_{n'}) (\langle \partial_{\mu} n | n' \rangle \langle n' | \partial_{\nu} n \rangle + \text{h.c.})$$

Inter-band (n, m)

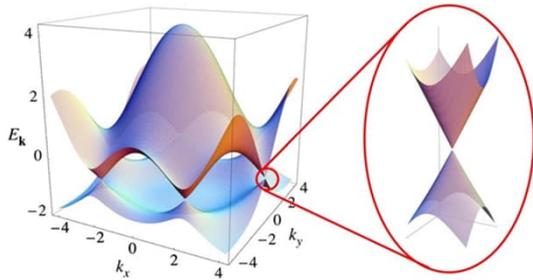
$$(\epsilon_n - \epsilon_m) \langle m | \partial_{\mu} n \rangle$$

$$\left[(\partial_{\mu} \epsilon_n - \partial_{\mu} \epsilon_m) \langle m | \partial_{\nu} n \rangle + \frac{1}{2} \epsilon_m \langle \partial_{\mu} \partial_{\nu} m | n \rangle + \frac{1}{2} \epsilon_n \langle m | \partial_{\mu} \partial_{\nu} n \rangle + \sum_{n'} \epsilon_{n'} (\langle \partial_{\mu} m | n' \rangle \langle n' | \partial_{\nu} n \rangle) \right] + (\mu \leftrightarrow \nu)$$

— 'classical' = determined by band dispersion

— 'geometric' = determined by Bloch states

QAHE in graphene



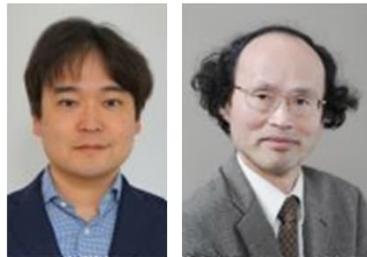
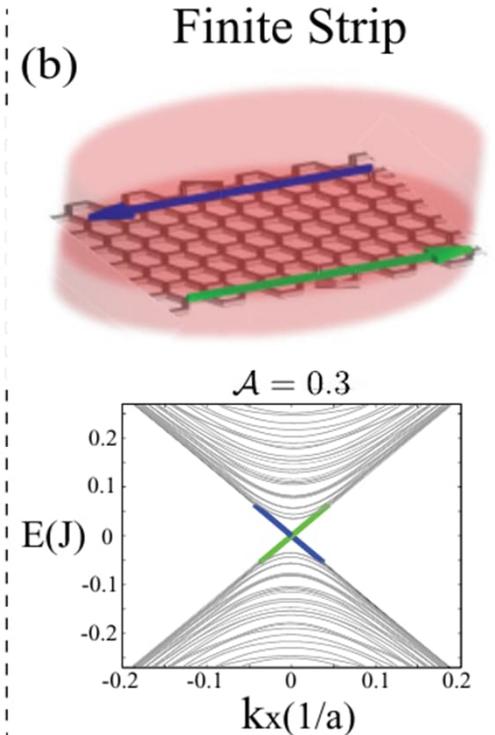
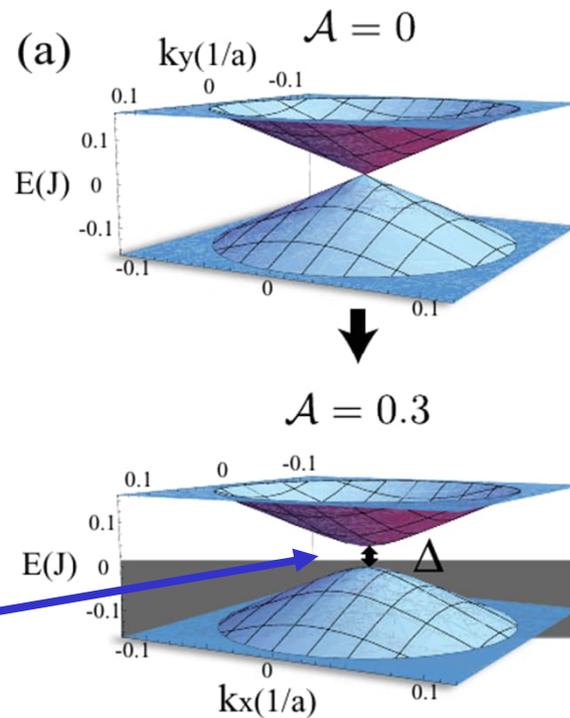
Neto, et al., Rev. Mod. Phys. 81 (2009)

circular drive:

$$H_{\text{eff}} \approx H_0 + \frac{[H_{-1}, H_1]}{\Omega} + O(\mathcal{A}^4)$$

$$\approx v_G(\sigma_y k_x - \sigma_x k_y \tau_z) \pm \frac{v_G^2 \mathcal{A}^2}{\Omega} \sigma_z \tau_z + O(\mathcal{A}^4)$$

breaks TRS

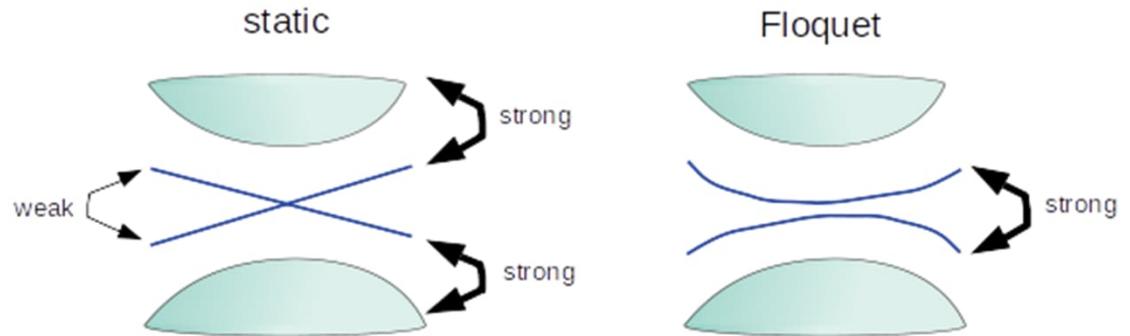
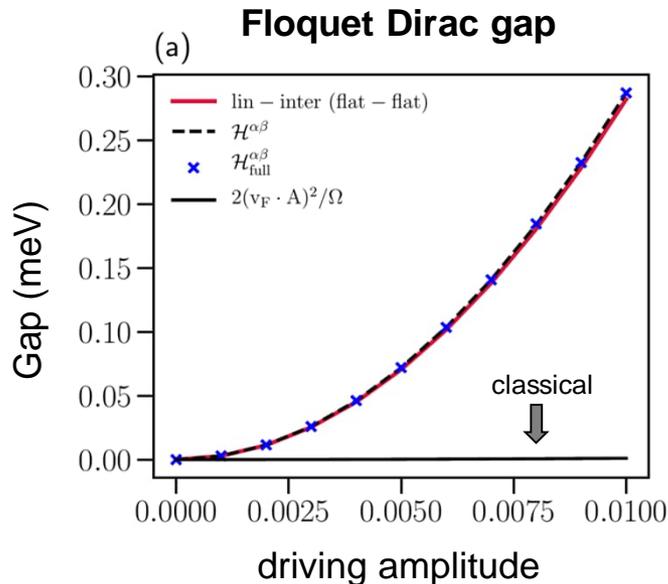
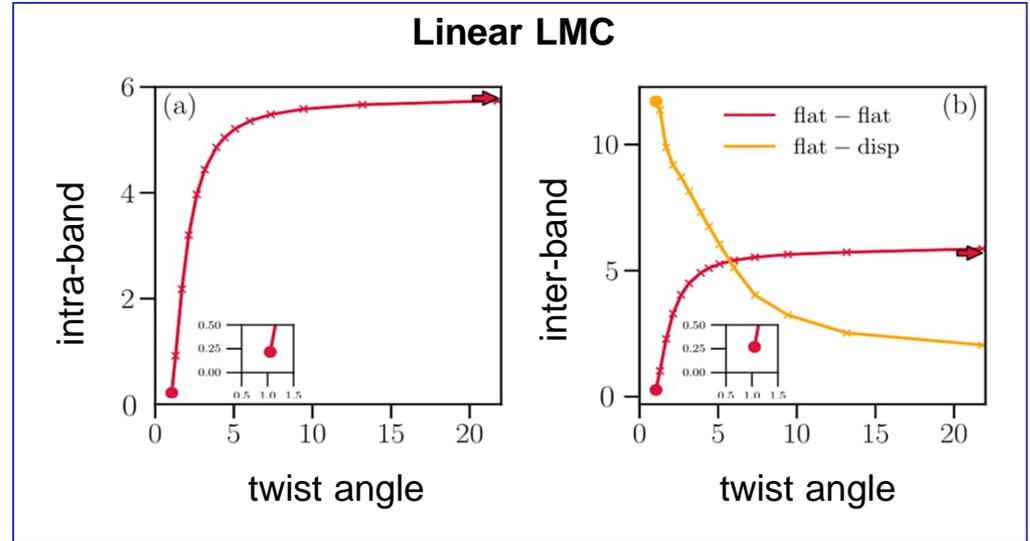
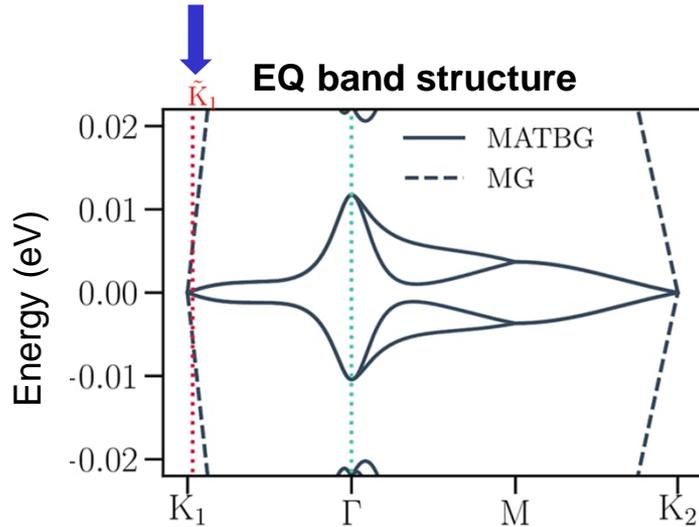


T. Oka & H. Aoki, PRB 79, 081406 (2009)
Kitagawa et al. PRB 84, 235108 (2011)



Application: Light-induced Dirac gap in TBG

G. E. Topp, C. J. Eckhardt, D. M. Kennes, M. A. Sentef, and PT, PRB 2021



$$\langle m | H_{\text{FLOQ}}^A | n \rangle = \frac{iA_0^2}{2\Omega} \left[\sum_l \langle m | \frac{\partial H_0}{\partial k_x} | l \rangle \langle l | \frac{\partial H_0}{\partial k_y} | n \rangle - \langle m | \frac{\partial H_0}{\partial k_y} | l \rangle \langle l | \frac{\partial H_0}{\partial k_x} | n \rangle \right]$$

Summary

- Quantum geometry – geometric properties of eigenstates – is important
- Flat band superconductivity is given by quantum metric
 - Quantum metric of the bands tells much more than the band structure; even “remote” bands relevant
 - Quasiparticle movement quenched in flat band superconductivity
 - BEC is stabilized by quantum geometry; quantum distance restricts excitations
 - Light-matter coupling in flat bands has significant quantum geometric terms
 - Many-body quantum metric limits Drude weight
 - Be cautious with flat bands: Fermi surface is missing! (DC conductivity)

