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Quantum Geometry of the Quantum Hall Effect

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- Quantum Hall effect as a "Středa Anomaly"
- quadrupole density and Hall viscosity
- Laughlin state, reinterpret "flux attachment" as "orbital attachment"
- Berry Curvature of Bloch states and embedding in Euclidean space
- **Irrelevance** of k-space geometry for FCI, use real-space quantum geometry instead. FQH/FCI derives from short-distance real-space repulsive interactions

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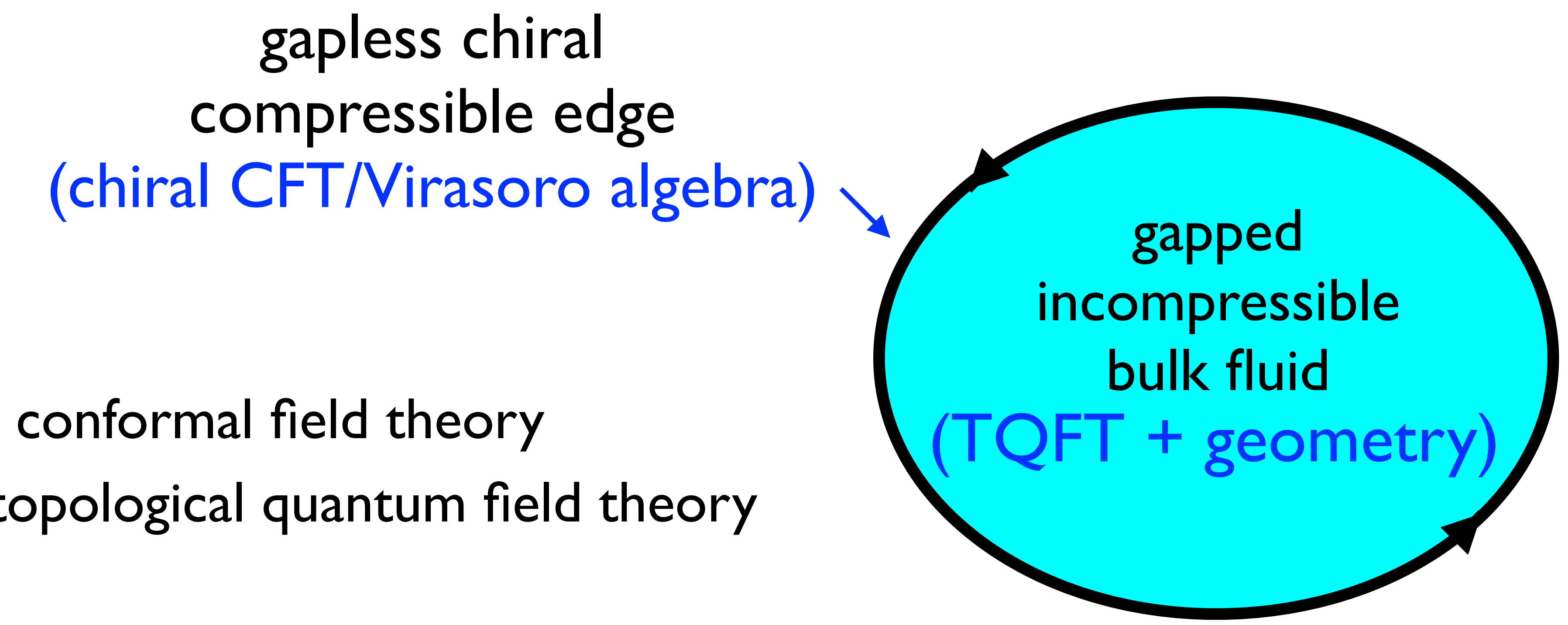
Electrons carry electric charge: this has two consequences

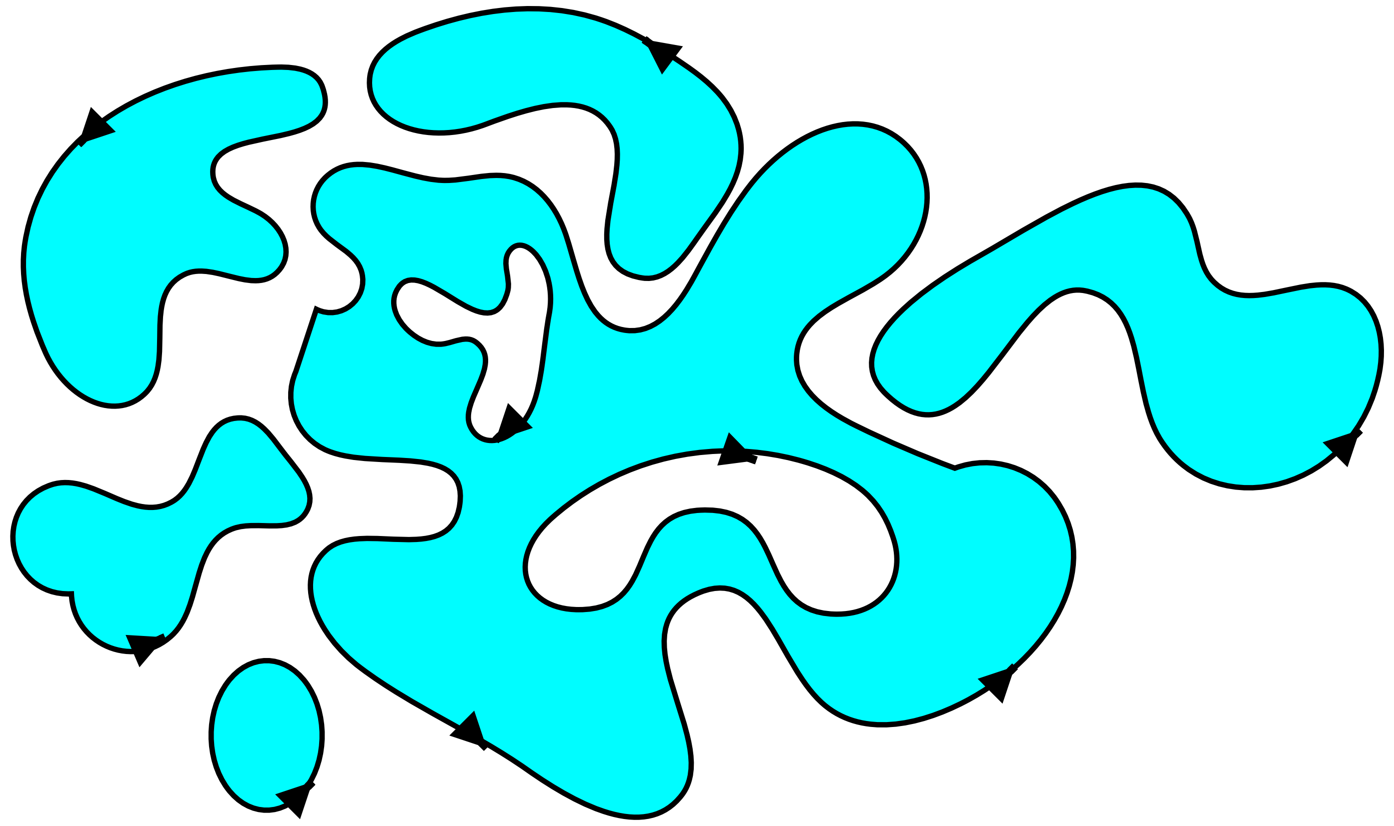
- they are sources of electric and magnetic fields
- they react to external electric and magnetic fields

I will describe a "theorists model" of the "clean limit"of
the quantum Hall fluid that largely ignores the first of these properties



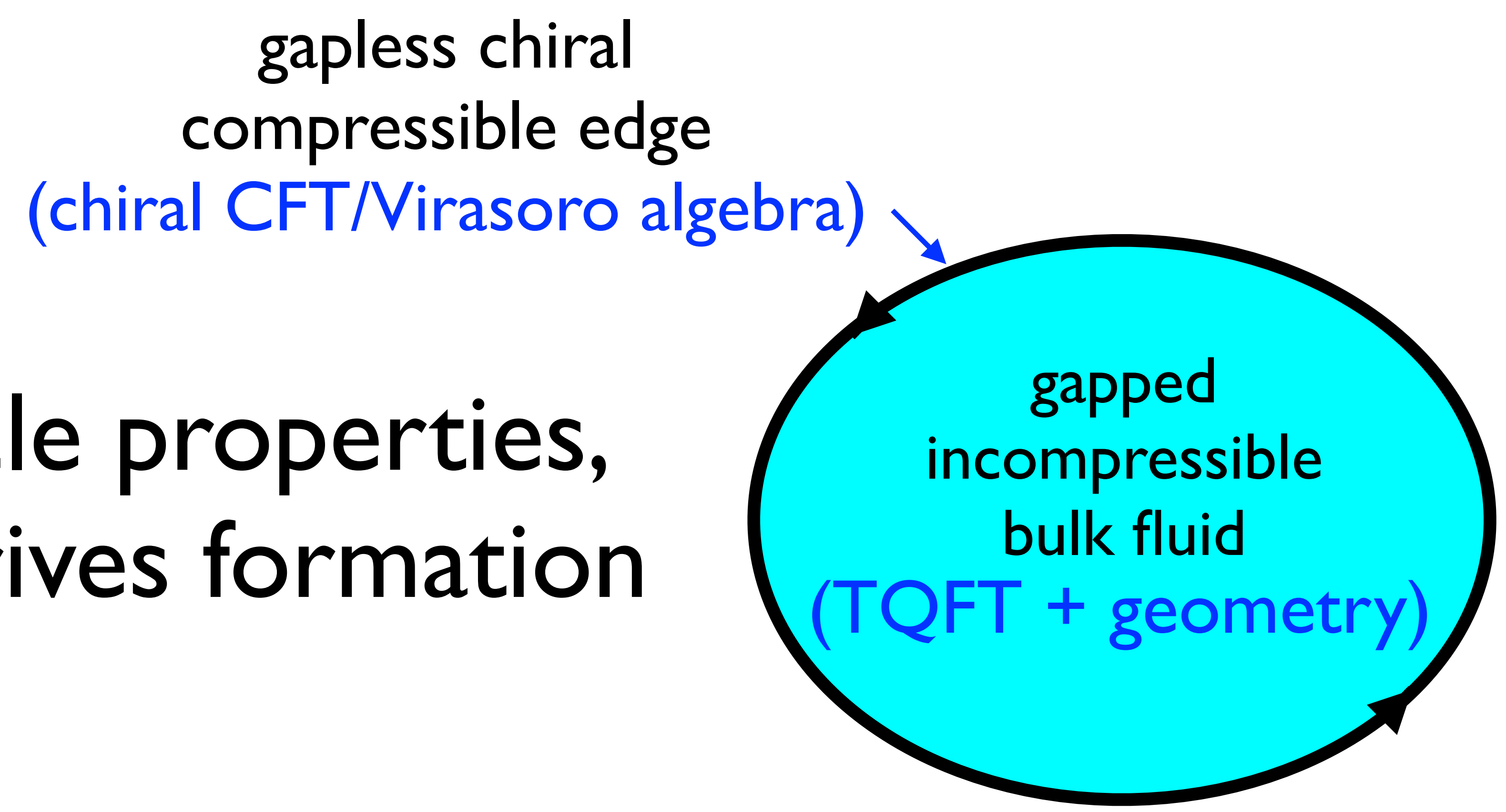
CFT = conformal field theory
TQFT = topological quantum field theory

Theorists' model suppresses long-range part
of Coulomb interaction as in Hubbard models



long-range part of Coulomb interaction
wants approximate local charge neutraility,
so edges percolate through system

- We already heard a bit about QHE topology
- Topology classifies a gapped state and its quasiparticle properties, but has nothing to say about energetics and what drives formation of such states. I will argue that geometry does.
- The quantum Hall effect was first seen in two-dimensional electron systems with Landau quantization by high magnetic fields. Most of our ideas about it were developed in that context
- More recently, the fractional QHE hs been found in "flat band" Chern-band systems with ferromagnetism, but no magnetic field or Landau levels. This provides an opportunity to reexamine our ideas about it, and discard Landau-level specific ideas that do not apply in Chern bands.



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Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations

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(Received 22 February 1983)

This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.

- Laughlin told us the the (fractional) QHE was exhibited by an incompressible quantum fluid, by the "fluid"(= something that flows) aspects have never really been followed up on.
- We also know (from numerics) that the Laughlin state is the correct description of e.g the incompressible state at 1/3 Landau level filling, but why this has also not really been explained ("it's a clever wavefunction" is NOT an explanation!)

$$\Psi_L^{(q)} \propto \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i} \qquad z_i^* z_i = \frac{1}{2}\left|\frac{eB}{\hbar}\right| \delta_{ab} x_i^a x_i^b$$

odd integer ($q = 1$ is Slater determinant)

- **I have learned that it is very useful to use consistent upper/lower spatial indices**

- **Cartesian coordinate system (inertial "Laboratory frame")**

$$\boldsymbol{x} = x^a \boldsymbol{e}_a \quad \equiv x^1\boldsymbol{e}_1 + x^2\boldsymbol{e}_2 + \ldots x^d\boldsymbol{e}_d$$

Summation convention only on upper/lower index pairs

$\boldsymbol{e}_a, a = 1, \ldots d$ are tangent unit vectors of d-dimensional space

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

$$\partial_a \equiv \frac{\partial}{\partial x^a}$$

distinguish Euclidean metric δ_{ab} of flat space-time from Kronecker symbol δ^a_b and inverse metric δ^{ab}

- spatial displacements, velocities have "upper" (contravariant) indices, spatial derivatives, tangent unit vectors momentum forces, wavevectors have "lower" (covariant) indices.

- Physically-meaningful equations should be the same in ANY coordinate system. Using consistent indices, the Euclidean metric δ_{ab} should not appear in any non-gravitational equation.

- Newtonian particles (nucleus of atom, ionic cores) have inertial mass tensors $m_{ab} = m_{\mathrm{grav}}\delta_{ab}$.
Bloch electrons are NOT Newtonian particles

- Maxwell equations are coordinate-independent. Euclidean metric of flat space only appears in constitutive equations of the vacuum

$$E_a = \partial_a A_0 - \partial_t A_a$$

$$F_{ab} = \partial_a A_b - \partial_b A_a = \epsilon_{abc} B^c$$

Faraday tensor

$$\partial_a D^a = J^0$$

Prefer F_{ab} to B^a (no right-hand rule ϵ^{abc})

$$\partial_a B^a = 0 \qquad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

$$\epsilon^{abc} \partial_b E_c = -\partial_t B^a \qquad \partial_a E_b - \partial_b E_a = -\partial_t F_{ab}$$

$$\partial_b (\epsilon^{abc} H_c) = J^a + \partial_t D^a \qquad \partial_b H^{ab} = J^a + \partial_t D^a$$

$$\partial_t J^0 + \partial_a J^a = 0 \qquad \pi_a = \epsilon_{abc} D^a B^c$$

$$D^a = \left.\frac{\partial J_U^0}{\partial E_a}\right|_B \quad H_a = \left.\frac{\partial J_U^0}{\partial B^a}\right|_E \quad J_U^a = \epsilon^{abc} E_a H_b$$

Linear constitutive equations

$$D^a = \varepsilon^{ab} E_a$$

$$B^a = \mu^{ab} H_b$$

In vacuum

$$\varepsilon^{ab} = \varepsilon_0 \delta^{ab}$$

$$\mu^{ab} = \mu_0 \delta^{ab}$$

$$U = \tfrac{1}{2}\left(D^a E_a + B^a H_a\right)$$

- stress is a mixed-index tensor

$$\partial_t \pi_a - \partial_b \sigma_a^b = f_a$$

Pressure

$$P\delta_a^a = \sigma_a^a$$

- $-\sigma_b^a$ is the current J_b^a of component b of momentum in direction a.
  f_a is the body force

- in Maxwell equations

$$\pi_a = \epsilon_{abc} D^a B^c$$

energy density

$$\sigma_b^a = D^a E_b + B^a H_b - \frac{1}{2} J_U^0 \delta_b^a$$

$$f_a = E_a J^0 + \epsilon_{abc} J^b B^c$$

- Viscosity in a fluid: linear response of stress tensor to a gradient of flow velocity $\mathrm{v}(x)$

$$\sigma^a_b = \eta^{ac}_{bd}\partial_c \mathrm{v}^d(x) + O(\mathrm{v}^2)$$

- rate of dissipation:

$$\frac{dU}{dt} = -\,\sigma^a_b \partial_a \mathrm{v}^b$$

$$\boxed{\begin{array}{l}\text{Vanishes if } \eta^{ac}_{bd} = -\,\eta^{ca}_{db} \text{ (antisymetric)} \\ \bullet \text{ The antisymmetric part of the viscosity} \\ \text{tensor is the ``odd'' or ``Hall'' viscosity}\end{array}}$$

- what is an electric quadrupole? (most people don't know!!!!

- in Physics 101 class we usually learn that a "point quadrupole" is two dipoles back-to-back that is the source of an electric field falling of as $1/r^4$, and is a "traceless tensor". NO, THIS IS WRONG!

- The electric quadrupole of a charge distribution is its second moment, which unlike "dipole moment", is unambiguously defined:

$$q^{ab} = \frac{1}{2}\sum_i q_i(x_i^a - \bar{x}^a)(x_i^b - \bar{x}^b) \qquad \sum_i q_i(x_i^a - \bar{x}^a) = 0$$

$$U = -\, q^{ab}\partial_a E_b(x)$$ energy in an electric field

dielectric of permittivity tensor

- Laplace equation $\quad -\varepsilon^{ab}\partial_a\partial_b V = \rho$

- Fourier transform

quadrupole

$$\epsilon^{ab}k_a k_b \tilde{V}(k) = \tilde{\rho}(k) = q^{ab}k_a k_b$$

$$\tilde{V}(k) = \frac{q^{ab}k_a k_b}{\epsilon^{ab}k_a k_b}$$

- long range field comes from singular part as $|k| \to 0$

- Part of $q^{ab} \propto \varepsilon^{ab}$ does not contribute to long rage field, but this does not mean it can be neglected!

- a uniform sphere of charge has a (primitive) quadrupole that distinguised it from a point charge (e.g. proton, neutron) , where it is called the “radius of charge”

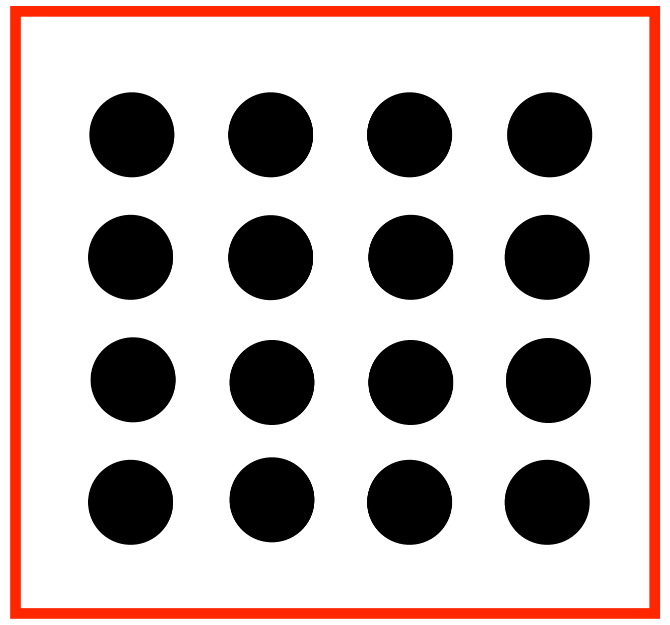
- Ideal Quantum Hall Fluids:
  - Are incompressible charged 2D fluids localized on a lattice plane that obey the Streda relation for the electric charge density $J^0(x) = \frac{1}{2}\sigma_H^{ab}F_{ab}(x)$ where $F_{ab}(x)$ are the in-plane components of the Faraday (magnetic flux) tensor. $\sigma_H^{ab} = (e^{*2}/2\pi\hbar)k\epsilon^{ab}$ is quantized.
  - Have an elementary unit with charge e_b that is a multiple of the electron charge e, and fractionally-charged excitations with charges that are multiples of $e^* = e_b/|k|$, where k is the level of an Abelian $U(1)_k$ Chern-Simons gauge field that couples to the elementary unit.
  - Have a traceless stress tensor (do not support pressure)
  - Are dissipationless (gapped, have antisymmetric conductivity and viscosity tensors at T=0)
  - Have a (primitive) electric quadrupole density $Q^{ab}(x)$ where $(ke^*/\hbar)^2 \det Q \geq \frac{1}{4} \det \sigma_H$
  - Have gapless edges where the momentum density (generator of edge diffeomorphisms) obeys the Virasoro algebra

$$[\pi(x), \pi(x')] = i\hbar\left(\pi(x)\delta'(x-x') + \frac{1}{12}c\hbar\delta'''(x-x')\right)$$

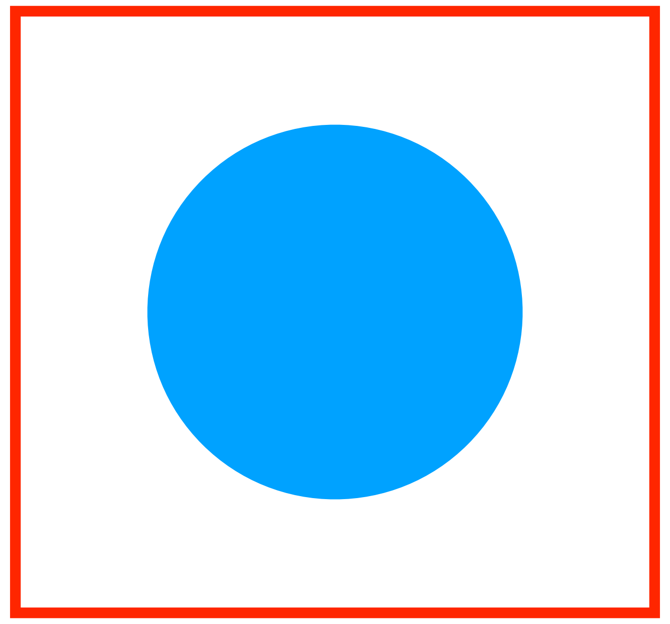
$$P = \int dx\, \pi(x)$$

(signed) chiral central charge

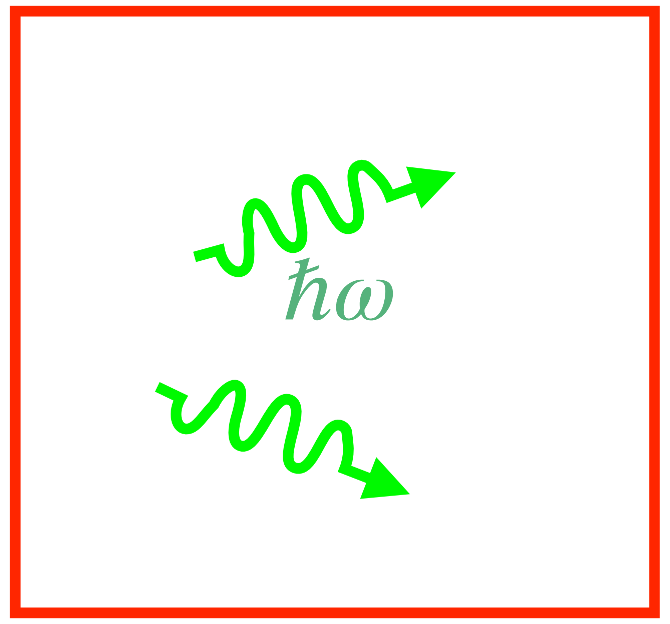
Condensed matter has three subsystems:



Lattice (elastic)
Nuclear coordinates, phonons



Electronic
(Fermi surface or superconducting condensate)

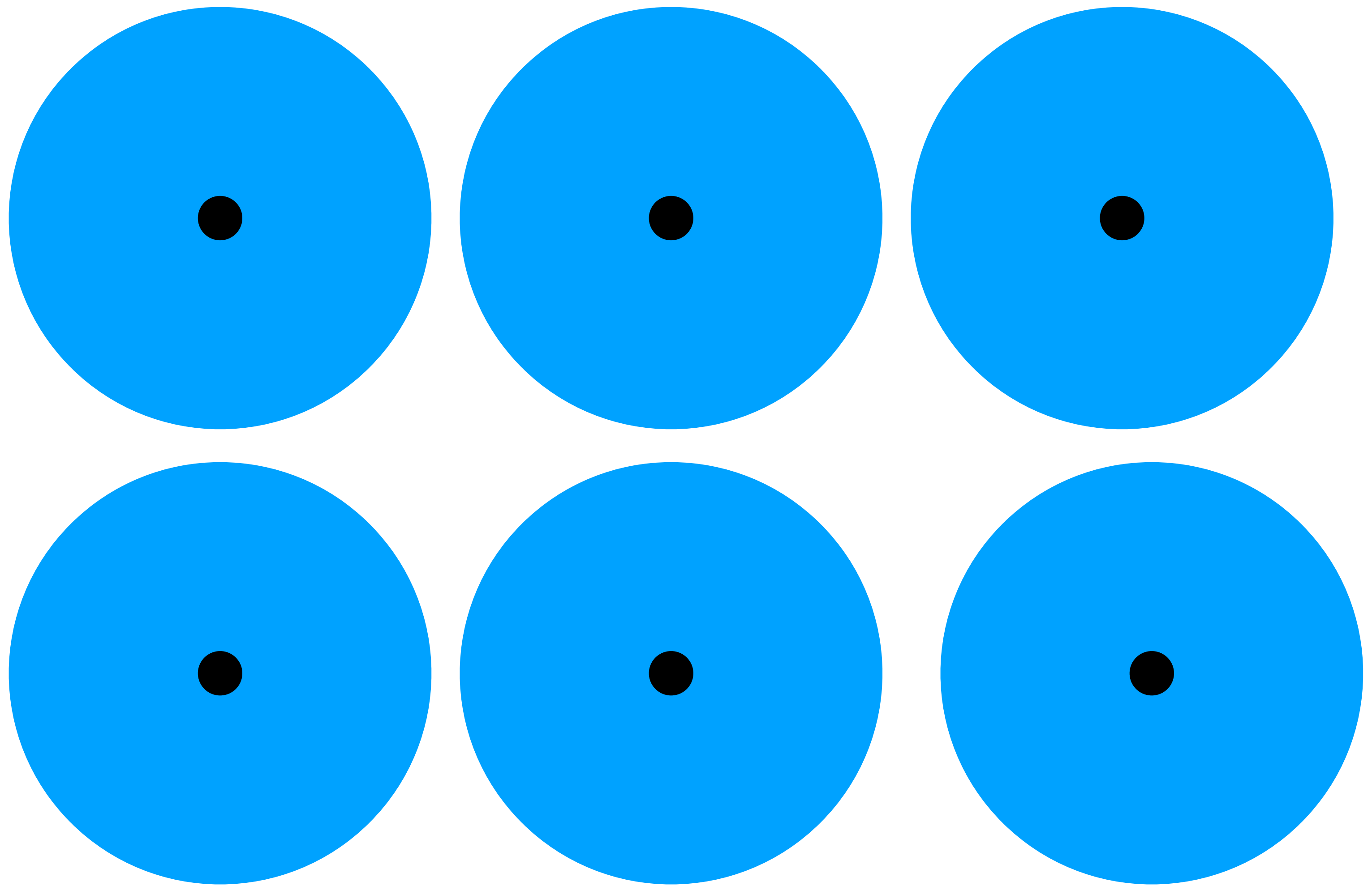


Electromagnetic
(Photons/polaritons)

- When there is no (bulk) Fermi surface, the electronic subsystem is incompressible, with no autonomous low-energy degrees of freedom

- Usually this means the system is a band insulator, where the electron density is fixed by the local Bragg vector field, which determines the local volume of the Brillouin zone.
- The low-energy excitations (phonons) are fluctuations of the Bragg-vector fields

- In a band insulator, the electrons cannot move relative to the lattice
- The electronic state is essentially described by a Slater determinant of local filled Wannier (atomic-like) orbitals, which have fixed positions relative to the nuclear coordinates that define the lattice



In band insulators, the electronic charge density on lengthscales larger than the lattice scale is quantized in units given by the Brillouin zone volume, given by the [(local) Bragg vector field]

$$J^0(x) = \frac{ne}{(2\pi)^3} \left| \boldsymbol{G}_1(x) \cdot \boldsymbol{G}_2(x) \times \boldsymbol{G}_3(x) \right|$$

- In band insulators (represented as Slater determinants of Wannier orbitals) all the electrons are "owned" by the lattice, and are not fluid.

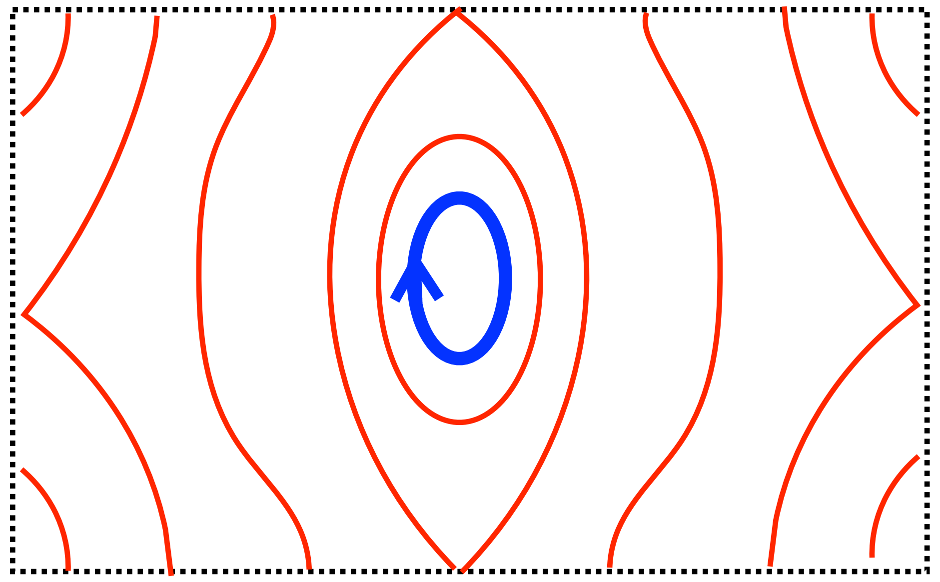
- In quantum Hall systems, some of the electrons on lattice planes are "captured" by the magnetic flux (Faraday tensor) and can flow parallel to the plane with the electromagnetic drift velocity defined by $E + v \times B = 0$,

$$\sigma_H^{ab} = \frac{e^2 \epsilon^{abc}}{2\pi h} \frac{G_c}{2\pi}$$

Reciprocal lattice vector normal to Hall lattice planes

3D integer quantum Hall conductivity

- The mechanism by which the magnetic flux captures electrons is Landau quantization to form Landau levels, with one orbital per quantum h/e of magnetic flux through the plane.



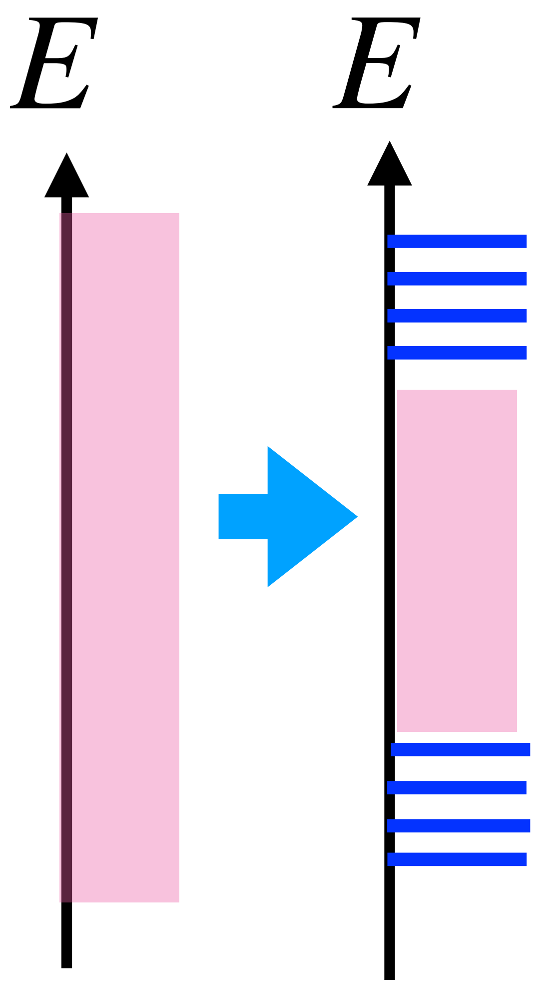
Closed orbit in k-space suppresses Umklapp

$$\hbar \partial_t k_a = e F_{ab} \partial_t x^a$$
$$\hbar \partial_t x^a = v_g^a(k) - \hbar \mathscr{F}^{ab}(k) \partial_t k_b$$

group velocity **Berry curvature**

Semiclassical Bloch dynamics

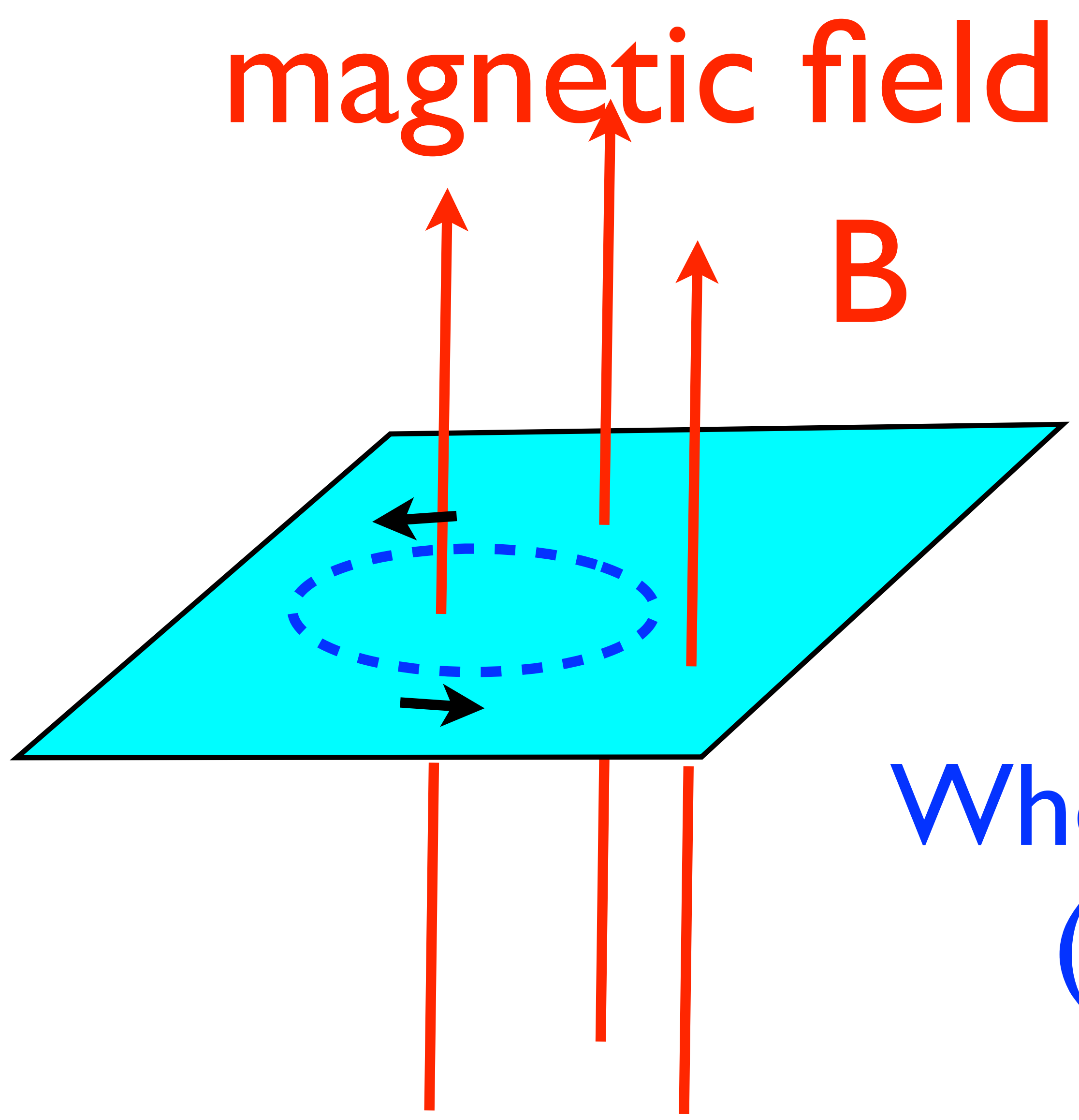
- **suppression of Umklapp means the captured electrons in the Landau level no longer "know" about the Bragg vector field, crystal momentum becomes true momentum, and they can be described in an effective continuum theory that ignores the lattice**



One orbital on the lattice plane per flux quantum h/e in each level

- **The quantum Hall fluids are perhaps the clearest examples of topological states**

- Originally found in very large magnetic fields, where circular orbits of electrons moving on a 2D surface are quantized (Landau levels)



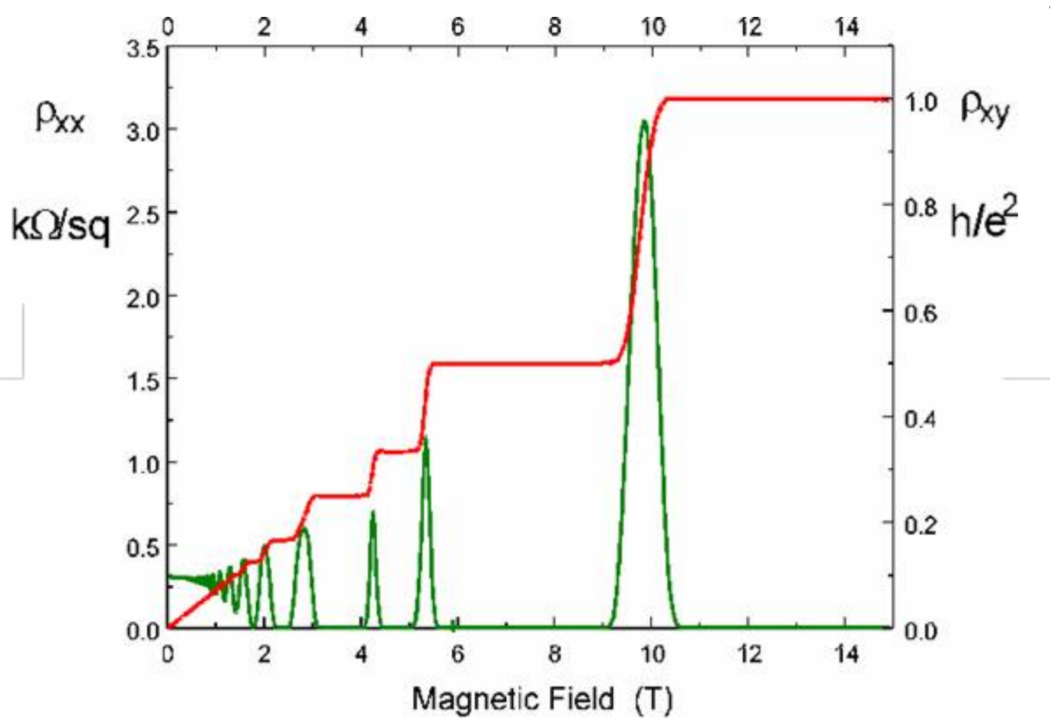
Hall conductance

$$\sigma_H = n\frac{e^2}{h}$$

Whole number (integer)

fundamental constants

Klaus von Klitzing (Nobel 1985)



"French Imperialism" Napoleon I

Old Kilogram (platinum weight kept in Paris)

Now part of the new system of units since 10th November 2018 (replacing the kilogram in Paris)

New Kilogram ("Kibble balance" using quantum Hall effect)

"New boss" is NOT the same as the "old boss"

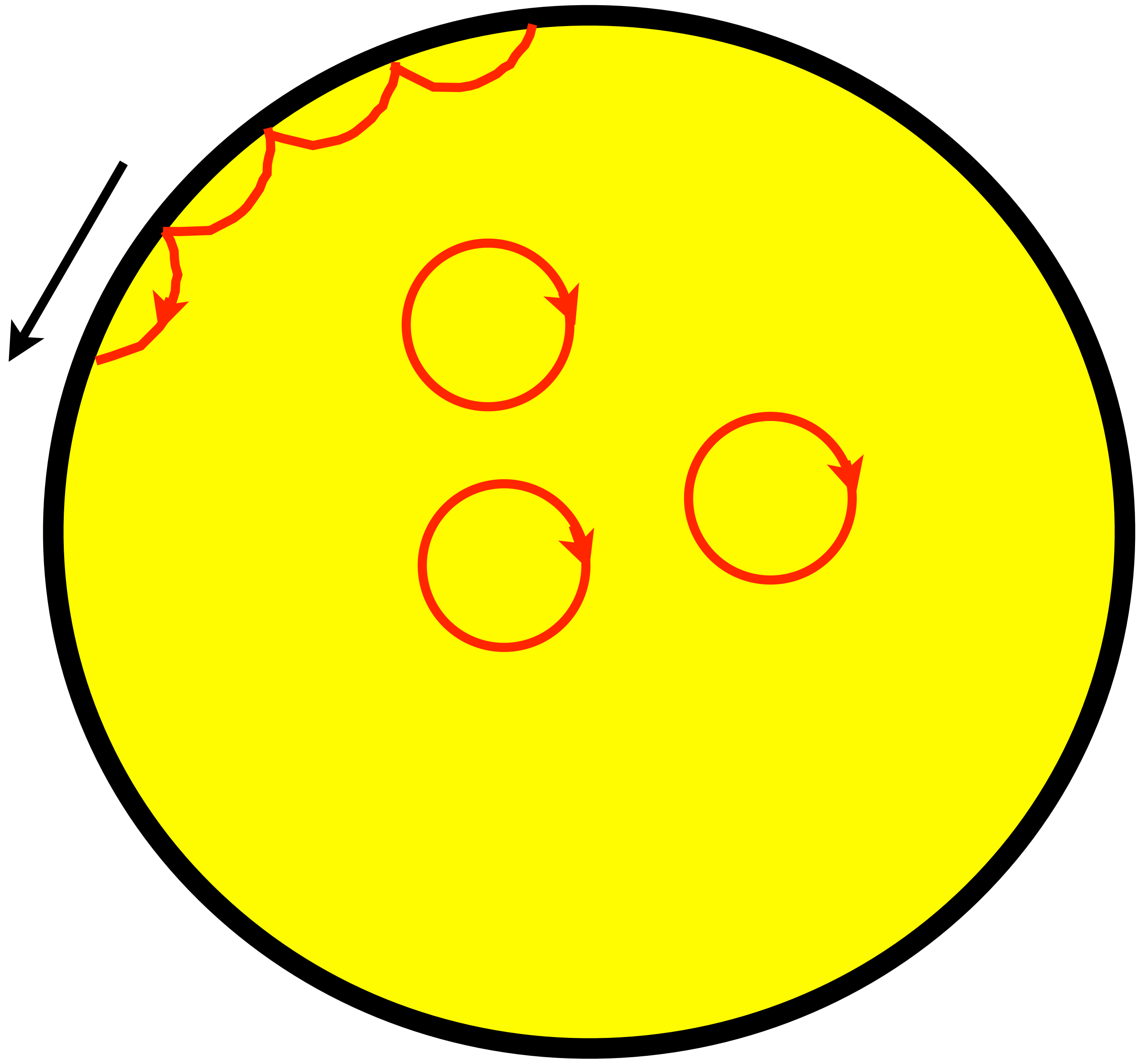
(more democratic, everyone can build it)

Their topological nature took some time to become clear

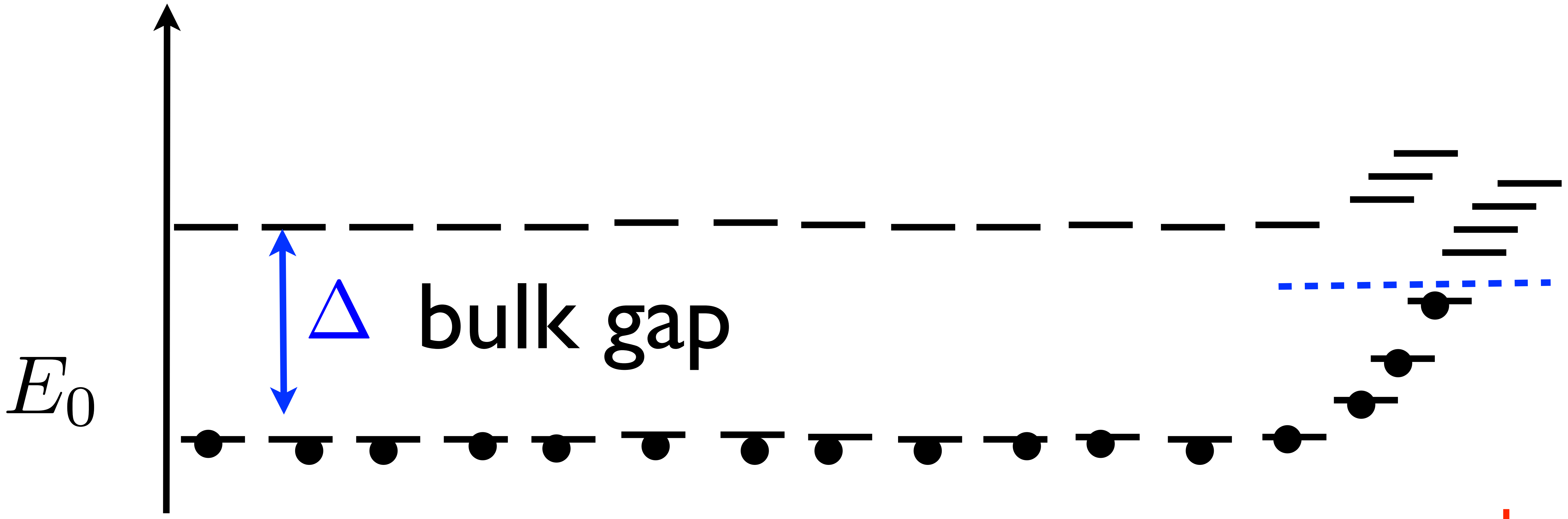
Bert Halperin

- counter-propagating "one-way" edge states (Halperin)
- confined system with edge must have edge states!

This is topological!



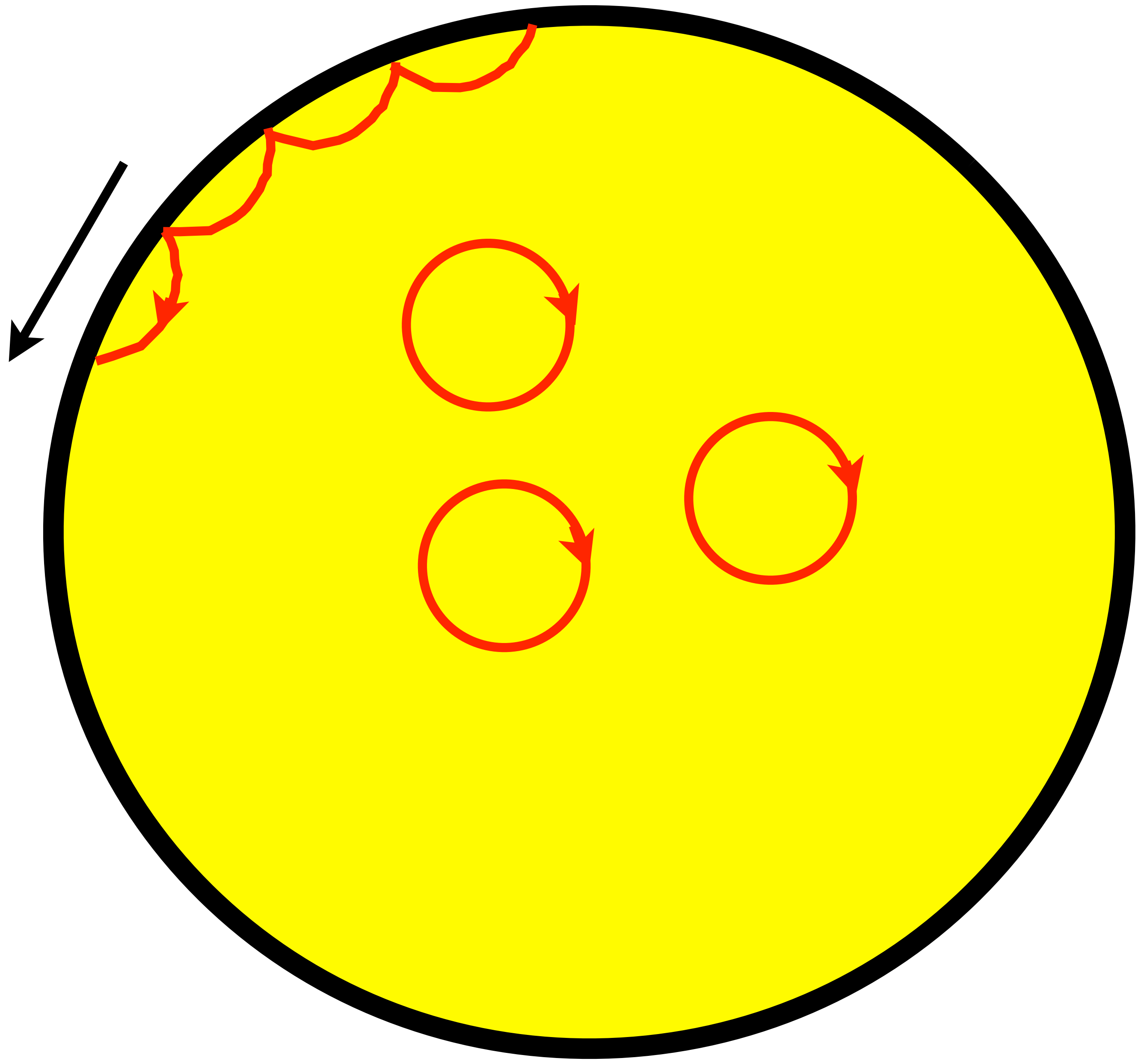
One-way transport at edges!!!



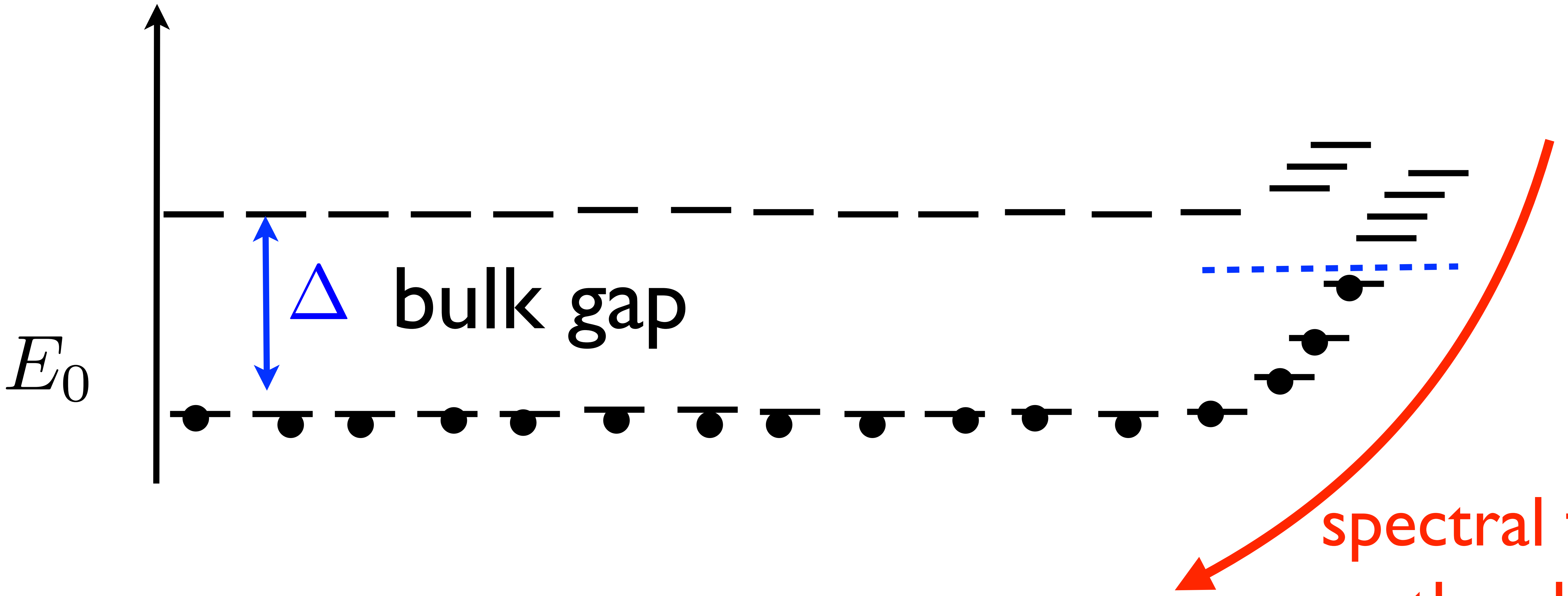
"Fermi level" pinned at edge

don't need to fine-tune magnetic field

- The topological "chiral" (directional) edge states are the key property of quantum Hall systems:
- Their "one-way" character derives for broken time reversal symmetry, and allows the "anomalous" quantum Hall effect to occur in ferromagnetic systems, even in the absence of magnetic flux.



$$kL = 2\pi n + e\Phi_B/\hbar$$



Periodic boundary conditions around edge change with magnetic flux through bulk

spectral flow of electron orbitals through the chiral edge state as Φ_B changes

- Laughlin described the (fractional) quantum Hall effect as being due to an "incompressible quantum fluid" of electrons

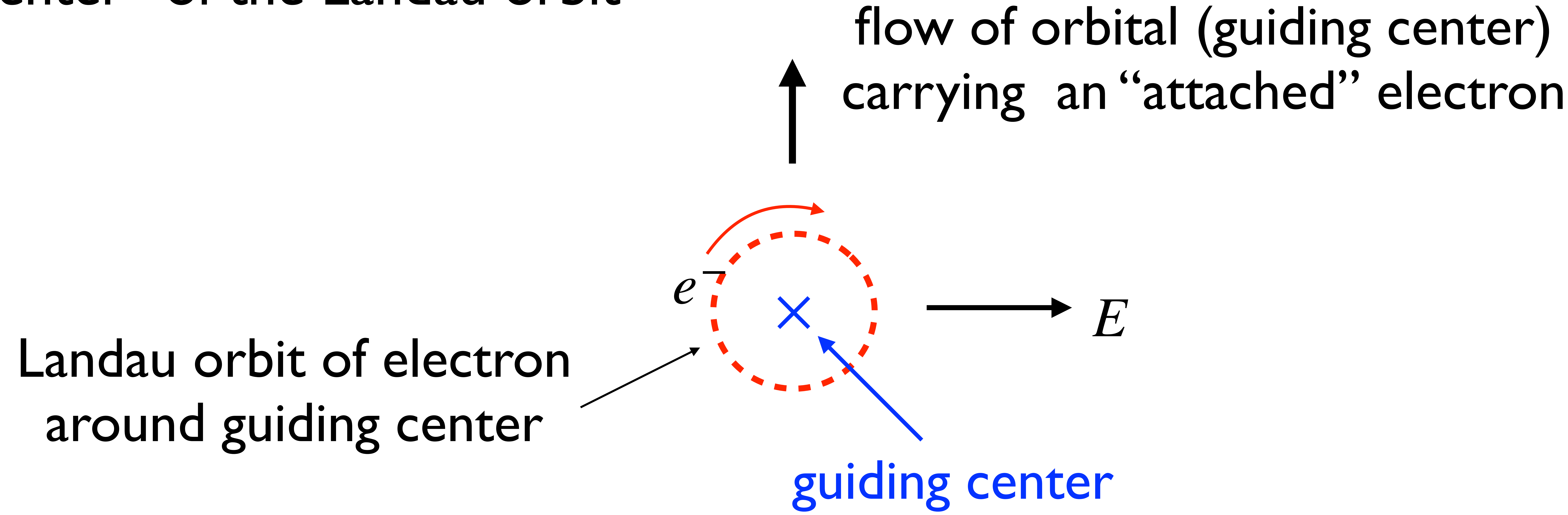
- This fluid character is very different to the solid character of another "incompresssible" electronic state: the band insulator:

- In the band insulator the (local real space Wannier) electronic orbitals are locked to the crystal lattice, at fixed points in the unit cell, one orbital per sublattice per unit cell

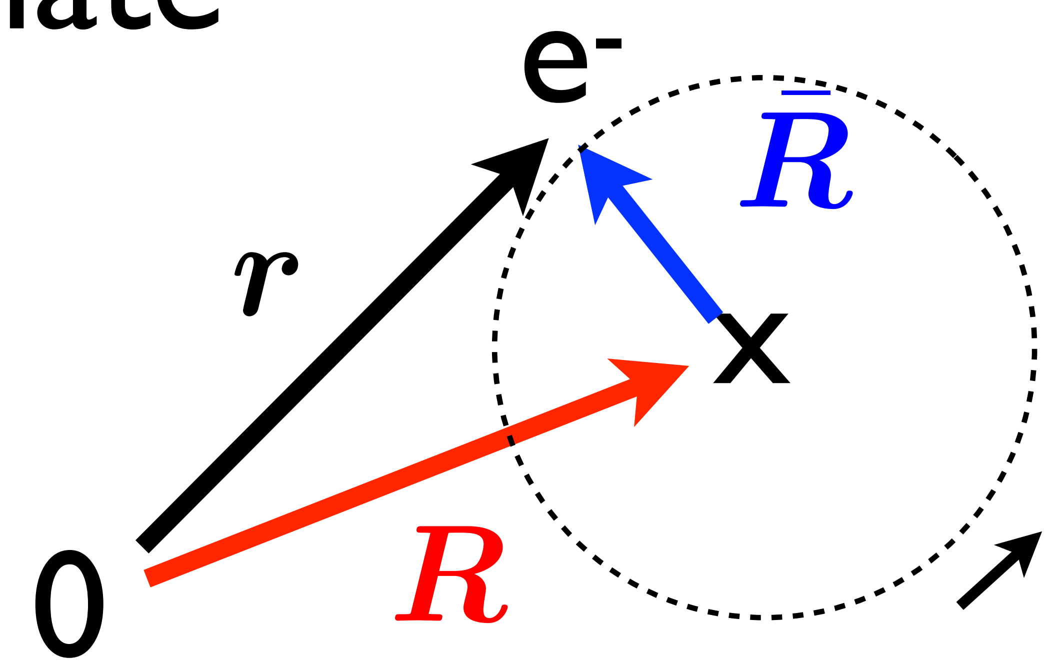
- In the quantum Hall effect, Umklapp is suppressed, and the one-electron orbitals are detached from the lattice, and are free to flow, carrying any electrons that occupy them "along for the ride"

- In Landau levels, the local orbitals that electrons occupy are characterized as "Guiding centers"

- The orbital an electron occupies ("is attached to") is centered on the "guiding center" of the Landau orbit



- Landau level decomposition of the spatial coordinate



$$p_a = -i\hbar\nabla_a - eA_a(\boldsymbol{x})$$

$$[p_x, p_y] = i\hbar eB$$

- Landau orbit radius vector

$$\bar{\boldsymbol{R}} = \frac{1}{eB}(p_y, -p_x)$$

- Landau orbit guiding center

$$\boldsymbol{R} = \boldsymbol{r} - \bar{\boldsymbol{R}}$$

$$\boldsymbol{r} = \boldsymbol{R} + \bar{\boldsymbol{R}} \quad [R^a, \bar{R}^b] = 0$$

$$[r^x, r^y] = 0$$

$$[\bar{R}^x, \bar{R}^y] = i\ell_B^2$$

$$[R^x, R^y] = -i\ell_B^2$$

after Landau-level quantization, only
the guiding centers remain as
dynamical variables

- An orthonormal basis of eigenstates of the one-body Hamiltonian has the form

n is Landau level index

$$h(p - eA)|n, m\rangle = \varepsilon_n |n, m\rangle$$

m labels degeneracy of Landau level

$$\frac{1}{2\ell^2} g_{ab}(R^a - x_0^a)(R^b - x_0^b)|m, n\rangle = (m+\frac{1}{2})|n, m\rangle$$

arbitrary positive-definite metric with $\det g = 1$

arbitrary choice of origin

$$\boxed{\begin{aligned} &a^\dagger |n, m\rangle = \sqrt{(m+1)}|n, m+1\rangle \\ &a|n,0\rangle = 0 \end{aligned}}$$

- A Laughlin state parametrized by a metric can now be written in any Landau level:

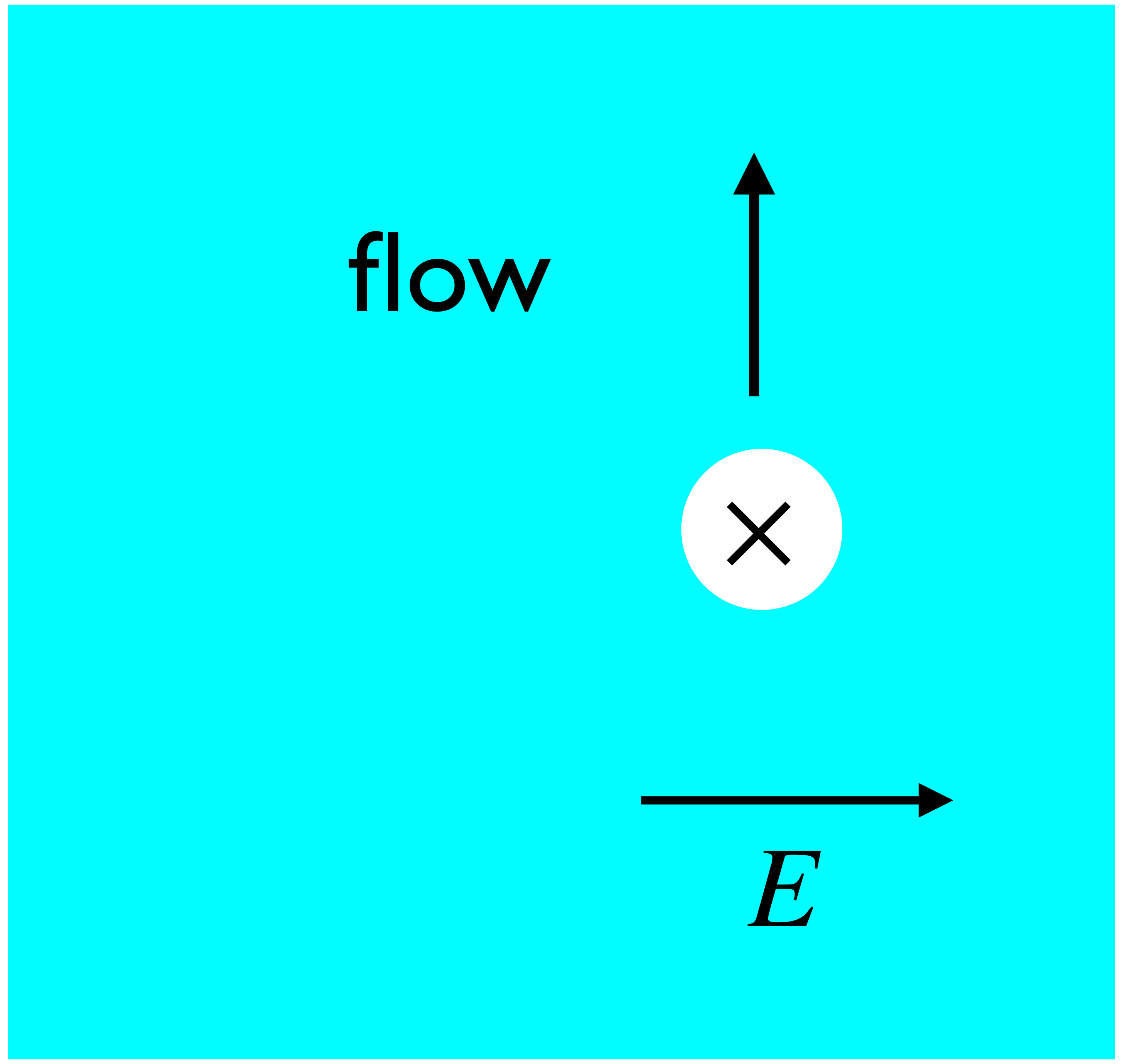
$$|\Psi_L^{(q)}(n,g,x_0)\rangle \propto \prod_{i<j} (a_i^\dagger - a_j^\dagger)^q |\Psi_0(n,g,x_0)\rangle$$

$$h(p_i - eA_i)|\Psi_0(n,g,x_0)\rangle = \varepsilon_n |\Psi_0(n,g,x_0)\rangle$$

$$a_i|\Psi_0(n,g,x_0)\rangle = 0$$

- The metric is a hidden geometric variational parameter of the Laughtin state:

- In a filled Landau level the charge carriers (holes) are **empty** local orbitals, which also flow in response to an electric field



- The most remarkable property of the QHE, the "Streda anomaly" is a direct consequence of orbital spectral flow:

Without edge states this would seem to violate local charge conservation in a gapped system!

electric charge density

$$\left.\frac{\partial J^0}{\partial F_{ab}}\right|_{\mu,T=0} = \sigma_H^{ab}$$

Faraday tensor

$$F_{ab} = \epsilon_{abc} B^b$$

(preferable, as no "right-hand rule" is invoked)

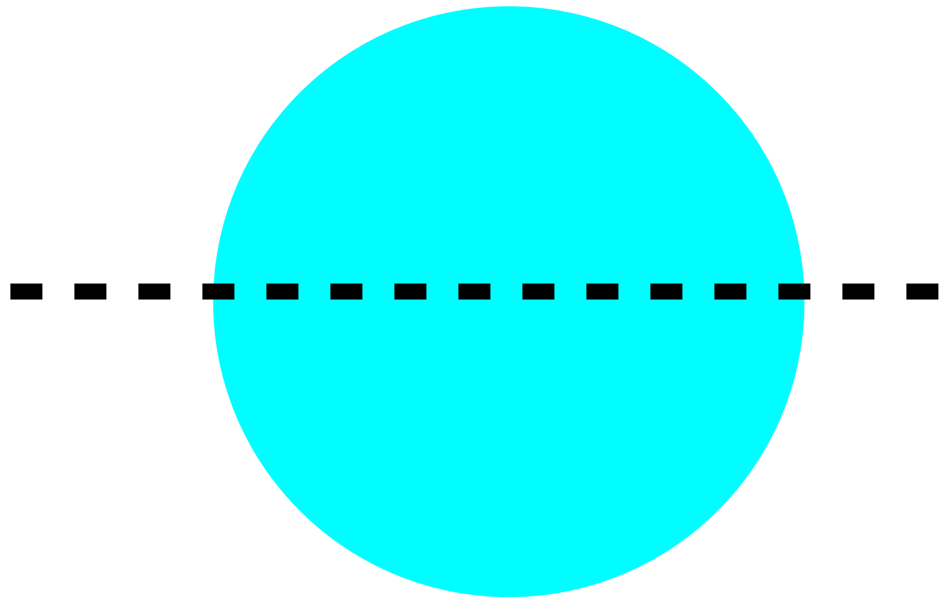
Purely antisymmetric, non-dissipative, topological

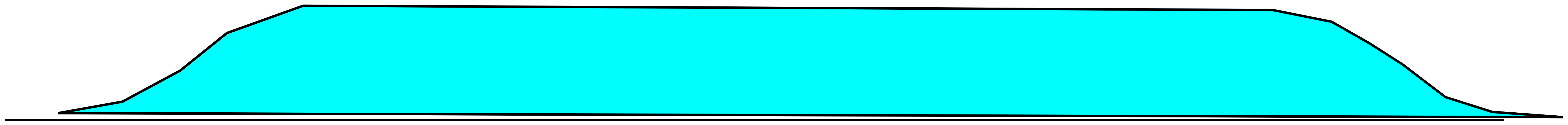
electric current density

$$J^a = (\sigma_\Omega^{ab} + \sigma_H^{ab}) E_b$$

contribution to conductivity from electrons bound to lattice (impurities etc)

- Unlike the fictional classical incompressible Euler fluid (which has infinite sound velocity, and instantaneous pressure equilibration) the FQHE is a true gapped incompressible quantum fluid.
- It does not support sound waves (has a quantum gap in the bulk) or hydrostatic pressure: no force is transmitted through its bulk, only around continuous edges.
  - A maximum-density droplet of QHE liquid does not need a confining potential to keep it from expanding!



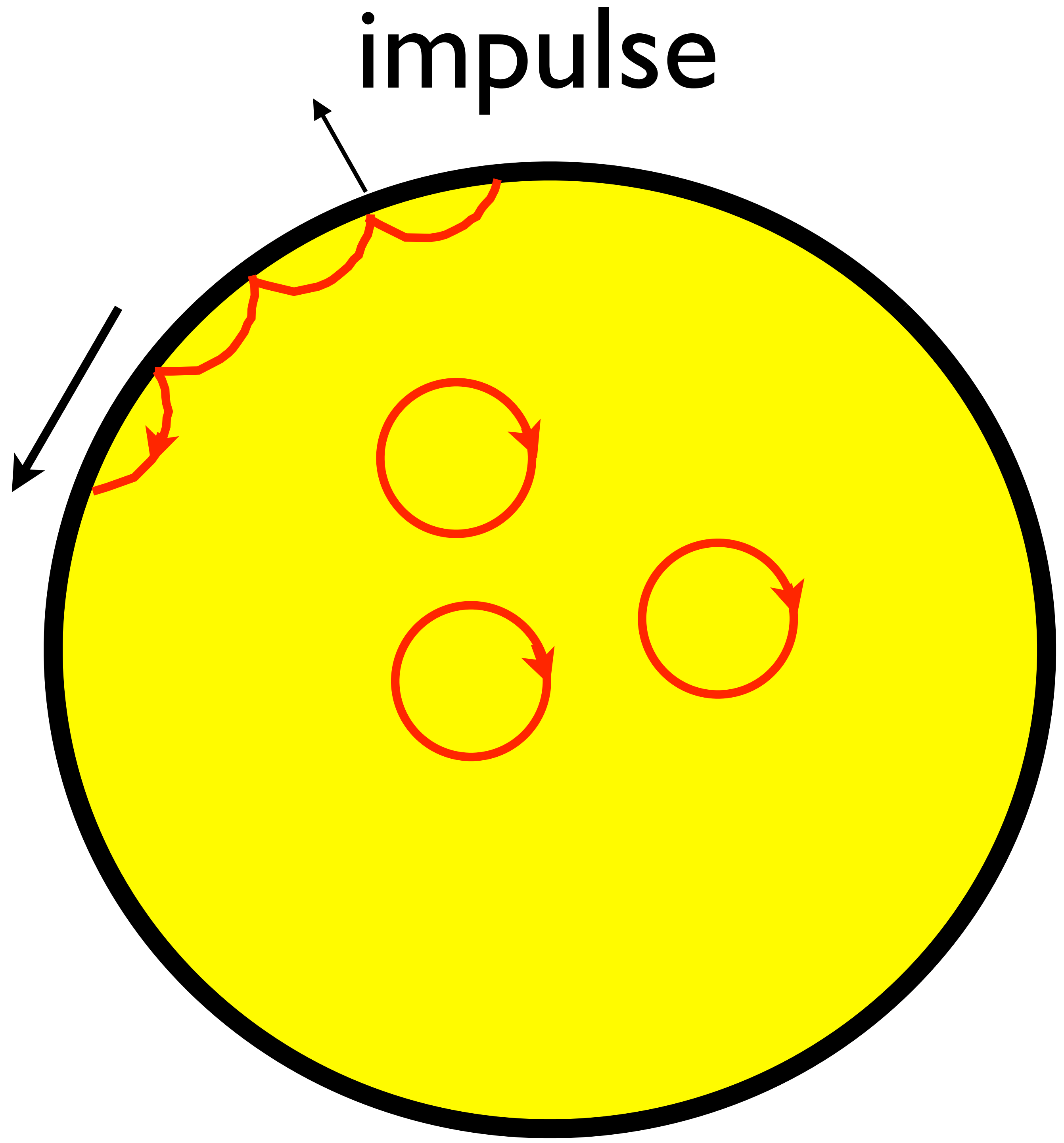


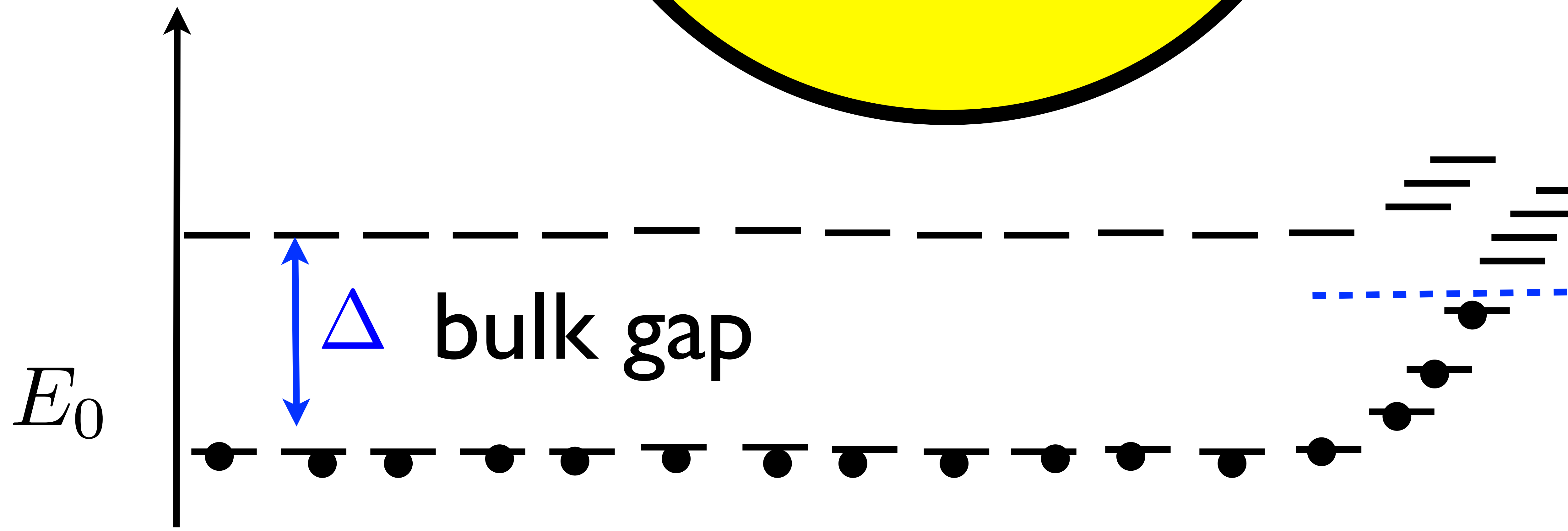
cross-section of QHE fluid droplet

- so if there is no pressure, what happens when a confining potential tries to compress the fluid?

- This familiar picture gives us a clue!

- In Landau levels, the confining potential generates an edge current, which grows until the outwards Lorentz force "$BI\ell$" balances the compression



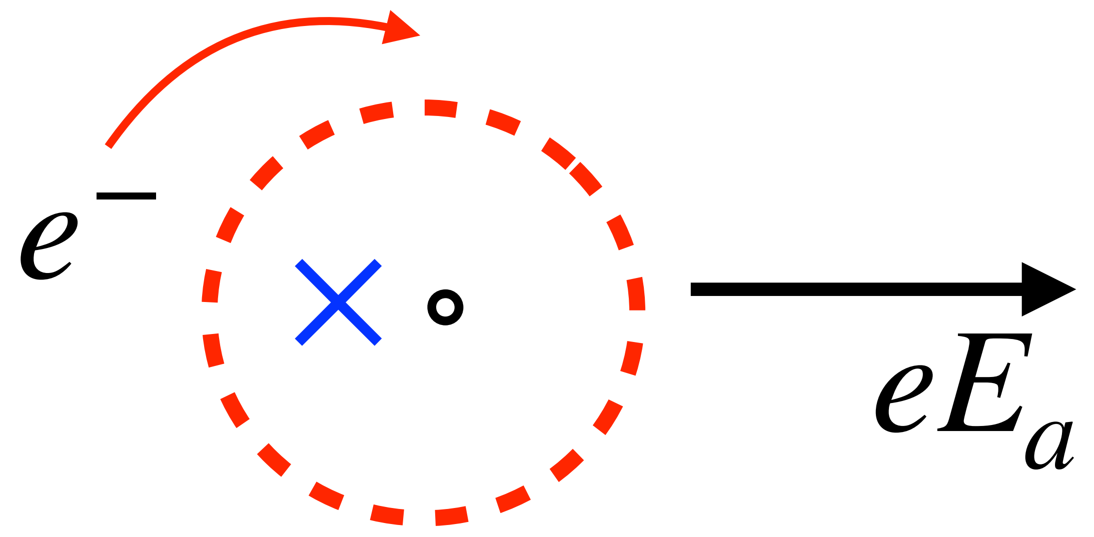


- A more careful analysis show that there are two distinct reponses to squeezing by the external confining potential:

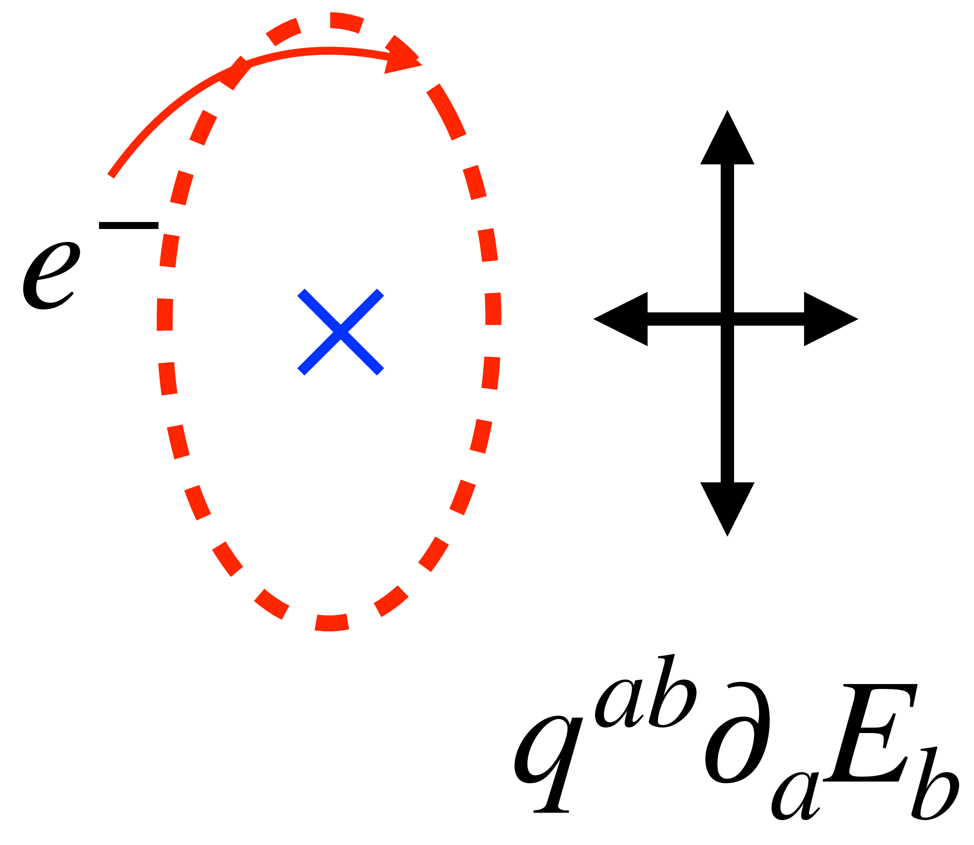
- The response to its first derivative (electric field) is a second order perturbation (Landau-level mixing) because the undeformed Landau orbit has no electric dipole moment relative to its guiding center.

- The response to its second derivative (electric field gradient) is a first order perturbation (Hall viscosity response) because the undeformed Landau orbit has a primitive electric quadrupole (second moment of charge distribution) relative to the guiding center.

center of charge
displaced from
guiding center



electric polarization of orbit



the second order quadrupolar response
the field gradient deforms the shape along the flow lines

- get local Landau level energy in non-uniform electric field

$$H = h(\tilde{R}) + V(\tilde{R} + R)$$

$$= h(\tilde{R}) + V(R) + \tilde{R}^a \partial_a V(R) + \frac{1}{4}\{\tilde{R}^a, \tilde{R}^b\}\partial_a\partial_b V(R) + \ldots$$

$$\varepsilon_n(R) = \varepsilon_n + V(R) + \underbrace{\langle n | \tilde{R}^a | n\rangle}_{=0}\partial_a V(R) + \underbrace{\frac{1}{4}\langle n | \{\tilde{R}^a, \tilde{R}^b\} | n\rangle\partial_a\partial_b V(R)}_{= -q^{ab}\partial_a E_b(R)}$$

$$\underbrace{-\frac{1}{2}\left(\sum_{n'(\neq n)} \frac{\langle n | \tilde{R}^a | n'\rangle\langle n' \tilde{R}^b | n\rangle}{\varepsilon_{n'} - \varepsilon_n}\right)\frac{1}{2}\{\partial_a V(R), \partial_b V(R)\}}_{= -\frac{1}{2}\chi_n^{ab} E_a(R) E_b(R)} + \ldots$$

$$= -E_a(R)P^a(R) + \frac{1}{2}\chi_n^{ab} E_a(R) E_b(R)$$

$$= -E_a P^a(R) + \frac{1}{2}(m_n)_{ab} v^a(R) v^b(R)$$

induced dipole "kinetic energy" $v^a = \epsilon^{ab} E_b / B$

- The electric polarization of the Landau orbit (Landau-level mixing) is interesting because it defines a "Gaililean" effective-mass tensor of the guiding centers when they flow

$$P^a = \chi^{ab} E_b \qquad U = -P^a E_a + \frac{1}{2}\chi^{ab} E_a E_b$$

$$v^a = \epsilon^{ab} E_b / B \qquad \frac{1}{2}\chi^{ab} E_a E_b \to \frac{1}{2} m_{ab} v^a v^b$$

- For Galileian Landau levels ($p^2/2m$ dispersion) m_{ab} defined above is precisely the Galileian mass tensor.

- Hall viscosity $\eta^{ab}_{cd} = -\eta^{ba}_{dc}$

  odd because fluid is dissipationless

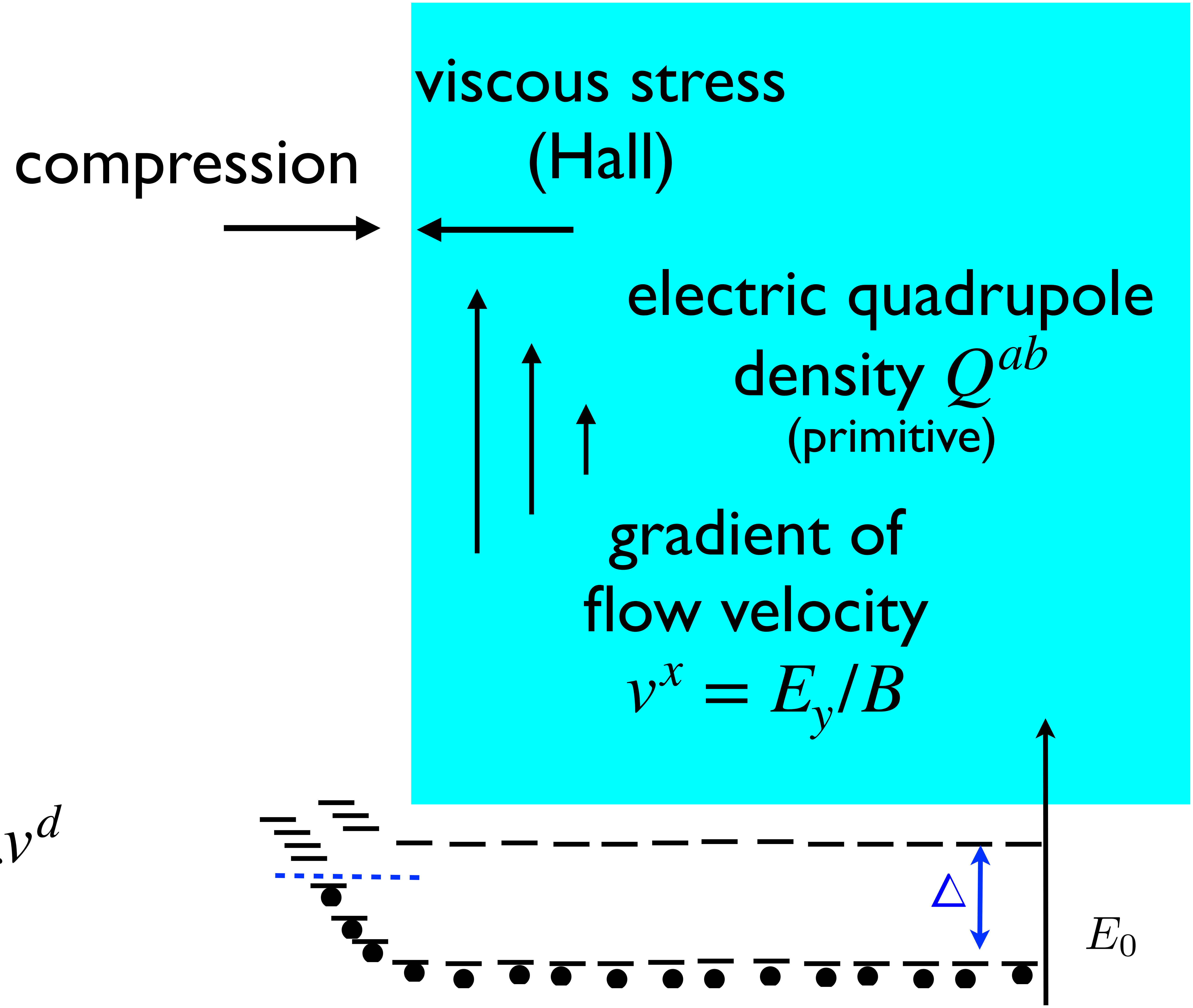
$$\eta^{ab}_{cd} = \delta^a_d F_{ce} Q^{be} - \delta^b_c F_{de} Q^{ae}$$

- note: Hall viscosity is traceless because fluid is incompressible: $\eta^{ab}_{ac} = 0$

- stress is traceless $\sigma^a_b = \eta^{ac}_{bd} \partial_c v^d$

  no pressure

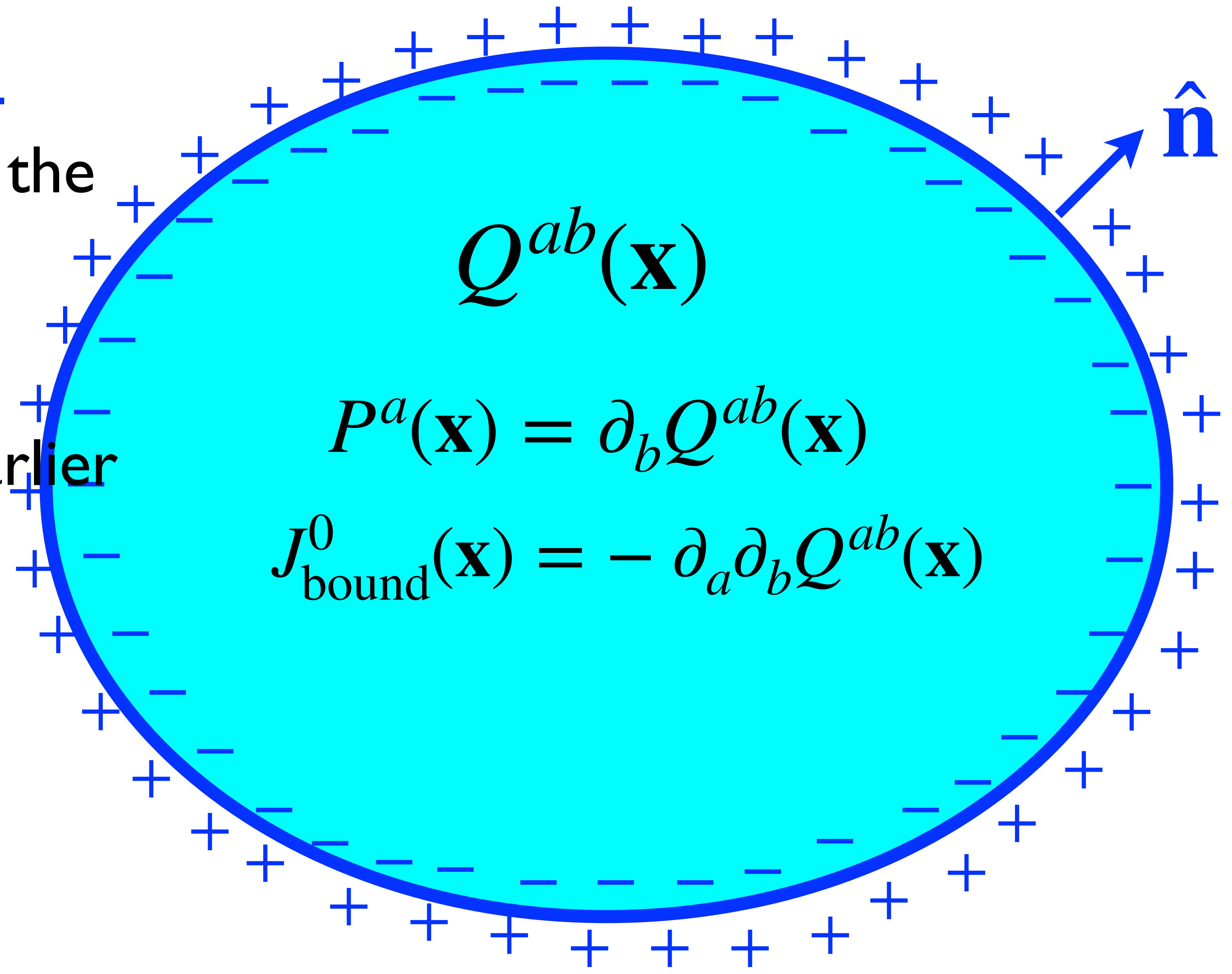
(in 1+1d CFT, stress-energy tensor is traceless)



• properties of a fluid with a quadrupole density

- in the interior, there is a bound-charge density given by (minus) the double divergence of the quadrupole density.

- This includes and generalizes earlier "Gaussian curvature" formulas derived on a sphere



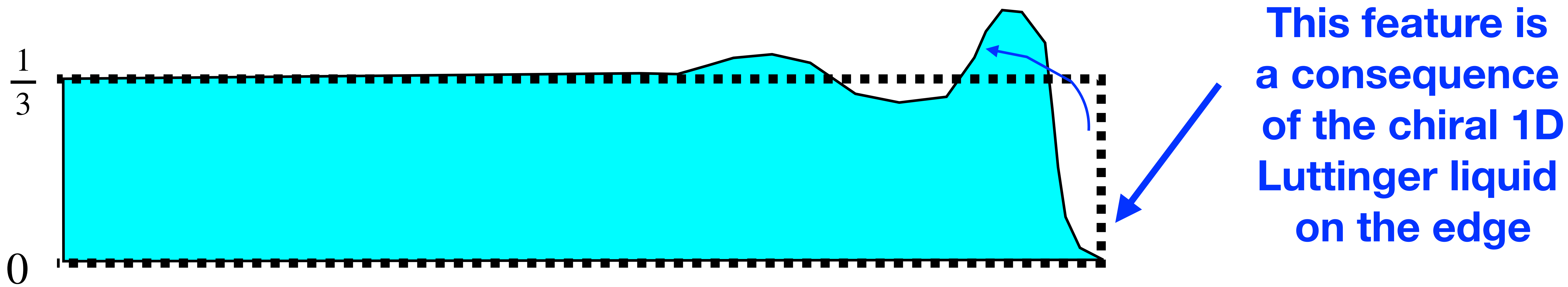
outward unit normal

- At the edge of the fluid, there is a surface polarization (dipole per unit length) that reveals the interior quadrupole density:

$$\hat{\mathbf{n}} \cdot \mathbf{P}_{\text{edge}} = Q^{ab}\hat{n}_a\hat{n}_b$$

- outwards/inwards edge polarization if Q^{ab} is positive/negative definite.

- electron density at edge of integer QHE state revealing the positive-definite Landau orbit quadrupole density (when divided by electron charge)



- guiding-center orbital occupations at edge of Laughlin 1/3 FQHE state revealing the negative-definite guiding-center quadrupole density

- Note that the "anti-Laughlin" 2/3 state has a positive-definite guiding-center quadrupole density (minus that of 1/3 Laughlin state).

- some new ideas and results:

- The electric polarization rigidly vanishes in the ground state of a quantum Hall fluid, (there is a gap for excitations that carry an electric dipole moment) but the ground state has a finite (primitive) electric quadrupole density

- This quadrupole density is a central feature in a new fundamental expression for the Hall viscosity

- In the FQHE, there is an emergent dynamical quadrupole field that accompanies "flux attachment", and the energetics of its formation is what stabilizes the elementary unit of the fluid, the "composite boson"

- This is the long-wavelength Girvin-MacDonald -Platzman mode

- The foundation of our understanding of the FQHE is the 1983 Laughlin wavefunction

$$\Psi \propto \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{4}|z_i|^2/\ell^2}$$

$q = 1$ is Slater determinant

- It explicitly exhibits key features, such as "flux attachment"

It's clever at keeping the electrons apart…???

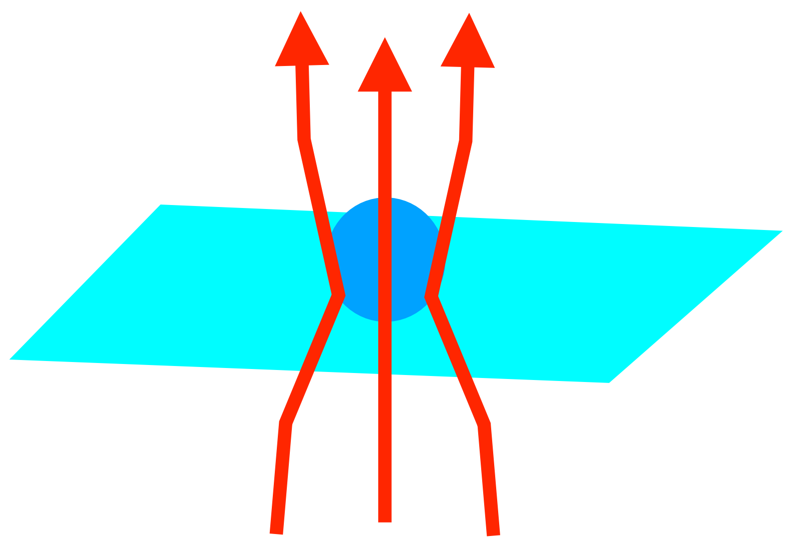
This is NOT an explanation!

- Numerical finite-size exact-diagonalization confirms it works, but "why" has never been precisely explained

- Discovery of FQHE in non-Landau-level zero-magnetic-flux lattice systems is an opportunity for a deeper understanding

Courtesy National Gallery of Art, Washington

- The Laughlin state has been the fundamental souce for interpretations of the (F)QHE
- A popular one has been the idea of "flux attachment" to form "composite particles"
- In this picture the Laughlin 1/3 state has two extra "flux quanta" (vortices) attached to it, and this is called "flux attachment"



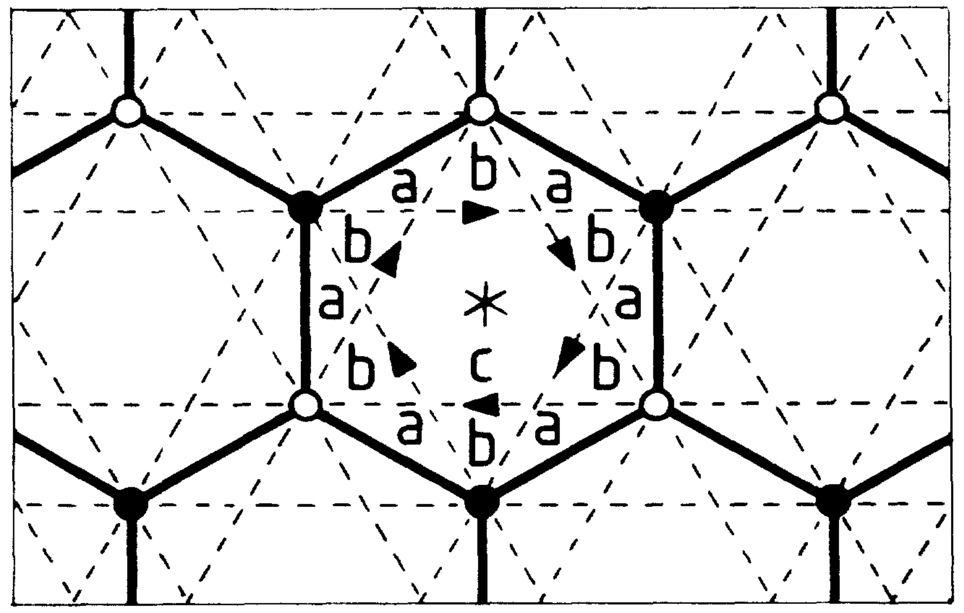
"flux attachment concept"

As a non-Slater-determinant state, with no "Wick's theorem" to allow Feynman diagrams etc, the Laughlin state has remained stubbornly intractable to analytic analysis, so is primarily described by theorists using heuristic cartoon pictures such as "flux attachment"

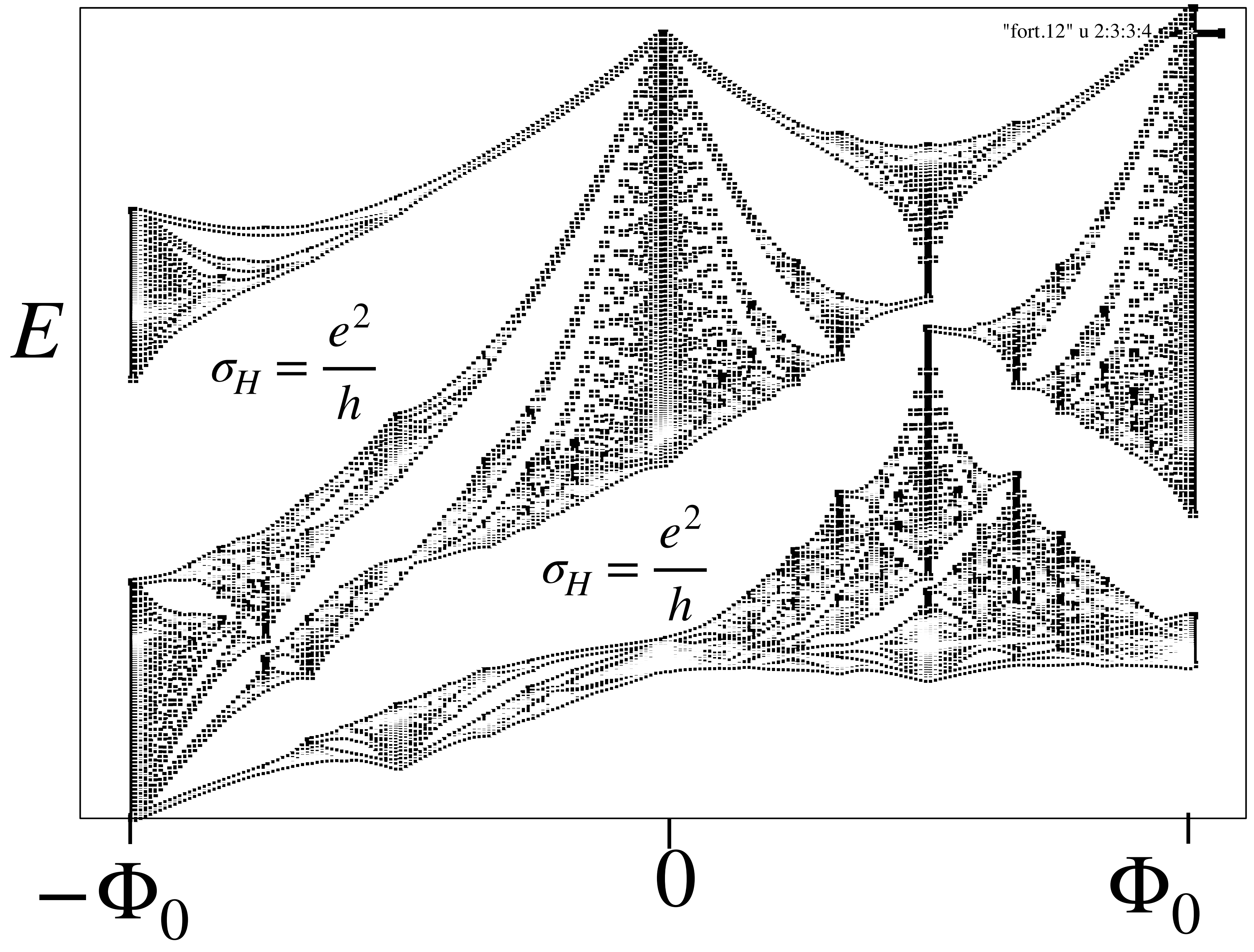
- The discovery (first from exact diagonalization 2011, then experimentally 2023) that a fractionally-filled Chern band (with a Streda anomaly) supports “anomalous” FQHE **WITHOUT ANY MAGNETIC FLUX** means that the “flux attachment” idea (which might seem plausible in Landau levels) needs a reworking.

- The new language I propose is (mobile) local **orbital attachment.** (In a Landau level there is one orbital per flux quantum). This applies equally to the FQHE systems with or without magnetic flux.

- The original "toy model" for the quantum anomalous Hall effect was the graphene-like model of spin-polarized electrons with (complex) second-neighbor hopping t_2



- In 2011, Chamon, Neupert et al. showed the lower band can be substantially-narrowed by tuning t_2. When this "flat band" is 1/3 filled, numerical exact diagonalization studies show it exhibits FQHE.



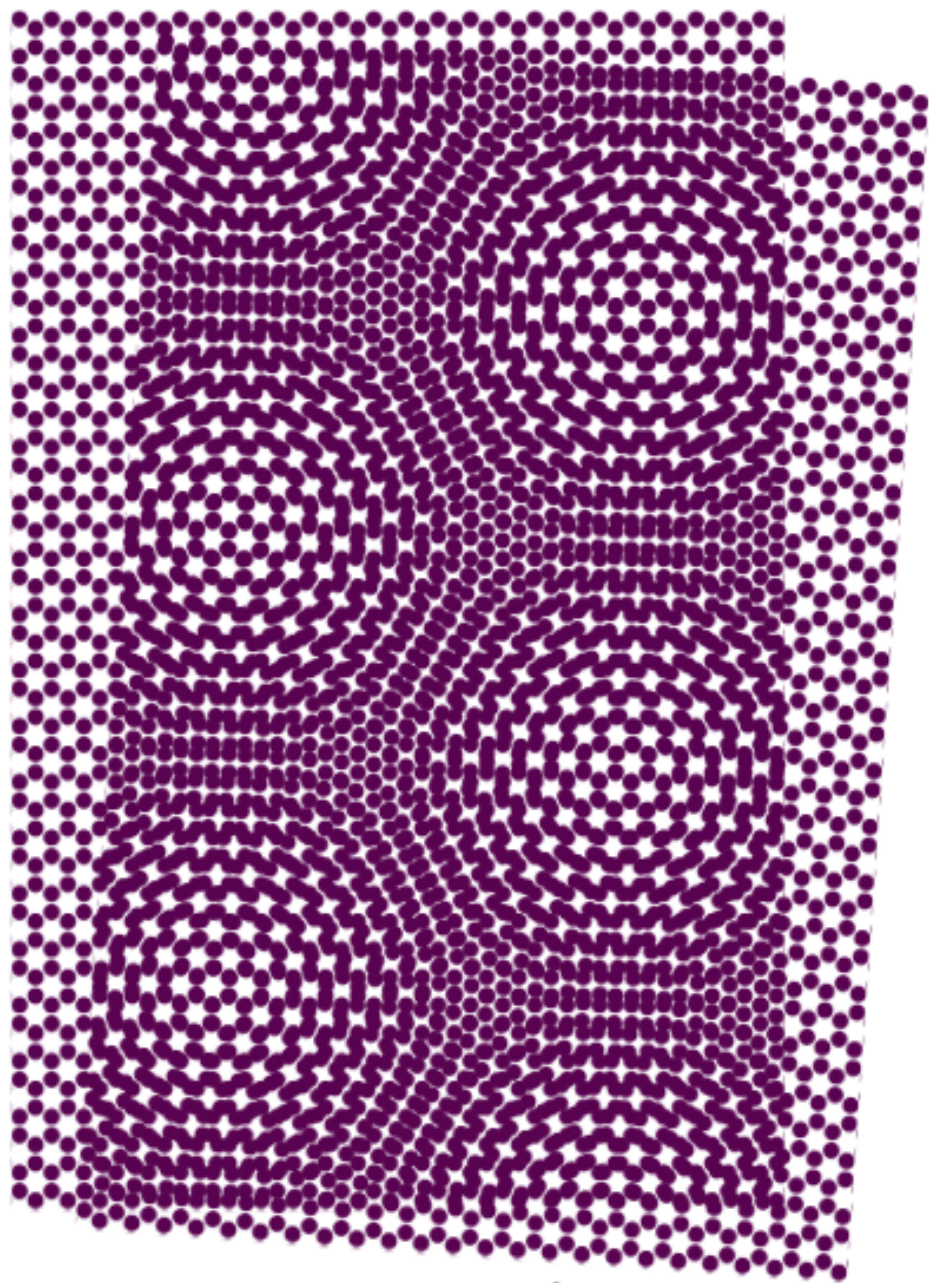
flux per unit cell

"Hofstadter" spectrum of the model with t_2 chosen to give flattest lower band (the embedding shown above is used: the spectum is then periodic in $\Phi \mapsto \Phi + 6\Phi_0$)

- Moiré patterns (e.g. twisted bilayer graphene at “magic angles”) support “flat bands” dominated by electron-electron interactions instead of kinetic energy

- Mid 2023: a number of groups have reported that fractional quantum Hall states can occur in these due to ferromagnetism without magnetic field and at higher temperatures!

- may lead to a new “platform" for FQH physics and topological quantum computing!



- what are the common features of FQH behavior in both lattices (anomalous) and Landau levels (regular)?

FQH in Landau levels

FCI in lattice models

no obvious place for holomorphic functions, etc

Opportunity to get a better understanding of FQH by removing Landau-level-specific ideas

- Laughlin state is parametrized by a Euclidean metric

arbitrary origin (c-number)

$$L = \frac{1}{2\ell^2} g_{ab}(R^a - r_0^a)(R^b - r_0^b) = \frac{1}{2}(a^\dagger a + a a^\dagger)$$

$$[R^a, R^b] = -i\ell^2$$

$$[a, a^\dagger] = 1$$

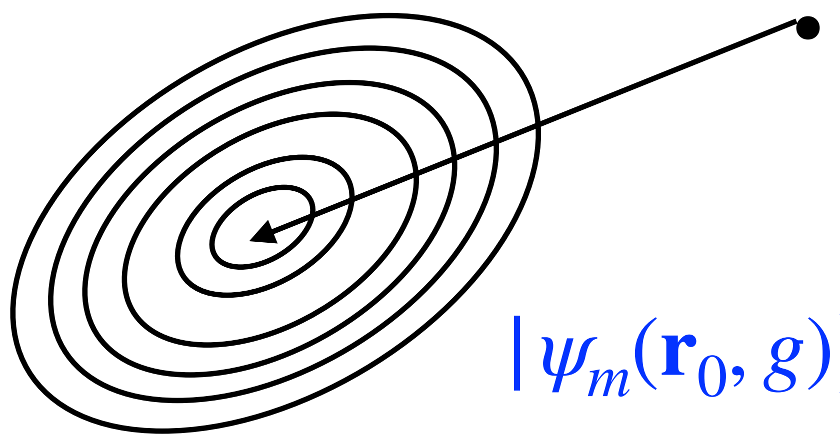
Heisenberg algebra (harmonic oscillators)

positive symmetric metric, $\det g = 1$

(g_{ab} does not have to be the usual metric δ_{ab})

- Laughlin state: $|\Psi_L^3(g)\rangle = \prod_{i<j}(a_i^\dagger - a_j^\dagger)^3|0\rangle$ $\qquad a_i|0\rangle = 0$

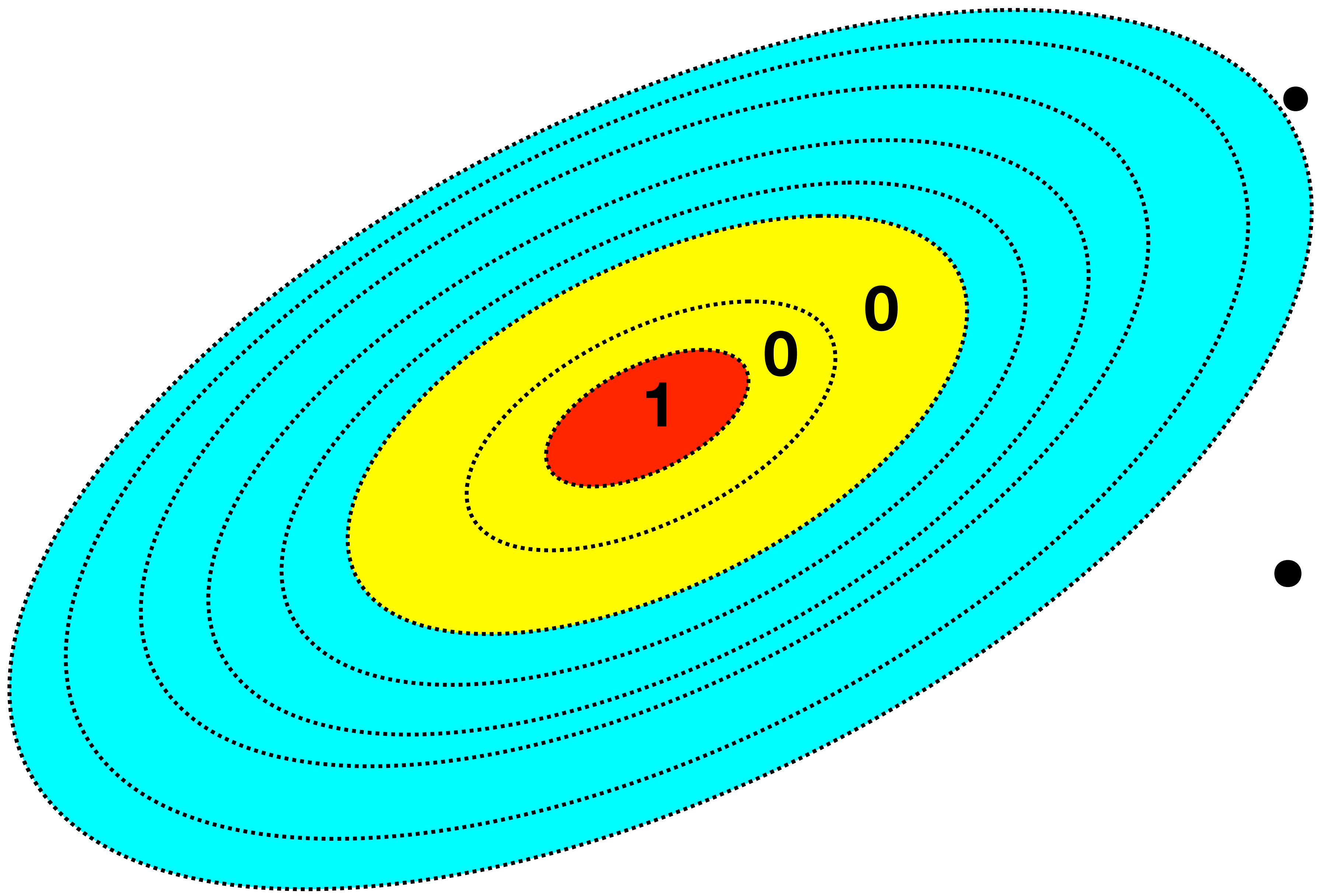
- "Onion ring" orthonormal basis of one-electron states



centered at $\mathbf{r}_0$)

$$|\psi_m(\mathbf{r}_0, g)\rangle = \frac{1}{\sqrt{m!}}(a^\dagger)^m|0\rangle$$

area $2\pi\ell^2$ **between rings**

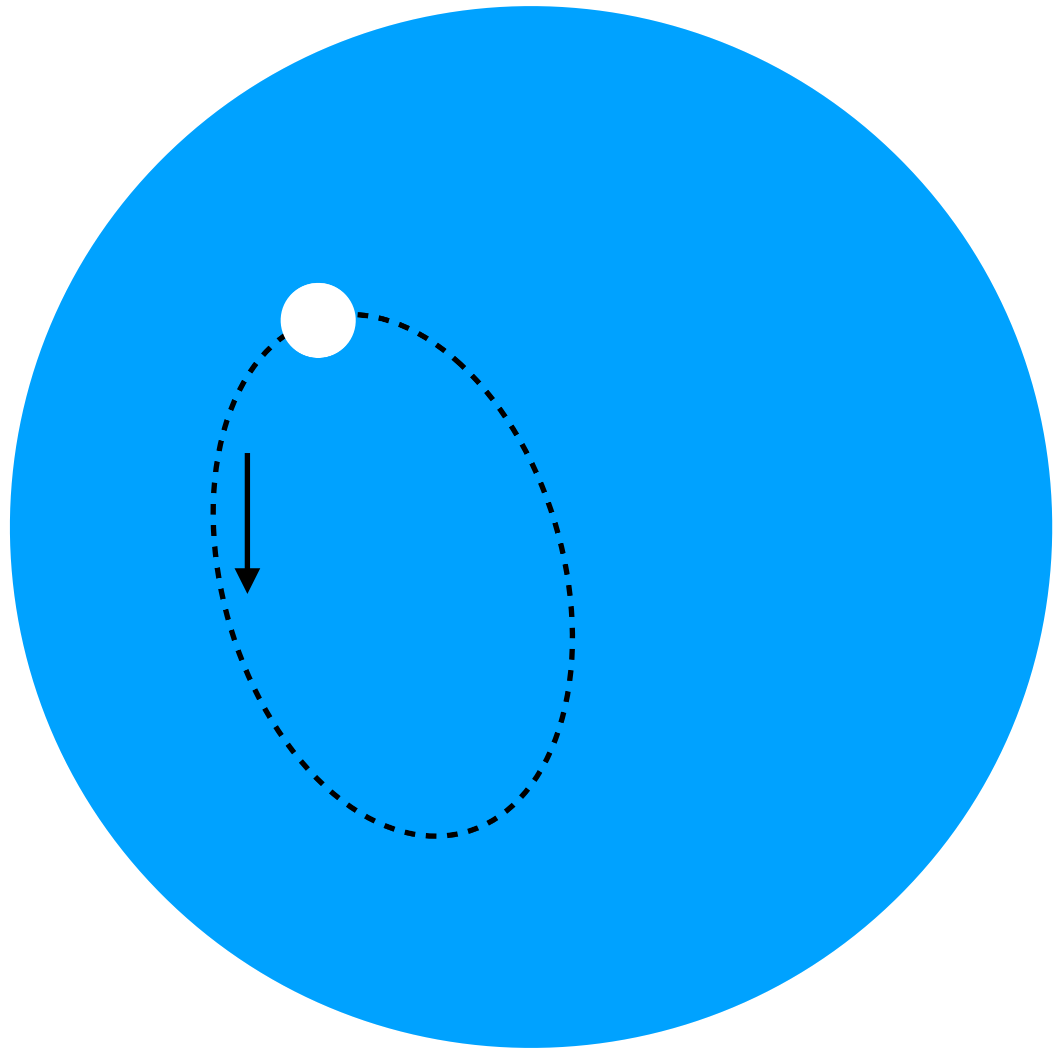


- The “onion ring” basis in the Landau level is crucial for understanding the Laughlin state.
- Each ring is a chiral topological edge state of the region it encircles.

- If central orbital is occupied, the net two orbitals are empty in the 1/3 Laughlin state.
- We can say that each electron “occupies three orbitals”, or that “three orbitals are attached to each electron”
- This language change (“orbital-” instead of “flux-attachment” can also apply to FCI, where there is no “flux”

- The key difference between local orbitals in a Landau level (centered on a guiding center) and (maximally localized) Wannier orbitals is that Landau orbitals can be moved adiabatically in the Hall plane.

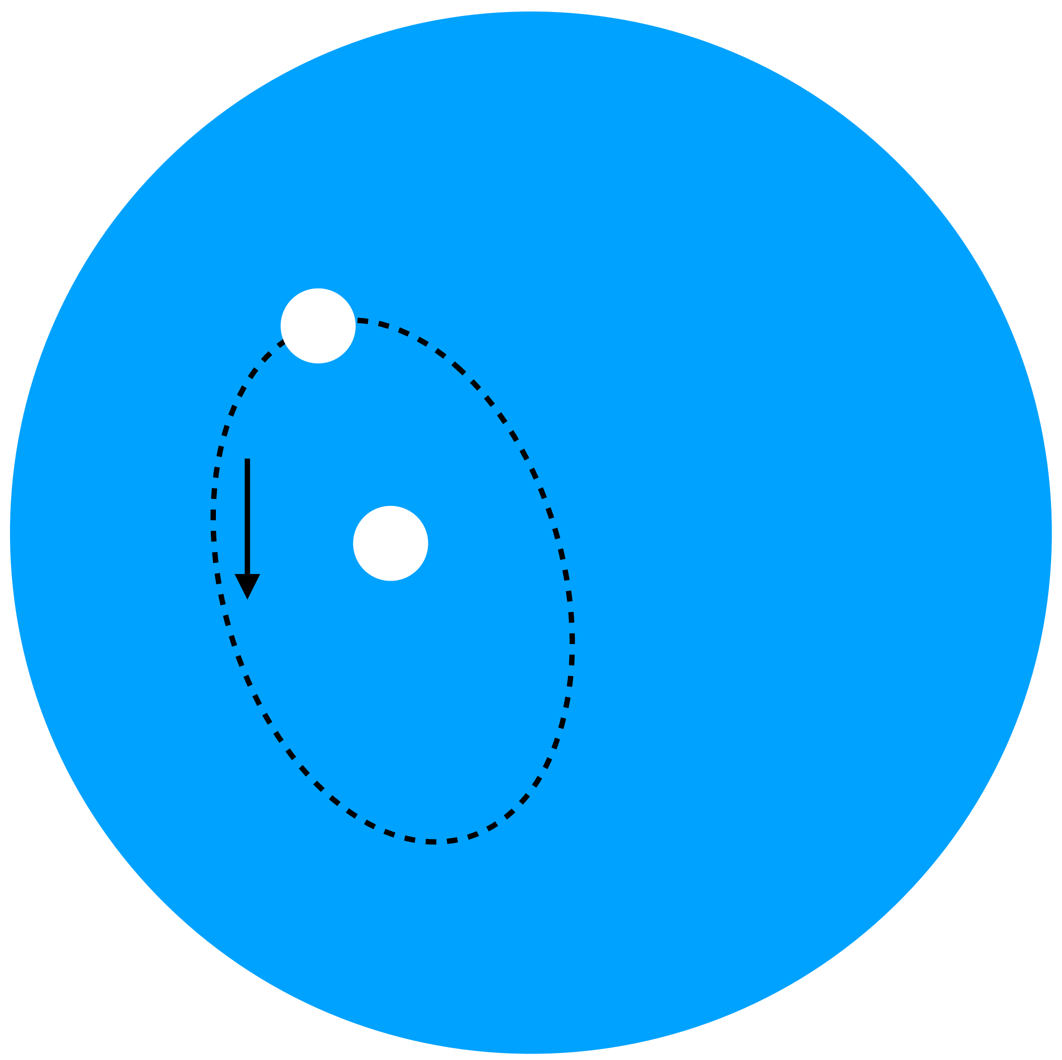
- This leads to the "composite boson picture" and Abelian Chern-Simons gauge field.



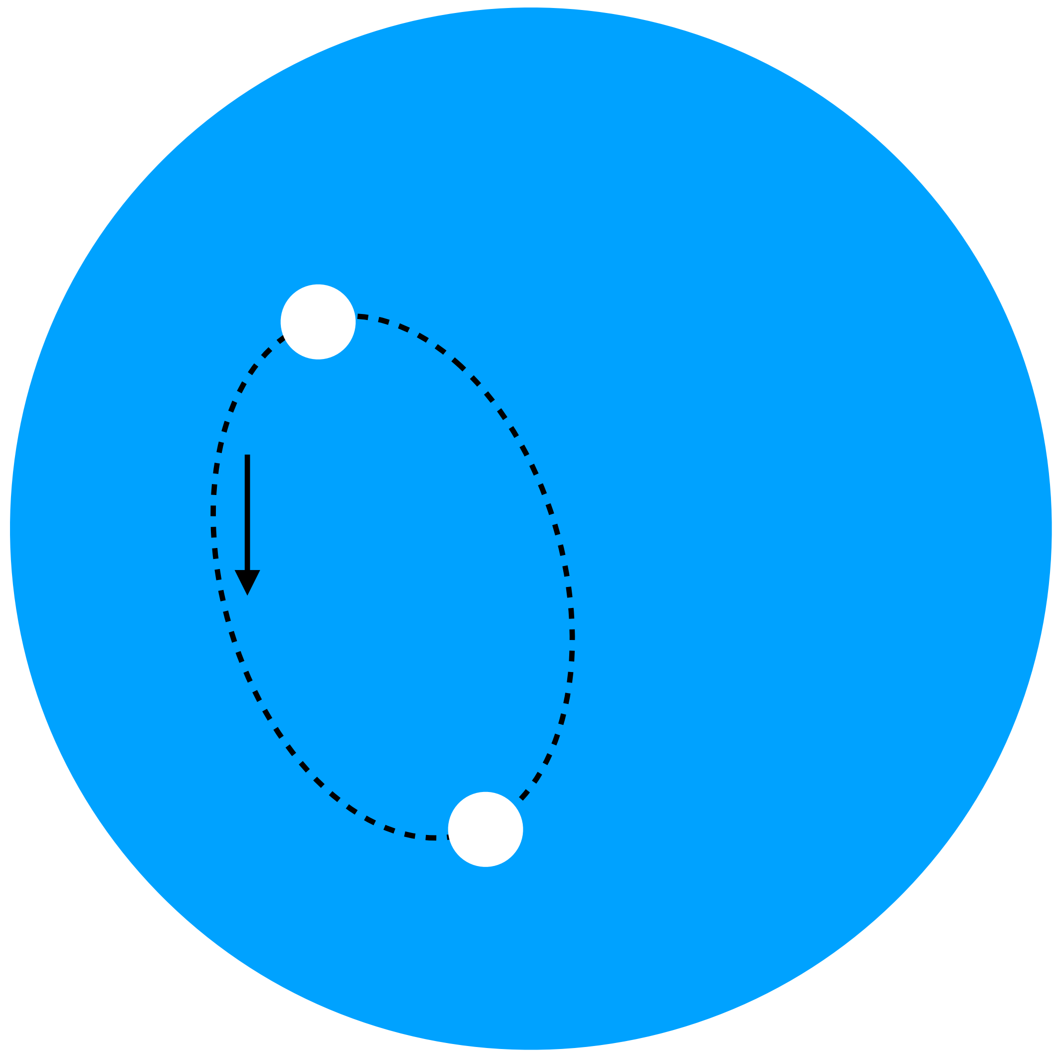
- Adiabatically drag a hole (empty orbital in an otherwise-filled Landau level) around a closed path.

- There is a Berry phase equal to the number of electrons enclosed by the path (Arovas Wilczek Schreiffer 1984)

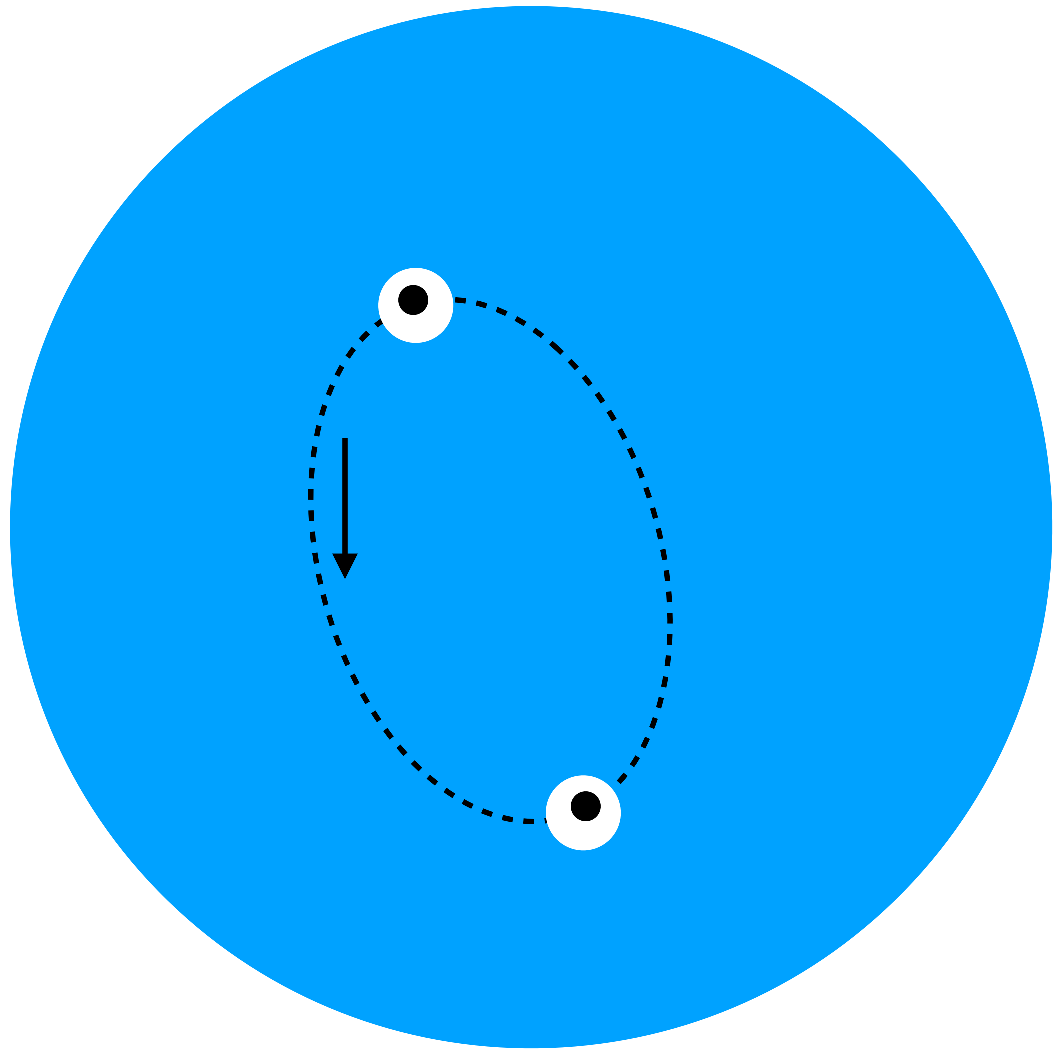
$$\Psi = \prod_i (z_i - w) \prod_{i<j} (z_i - z_j) \prod_i e^{-\frac{1}{2} z_i^* z_i}$$



- Because there is one electron per flux quantum, the Berry phase is equal in magnitude, opposite sign to the Bohm-Aharonov phase of an electron moving on the same path.



- if another empty orbital is present inside the path, the Berry phase is reduced by 2π
- The closed path is equivalent to two exchanges. The exchange factor is π and the holes are fermions as expected.



- Filled orbitals (fermionic hole + fermionic electron) are bosons
- The exchange of two filled orbitals is a row exchange plus a column exchange in the Slater determinant.
- The Berry phase cancels the Bohm-Aharonov phase, and the composite boson is effectively neutral so can "condense"
- This "composite boson" picture generalizes naturally to the FQHE

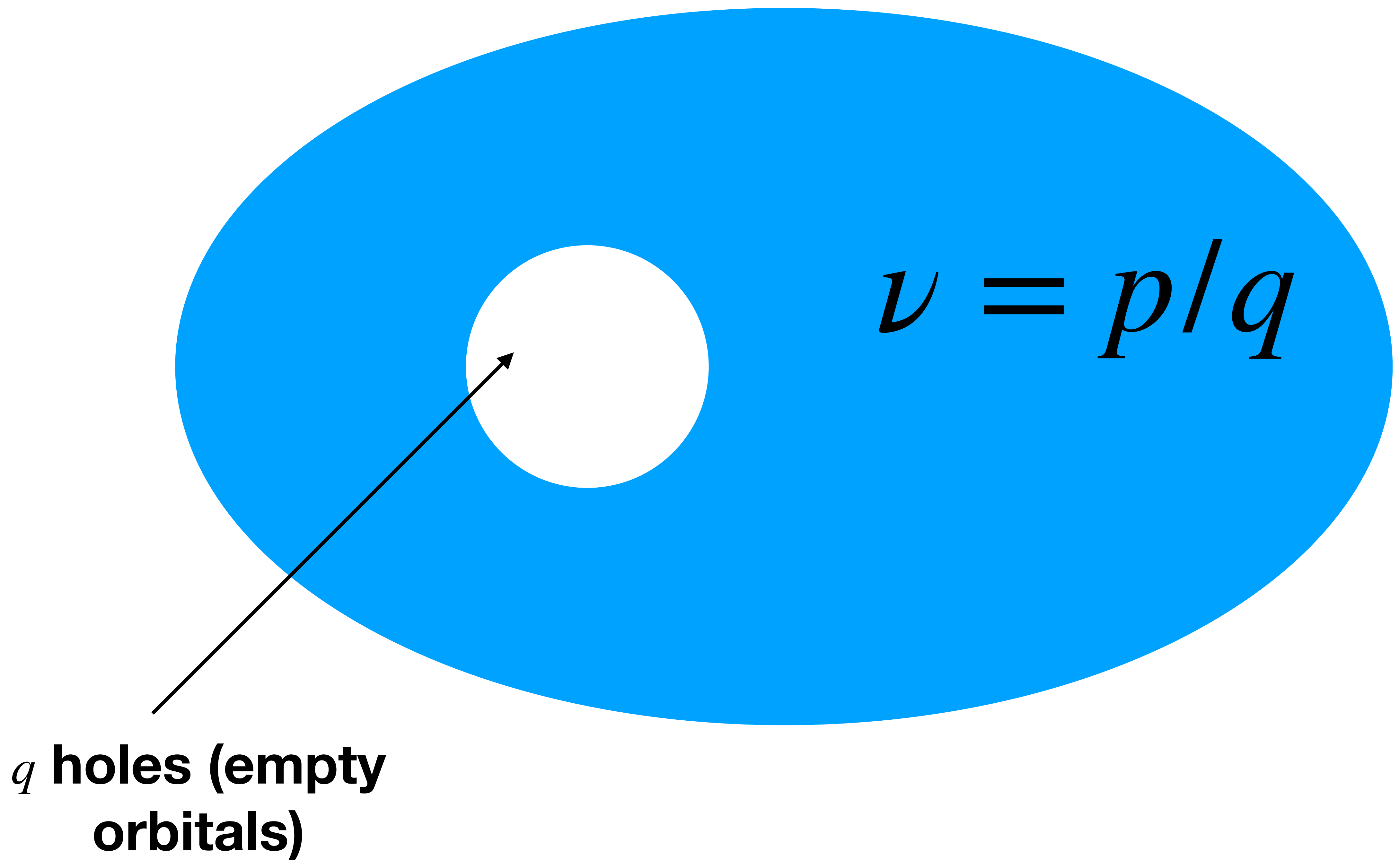
exchange particles -1

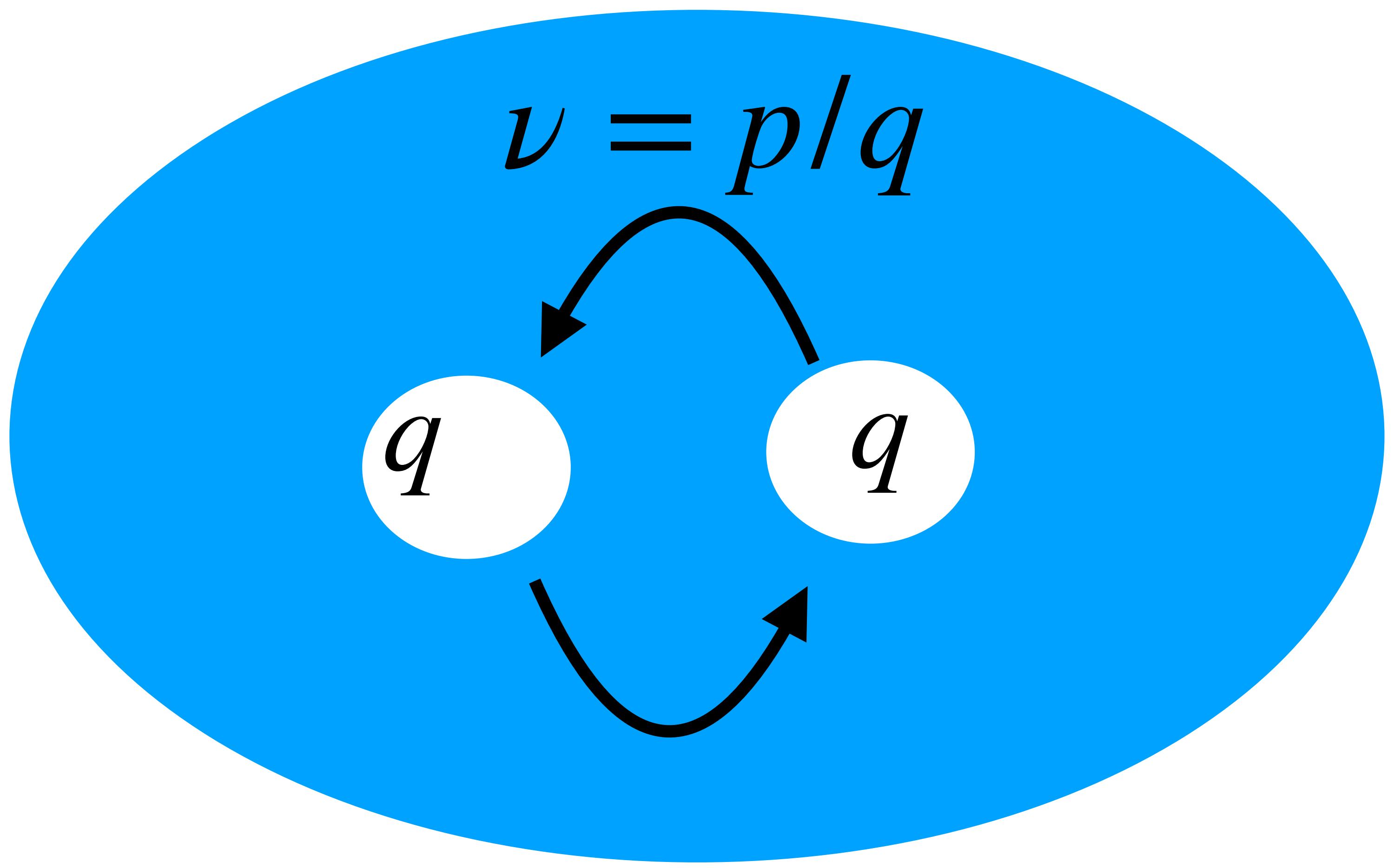
$$\begin{vmatrix} \cdots & \psi_m(x_i) & \cdots & \psi_m(x_j) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \psi_n(x_i) & \cdots & \psi_n(x_j) & \cdots \end{vmatrix}$$

-1 **exchange orbitals**

Slater determinant (filled Landau level)

- The initial discussion of composite bosons in the integer QHE generalizes to the fractional case

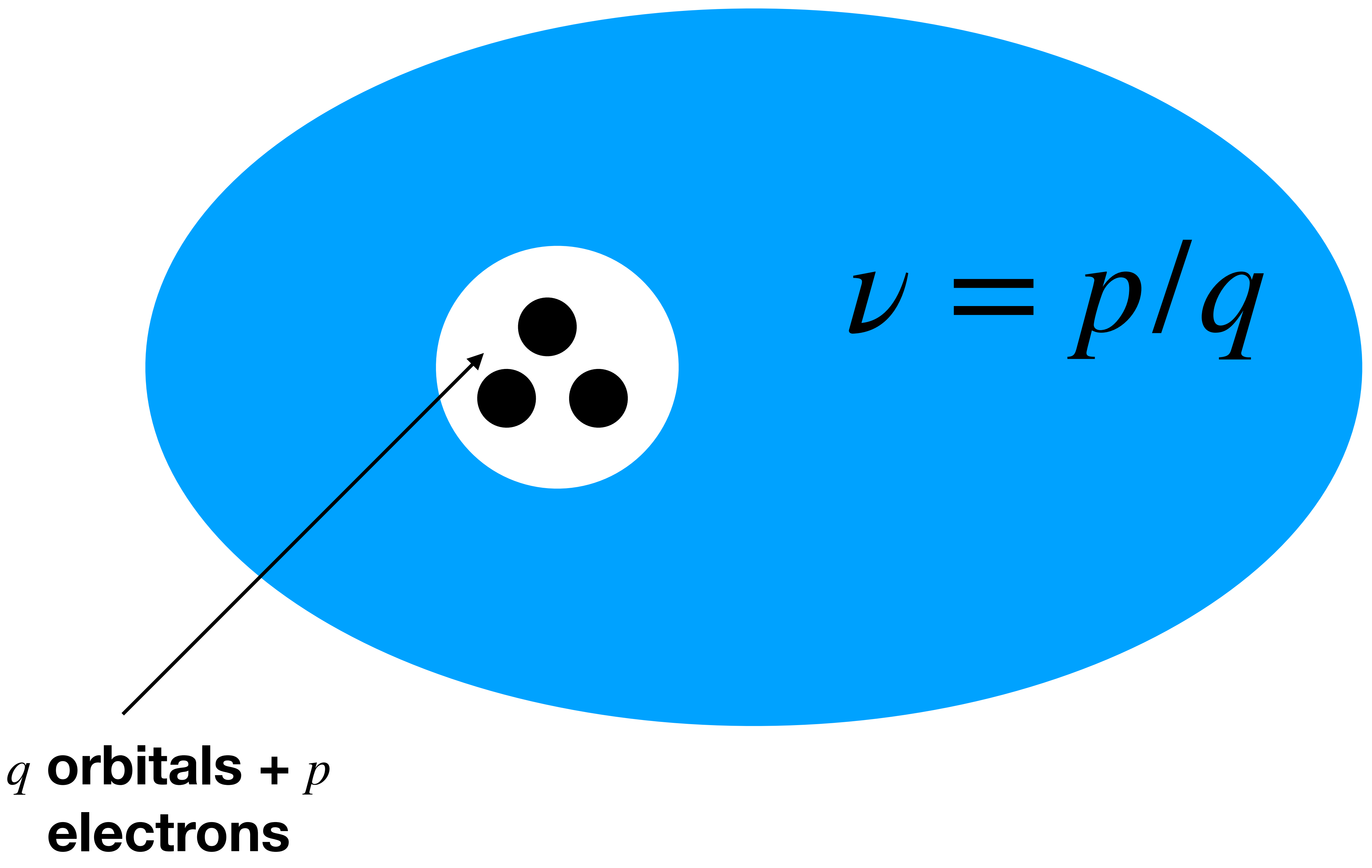




Chern-Simons index k = pq

Two-q-hole exchange factor $(-1)^k = \pm 1$

(the condition that this is regular Fermi/Bose statistics quantizes k in Abelian CS theory)



$$e_b = pe$$

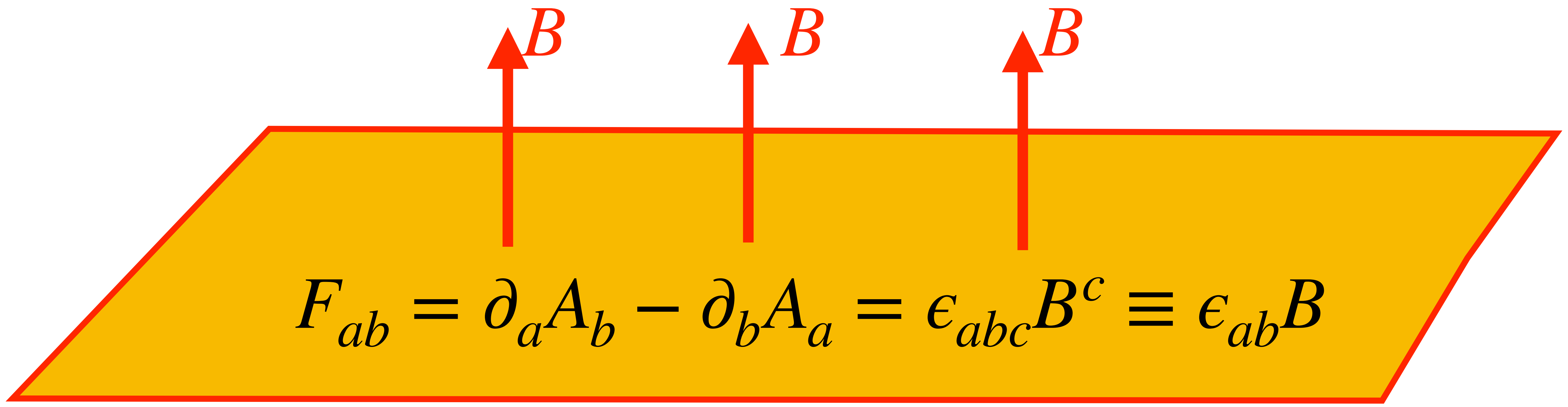
$$k = pq$$

Now the p particles will cluster at the center of the “bubble” of q orbitals

This will lower the correlation energy from repulsive short-range interactions with particles not in the “bubble “, and create a (primitive) quadrupole

composite object is a boson if $(-1)^k = (-1)^p$

- In the (clean-limit) quantum Hall effect, the electronic subsystem remains incompressible, but some of the electrons (on a 2d lattice plane) are released from control by the lattice, and form an incompressible fluid that is controlled by the electromagnetic degrees of freedom, the Faraday tensor in the 2d lattice plane



- The 2d Faraday tensor is used in preference to B because it does not depend on an (arbitrary) handedness (chirality) convention

Faraday in 2D plane (not chiral)

$$F_{ab} = \epsilon_{ab} B$$

2D antisymmetric Levi-Civita symbol times normal flux (both chiral)

- the fundamental QHE property is not $J^a = \sigma_H^{ab} E_b$, but the Středa relation

$$\left.\frac{\partial J^0}{\partial F_{ab}}\right|_\mu = \sigma_H^{ab}$$

- We can reinterpret this as defining the density of "captured" electrons in the Hall fluid:

$$J^0_{\rm em} = \frac{1}{2}\sigma_H^{ab} F_{ab}$$

- The flow velocity is just the electromagnetic drift velocity:

$$E_a + F_{ab} v_D^b = 0 \qquad v_D^a = \epsilon^{ab} E_b / B$$

- Then the Hall current is

$$J^a = J^0_{\rm em} v_D^a = \sigma_H^{ab} E_b$$

$$v_D^a = \epsilon^{ab} E_b / B$$

- Since (for constant B) the gradient of the drift velocity is proportional to the gradient of the electric field, which couples to the quadrupole density, the dependence of the new formula for the Hall viscosity is seen to be very natural!

$$(\eta_H)^{ac}_{bd} = \delta^a_d F_{be} Q^{ce} - \delta^c_b F_{de} Q^{ae}$$

- **The quantized Hall conductivity has the form**

$$\sigma_H^{ab} = \frac{(e_b)^2}{2\pi\hbar} k^{-1} \epsilon^{ab}$$

Here e^* is the charge of the elementary unit (composite boson) of the incompressible fluid, and k is the integer index of a $U(1)_k$ Abelian Chern-Simons effective topological field theory with

$$\boxed{\mathscr{L}_{\text{cs}} = \frac{\hbar k \epsilon^{ab}}{4\pi} a_a \partial_t a_b} \quad \boxed{J^0_{\text{em}} = \frac{e_b}{2\pi} \epsilon^{ab} \partial_a a_b} \quad \boxed{J^a_{\text{em}} = -\frac{e_b}{2\pi} \epsilon^{ab} \partial_t a_b}$$

electric charge density "owned" by e.m. field

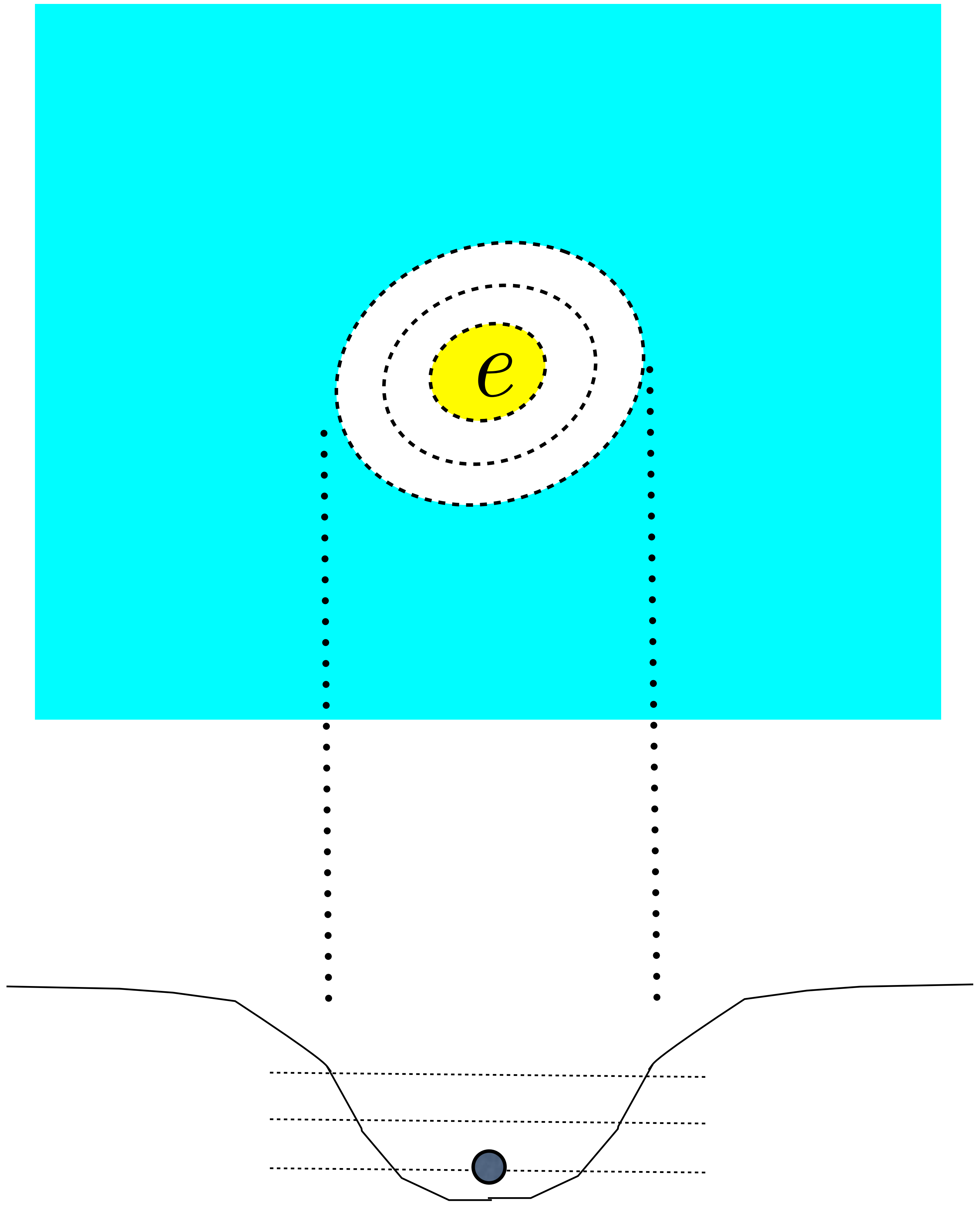
electric current density

$$S_{CS} = \int dt \int d^2x \mathscr{L}_{\text{cs}} + A_0 J^0_{\text{em}} + A_a J^a_{\text{em}}$$

The elementary charge of topological excitations is $\pm e_b/k$

$1/3$ Laughlin state

If the central orbital is filled, the next two are empty



The composite boson has inversion symmetry about its center

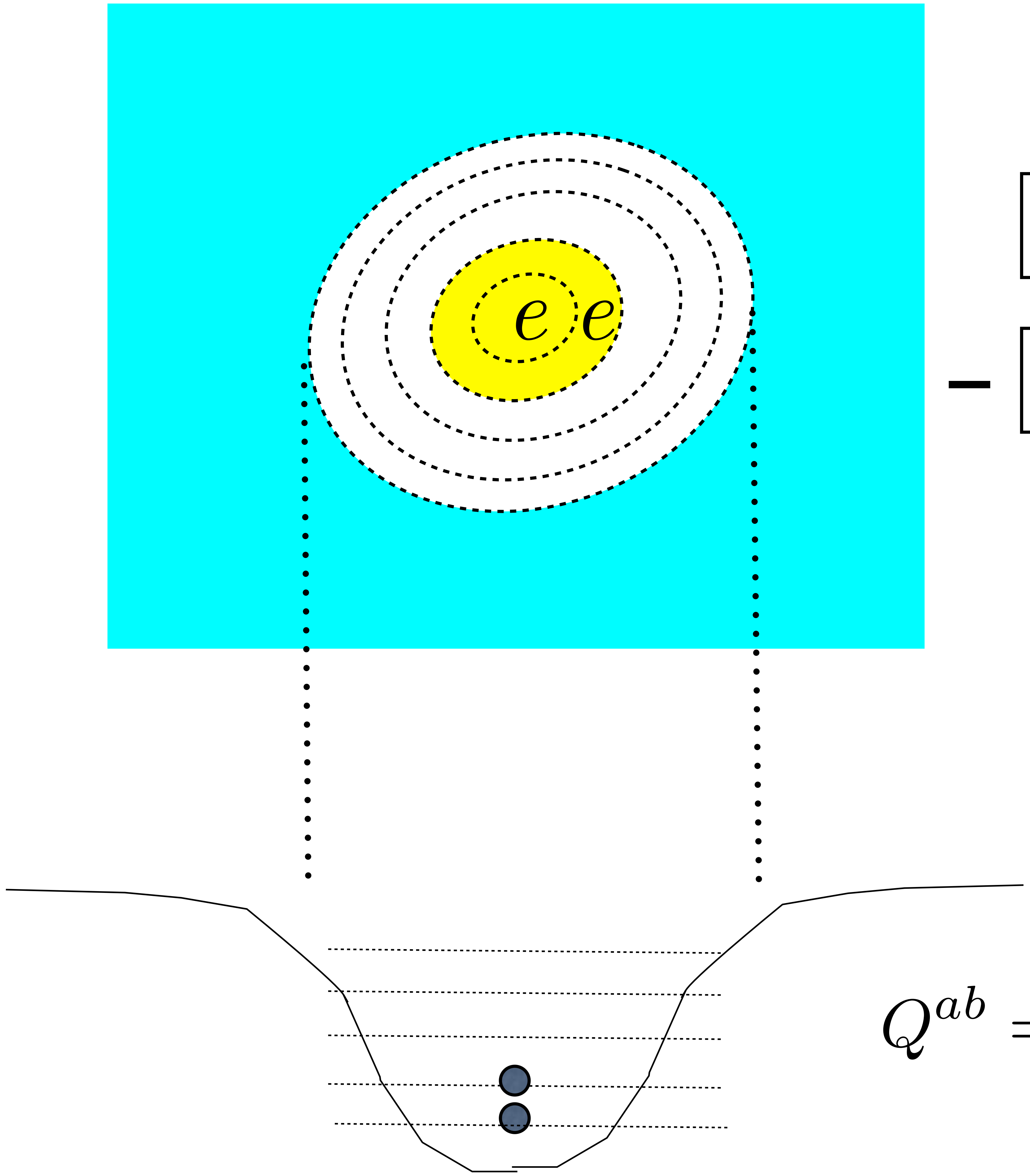
It has a "spin"

	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
	1	0	0	.....	$L = \frac{1}{2}$
−	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$- L = \frac{3}{2}$

$s = -1$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$$\begin{array}{ccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & & \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 0 & 0 & 0 \\ \hline \end{array} \;\cdots\cdots \qquad L = 2$$

$$- \begin{array}{|c|c|c|c|c|}\hline \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \hline \end{array} \;\cdots\cdots \qquad - L = 5$$

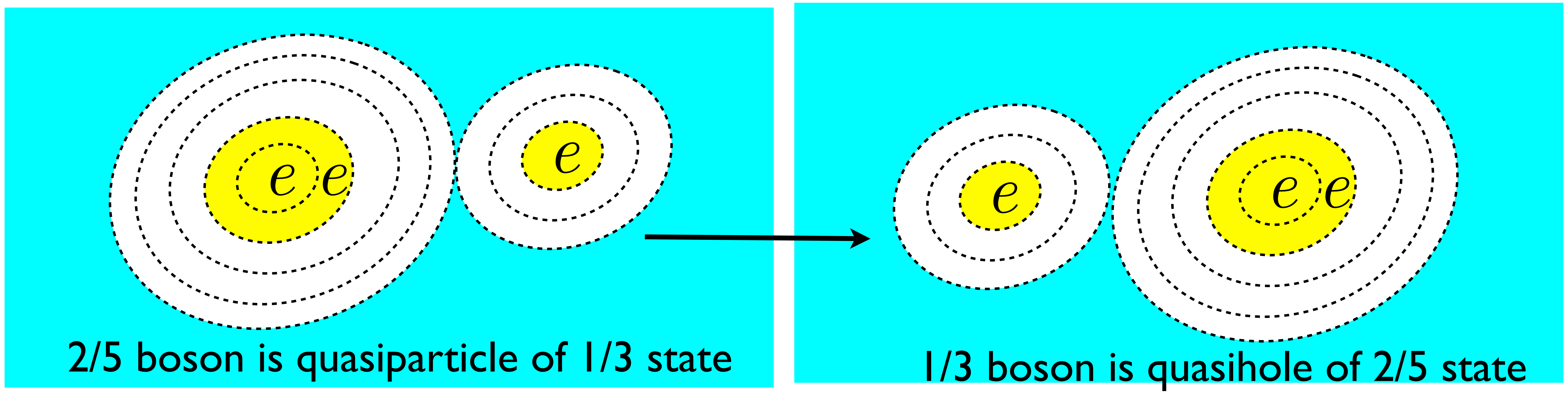
$$s = -3$$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

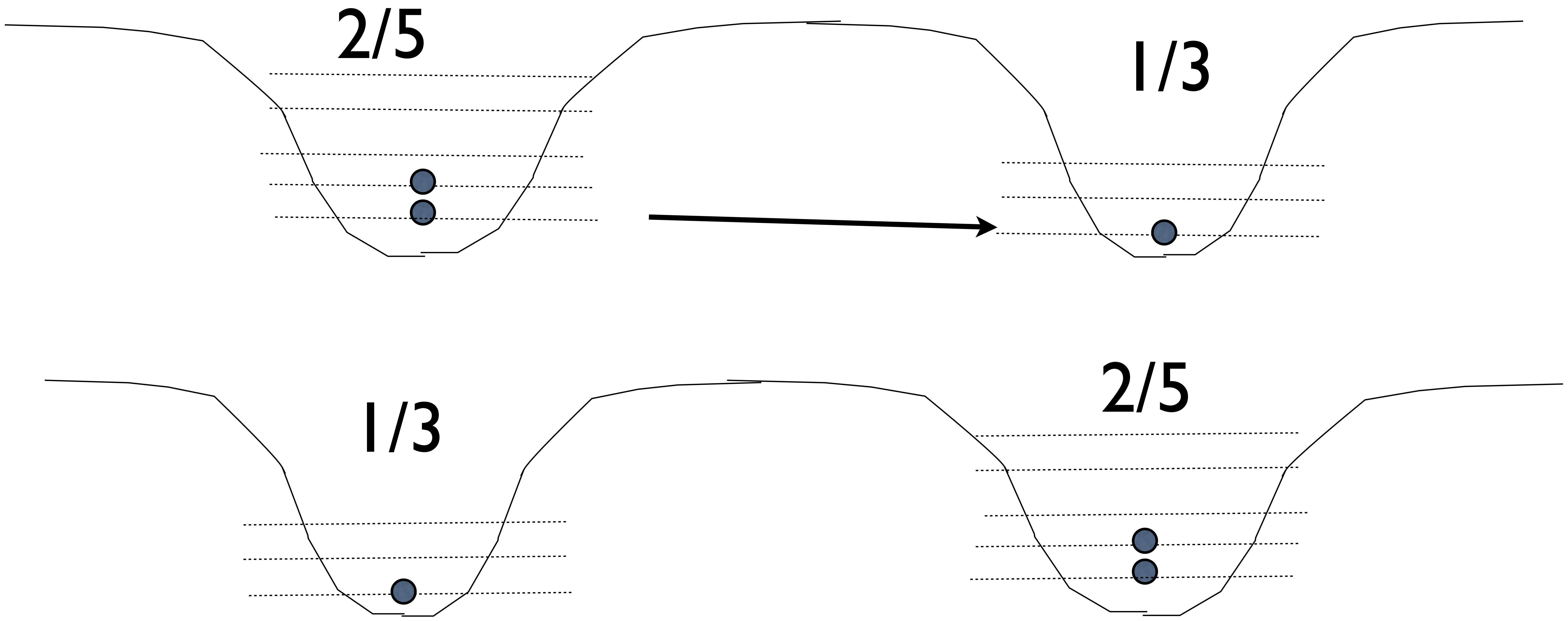
$$Q^{ab} = \int d^2r\, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

hopping of a "composite fermion" (electron + 2 flux quanta)

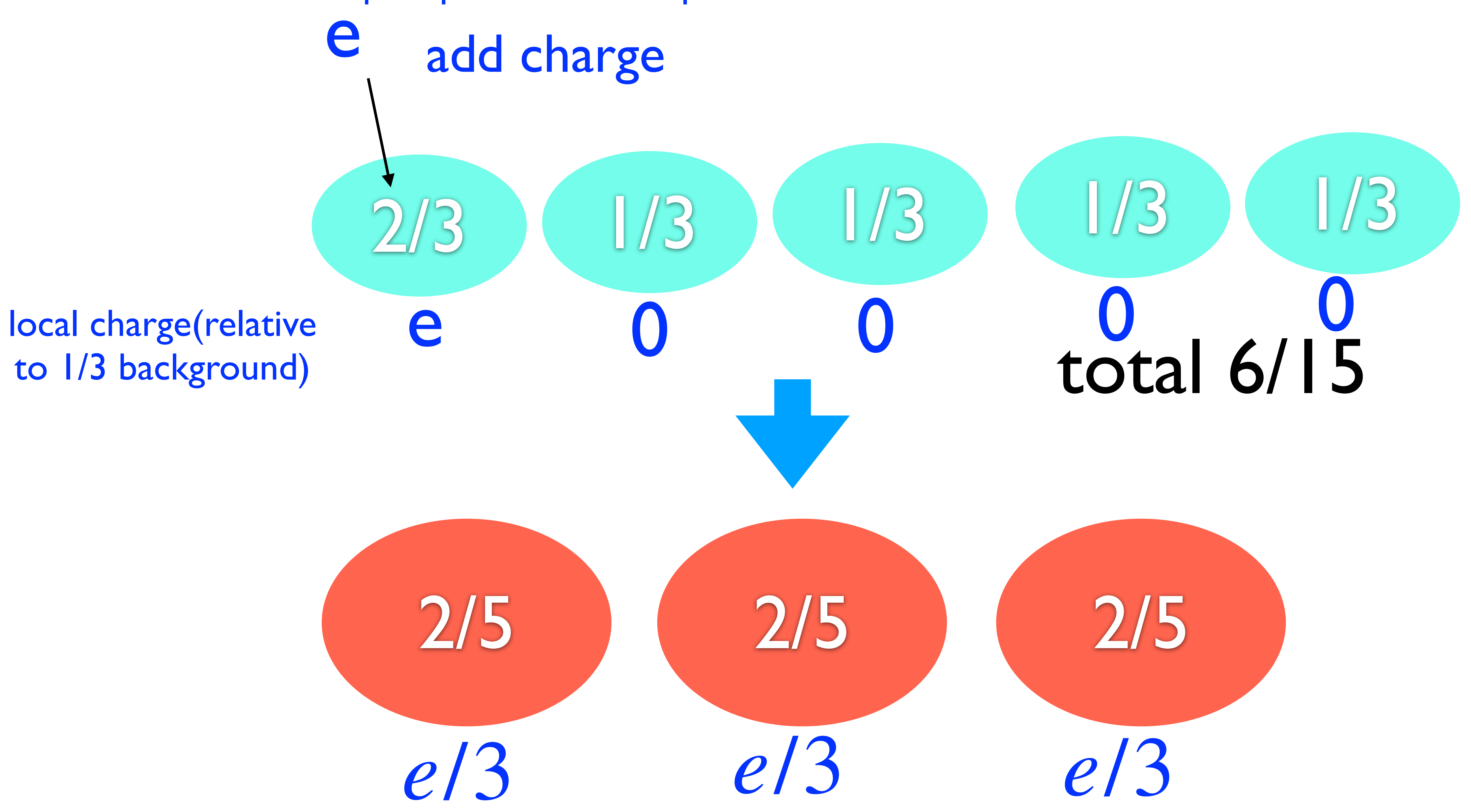


Jain's "pseudo Landau levels"

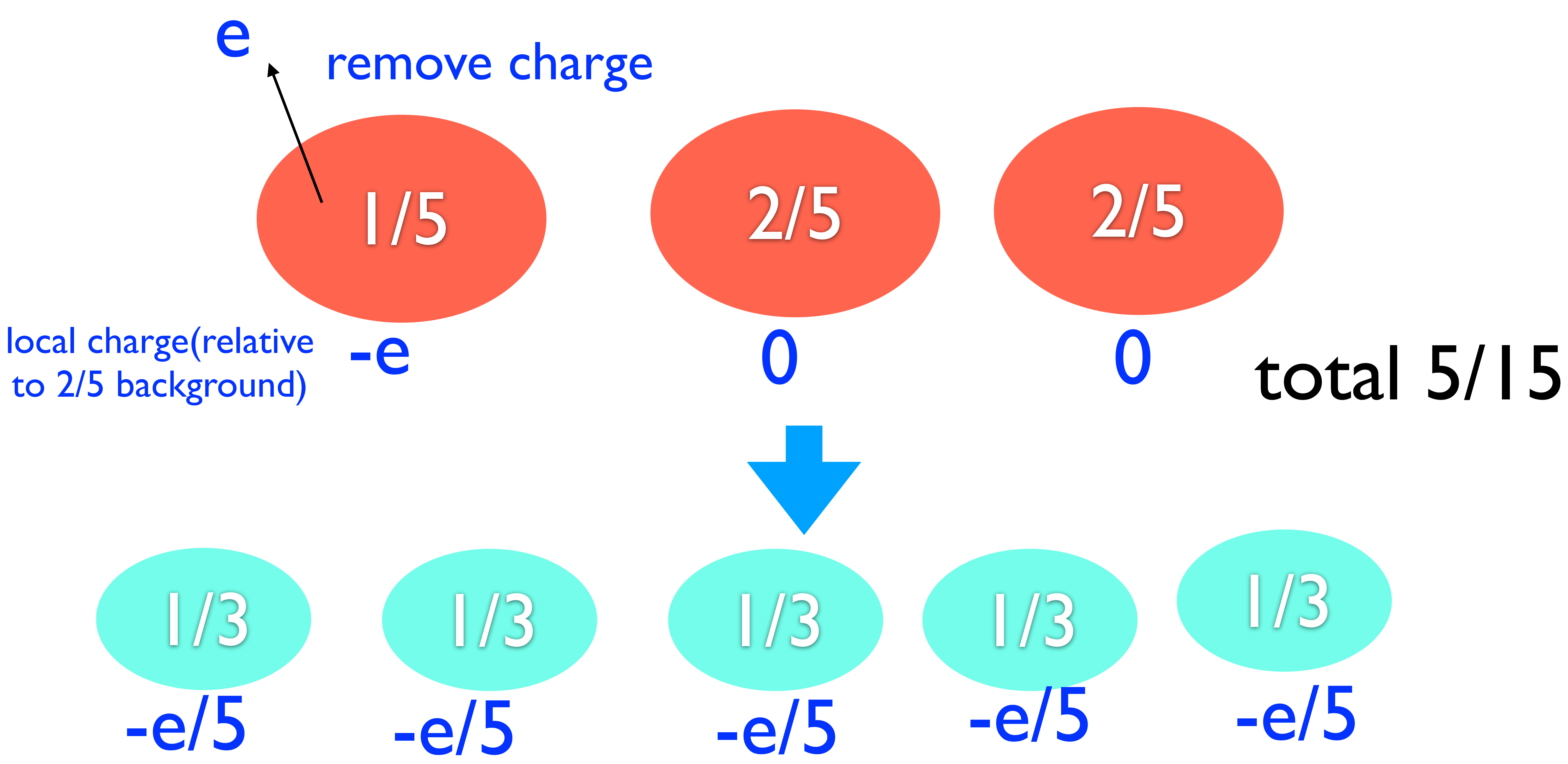


• adding charge to the 1/3 state

- quasi particles = composite bosons of 2/5 state



- similarly, quasiholes of 2/5 state are composite bosons of 1/3:



- Jain sequences

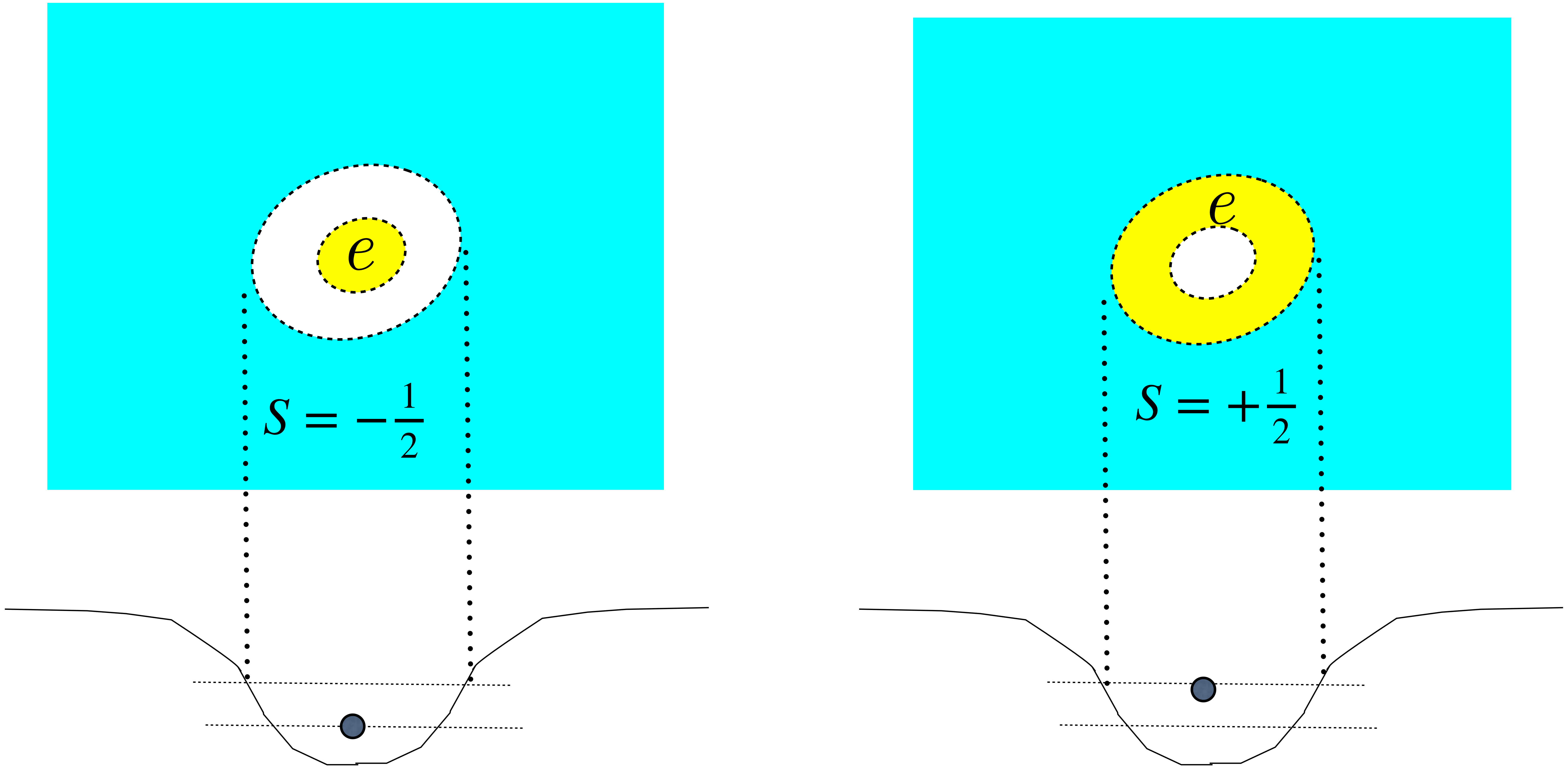
composite fermion $\frac{1}{2}$

$$\frac{1}{3} + \frac{1}{2} = \frac{2}{5} + \frac{1}{2} = \frac{3}{7} + \ldots$$

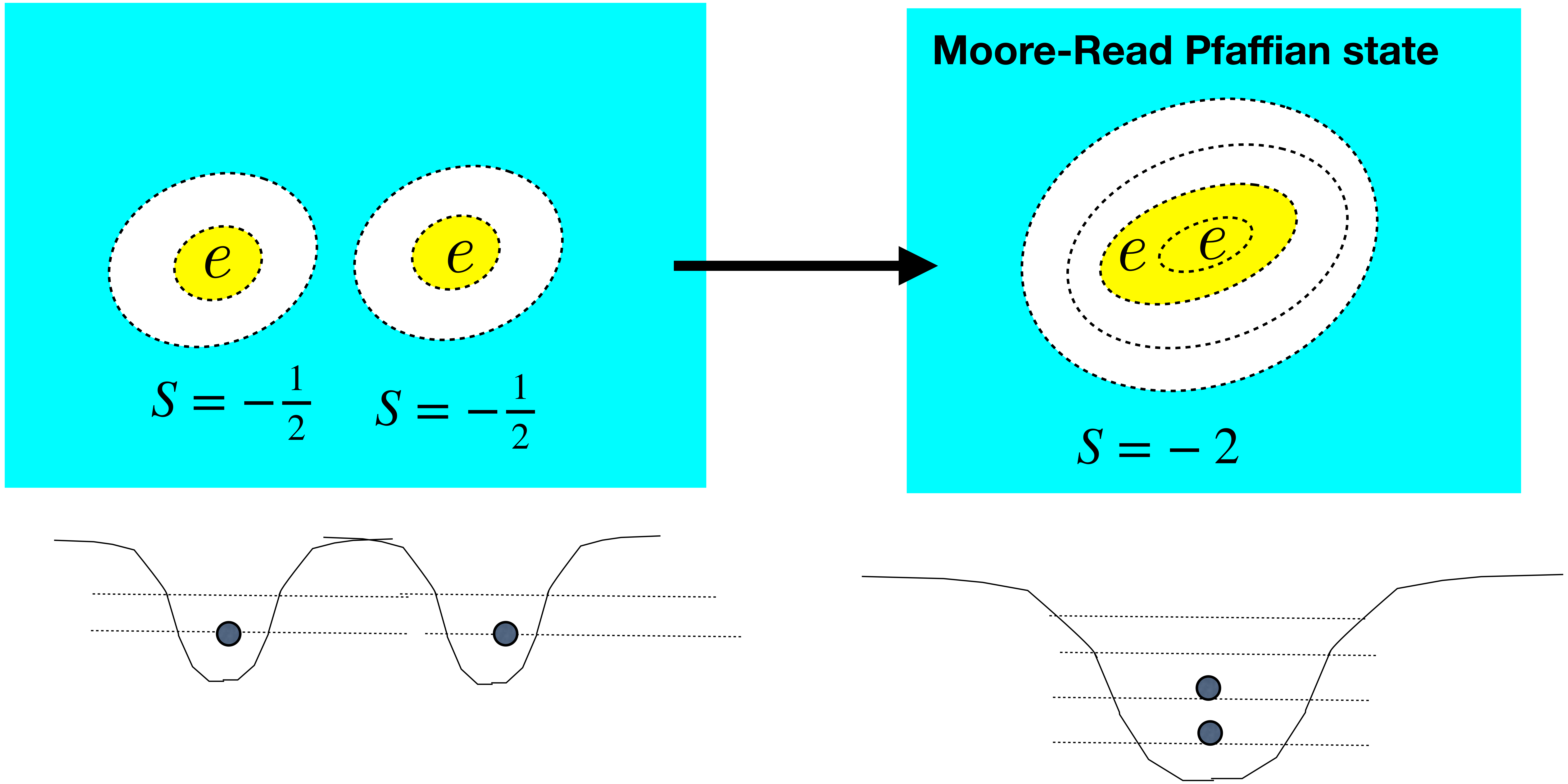
composite fermion $\frac{1}{4}$

$$\frac{1}{3} + \frac{1}{4} = \frac{2}{7} + \frac{1}{4} = \frac{3}{11} + \ldots$$

- quadrupole of composite fermions is also important
- at $\nu = 1/2$, there are two particle-hole conjugate species

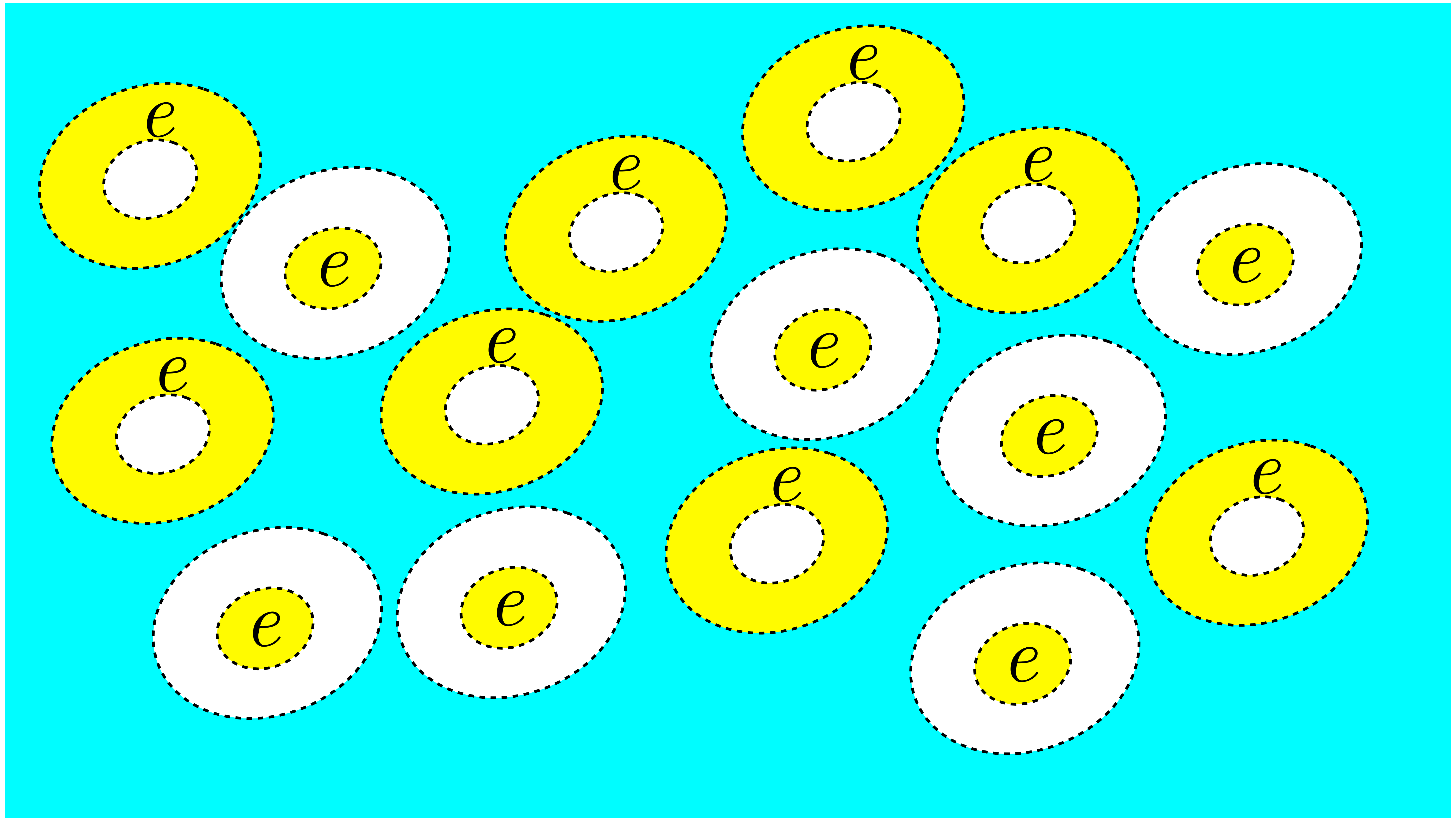


the electron (or hole) excludes other particles from a region containing 2 flux quanta, creating a potential well in which it is bound

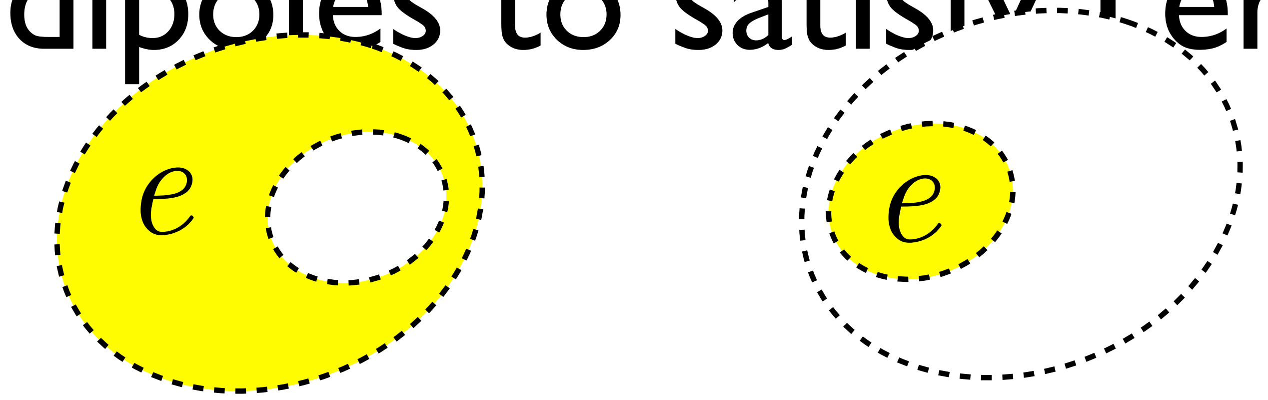


pairing of equal-type composite fermions produces composite boson with larger quadrupole

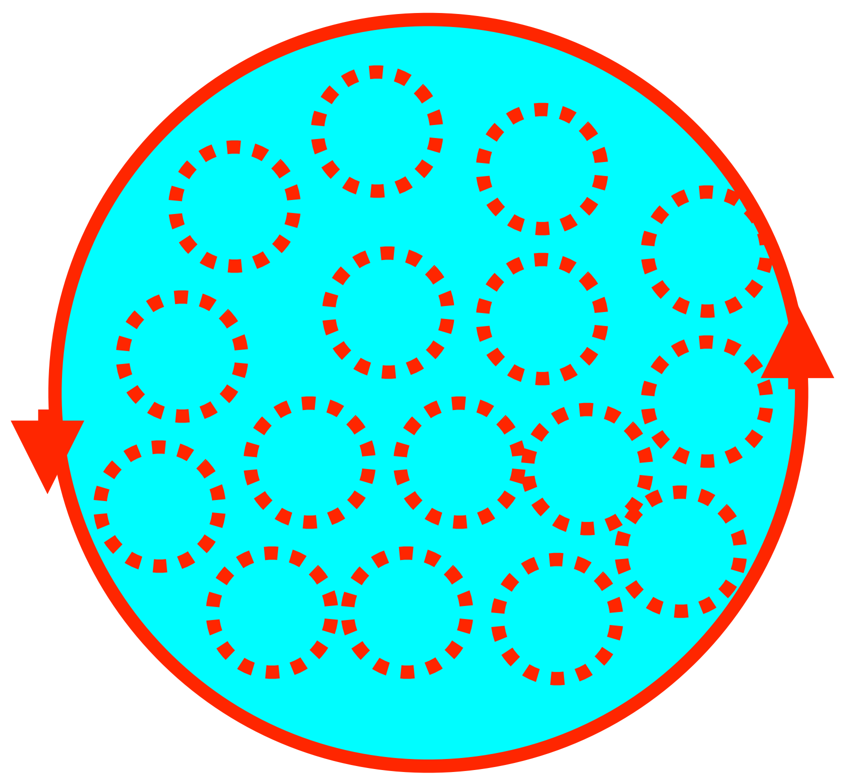
pairing of opposite-type composite fermions produces no extra quadrupole, so does not occur in single LL



- PH-symmetric CFL has equal numbers of both types of cf's, they must mix to all carry different dipoles to satisfy Fermi statistics



- Rotationally invariant (TOY) models



- Circular 2D droplet of N composite bosons with an unexcited chiral Luttinger liquid on its edge

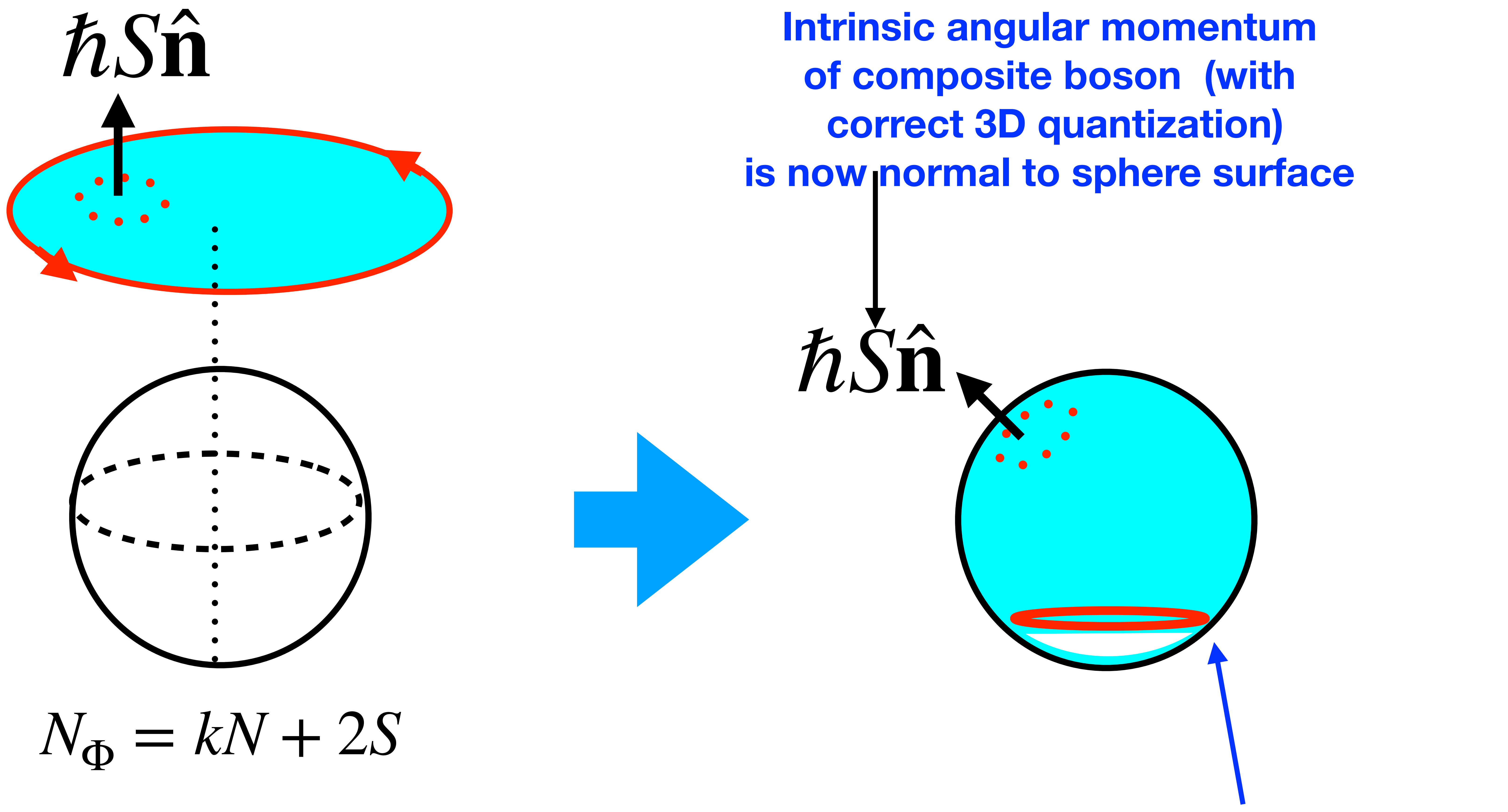
- (Planar) angular momentum about center is

$$L = \hbar\left(\frac{1}{2}kN^2 + SN\right)$$

Chern-Simons index (a chiral integer)

"intrinsic orbital angular momentum" of the composite boson (a chiral half-integer)

- The rotationally-invariant disk can be put on a sphere (not possible without rotational invariance)



- The spin -S Berry phase as the composite boson moves on the sphere surface matches the intrinsic Wen-Zee "spin-connection" picture

- For rotationally-invariant TOY MODELS, the quadupole density is quantized

$$Q^{ab} = \frac{e_b S \delta^{ab}}{4\pi k}$$

inverse of Euclidean metric tensor that defines rotational invariance

$2S =$ integer

- note that $e^* = e_b/k$ is elementary fractional charge

- On the sphere, a Cartesian coordinate system cannot be used, but a coordinate system with spatially-varying determinant-1 metric $g_{ab}(\mathbf{x})$ should be used.

- To evaluate the Gaussian curvature κ at a point $\mathbf{x}_0$, choose a curvilinear coordinate system on the sphere which is "inertial" at that point:

$$g_{ab}(\mathbf{x}_0) = \delta_{ab} \quad \Gamma^a_{bc}(\mathbf{x}_0) = 0 \quad \det g(\mathbf{x}) = 1$$

$$\kappa(\mathbf{x}_0) = -\frac{1}{2}\partial_a\partial_b g^{ab}(\mathbf{x}_0)$$

- *The apparently-impressive success of the "Hall viscosity = coupling to background geometry" Gaussian curvature model is seen to just be a special limit of the quadrupole density bound-charge formula* $J^0_{\text{bound}} = -\partial_a\partial_b Q^{ab}$

- In "toy models" with SO(2) symmetry, the composite boson carries a quantized planar (azimuthal) angular momentum S, where

$$(-1)^{2S} = (-1)^k$$

- The parity under 2D inversion is

$$(-1)^{S+\frac{1}{2}k} = \pm 1$$

Note that the ratio S/k is orientation independent, even though individually S and k depend on orientation choice

- The correspondence to previous formulas is

$$S = p\bar{s} = -\frac{1}{2}p\mathcal{S} \longleftarrow$$

"shift on sphere" (Wen and Zee, 1992)

"intrinsic orbital angular momentum per electron" (Read, 2009)

(note: $2\bar{s}$ and $\mathcal{S}$ are **not** generically integers)

- Then the quadrupole density is also quantized:

$$Q^{ab} = \frac{e_b S}{4\pi k}\delta^{ab}$$

inverse of Euclidean metric

  - This provides full agreement with Reed's (2009) formula for Hall viscosity of systems with SO(2) rotational invariance.

- In pseudo-isotropic cases, with only discrete three-fold, four-fold or six-fold rotation symmetry, Q^{ab} will still have this form, but with unquantized S

- I will now take a rapid trip though the geometry and topology of Bloch-state Berry curvature

- Geometry and topology were first connected by the Gauss-Bonnet theorem:

$$\int_{\text{2D surface}} d^2x \; \text{(intrinsic curvature)} \propto \begin{array}{c}\text{Euler characteristic} \\ \text{or } (1\text{- genus})\end{array}$$

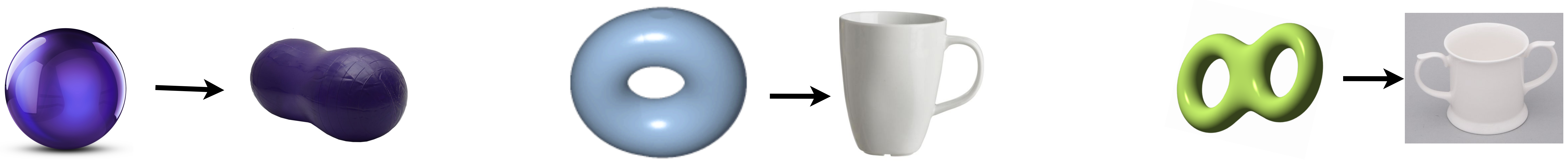
local geometry

global topology

$$\int d^2x \frac{1}{R^2} = 4\pi \times 1$$

seems trivial for a sphere, but still true for any genus-0 closed surface

- Integers
- invariant under smooth local deformations of the surface



genus 0

genus 1

genus 2

- This remarkable relation evolved through mathematical abstraction to the Chern classes, in particular

$$\int_{\mathcal{M}_2} dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x}) = 2\pi\mathbb{C}_1$$

integral over a closed orientable 2-manifold

"Chern number"

first Chern class (an integer) replaces Euler's characteristic

Berry curvature

mathematically, this is a "U(1) fiber bundle"

$$|\Psi(\boldsymbol{x})\rangle$$

quantum state that depends on a set of continuous parameters x

$$\left\langle \frac{\partial\Psi(x)}{\partial x^\mu} \middle| \frac{\partial\Psi(x)}{\partial x^\nu} \right\rangle - \left\langle \frac{\partial\Psi(x)}{\partial x^\nu} \middle| \frac{\partial\Psi(x)}{\partial x^\mu} \right\rangle = i\mathcal{F}_{\mu\nu}(x)$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

$$e^{i\oint_\Gamma dx^\mu \mathcal{A}_\mu} = e^{i\Phi_\Gamma} \longleftarrow$$ Berry phase of boundary

- on a 2-d manifold $\mathcal{M}$ with boundaries $\partial\mathcal{M}_i$

$$\boxed{\exp\left(i\int_{\mathcal{M}}\mathcal{F}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\right) = \prod_i \exp i\oint_{\partial\mathcal{M}_i}\mathcal{A}_{\mu}dx^{\mu}}$$

Integrated Berry curvature("flux") in interior

product of Berry phase factors on boundaries

"punctures"

Stokes theorem

The first Chern invariant.

- for a compact 2d manifold with no boundaries

$$\exp\left(i\int_{\mathcal{M}} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu\right) = 1$$

can take logarithm

Berry curvature

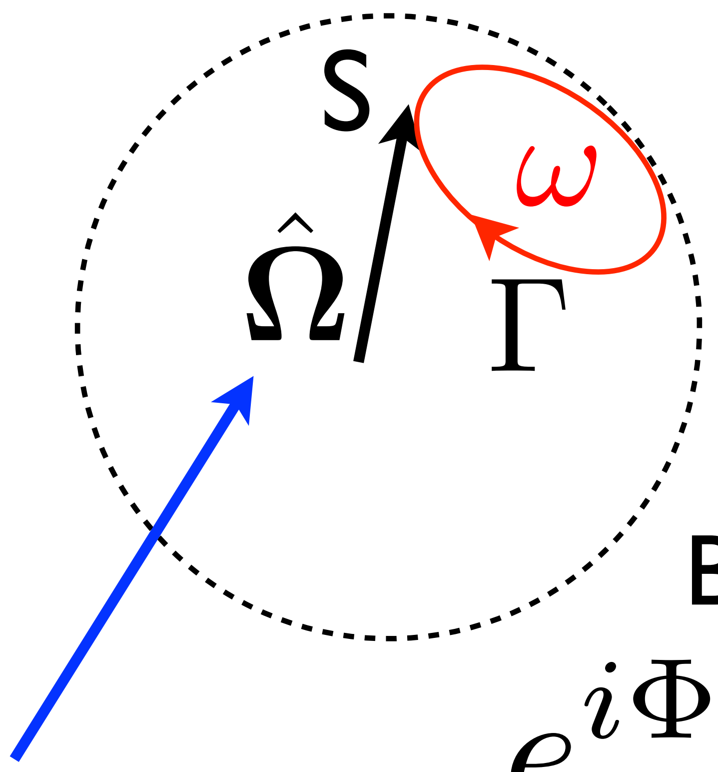
$$\int_{\mathcal{M}} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \equiv \int_{\mathcal{M}} d^2x \mathcal{F} = 2\pi C_1$$

integer "Chern number"

"first Chern class" topological invariant

- This topological invariant is central to systems with broken time-reversal symmetry (quantum Hall effect, Thouless et al. TKNN 1983, Simon 1983)

- The compact 2D manifold of the first Chern class is now a frequent ingredient in modern physics, and can occur in many different ways:



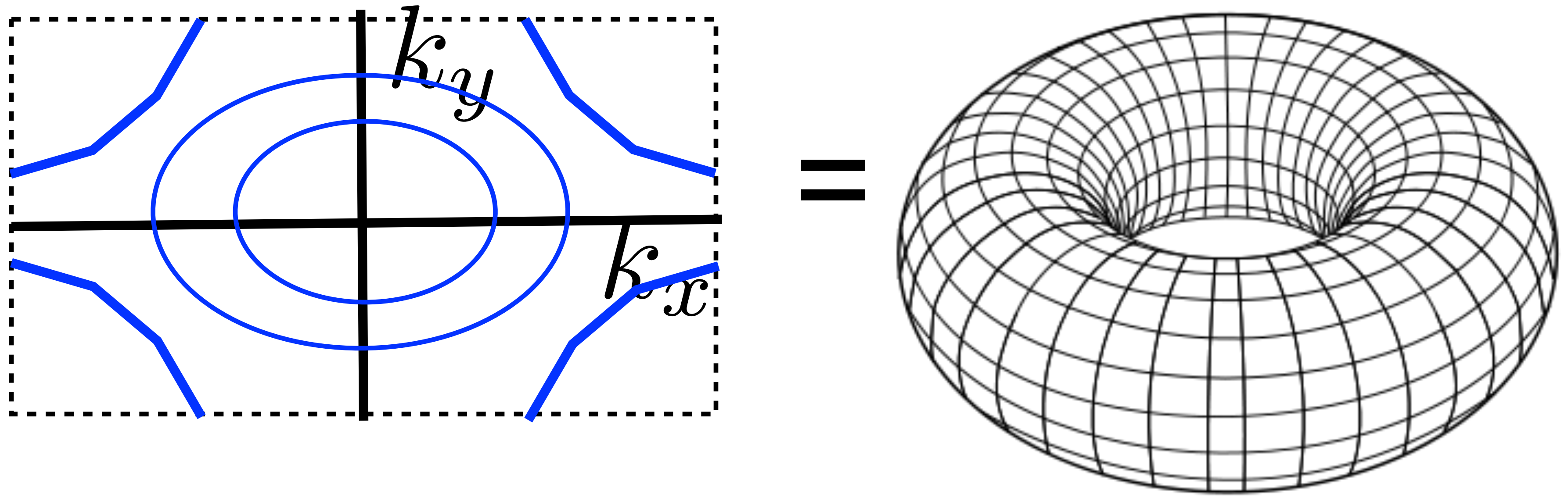
Berry phase:

$$e^{i\Phi_\Gamma} = e^{iS\omega}$$

Parameter space is unit sphere

solid angle enclosed is ambiguous modulo 4π
so 2S must be an integer

quantum spin
Chern number = 2S



electronic or photonic Bloch states in 2D Brillouin zone:

manifold is 2-torus

- The quantum state must be non-degenerate, so for the 2D bandstucture with spin-orbit coupling, either time-reversal or spatial inversion symmetry must be broken to get Berry curvature.

$$\boxed{\mathcal{F}_n(\boldsymbol{k}) = \mathcal{F}_n(-\boldsymbol{k})}$$

inversion

$$\boxed{\mathcal{F}_n(\boldsymbol{k}) = -\mathcal{F}_n(-\boldsymbol{k})}$$

Time-reversal

The Chern number vanishes unless time-reversal symmetry is broken

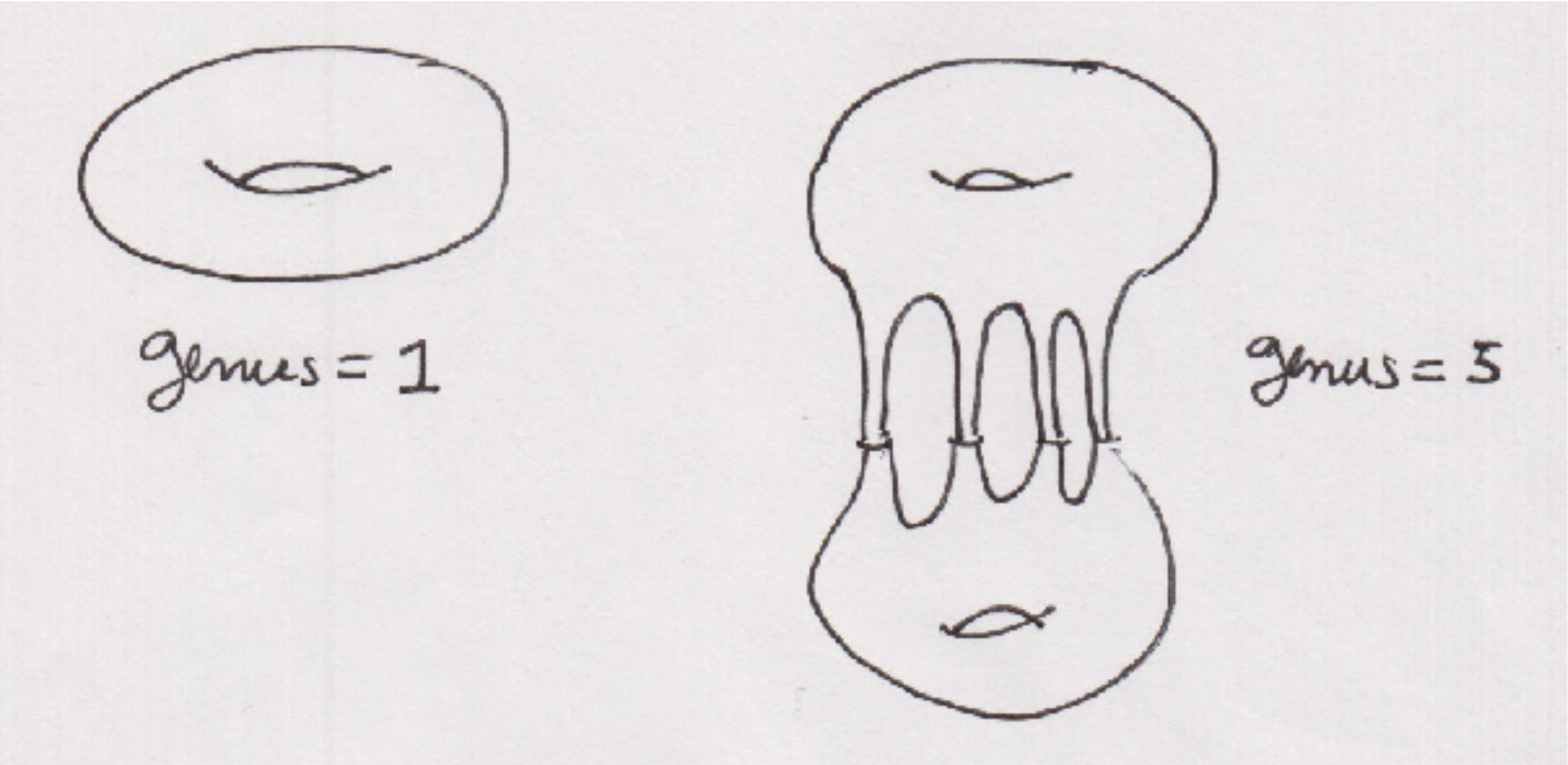
- Though it's not relevant for this talk, the recent "topological insulator" revolution started in 2005 when Kane and Mele discovered a new "Z_2" (as opposed to "U(1)") invariant in time-reversal-invariant 2D electronic bands with Kramers degeneracy.

- They first discovered the new invariant in systems with broken spatial inversion symmetry, when it can also be derived from the Berry curvature

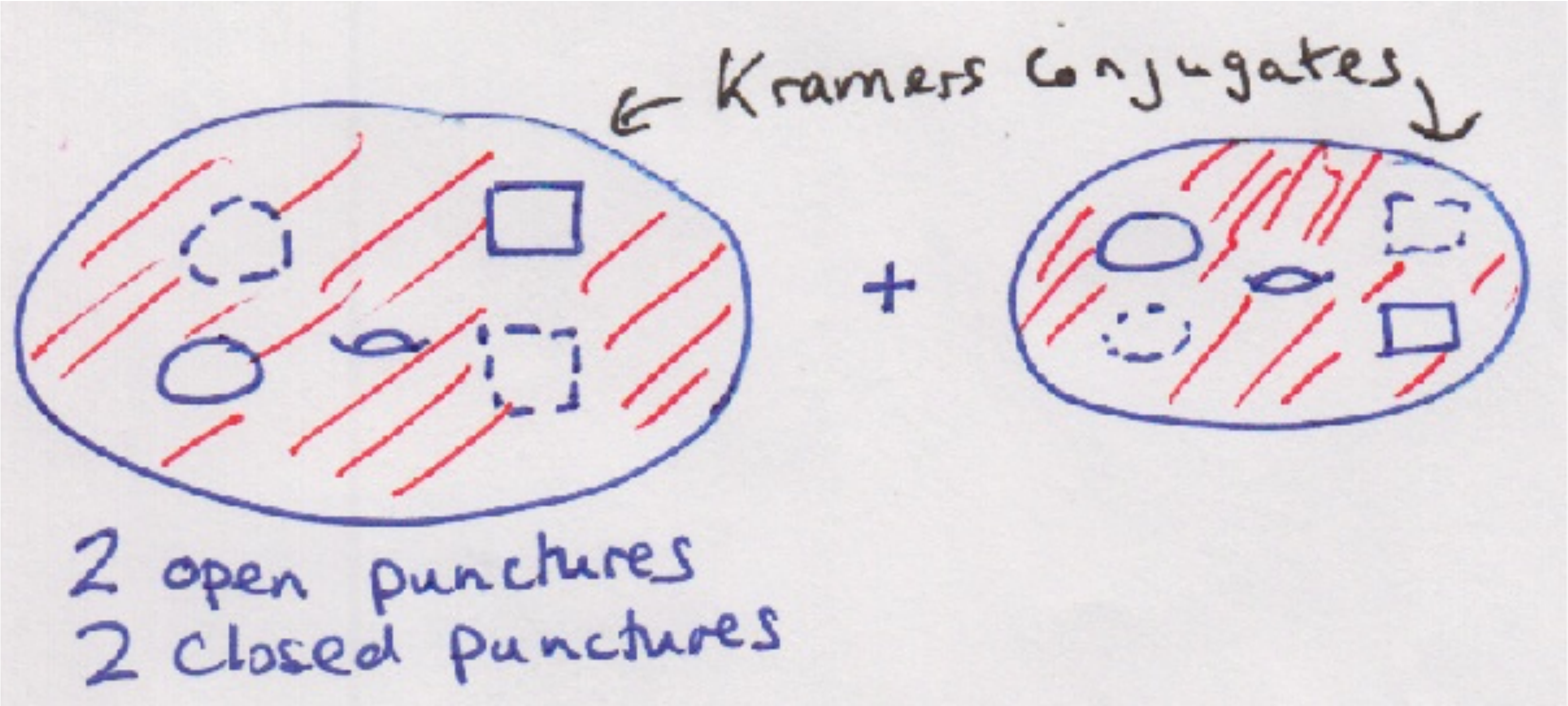
An explicitly gauge-invariant rederivation of the Kane-Mele Z_2 invariant

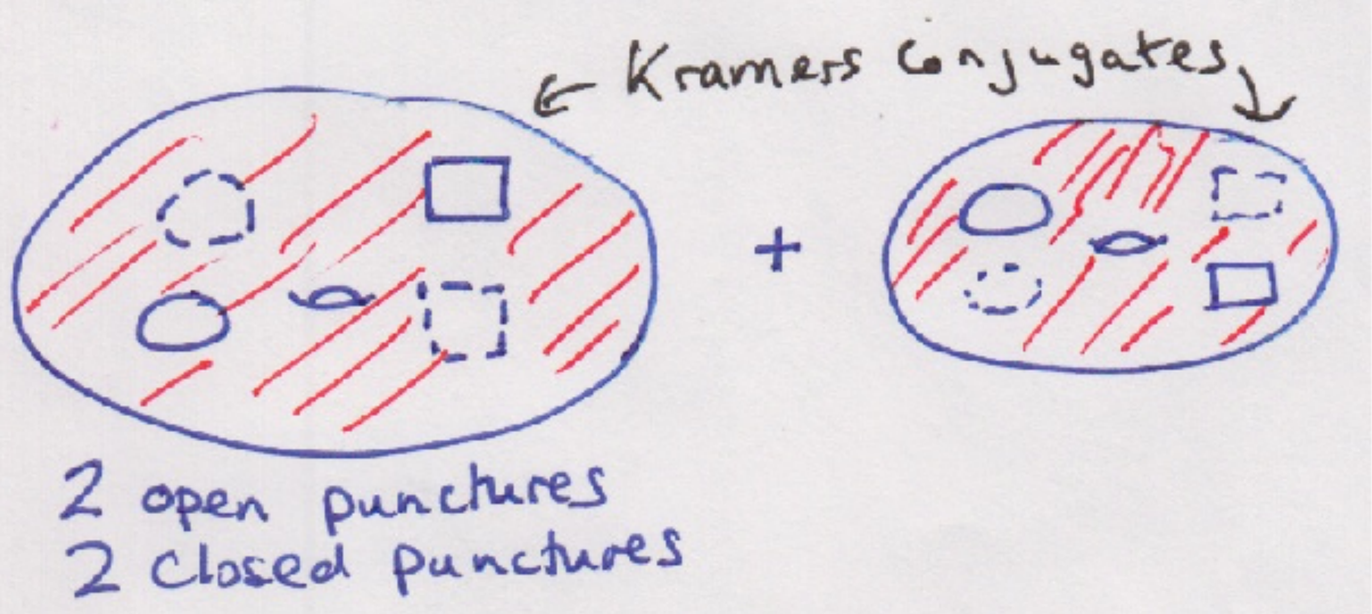
FDMH unpub.

- If inversion symmetry is absent, 2D bands with SOC split except at the four points where the Bloch vector is 1/2 x a reciprocal vector. The generic single genus-1 band becomes a pair of bands joined to form a genus-5 manifold



- This manifold can be cut into two Kramers conjugate parts, each is a torus with two pairs of matched punctures. In each pair, one puncture boundary is open one is closed.





- on a punctured 2-manifold

$$\exp i \int d^2\boldsymbol{k}\,\mathcal{F}^{12}(\boldsymbol{k}) = \prod_i e^{i\phi_i}$$

product of Berry phase-factors
of puncture boundaries

- in T-invariant electronic half-bands with SOC, punctures come in Kramers pairs:

$$\prod_{i=1}^{2n} e^{i\phi_i} = \left(\prod_{i=1}^{n} e^{i\phi_i}\right)^2$$

$$\boxed{\left(\exp i\frac{1}{2}\int d^2\boldsymbol{k}\,\mathcal{F}^{12}(\boldsymbol{k})\right)\prod_{i=1}^{n} e^{-i\phi_i} = \pm 1}$$

a perfect square, so we can take a square root!

- If inversion symmetry is present, the bands are unsplit and doubly-degenerate at all points in k-space, so the Berry curvature is undefined.
- For that case, Fu and Kane found a beautiful formula

$$\prod_{n} \prod_{k^*} I_{n,k^*} = \pm 1$$

= the Z_2 invariant

occupied bands

T+I-invariant k-points

Inversion quantum number ± 1 (about any inversion center)

(this changes sign at band-inversion transitions, as stressed by Bernevig, Hughes and Zhang)

- It may be useful to point out that the Berry curvature in k-space associated with Bloch states is slightly non-standard:

- The states from which the Berry curvature is obtained are **not** eigenstates of the Hamiltonian, but

$$|\Psi_n(\boldsymbol{k}, \{\boldsymbol{r}_i\})\rangle = U(-\boldsymbol{k}; \{\boldsymbol{r}_i\})|\boldsymbol{k}, n\rangle \qquad U(\boldsymbol{k}) = \sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}|i\rangle\langle i|$$

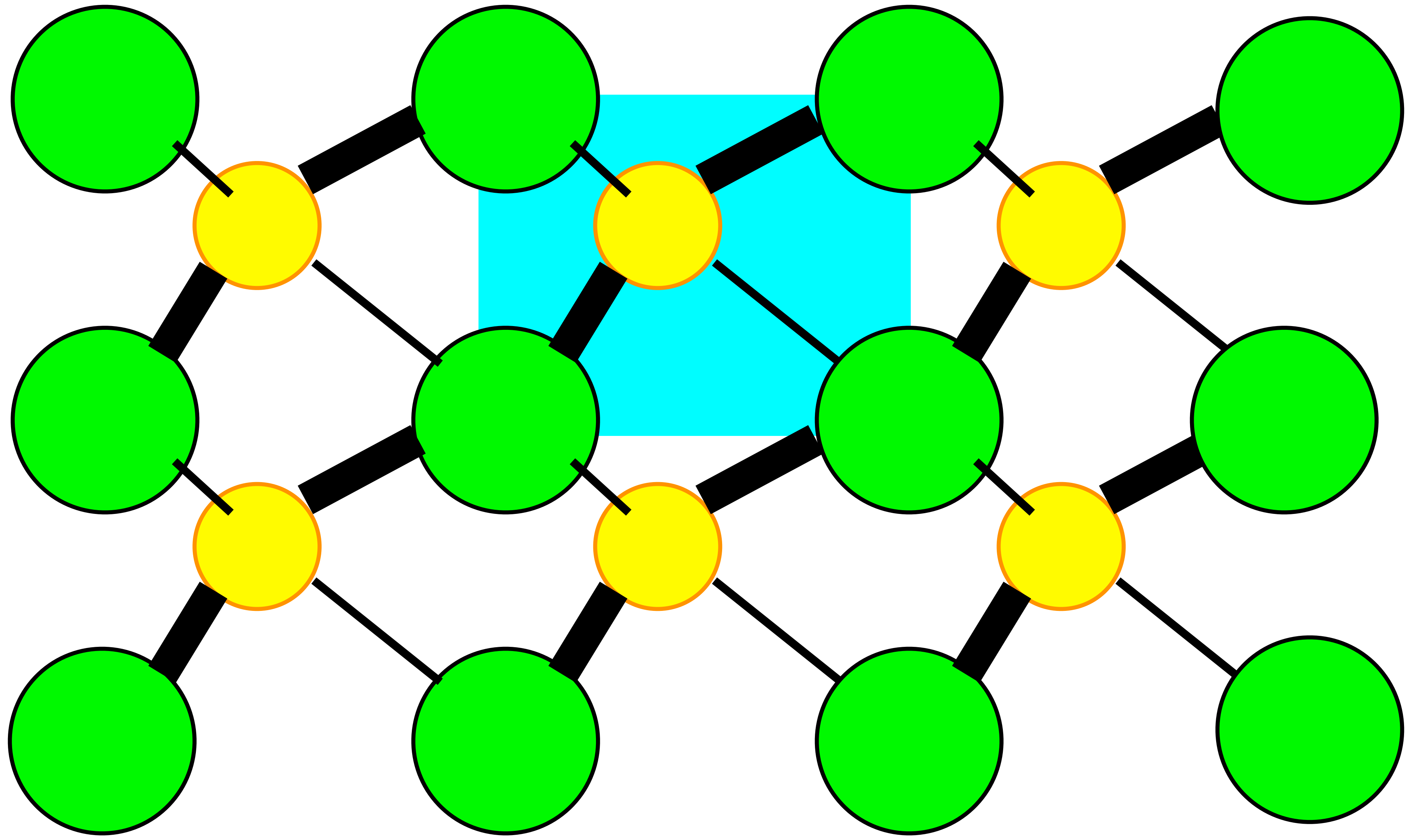
A periodic state that depends on the spatial embedding as well as k

The Bloch eigenstate, which is quasiperiodic, and independent of spatial embedding

Information on the embedding in Euclidean space

a basis of localized orbitals

- This extra feature becomes very clear in tight-binding models:



- the Bloch Hamiltonian only "knows" about the "hopping matrix elements" between orbitals, but **not** how the orbitals are embedded in space

- the Berry curvature in k-space of $|\Psi(\boldsymbol{k}, \{\boldsymbol{r}_i\})\rangle$ "knows" about the relative spatial locations of the orbitals, and allows the effect of perturbation by uniform electric and magnetic fields to be described

- theTopological invariants themselves do **not** depend on the geometry of the embedding

Semiclassical motion of a Bloch electron in weak quasi-uniform applied electromagnetic fields

$$H = \varepsilon_n(\boldsymbol{k}) - e\phi(\boldsymbol{r})$$

$$F_{ab}(\boldsymbol{r}) \equiv \epsilon_{abc}B^c(\boldsymbol{r})$$
$$E_a(\boldsymbol{r}) = -\nabla_a\phi(\boldsymbol{r})$$

$$\hbar\frac{d\boldsymbol{k}_a}{dt} = -e\left(\nabla_a\phi(\boldsymbol{r}) + F_{ab}(\boldsymbol{r})\frac{dr^a}{dt}\right)$$

Lorentz force

$$\frac{dr^a}{dt} = \frac{1}{\hbar}\nabla_k^a\varepsilon_n(\boldsymbol{k}) + \boxed{\mathcal{F}_n^{ab}(\boldsymbol{k})\frac{dk_b}{dt}}$$

group velocity + “anomalous velocity”

- Karplus and Luttinger (1954), Sundaram and Niu (1999)

full duality between r-space and k-space!

- The Karplus-Luttinger formula for the intrinsic band-structure component of the anomalous Hall effect of a 3D ferromagnetic metal is equivalent to

occupation number

$$\sigma_H^{ab} = \frac{e^2}{h}\left(\frac{1}{2\pi}\sum_n \int_{\mathrm{BZ}} d^3\boldsymbol{k}\, \mathcal{F}_n^{ab}(\boldsymbol{k}) n_n(\boldsymbol{k})\right)$$

- This is just the sum of the Berry curvature over all the occupied electron states in the band-structure, rediscovered by TKNN in the QHE.
- Only topological if all bands are completely filled or completely empty

Berry curvature symmary (generic)

classical parameter space

$x = \{x^\mu, \mu = 1,2,\ldots,d\}$

non-degenerate eigenstate $H(x)|\Psi_n(x)\rangle = E_n(x)|\Psi_n(x)\rangle$

expansion in fixed orthonormal basis $|\Psi_n(x)\rangle = \sum_n \psi_n(x)|n\rangle \qquad \langle n|n'\rangle = \delta_{nn'}$

$$|\partial_\mu \Psi_n(x)\rangle \equiv \sum_n \frac{\partial}{\partial x^\mu}\psi_n(x)|n\rangle$$

simple derivative in parameter space

$$|D_\mu \Psi_n(x)\rangle = |\partial_\mu \Psi_n(x)\rangle - |\Psi_n(x)\rangle\langle\Psi_n(x)|\partial_\mu \psi_n(x)\rangle$$

(Berry-gauge) covariant derivative

$$\langle\Psi_n(x)|D_\mu \Psi_n(x)\rangle = 0$$

$i\langle\Psi_n(x)|\partial_\mu \Psi_n(x)\rangle = \mathscr{A}^{(n)}_\mu(x)$ **Berry connection**

$\exp \gamma_C = \exp i\oint_C \mathscr{A}_\mu(x)dx^\mu$ **Berry phase**

$\partial_\mu A^{(n)}_\nu(x) - \partial_\nu A^{(n)}_\mu(x) = \mathscr{F}^{(n)}_{\mu\nu}(x)$ **Berry curvature**

$f^{(n)}_\mu = -\partial_\mu E_n(x) + \hbar\mathscr{F}^{(n)}_{\mu\nu}\dot{x}^\nu$ **quantum Lorentz force**

$\langle D_\mu \Psi_n(x)|D_\nu \Psi_n(x)\rangle \equiv \Gamma^{(n)}_{\mu\nu}(x)$ **quantum geometric tensor**

$$\Gamma^{(n)}_{\mu\nu}(x) = \mathscr{G}^{(n)}_{\mu\nu}(x) - \frac{1}{2}i\mathscr{F}^{(n)}_{\mu\nu}(x)$$

quantum metric (induced from Fubini-Study metric)

- quantum Lorentz force:

does work **does no work**

$$f_\mu^{(n)} = -\partial_\mu E_n(x) + \hbar \mathcal{F}_{\mu\nu}^{(n)} \dot{x}^\nu$$

- precise equivalent to

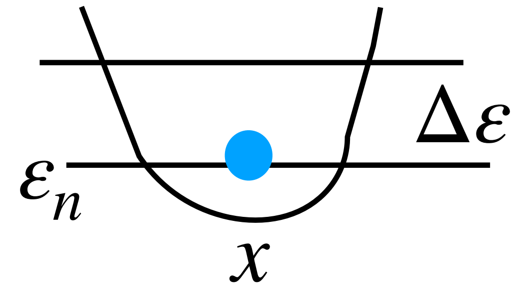
does work **does no work**

$$f_a = e(E_a + \epsilon_{abc}\dot{x}^b B^c) \;\; = e(E_a + F_{ab}\dot{x}^b)$$

- charge bound in movable potential at x (you control x and move it adiabatically by exerting a force on system)

$$\hbar \mathscr{A}_a = eA_a(x) \qquad E(x) = -eA_0(x) + \varepsilon_n$$

Berry connection



- quantum speed defined by quantum metric $\hbar\sqrt{(\mathscr{G}_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu)} \ll \Delta\varepsilon$

adiabatic motion speed limit

- FQHE occurs in "flat" Landau levels in a clean enough system so the repulsive two-body interaction dominates the (inhomogenous) one-body (potential) energy, but is small compared to the energy gaps separating partially-filled Landau levels from filled and empty ones

"bandwidth" (Landau level broadening)	**<<**	**interaction-energy**	**<<**	**"band gaps" (between Landau levels)**

- The same idea was applied to make "toy model" Bloch band "fractional Chern Insulator" systems in which exact numerical diagonalization revealed FQHE-like states

Neupert et. al and many others (Regnault, Sheng,....)

2023 update: NOW FOUND EXPERIMENTALLY!

pentalayer graphene arXiv:2309.17436 **$MoTe_2$**

- arXiv:2308.02657 (2023).
- Nature (2023), 10.1038/s41586-023-06452- 3.
- arXiv:2308.06177

- so there is a proof in principle that zero-field lattice systems can show Laughlin-like FQHE states.

- How does this fit in with the Laughlin picture of FQHE in a Landau level?

$$2\pi\ell_B^2 = \frac{B}{(h/e)}$$

$$\Psi \propto \prod_{i<j}(z_i - z_j)^m \prod_i e^{\frac{1}{4}|z_i|^2/\ell_B^2}$$

$$\boxed{z = x + iy}$$

- according to conventional wisdom the holomorphic structure of the Laughlin state has something to do with "being in the lowest Landau level": how can this translate to the lattice of a Chern insulator?

- In fact, the "holomorphic" structure of Laughlin and other "conformal block model wavefunctions" has **<u>nothing whatsoever to do with "being in the lowest Landau level"</u>**

- Instead, it derives from the non-commutative geometry of the "guiding centers" of Landau orbits, without any relation to the shape of those orbits around the center.

$$[R^x, R^y] = -i\ell_B^2$$

- thirty years after its experimental discovery and theoretical description in terms of the Laughlin state, the fractional quantum Hall effect remains a rich source of new ideas in condensed matter physics.
- The key concept is "**flux attachment**" that forms "**composite particles**" and leads to topological order.
- Recently, it has been realized that flux attachment also has interesting **geometric** properties

$$\Psi = \prod_{i<j}(z_i - z_j)^3 \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

Laughlin 1983

- elegant wavefunction, describes topologically-ordered fluid with fractional charge fractional statistics excitations

- exact ground state of modified model keeping only short range part of coulomb repulsion

- Validity confirmed by numerical exact diagonalization

30 years later:
unanswered question:
we know it works, but <u>why</u>?

my answer:
<u>hidden geometry</u>

some widespread misconceptions about the Laughlin state

- "it describes particles in the lowest Landau level"

No Landau level was specified: all specifics of the Landau level are hidden in the form of $U(\boldsymbol{r}_{12})$

- "It is a Schrödinger wavefunction"

Non-commutative geometry has no Schrödinger representation (this requires classical locality); it only has a Heisenberg representation.

- "Its shape is determined by the shape of the Landau orbit"

The interaction potential $U(\boldsymbol{r}_{12})$ determines its geometry (shape)

- "It has no continuously-tunable variational parameter"

Its geometry is a continuously-variable variational parameter

- In a 2D Landau level, we apparently start from a Schrödinger picture, but end with a "quantum geometry" which is no longer correctly described by Schrödinger wavefunctions in real space because of "quantum fuzziness" (non-locality)
- It remains correctly described by the Heisenberg formalism in Hilbert space.

- Landau quantization

$$\varepsilon(\boldsymbol{p})|\Psi_n\rangle = E_n|\Psi_n\rangle$$

discrete spectrum of macroscopically-degenerate Landau levels

- Project residual interaction in a single partially occupied "active" Landau level, all other dynamics is frozen by Pauli principle when gap between Landau levels dominates interaction potential

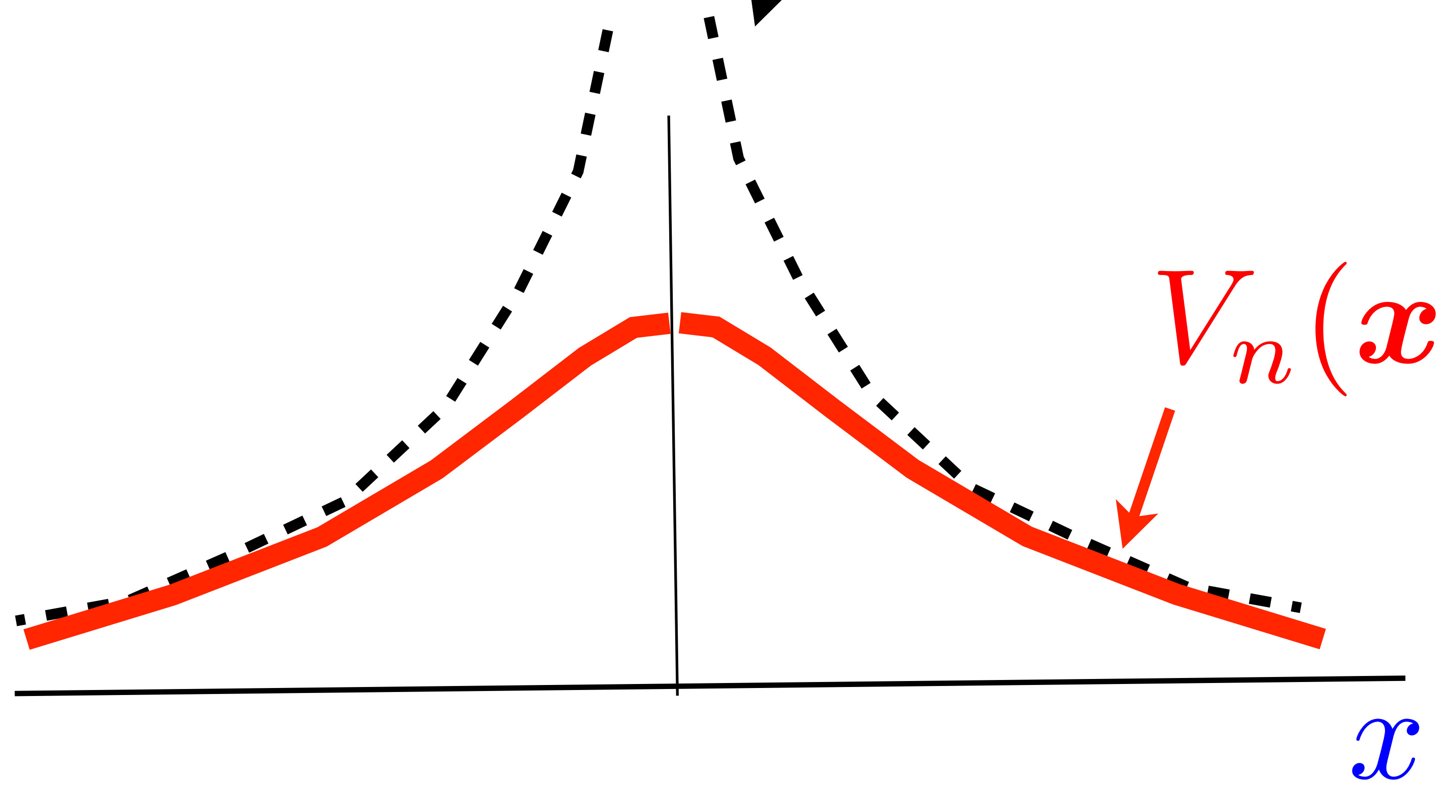
$$\boxed{\begin{aligned} [R^a, R^b] &= -i\ell_B^2\epsilon^{ab} \\ H &= \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j) \end{aligned}}$$

residual problem is non-commutative quantum geometry!

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

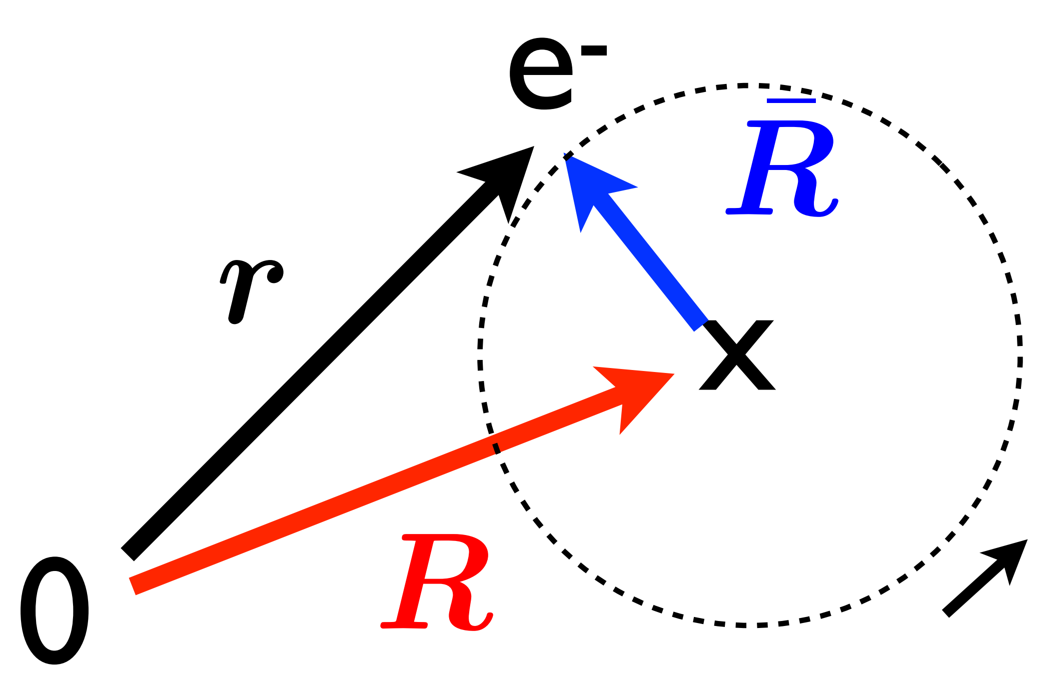
Identical quantum particles
(fermions or bosons)



We now have the final form of the problem:

- The potential $V_n(x)$ is a **very smooth** (in fact entire) function that depends on the form- factor of the partially-occupied Landau level
- The essential clean-limit symmetries are translation and inversion:

$$\boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$$



- Where did this come from?

$$p_a = -i\hbar\nabla_a - eA_a(\boldsymbol{x})$$

$$[p_x, p_y] = i\hbar e B$$

- Landau orbit radius vector

$$\bar{\boldsymbol{R}} = \frac{1}{eB}(p_y, -p_x)$$

- Landau orbit guiding center

$$\boldsymbol{R} = \boldsymbol{r} - \bar{\boldsymbol{R}}$$

$$\boldsymbol{r} = \boldsymbol{R} + \bar{\boldsymbol{R}} \quad [R^a, \bar{R}^b] = 0$$

$$[r^x, r^y] = 0$$

$$[\bar{R}^x, \bar{R}^y] = i\ell_B^2$$

$$[R^x, R^y] = -i\ell_B^2$$

after Landau-level quantization, only the guiding centers remain as dynamical variables

- Fundamental representions of the Heisenberg algebra are defined by **any** choice of a complex unit vector like

$$\boldsymbol{e} = \frac{1}{\sqrt{2}}(1, i) \quad \text{or} \quad \boldsymbol{e} = \frac{1}{\sqrt{10}}(5, 2+i) \qquad \boldsymbol{e}^* \times \boldsymbol{e} = i$$

$$\boxed{a^\dagger = \boldsymbol{e} \cdot \boldsymbol{R} \qquad [a, a^\dagger] = 1}$$

- this defines a (determinant 1) Euclidean-signature metric $g_{ab} = \frac{1}{2}(e_a^* e_b + e_b^* e_a)$

- the metric is a freely-choosable parameter of the representation.

- Any N=particle state has a representation

$$|\Psi\rangle = F(a_1^\dagger, a^\dagger, \ldots, a_N^\dagger)|0\rangle \qquad\qquad a_i|0\rangle = 0$$

A holomorphic function of N variable

- So we see that the "holomorphic" structure is a property of the non-commutative geometry of guiding center states after projection into a Landau level

- Where can we find non-commutative geometry on a lattice?

- A topologically-non-trivial bandstucture must have at least <u>two</u> orbitals in the unit cell, but if we project into that band, there is <u>only one independent state per unit cell</u>

- The overlap matrix between orbitals is then rank-deficient, with a kernel of null eigenvalues

$$\{c_i, c_j^\dagger\} = S_{ij} \quad = \langle i|P|j\rangle$$

Projection into band

Orbitals are renormalized after projection so that

$$\langle i|P|i\rangle = 1$$

$$\{c_i, c_i^\dagger\} = 1$$

$$\{c_i, c_j^\dagger\} = S_{ij}$$

- Because of this, an "onsite" Hamiltonian

$$H = \sum_i E_i n_i + \sum_{i<j} V_{ij} n_i n_j$$

will have non-trivial dynamics

- band topology is encoded in the complex phase of S_{ij}, and geometry in the quantum distance measure

$$d_{ij} = 1 - |S_{ij}|$$

- Sij define the fuzzy "quantum lattice" that generalizes the classical lattice S_{ij} = δ_{ij}

- A basis of orthogonal states of the projected band is obtained as the non-zero eigenstates of S

$$\sum_j S_{ij} u_{j\lambda} = s_\lambda u_{i\lambda} \qquad c_\lambda^\dagger = \frac{1}{\sqrt{s_\lambda}} \sum_i u_{i\lambda} c_i^\dagger$$

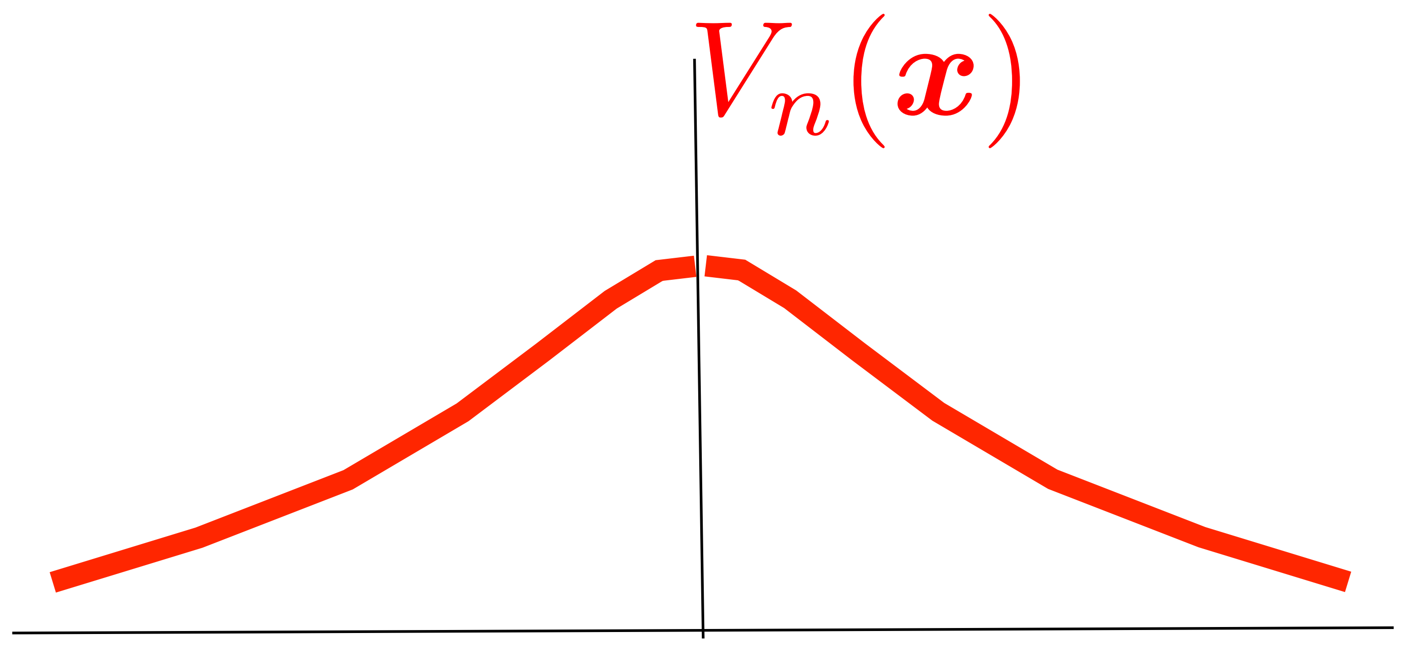
- For the basis of coherent (Gaussian) Landau level states, this leads to the holomorphic states, which are the non-zero eigenstates of

$$S(\boldsymbol{x}, \boldsymbol{x}') = e^{-\frac{1}{4}(z^* z - 2z^* z' + z'^* z')/\ell_B^2}$$

- the basic physics and energetics of the FQHE involves flux attachment
- What is missing so far is a detailed physical understanding of the energetics that drives the different way flux is attached in different FQH states

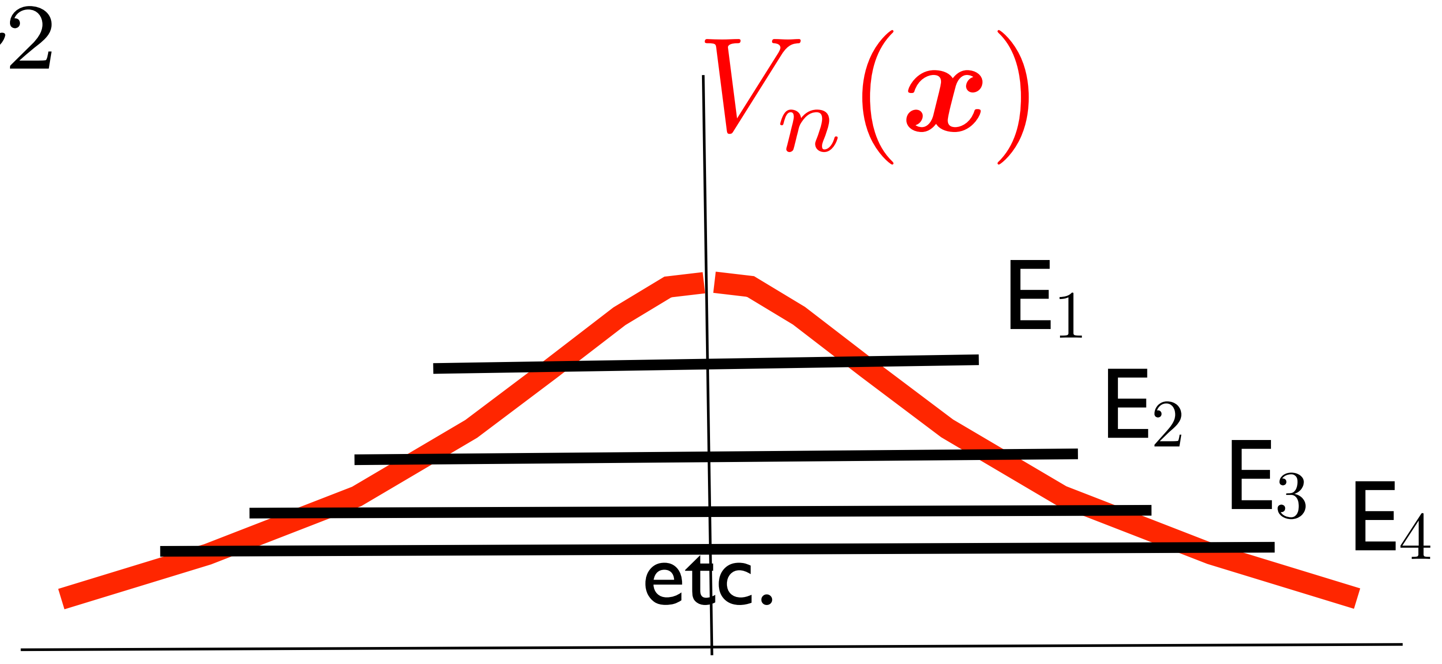
$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$



- The quadratic expansion of this even function around the origin defines a natural "interaction metric"
- The problem is often simplified by giving it a continuous rotation symmetry that respects this metric, but this is non-generic, and not necessary.
- This metric and a rotation symmetry are important in model FQH wavefunctions based on cft, which have a stronger conformal invariance property.

- It is straightforward to solve the two-body Hamiltonian: $\boldsymbol{R}_{12} = \boldsymbol{R}_1 - \boldsymbol{R}_2$



$$[R^a_{12}, R^b_{12}] = 2i\ell_B^2\epsilon^{ab}$$

equivalent to a one-particle problem

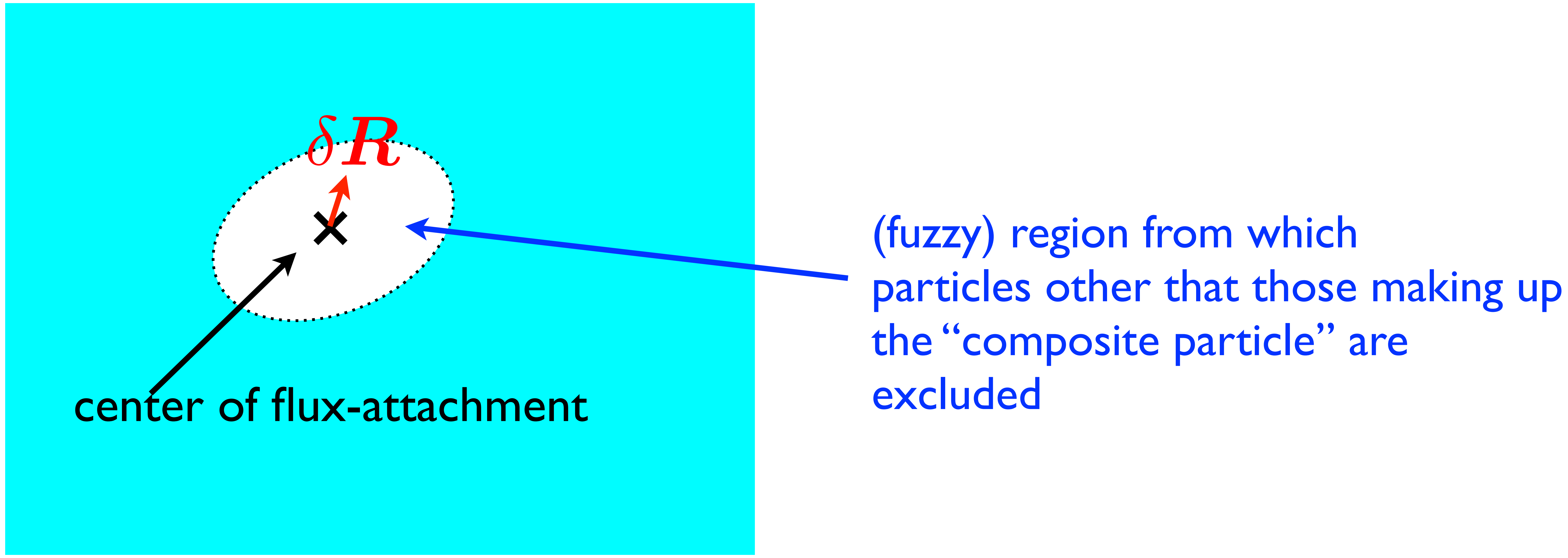
$$H = V_n(\boldsymbol{R}_{12})$$

- If there is a rotational symmetry, the energy levels (called "pseudopotentials") completely characterize the interaction potential.

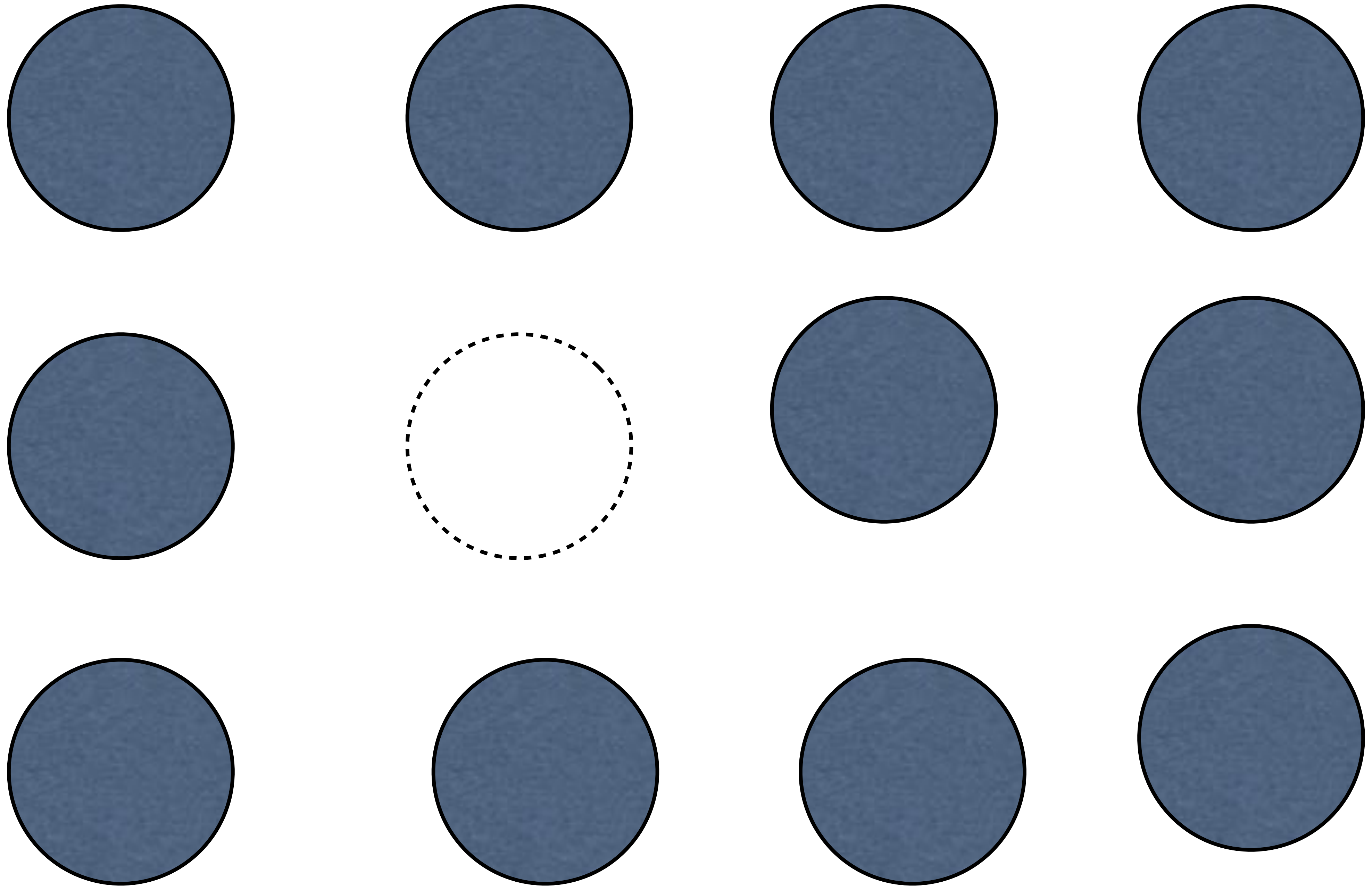
- a large gap between energy levels favors **flux attachment** with a shape close to that of the "interaction metric"

- Flux attachment is a gauge condensation that removes the gauge ambiguity of the guiding centers, giving each one a "natural" origin, so they define a physical electric dipole moment of the "composite particle" in which they are bound by the "attached flux".

- This is analogous to how the "the vector potential becomes an observable" (in a hand-waving way) in the London equations for a superconductor.



- quantum solid
- unit cell is correlation hole
- defines geometry



- repulsion of other particles make an attractive potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

- In Maxwell's equations, the momentum density is

$$\pi_i = \epsilon_{ijk} D^j B_k \qquad D^i = \epsilon_0 \delta^{ij} E_j + P^i$$

- The momentum of the condensed matter is

$$\boldsymbol{p} = \boldsymbol{d} \times \boldsymbol{B}$$

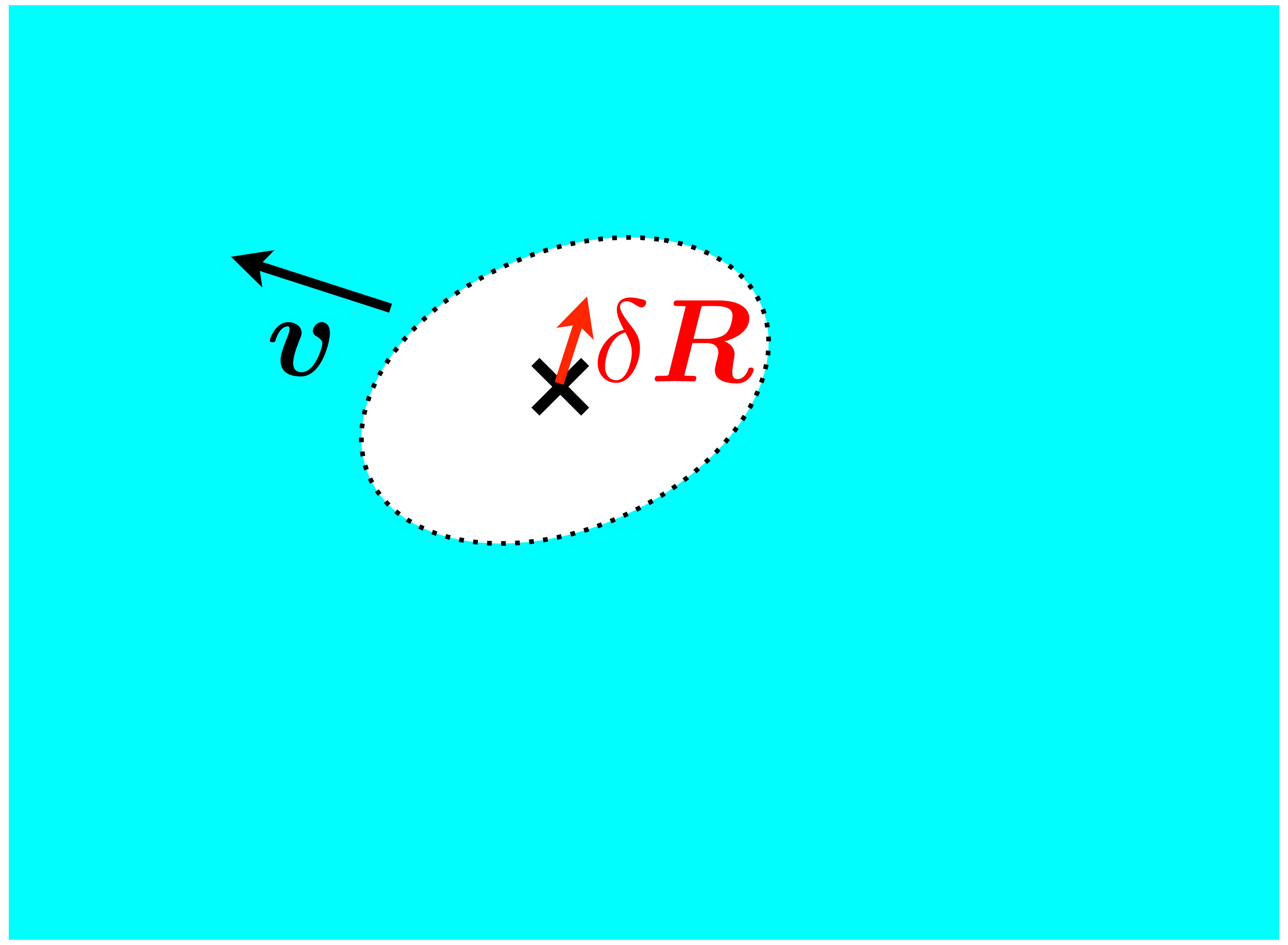
electric dipole moment

- in 2D the guiding-center momentum then is

$$p_a = eB\epsilon_{ab}\delta R^b$$

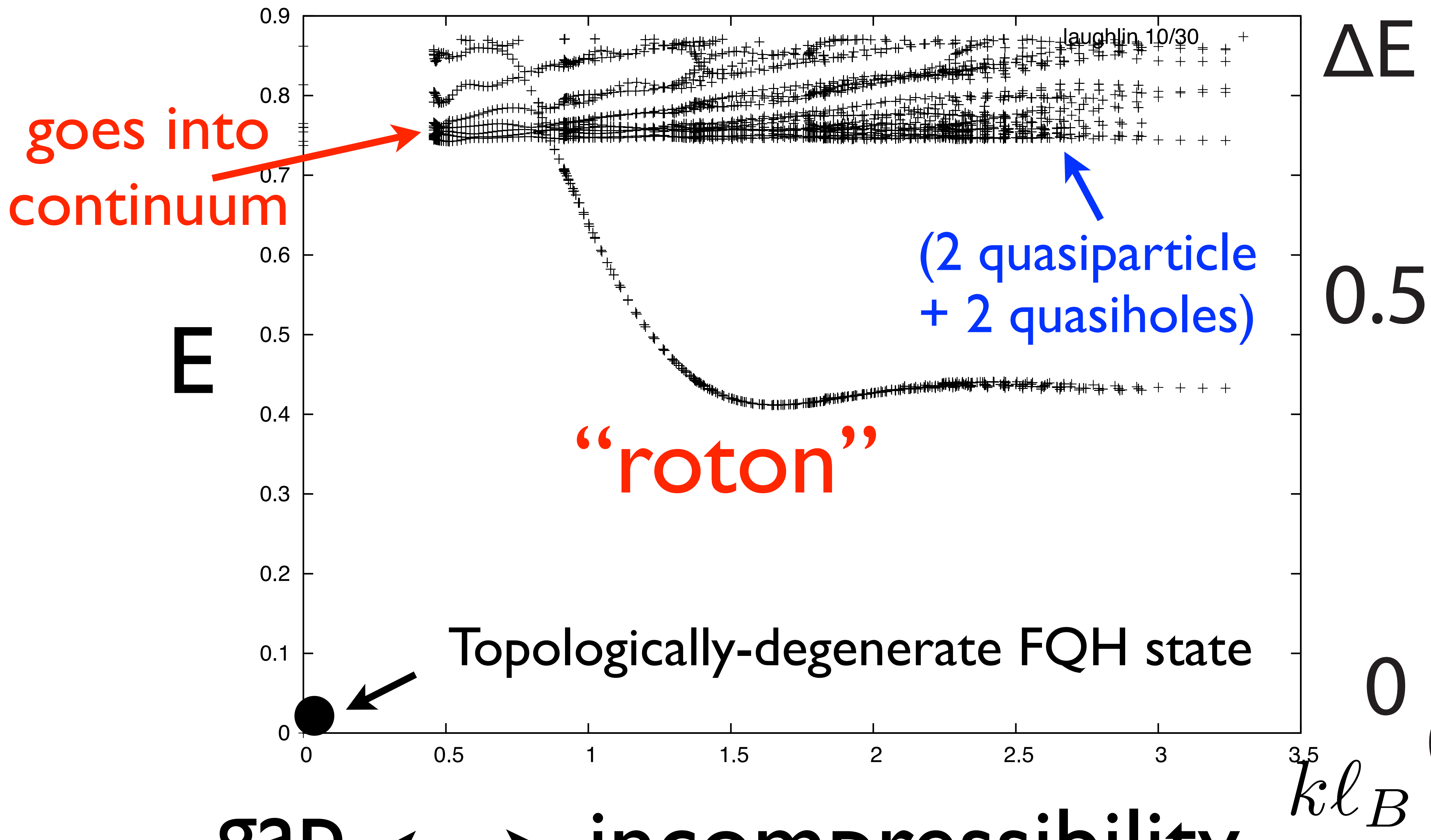
- The electrical polarization energy of the dielectric composite particle then gives its energy-momentum dispersion relation, with no involvement of any "Newtonian inertia" involving an effective mass

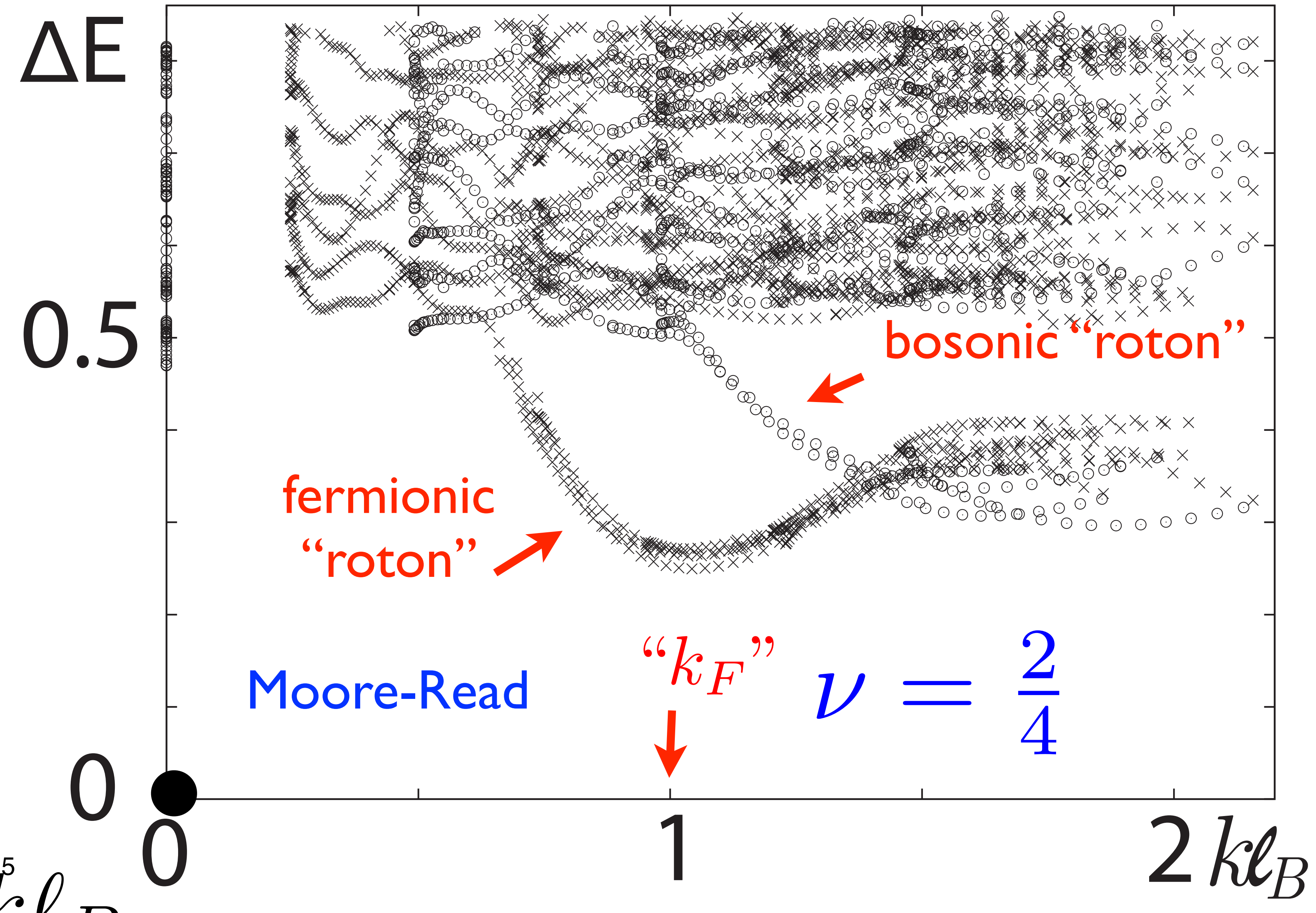
- The Berry phase generated by motion of the "other particles" that "get out of the way" as the vortex-like "flux-attachment" (orbital-attachment) moves with the particle(s) it encloses can be formally-described as a Chern-Simons gauge field that cancels the Bohm-Aharonov phase, so that the composite object propagates like a neutral particle.



- If the the composite particle is a boson, it condenses into the zero-momentum **(zero electric dipole-moment)** inversion-symmetric state, giving an incompressible-fluid Fractional Quantum Hall state, with an energy gap for excitations that carry momentum or electric dipole moment ("quantum incompressibility", no transmission of pressure through the bulk).

- All FQH states have an elementary unit (analogous to the unit cell of a crystal) that is a composite boson under exchange.
- It may be sometimes be useful to describe this boson as a a bound state of composite fermions (with their own preexisting flux attachment) bound by extra flux (Jain's picture)





gap ⟷ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment p$_e$**

$$\hbar k_a = \epsilon_{ab} B p_e^b$$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: fluid **does not transmit pressure through bulk!**

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"
- This is the analog of a unit cell in a solid....

- The 2D in-plane quadrupole density of the QHE fluid is important because the fluid ***rigidly*** has no electric polarization tangent to the Hall plane when it is not flowing. (There is a gap for excitations carrying electric dipole moment)

- The model for the FQHE in a single partially-occupied Landau level is

$$H = \sum_{i<j} V(\mathbf{R}_i - \mathbf{R}_j) \qquad [R_i^a, R_j^b] = -i(\hbar/eB)\epsilon^{ab}\delta_{ij} \qquad F_{ab} = B\epsilon_{ab}$$

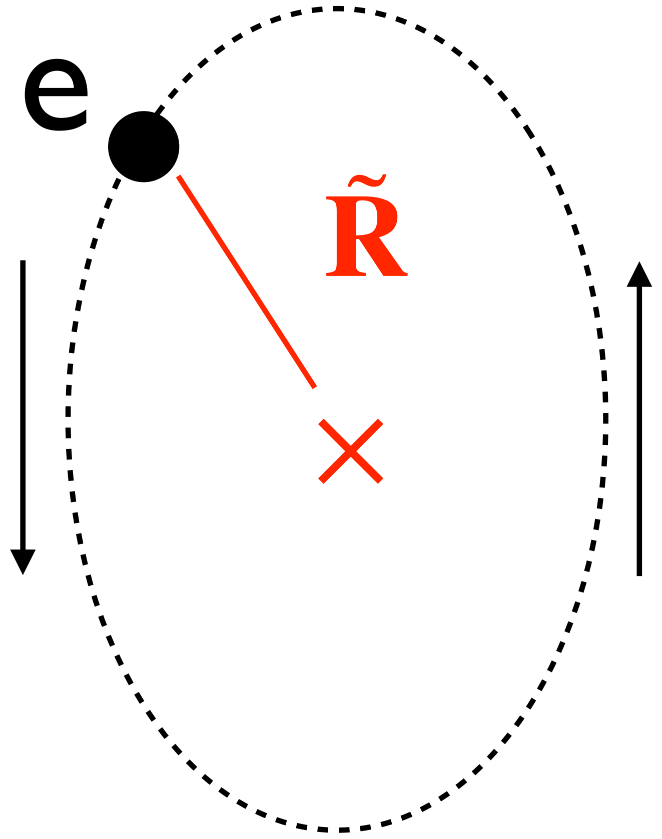
guiding centers

- This naturally has translation and 2D inversion symmetry ($R_i^a \mapsto -R_i^a$) but not unphysical SO(2) continuous rotational symmetry, which is an extra "toy model" feature used by most authors.

- The use of "extra symmetries" such as SO(2) is "dangerous" as it allows irrelevant features of the extra symmetry to be confused with generic features of the problem. However 2D inversion symmetry is a "natural" symmetry to keep, giving the composite boson a parity but not an angular momentum.

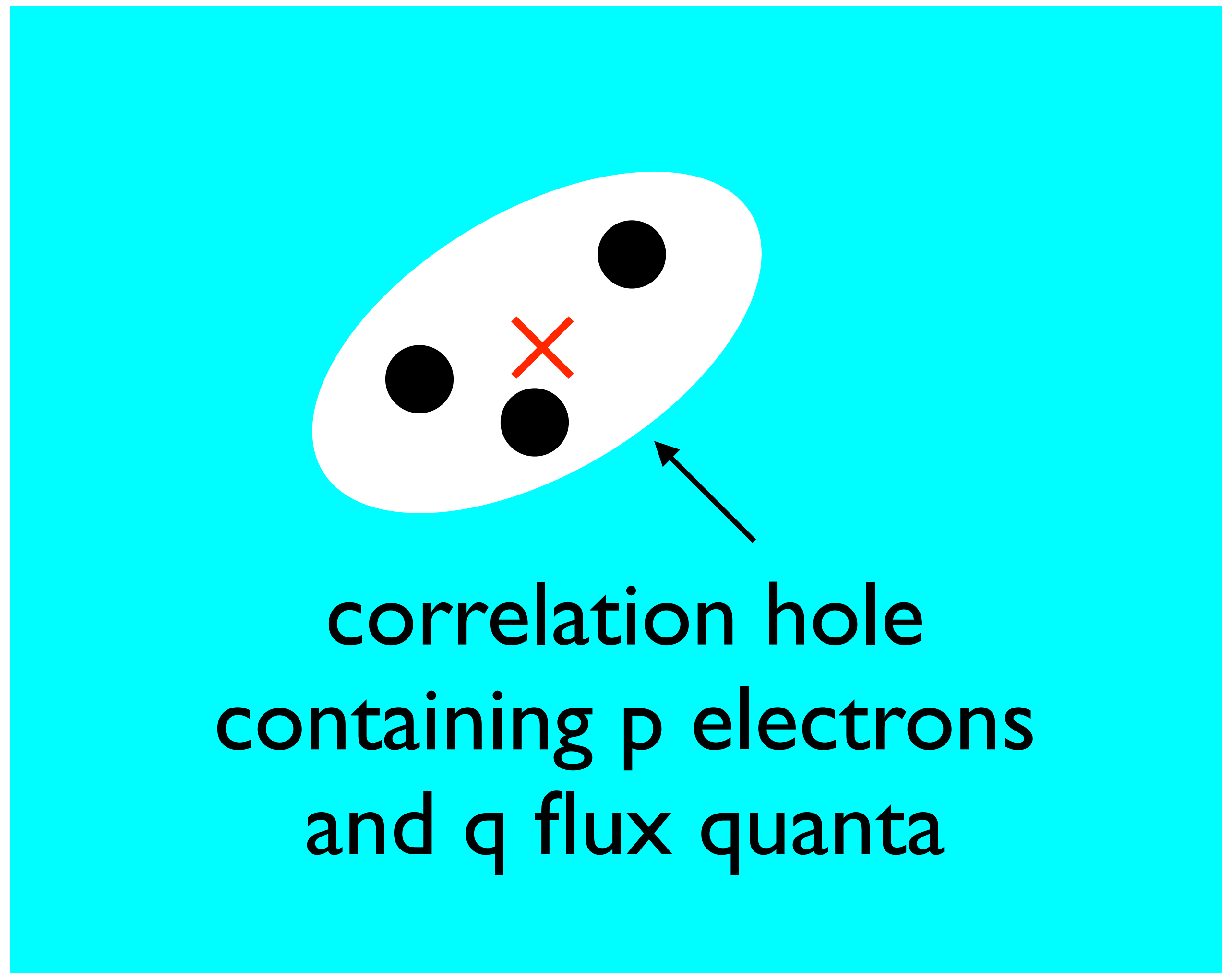
- quadrupoles in the QHE are positive- or negative-definite symmetric contravariant tensors.
- There are two type of quadrupoles:

  - Landau orbits have a (static) electric quadrupole moment relative to the guiding center



by definition, the (static) guiding center is the time-averaged center of the orbit

  - In the FQHE, "flux attachment" to form "composite bosons" also generates a (dynamic) "guiding-center" quadrupole moment



- "Berry curvature of Bloch states" is NOT a "standard" Berry curvature, but enters in the semiclassical dynamics of a Bloch electron wavepacket subjected to electromagnetic fields

electric field

Faraday tensor

electromagnetic vector potential

$$\hbar \dot{k}_a = e(E_a(\mathbf{x}) + F_{ab}(\mathbf{x})\dot{x}^b)$$

$$F_{ab}(\mathbf{x}) \equiv \frac{\partial}{\partial x^a} A_b(\mathbf{x}) - \frac{\partial}{\partial x^b} A_a(\mathbf{x})$$

$$\dot{x}^a = v^a(\mathbf{k}) - \mathscr{F}^{ab}(\mathbf{k})\dot{k}_b$$

$$\mathscr{F}^{ab}(\mathbf{k}) \equiv \frac{\partial}{\partial k_a} \mathscr{A}^b(\mathbf{k}) - \frac{\partial}{\partial k_b} \mathscr{A}^a(\mathbf{k})$$

Group velocity

Berry curvature

Berry connection

- The wavepacket has a Gaussian form centered at $\mathbf{x}$ in Euclidean space and at $\mathbf{k}$ in Bloch space (Brillouin zone). The Bloch band is non-degenerate at this $\mathbf{k}$

- Luttinger discovered this in 1957 (no-one believed him!)

$$\dot{k}_b = (e/\hbar)E_b$$

$$\dot{x}^a = v^a(\mathbf{k}) - \mathscr{F}^{ab}(\mathbf{k})\dot{k}_b$$

"anomalous velocity"

"current density"

$$\frac{e^2}{\hbar}\left(\int_{\mathrm{BZ}} \frac{d^dk}{(2\pi)^d} \sum_n \mathscr{F}_n^{ab}(k) n_n(k)\right) E_b$$

$$J^a = e\sum_{nk} v_n(k)^a n_{nk} - \frac{e^2}{\hbar}\left(\sum_{nk} \mathscr{F}_n^{ab}(k) n_{nk}\right) E_b$$

- In 2D when all bands are either filled or empty,

$$\sigma_H^{ab} = \frac{e^2}{2\pi\hbar}\left(\frac{1}{2\pi}\int_{BZ} d^2k \sum_n n_n \mathscr{F}_n^{ab}(k)\right)$$

Integer Chern invariant (TKNN)

- In an isolated (everywhere non-degenerate) two-dimensional band, the Berry curvature obeys a sum rule

$$\mathscr{F}_n(\mathbf{k}) = \frac{1}{2}\epsilon_{ab}\mathscr{F}_n^{ab}(\mathbf{k})$$

$$\frac{1}{2\pi}\int_{\mathrm{BZ}} d^2k\, \mathscr{F}_n(\mathbf{k}) = C_n \longleftarrow$$

Chern invariant of the band (integer)

- This is a sum rule, **<u>not</u>** a definition of C_n, which is defined independently of the Berry curvature.

- The orientation-independent integer-valued antisymmetric **<u>"Chern tensor</u>"** $C_n^{ab} = C_n\epsilon^{ab}$ **<u>is</u>** a property of the Bloch Hamiltonian H_0, and can be determined by examining its edge states, without reference to the embedding-dependent k-space Berry curvature.

- For a Bloch band to exhibit Berry curvature, there must be $n > 1$ sublattices so there are n orbitals in the unit cell (tight-binding picture)

- Let $\rho_i(m, \mathbf{k})$ be the weight of Bloch state $\mathbf{k}$ of band m on sublattice i

$$\rho_i(m, \mathbf{k} + \mathbf{G}) = \rho_i(m, \mathbf{k}) \qquad \sum_i \rho_i(m, \mathbf{k}) = 1$$

- Let the embeddings of each orbital on sublattice i be changed by $\delta \mathbf{x}_i$. The change in Berry curvature of band m is

$$\delta \mathcal{F}_m^{ab}(\mathbf{k}) = \sum_i \delta x_i^c \left(\delta_c^a \frac{\partial}{\partial k_b} - \delta_c^b \frac{\partial}{\partial k_a} \right) \rho_i(m, \mathbf{k})$$

- This leaves the Chern invariant unchanged but moves Berry curvature around in the Brillouin zone

- does the Berry curvature on a 2D compact manifold DEFINE its Chern invariant????

$$\mathscr{F}_{\mu\nu}(\mathbf{x}) = -i\langle\partial_\mu\Psi(\mathbf{x})|\partial_\nu\Psi(\mathbf{x})\rangle - (\mu \leftrightarrow \nu)$$

$$\int_S dx^\mu \wedge dx^\nu \mathscr{F}_{\mu\nu}(\mathbf{x}) = 2\pi C$$

- NO: this is just a sum rule that any non-singular Berry curvature on the manifold must satisfy.

- **The Chern number of a 2D Bloch band is always well-defined, but its Berry curvature is not defined until its embedding in the background Euclidean space (with supports the electromagnetic field) is specified.**

- Integer QHE seen in 2D Slater-Determinant filled-band systems with a Streda anomaly (Landau levels, Chern insulators)
- Fractional QHE seen in flat-band systems dominated by short-distance Coulomb repulsion. Incompressible states are due to Coulomb repulsion, are NOT Slater-determinant states.

Laughlin 1/3 state in Landau level (first-quantized form, second quantization is not useful!)

$$|\Psi_L^{1/3}(g)\rangle = \prod_{i<j}(a_i^\dagger - a_j^\dagger)^3|0\rangle \qquad a_i|0\rangle = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

a Euclidean signature metric, $\det g = 1$

$$\boxed{\begin{array}{l} R^a = P_n x^a P_n \quad \text{projection of particle coordinate into a Landau level} \\ [R^a, R^b] = -i(\hbar/eB)\epsilon^{ab} \\ F_{ab} = B\epsilon_{ab} \quad \sqrt{\det(eF/\hbar)} = \ell^{-2} \end{array}}$$

$$a_i^\dagger = \frac{\omega_a R_i^a}{\sqrt{2}\ell}$$

$$-ieF_{ab}\omega^b = (\hbar/\ell^2)g_{ab}\omega^b$$

$$g_{ab}\omega^{*a}\omega^b = 1$$

$$g_{ab} = \tfrac{1}{2}(\omega_a^*\omega_b + \omega_b^*\omega_a)$$

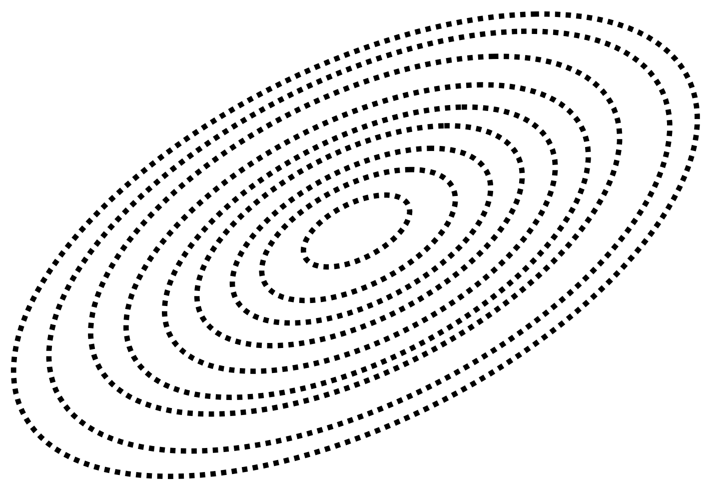
$$ieF_{ab} = \tfrac{1}{2}(\hbar/\ell^2)(\omega_a^*\omega_b - \omega_b^*\omega_a)$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow a^\dagger = \frac{(X_i + iY_i)}{\sqrt{2}\ell} \qquad g = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \longrightarrow a^\dagger = \frac{(3X_i + i2Y_i)}{\sqrt{12}\ell}$$

- The Laughlin state $|\Psi_L^{1/m}(g)\rangle$ has a simple form in the single-particle basis defined by its metric g_{ab}:

$$\tfrac{1}{2}g_{ab}R^aR^b|m\rangle = (m+\tfrac{1}{2})\ell^2|m\rangle$$

$$m = 0,1,2,\ldots$$



- This basis divides the 2D plane into a concentric system of elliptical "onion ring" annuli. Each annulus covers an area $2\pi\ell^2$ through which one quantum $\Phi_0 = h/e$ of magnetic flux passes, and supports one single-particle state which falls off exponentially outside it.

- The central state $m = 0$ is a minimum-uncertainty Gaussian coherent state satisfying

$$\langle\Psi|\tfrac{1}{2}\{R^a,R^b\}|\Psi\rangle - \langle\Psi|R^a|\Psi\rangle\langle\Psi|R^b|\Psi\rangle = \tfrac{1}{4}g^{ab}\ell^2$$

- before moving on to Bloch systems, there is one interesting observation to make about the maximally-localized coherent states $|\Psi_n(\mathbf{x}, g)\rangle$ in Landau levels:
- Their localization around their center falls off as a Gaussian: **i.e., more rapidly than any exponential: "hyperlocalized"**
- This is atypical, and stems from an underlying holomorphic structure not present in lattice models.

The Landau-level system can be regarded as the limit in which the generic exponential decay length goes to zero

- Numerical studies of the (spin-polarized) flat-band model show that, for some values of the parameters, a zero-field fractional anomalous Hall effect occurs at 1/3 band filling in a model

Projection into Bloch band n

$$P_n \left(\sum_{ij} t_{ij} c_i^{\dagger} c_j + \sum_{ij} V_{ij} n_i n_j \right) P_n$$

Chern band

short-distance (nearest-neighbor) gauge-invariant repulsive interaction

- It appears necessary that the band is a "Chern band" with a non-zero Chern index (i.e., a Streda anomaly).

- Prior to the FCI discovery, our understanding of FQHE has been Landau-level based

- The numerical observation of the FCI state in “toy models” followed by the recent experimental observation in Moiré flat-band systems provides an opportunity to reassess FQHE theory, and drop Landau-level-specific features such as holomorphic functions and rotational symmetries which have NO PLACE in an extension of FQHE theory to the FCI state.

- The question is not “how can a flat band Bloch system mimic a Landau level”,

- but instead: “what are the common features between Landau level FQHE systems and Bloch FCI systems”

- Various authors have suggested that the k-space distribution in the Brillouin zone of single particle Berry curvature $\mathscr{F}_n^{ab}(\mathbf{k})$ or the quantum geometric tensor $\Gamma_n^{ab}(\mathbf{k})$ may play a role in the explanation.

- Based on the existence of FCI behavior in the toy models, these can be immediately ruled out as conceptually invalid ideas: The toy models are network models with no Euclidean-space embedding to define a physically-meaningful k-space Berry curvature

k-space behaviors proposed to give rise to FCI:

~~quasi-uniform Berry curvature~~

$$\mathscr{F}_n^{ab}(\mathbf{k}) = \mathscr{F}_n(\mathbf{k})\epsilon^{\mathbf{ab}}$$

$F_n(\mathbf{k}) \approx$ **constant**

~~“vortexability”~~

$$\Gamma_n^{ab}(\mathbf{k}) = |\mathscr{F}_n(\mathbf{k})|(g_n^{ab}(\mathbf{k}) + is\epsilon^{ab})$$

$g_n^{ab}(\mathbf{k}) \approx$ **constant** $s = \pm 1, \det|g| = 1$

nearest-neighbor
interactions
(in local basis)

- After projection into the narrow band, this model with nearest-neighbor repulsion appears to show FCI behavior at 1/3 filling of the narrow band!

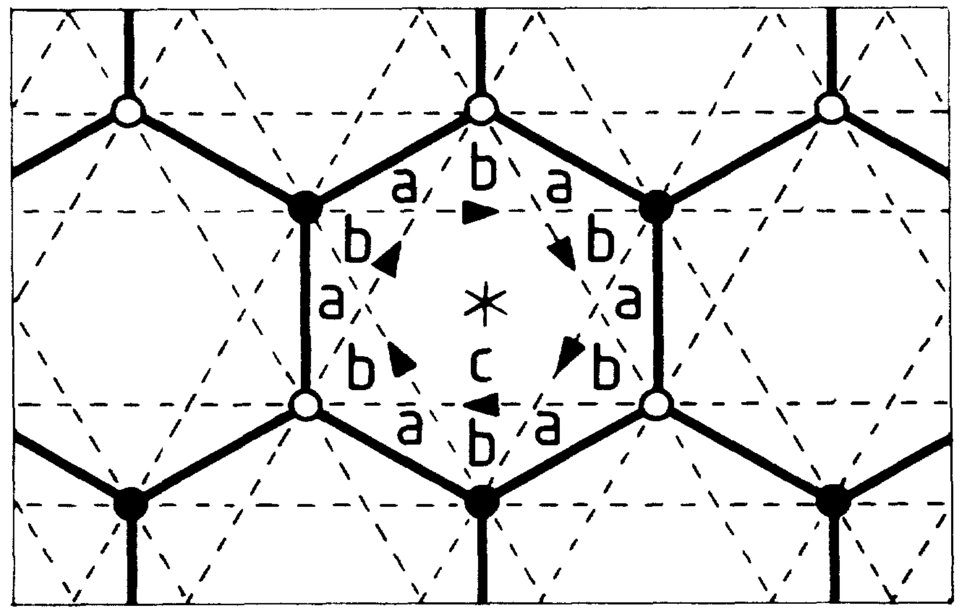
$$H = H_1 + H_2 \qquad H_2 = \sum_{\langle i,j \rangle} V n_i n_j$$

- Proposed way to study this: first understand when and why FCI behavior occurs in the band-projected model

$$H_0 = P H_1 P$$

- Next examine stability of FCI behavior with respect to one-particle dispersion

$$H = H_0 + P H_1 P$$



Model is purely a
network model

No embedding specified

FCI occurs in absence
of EM fields

No "Berry curvature" or
"quantum geometric tensor"
in Brillouin zone is defined for
this model that exhibits FCI

- Real-space quantum Geometry of Bloch bands without Euclidean embedding:

Local real-space basis

$$|R, i\rangle$$

unit cell label **sublattice label**

$$\langle R, i | R', j\rangle = \delta_{RR'}\delta_{ij}$$

orthonormal

$$H = \sum_{ij} t_{ij}(R - R') | R, i\rangle\langle R', j |$$

Bloch Hamiltonian

$$T(R) = \sum_{R',i} | R' + R, i\rangle\langle R' i |$$

Lattice translation operator

$$P_n = \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} | k, n\rangle\langle k, n |$$

Band projection operator

$$P_n P_{n'} = \delta_{nn'} P_n$$

- projected local basis (overcomplete, non-orthogonal)

$$|\Psi_n(R,i)\rangle = P_n|R,i\rangle$$

$$\langle\Psi_n(R,i)|\Psi_n(R',j)\rangle = S_{ij}^{(n)}(R-R')$$

- The fundamental quantum-geometry property:

$$\langle R,i|P_n|R',j\rangle = S_{ij}^{(n)}(R-R')$$

falls off exponentially for large |R-R'|:
(decay length diverges at transitions where Chern index changes)

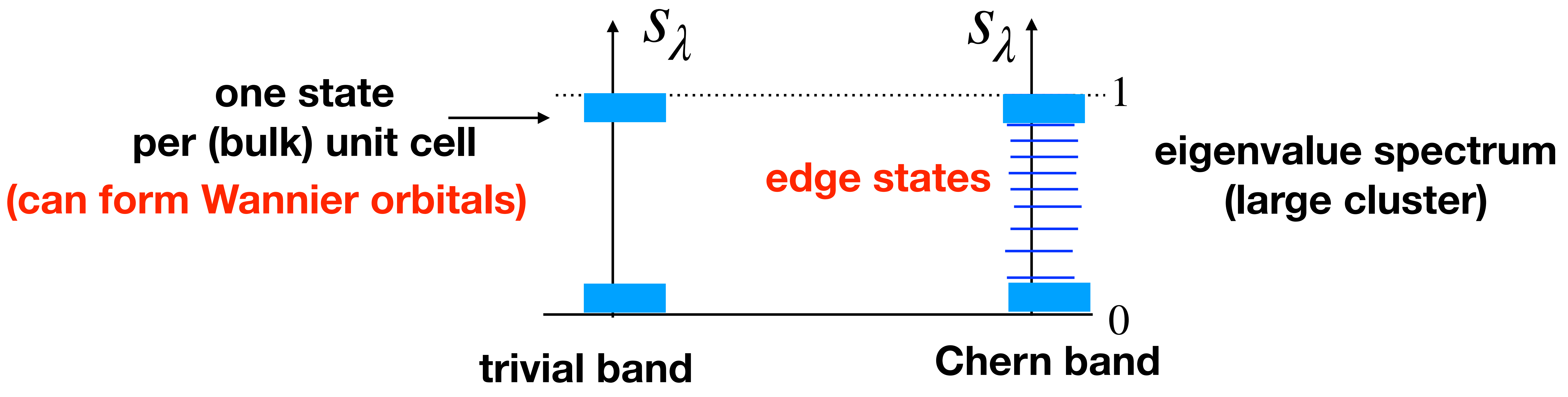
- conjecture: rapid decay of $S_{ij}^{(n)}(\mathbf{R}-\mathbf{R}')$ with distance on the Bravais lattice favors FCI.

$$\sum_{(R',j)\in C} S_{ij}(R-R')w_\lambda(R',j) = s_\lambda w_\lambda(R,i) \qquad 0 < s_\lambda < 1$$

$$\sum_{(R,i)\in C} w^*_\lambda(R,i)w_{\lambda'}(R,i) = \delta_{\lambda\lambda'}$$

- The orthonormal basis is

$$|\Psi_\lambda\rangle = \frac{1}{\sqrt{s_\lambda}} \sum_{(R,i)\in C} w_\lambda(R,i)\,|\Psi_n(R,i)\rangle \qquad \langle\Psi_\lambda|\Psi_{\lambda'}\rangle = \delta_{\lambda,\lambda'}$$



- spectral evolution across the phase transition

$C = 1$ $C = 0$

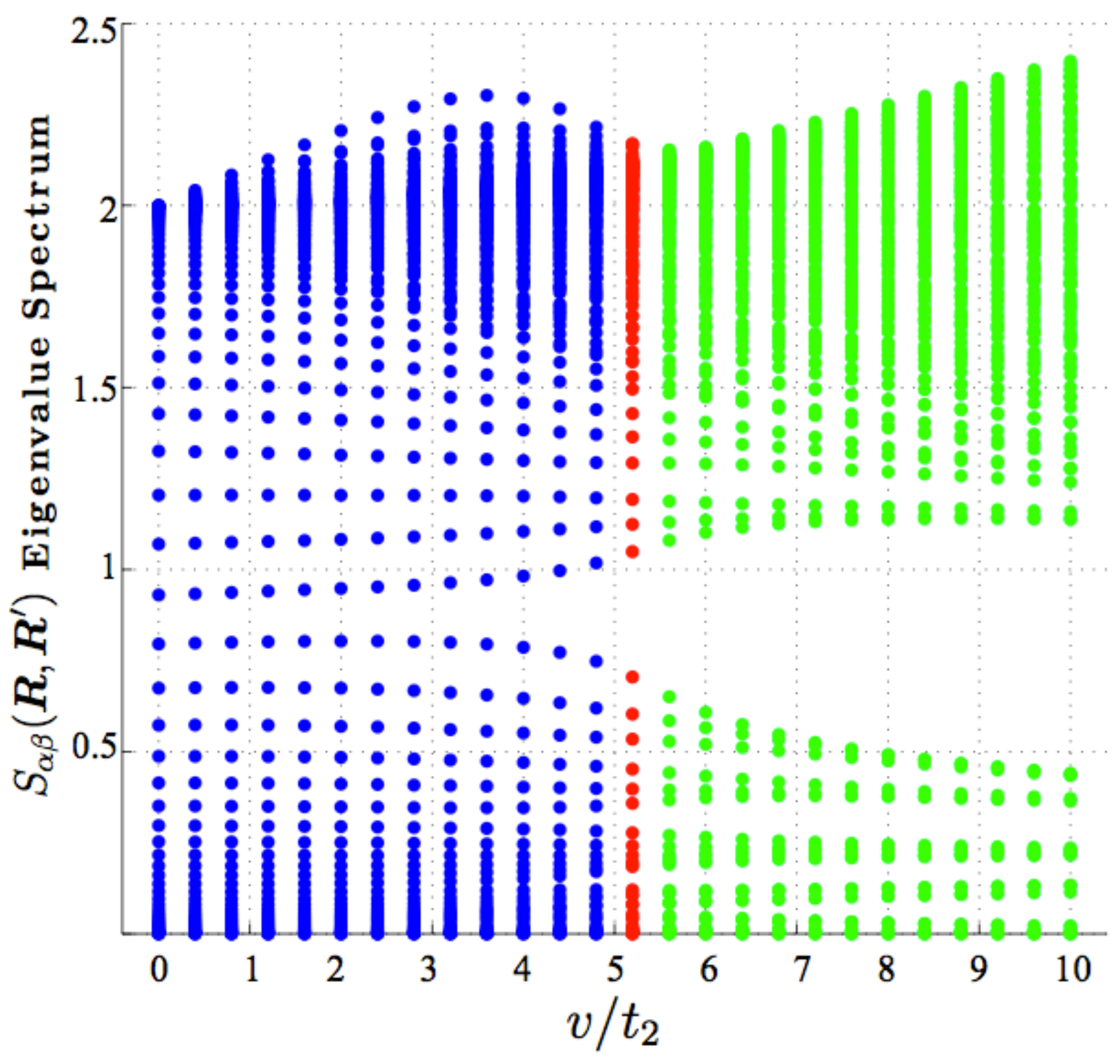


FIG. 4: Eigenvalue spectrum of the overlap matrix after diagonalizing over a circular region of the lattice with a ten-bond radius.

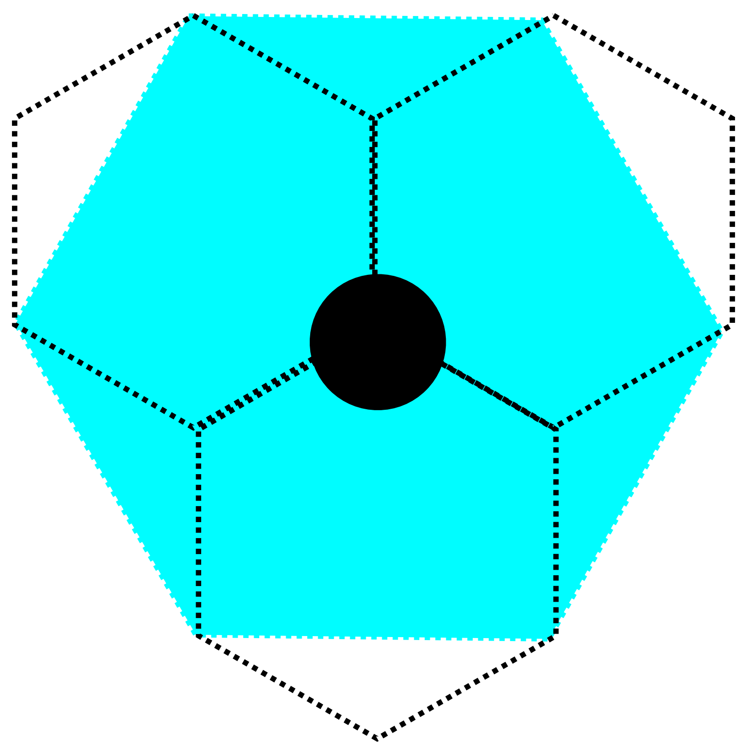
- In the case of the Chern band, the maximum eigenvalue s_λ indicates the basis set member localized nearest the center of the cluster, the “central state”, analogous to a coherent state.

- In order of decreasing s_λ the set of “concentric” edge states of the “onion rings” are denumerated.

- The key question for e.g., the 1/3 FQHE state is, how compact is the region on which the top three states are supported. This depends how far from a critical point the system is.

- The “lattice uncertainty principle” says that the area covered by the three orbitals cannot be less than three unit cells.

  - If sufficiently compact, exclusive occupation of this region could prevent nearest-neighbor interactions



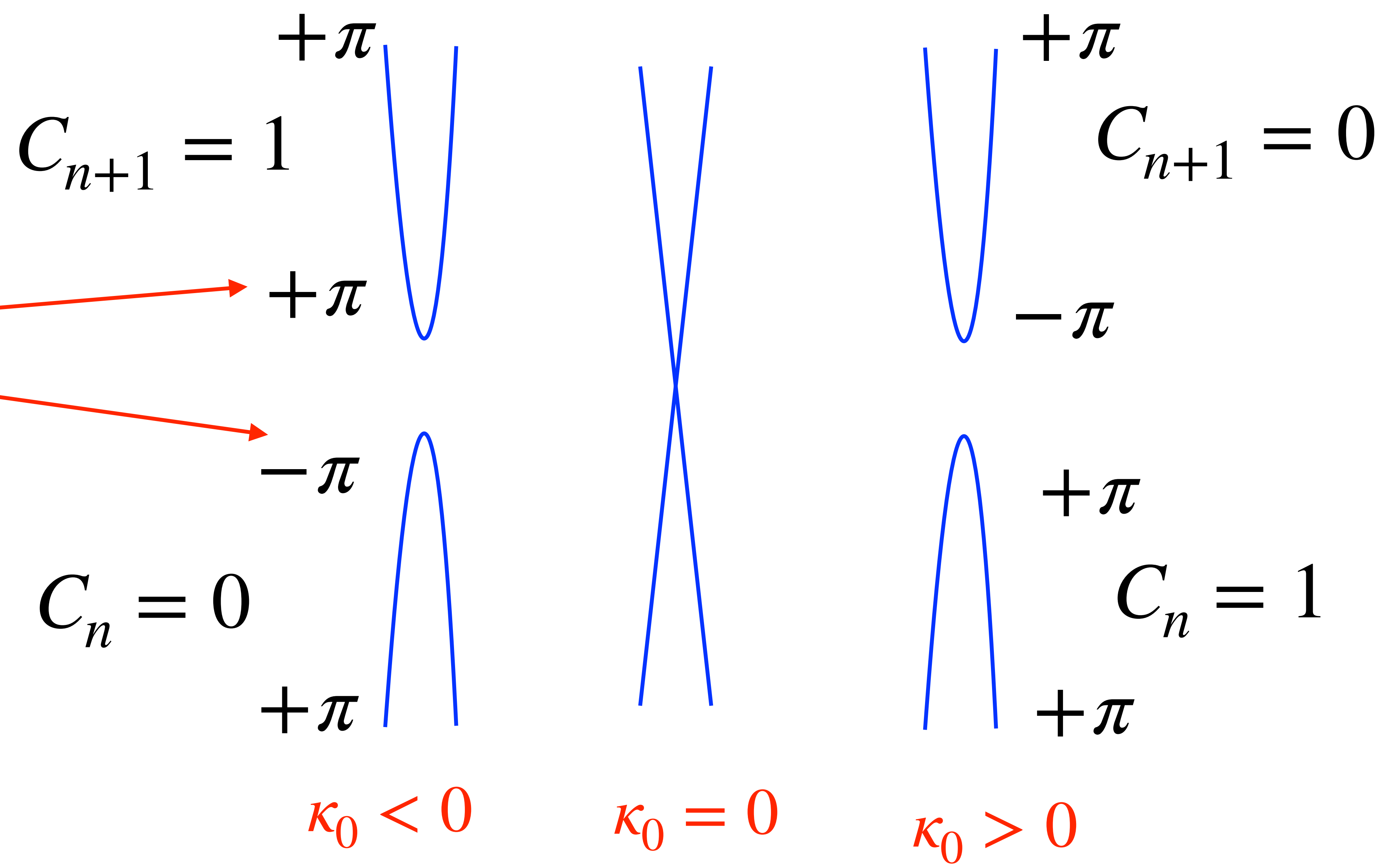
- At a critical point where the Chern invariant changes, a massive Dirac/Weyl point gap closes, and reopens

energy gap $\Delta(\vec{k}_0 + \delta\vec{k}) \propto \sqrt{(\kappa_0^2 + g^{ab}\delta k_a \delta k_b)}$

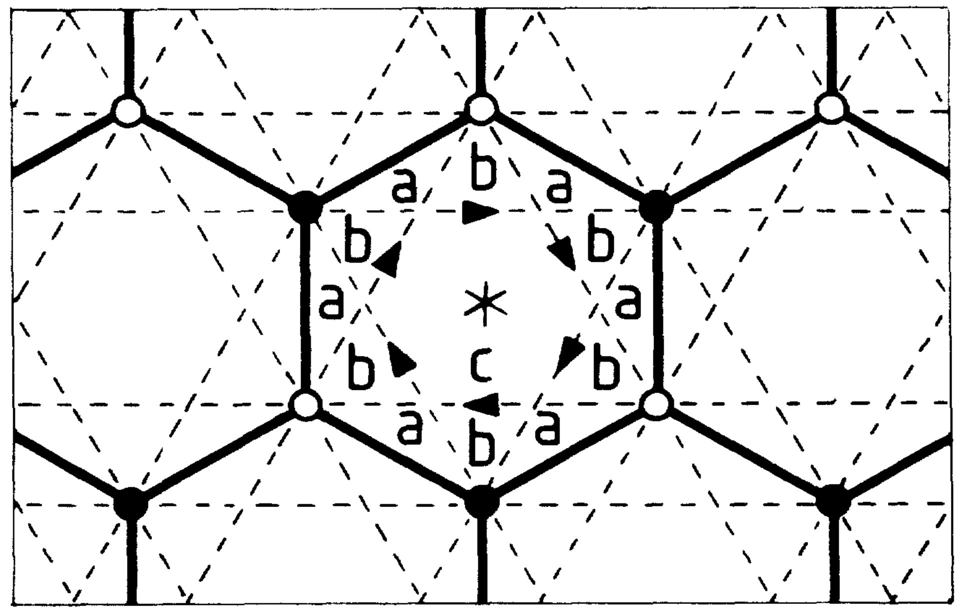
(spatial) conformal metric of Dirac point

band n+1

contributions
to TKNN integral
from near Dirac point



band n



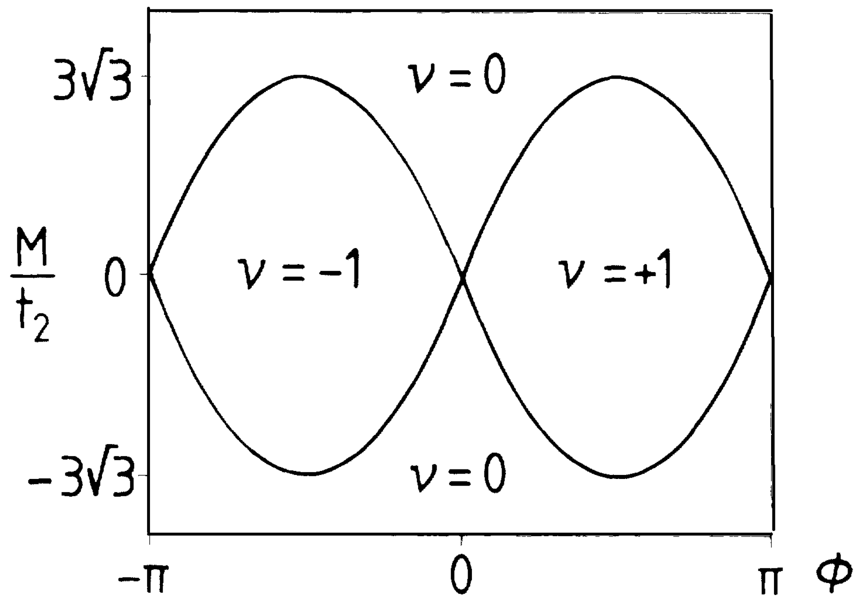


FIG. 2. Phase diagram of the spinless electron model with $|t_2/t_1| < \frac{1}{3}$. Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}|\sin\phi|$. This figure assumes that t_2 is positive; if it is negative, ν changes sign. At the phase boundaries separating the anomalous and normal ($\nu = 0$) semiconductor phases, the low-energy excitations of the model simulate undoubled massless chiral relativistic fermions.

- as the critical point is approached $(\kappa \to 0)$

$$\lim_{|R| \longrightarrow \infty} |S_{ij}(R)| \propto \frac{1}{|R|} e^{-\kappa|R|}$$

- here $|R|^2 = g_{ab}R^aR^b$ is measured with the emergent metric of the Dirac point, which characterizes the conformal invariance of the critical point.

- note that a quantum distance between the projected orbitals is defined by

$$d_{R,i;R',j} = 1 - \frac{|S_{ij}(R - R')|}{\sqrt{(S_{ii}(0)S_{jj}(0))}}$$ **(pure-state Bures distance)**