Gravitational Wave Emission: Derivation, Back-Reaction, and Waveforms

1 Derivation of the Quadrupole Formula

In the Lorenz gauge, the linearized Einstein field equations are:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

In the wave zone, the solution is:

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

Assuming $d \ll r$, expand in multipoles. The leading contribution is the quadrupole:

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = \frac{2G}{r} \, \ddot{Q}_{ij}^{\mathrm{TT}}(t - r)$$

2 Back-Reaction and Orbital Decay

The Newtonian orbital energy is:

$$E = -\frac{G m_1 m_2}{2a}$$

and the quadrupole power loss is:

$$\frac{dE}{dt} = -\frac{32 G^4}{5 c^5} \frac{(m_1 m_2)^2 (m_1 + m_2)}{a^5}$$

Using energy balance, this yields:

$$\frac{da}{dt} = -\frac{64 G^3}{5 c^5} \frac{m_1 m_2 (m_1 + m_2)}{a^3}$$

3 Post-Newtonian Expansion Overview

The post-Newtonian (PN) expansion organizes corrections in powers of v/c:

- 0PN: Newtonian dynamics

- 1PN: First relativistic correction to orbits
- 2.5PN: First radiation-reaction term
- Higher orders: improve inspiral phase accuracy

Orbital frequency:

$$\omega^2 = \frac{G(m_1 + m_2)}{a^3}$$

4 Waveform for Circular Inspirals

For a quasicircular binary:

$$h_+(t) = \frac{4G\mu}{c^4 D}(GM\omega(t))^{2/3}\cos[2\phi(t)]$$

$$h_\times(t) = \frac{4G\mu}{c^4 D}(GM\omega(t))^{2/3}\sin[2\phi(t)]$$

where:

- $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass
- $M = m_1 + m_2$ is the total mass
- D is the luminosity distance
- $\phi(t)$ is the orbital phase
- $\omega(t)$ is the orbital frequency