Foundational Equations in General Relativity for Gravitational Wave Analysis

1 Einstein-Hilbert Action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R + S_{\text{matter}}$$

2 Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu} \quad \text{or} \quad G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

3 Linearized Gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda} \quad \text{with} \quad \partial^{\mu}\bar{h}_{\mu\nu} = 0$$

$$\Box \bar{h}_{\mu\nu} = 0$$

4 Quadratic Action for Linearized Gravity

$$S^{(2)} = \frac{1}{64\pi G} \int d^4x \left(\partial_{\lambda}\bar{h}_{\mu\nu}\, \partial^{\lambda}\bar{h}^{\mu\nu} \right)$$

or more generally:

$$S^{(2)} = \frac{1}{64\pi G} \int d^4x \Big[\partial_{\lambda} h_{\mu\nu} \partial^{\lambda} h^{\mu\nu} - 2\partial^{\mu} h_{\mu\nu} \partial_{\lambda} h^{\lambda\nu} + 2\partial^{\mu} h_{\mu\nu} \partial^{\nu} h - \partial_{\lambda} h \partial^{\lambda} h \Big]$$

5 Transverse-Traceless (TT) Gauge

$$h^{\text{TT}}_{0\mu} = 0, \quad \partial^i h^{\text{TT}}_{ij} = 0, \quad h^{\text{TT}} = h^{\text{TT}}_{ii} = 0$$

$$\Box h^{\text{TT}}_{ij} = 0$$

$$S^{\text{TT}} = \frac{1}{64\pi G} \int d^4x\, \partial_{\lambda} h^{\text{TT}}_{ij}\, \partial^{\lambda} h^{ij}_{\text{TT}}$$

6 Quantization of Gravitational Waves

$$\hat{h}_{ij}^{\mathrm{TT}}(x) = \sum_{A=+,\times} \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{16\pi G}{2\omega_k}} \left[e_{ij}^{A}(\vec{k})\, \hat{a}_{\vec{k}}^{A}\, e^{-ik\cdot x} + e_{ij}^{A*}(\vec{k})\, \hat{a}_{\vec{k}}^{A\dagger}\, e^{ik\cdot x} \right]$$

7 Energy Density of Gravitational Waves

$$\left\langle T_{00}^{\mathrm{GW}} \right\rangle = \frac{1}{32\pi G} \left\langle \dot{h}_{ij}^{\mathrm{TT}}\, \dot{h}_{ij}^{\mathrm{TT}} \right\rangle$$

$$\left\langle T_{00}^{\mathrm{GW}} \right\rangle = \frac{\omega^2}{32\pi G} \sum_{A} |A_A|^2$$

8 Geodesic Deviation

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^{\mu}{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma$$

$$\frac{d^2\xi^i}{dt^2} = -\frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}} \xi^j$$

9 Detector Response

$$h(t) = \frac{\Delta L(t)}{L} \approx \frac{1}{2}\, \hat{e}_i \hat{e}_j\, h_{ij}^{\mathrm{TT}}(t)$$

$$h(t) = \frac{1}{2} \left(h_{xx}^{\mathrm{TT}}(t) - h_{yy}^{\mathrm{TT}}(t) \right)$$

10 Quadrupole Radiation Formula

In the wave zone, the leading-order transverse-traceless metric perturbation is:

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = \frac{2G}{r}\, \ddot{Q}_{ij}^{\mathrm{TT}}(t - r)$$

Here, Q_{ij}^{TT} is the transverse-traceless part of the mass quadrupole moment:

$$Q_{ij}(t) = \int d^3x\, \rho(t, \vec{x}) \left(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^2 \right)$$

11 Gravitational Wave Energy Loss (Back-Reaction)

The total power radiated in gravitational waves is:

$$\frac{dE}{dt} = -\frac{G}{5} \left\langle \dddot{Q}_{ij}\, \dddot{Q}^{ij} \right\rangle$$

Example: Binary System in Circular Orbit

For a Newtonian binary system with masses m_1 and m_2 in circular orbit:

$$\frac{dE}{dt} = -\frac{32G^4}{5c^5}\frac{(m_1 m_2)^2(m_1 + m_2)}{r^5}$$

This formula describes the energy loss due to gravitational radiation and leads to orbital decay, as observed in binary pulsar systems.