Gravitational radiation is often treated classically, but the world is quantum mechanical.

What is its quantum-mechanical state? How do we measure it? How do we predict it?

The “most classical” states, as we’ll discuss, are the so-called coherent states.

The *coherent state hypothesis* is a precise form of the statement “gravitational radiation is classical”. That hypothesis has sharp experimental consequences.

We’ll be exploring how to test the coherent state hypothesis: What it is, why it often holds, why we expect it sometimes to fail, and how to tell the difference experimentally.

Radiation Field States

Paradigms and Properties

- Coherent states
- Thermal and Fock states
- Generating coherent states
- Squeezing
- Generating squeezed states
- Radiation from afar
- Wigner functions

Coherent States

“Maximally Classical”

Warm-up: harmonic oscillator gymnastics

$$H = \frac{1}{2}(p^2 + q^2) \; ; \; [q, p] = i$$

$$a = \frac{1}{\sqrt{2}}(q + ip) \; ; \; a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$

$$aa^\dagger - a^\dagger a = 1 \qquad H = a^\dagger a + \frac{1}{2}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \; ; a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \; ; a^\dagger a|n\rangle = n|n\rangle$$

$$H|n\rangle = (n + \frac{1}{2})|n\rangle$$

$$a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$

$$|0\rangle = \int dq \, \frac{e^{-q^2/2}}{\pi^{\frac{1}{4}}} |q\rangle$$

(Hermite polynomials emerge …)

Coherent states defined

$$|\alpha\rangle \;=\; \sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \;=\; D(\alpha)\,|0\rangle$$

$$a\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle$$

Displacement operators

$$D(\alpha) \;=\; e^{(\alpha a^\dagger - \alpha^* a)} \;=\; e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} \;=\; e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$\boxed{D(\alpha)\,D(\beta) \;=\; e^{i\mathrm{Im}(\alpha\beta^*)} D(\alpha+\beta)}$$

$$D(\alpha)\,D(-\alpha) \;=\; 1 \;;\; D(\alpha)^\dagger \;=\; D(-\alpha)$$

BCH formula (semi-trivial version)

If $[A,[A,B]] \;=\; 0$ and $[B,[A,B]] \;=\; 0$, then

$$e^{A+B} \;=\; e^A\, e^B\, e^{-\frac{1}{2}[A,B]} \;=\; e^B\, e^A\, e^{\frac{1}{2}[A,B]}$$

Evolution (from expression in number basis)

$$e^{-iHt}\,|\,\alpha\rangle \;=\; e^{-it/2}\,|\,e^{-it}\alpha\rangle$$

Wave function

$$|\,\alpha\rangle \;=\; \int dq\, e^{-(q-q_0)^2/2} e^{ip_0 q}\,|\,q\rangle \;\; ; q_0 = \sqrt{2}\,\mathrm{Re}\,\alpha,\;\; p_0 = \sqrt{2}\,\mathrm{Im}\,\alpha$$

$$\langle\alpha(t)\,|\,q\,|\,\alpha(t)\rangle \;=\; \sqrt{2}\,|\,\alpha(0)\,|\;\cos(\sigma - \omega t)$$

with $\alpha(0) \;=\; |\,\alpha(0)\,|\,e^{i\sigma}$

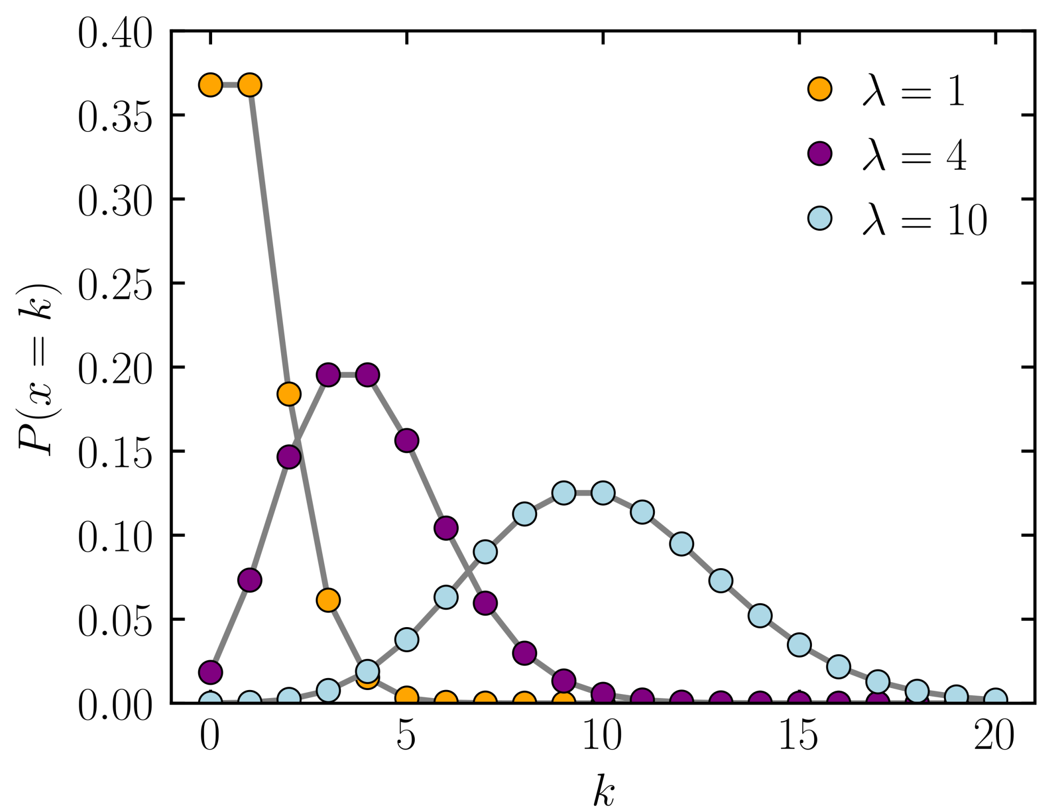
$$\langle\alpha(t)\,|\,p\,|\,\alpha(t)\rangle \;=\; \sqrt{2}\,|\,\alpha(0)\,|\;\sin(\sigma - \omega t)$$

Poisson distribution in number

$$P(n) \;=\; |\langle \alpha | n \rangle|^2 \;=\; e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

$$\langle \alpha | n | \alpha \rangle \;=\; |\alpha|^2 \;\;;\;\; \langle \alpha | n^2 | \alpha \rangle \;=\; |\alpha|^2 + |\alpha|^4$$

$$\overline{n^2} - \bar{n}^2 \;=\; \bar{n} \;\;;\;\; \frac{\Delta n}{\bar{n}} \;\equiv\; \frac{\sqrt{\overline{n^2} - \bar{n}^2}}{\bar{n}} \;=\; \frac{1}{\sqrt{\bar{n}}}$$



Position and Momentum

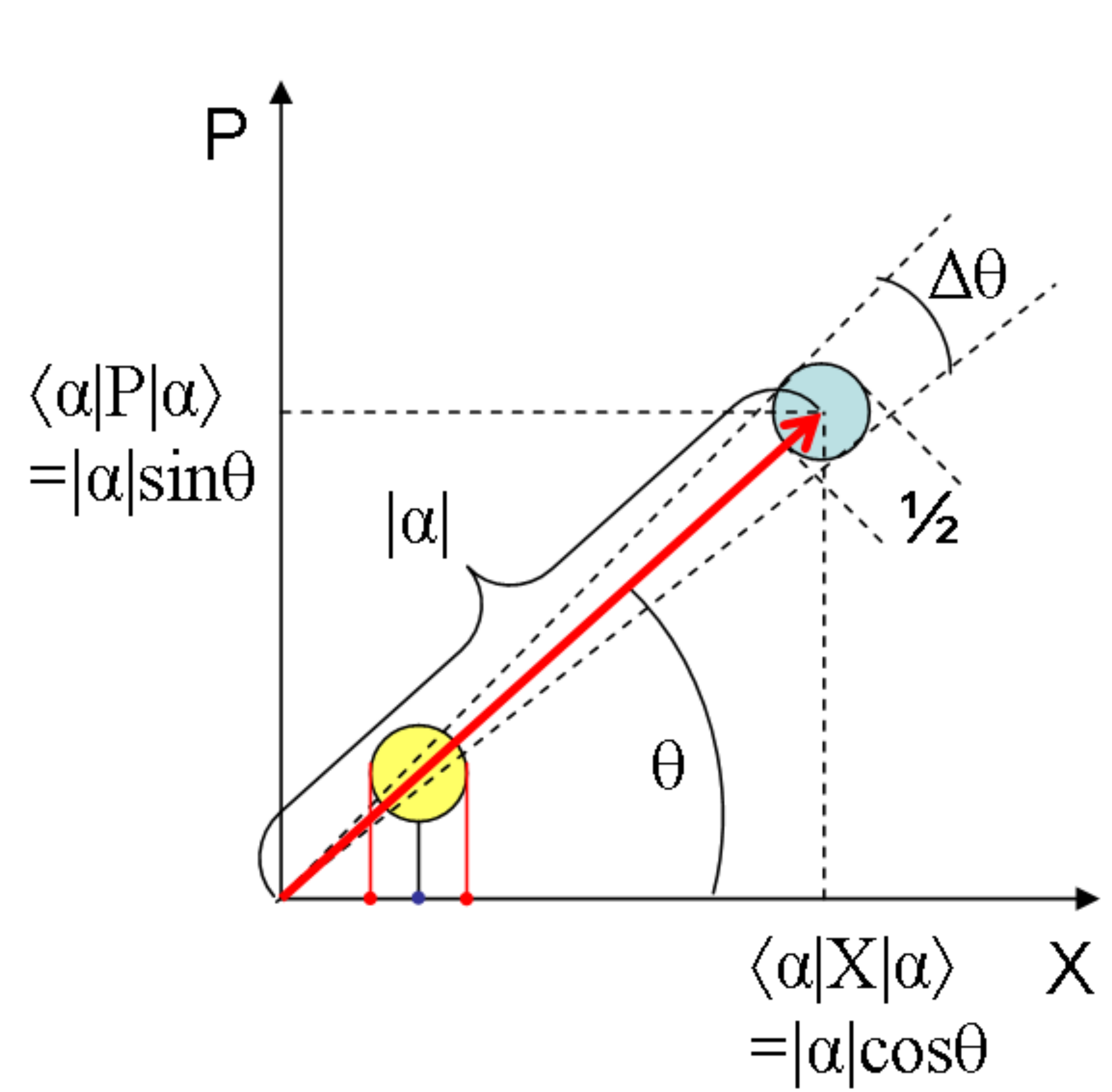
$$\langle\alpha|\, q = \frac{a^\dagger + a}{\sqrt{2}}\, |\alpha\rangle \;=\; \sqrt{2}\,\mathrm{Re}\,\alpha \qquad \langle\alpha|\, p = \frac{a - a^\dagger}{\sqrt{2}i}\, |\alpha\rangle \;=\; \sqrt{2}\,\mathrm{Im}\,\alpha$$

In the ground state $|\alpha = 0\rangle$ we have equal variances $\frac{1}{2}$ in q, p. The product of those uncertainties saturates the Heisenberg bound.

We also get the same variance for $\cos\phi\, q \;+\; \sin\phi\, p$, with any ϕ.

Thus we have a "circle of uncertainty" around the origin in the complex α plane. Applying displacement operators, we find coherent states describing a translated circle surrounding its central point.

All this justifies the general feeling that coherent states are maximally classical, and the picture that follows:



“Vacuum noise only”

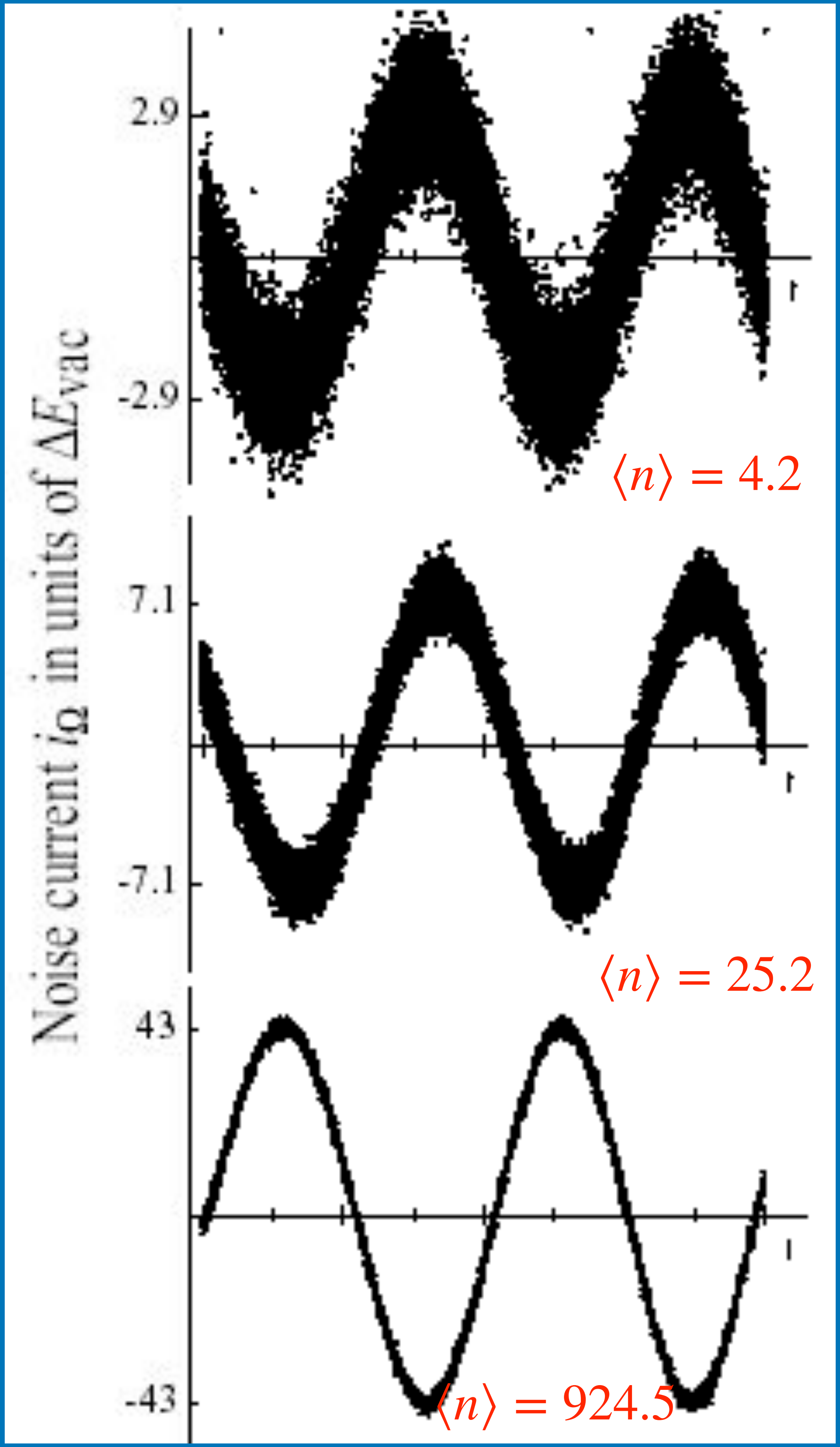
Electric (or “Electric”) field modes

In the quantum realization of free electromagnetic fields, each mode of the electric field occurs as a “q” (or a p), *e. g.* :

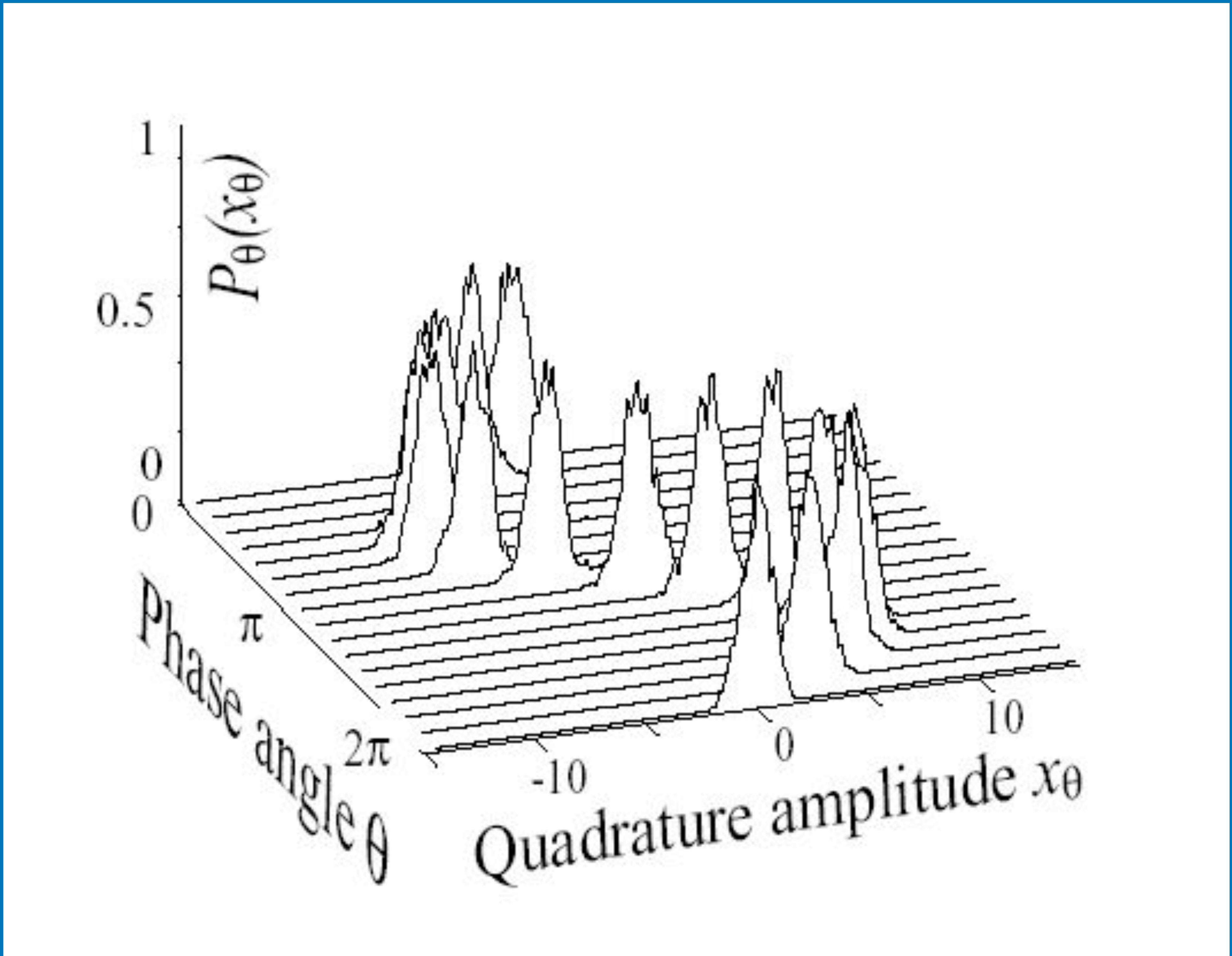
$$E_x(z,t) \;=\; \mathscr{E}_0\,(a \;+\; a^\dagger)\;\;;\, B_y(z,t) \;=\; \mathscr{E}_0\,\frac{1}{i}(a \;-\; a^\dagger)$$

Thus, we can speak of quantum “coherent states”* of a mode, and discuss their observable characteristics.

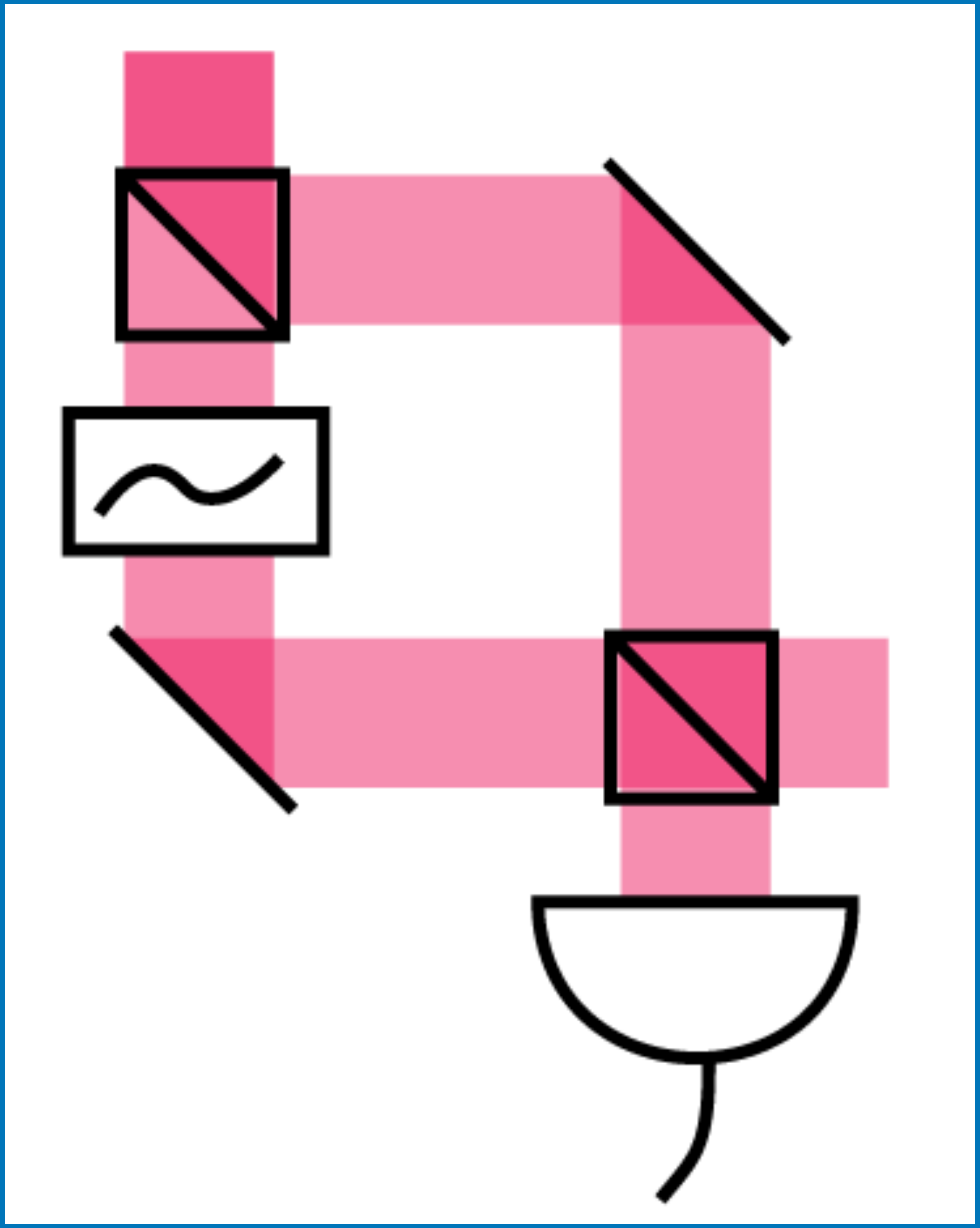
*Note that the language of “coherence” is also used to describe other, quite distinct phenomena in optics.



Electric field in coherent states (*homodyne measurement*)



Wave packet evolution, $\langle n \rangle = 25.2$



Homodyne detection principle

Coherent State Bases

Coherent states have such pleasant properties that it can be very convenient to use them as a basis for expansions.

The simplest completeness relation is

$$1 \;=\; \int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|$$

Note however that

$$\langle\beta|\alpha\rangle \;=\; e^{i\,\mathrm{Im}\,\beta^*\alpha}\,e^{-\frac{1}{2}|\beta-\alpha|^2}$$

so this basis is highly over-complete.

An alternative way to express density matrices (pure or not) is the so-called P representation. It will be extremely useful for us!

$$\rho \;=\; \int d^2\alpha \, P(\rho, \alpha) \, |\alpha\rangle\langle\alpha|$$

Since the real and imaginary parts of α encode q, p, the P representation is a kind of phase space representation. P is real, but it is not necessarily positive and it can be quite singular.

For a coherent state density matrix $\rho_\alpha = |\alpha\rangle\langle\alpha|$ we have of course $P(\rho_\alpha, \beta) \;=\; \delta^2(\alpha - \beta)$.

Thermal and Fock States

Other Reference Cases

Thermal "state", temperature T: $\rho_T = \dfrac{e^{-\frac{H}{T}}}{\mathrm{Tr}\, e^{-\frac{H}{T}}}$; $\bar{n} = \dfrac{1}{e^{\frac{1}{T}} - 1}$

$$P(\rho_T, \alpha) = \frac{1}{\pi \bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}}$$

Fock state $|n\rangle$; $\rho_n = |n\rangle\langle n|$

$$P(\rho_n, \alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial^n \alpha \partial^n \alpha^*} \delta(\mathrm{Re}\alpha)\delta(\mathrm{Im}\alpha)$$

Generating Coherent States

Lasers, Linear GW Sources

The main result here is that ***linear*** *interaction with a* ***classical source**** *produces coherent states of the radiation field*.

* $\approx$ a source for which we can ignore back-reaction.

Using the machinery of displacement operators, we can prove this in a few lines: (next slide)

0. $U(t_f, t_i) \;=\; T\exp\left(-i\int_i^f dt\, H(t)\right)$; $H(t) \;=\; \int d^3x\, A(x,t)\cdot J(x,t)$

1. It is legitimate to analyze mode by mode:

$$A(x,t) \;\to\; \delta^3(k-k_0)\left(a_{k_0}e^{i(k_0x-\omega t)} + a^\dagger_{k_0}e^{-i(k_0x-\omega t)}\right)$$

2. Fourier: $H(t) \;=\; \tilde{J}(k_0\,,t)\left(a_{k_0}e^{-i\omega t} + a^\dagger_{k_0}e^{i\omega t}\right)$

3. Newton (analysis):

$$T\exp\left(-i\int_\tau^{\tau+\epsilon} dt\, H(t)\right) \;\approx\; T\exp\big(-i\epsilon H(\tau)\big) \;=\; TD(\epsilon e^{i\omega\tau J(k_0,\tau)})$$

4. Newton (synthesis): Using the algebra of displacements, the product over time intervals becomes - up to phases - a sum that converts to an integral.

5. Result: $U(t_f, t_i) \;=\; \text{phase} \times D\left(\int_i^f dt\, \tilde{J}(k_0,t)\, e^{i\omega t}\right)$

$\Rightarrow$ From $|\,0\rangle$ to a coherent state!

Lasers operating well above threshold produce light in a quantum coherent state (possible involving several modes), and we should also anticipate this for gravitational radiation produced whenever a linearized theory coupled to a robust source is an appropriate model.

But known sources of gravitational radiation sometimes involve strongly non-linear effects, and other “known” sources involve pair creation. Then we might expect - and will find - departures from the coherent state hypothesis.

Squeezing

A Classic "Non-Classicality"

Intuition: Commuting the creation operators through the annihilation operators just generates phases, so (up to a phase) we can we can ignore time ordering.

The next level of complexity is to bring in (exponentials of) expressions quadratic in creation and annihilation operators. Several possibilities occur naturally in applications and have been studied intensely, including quadrature squeezing, number squeezing, and 2-mode squeezing. I'll introduce them briefly here. We'll use them a lot.

squeezing operator

$$S(\xi) \;=\; \exp\Big(\frac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2})\Big) \;\; ; \; \xi \;\equiv\; r\, e^{i\theta}$$

$$\begin{aligned} S^\dagger(\xi)\, a\, S(\xi) \;&=\; a \cosh r - a^\dagger e^{i\theta} \sinh r \\ S^\dagger(\xi)\, a^\dagger\, S(\xi) \;&=\; a^\dagger \cosh r - a e^{-i\theta} \sinh r \end{aligned}$$

Uncertainty Perspective

An interesting aspect of squeezed states is their re-arrangement of uncertainty.

$$[A, B] \;=\; iC$$

$$\langle s|(\tilde{A} + i\lambda\tilde{B})(\tilde{A} - i\lambda\tilde{B})|s\rangle \;=\; \langle s|\tilde{A}^2 + \lambda C + \lambda^2\tilde{B}^2|s\rangle \geq 0 \text{ for all real } \lambda,$$

$$\text{with } \tilde{A} = A - \langle s|A|s\rangle,\ \tilde{B} = B - \langle s|B|s\rangle$$

$$\Rightarrow \langle s|\tilde{A}^2|s\rangle\langle s|\tilde{B}^2|s\rangle \;\geq \frac{1}{4}\langle s|C|s\rangle^2 \qquad (*)$$

The canonical case is $A = q, B = p, C = 1$. The bound is saturated by coherent states, with both factors on the left $= \dfrac{1}{2}$. (This property singles out coherent states.)

Squeezed states "squeeze" something in the sense that one of the factors in (*) becomes smaller than the square root. Of course, that means that the other must be larger! I suggest that we call them "ballooned" directions.

“squeezing the vacuum”

With $|r,\theta\rangle = S(\xi)|0\rangle$

$$\langle r,\theta|\tilde{q}^2|r,\theta\rangle = \frac{1}{2}(\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta)$$

$$\langle r,\theta|\tilde{p}^2|r,\theta\rangle = \frac{1}{2}(\cosh^2 r + \sinh^2 r + 2\sinh r\cosh r\cos\theta)$$

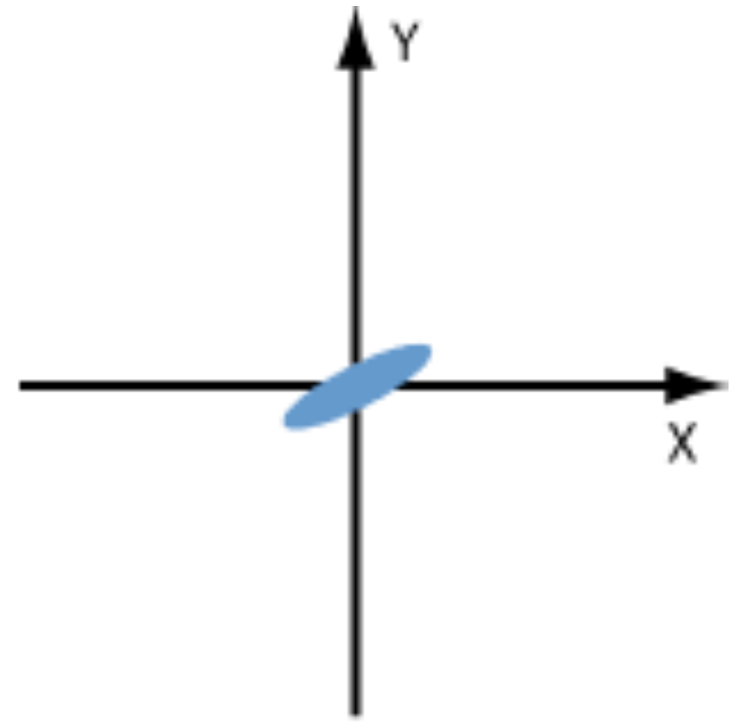
For $\theta = 0$, $\langle r,0|\tilde{q}^2|r,0\rangle = \frac{1}{2}e^{-r}$, $\langle r,0|\tilde{p}^2|r,0\rangle = \frac{1}{2}e^{r}$, so we have squeezing in the q direction, ballooning in the q and a minimum uncertainty product. For $\theta = \pi$, the roles are reversed. In general, neither $\tilde{q}$ nor $\tilde{p}$ squeezed and the product is non-minimal.

In the (q,p) plane, we have an error ellipse that rotates as a function of $\theta/2$.

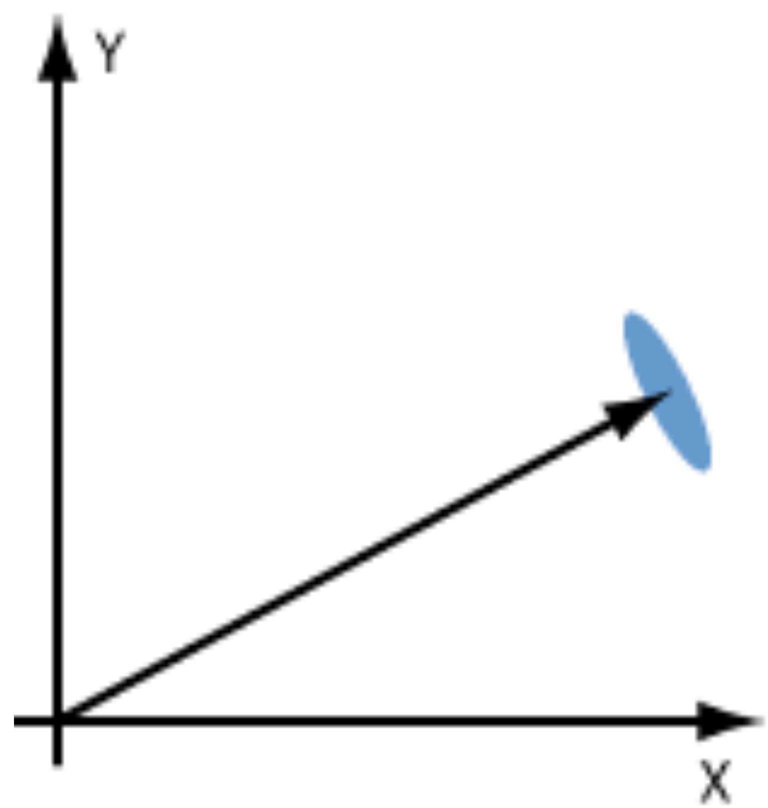
Squeezing coherent states

One can displace the vacuum squeezed states, according to
$D(\alpha)S(\xi)\,|\,0\rangle \;\equiv\; |\,\alpha, r, \theta\rangle$. These states have rigidly displaced error ellipses.

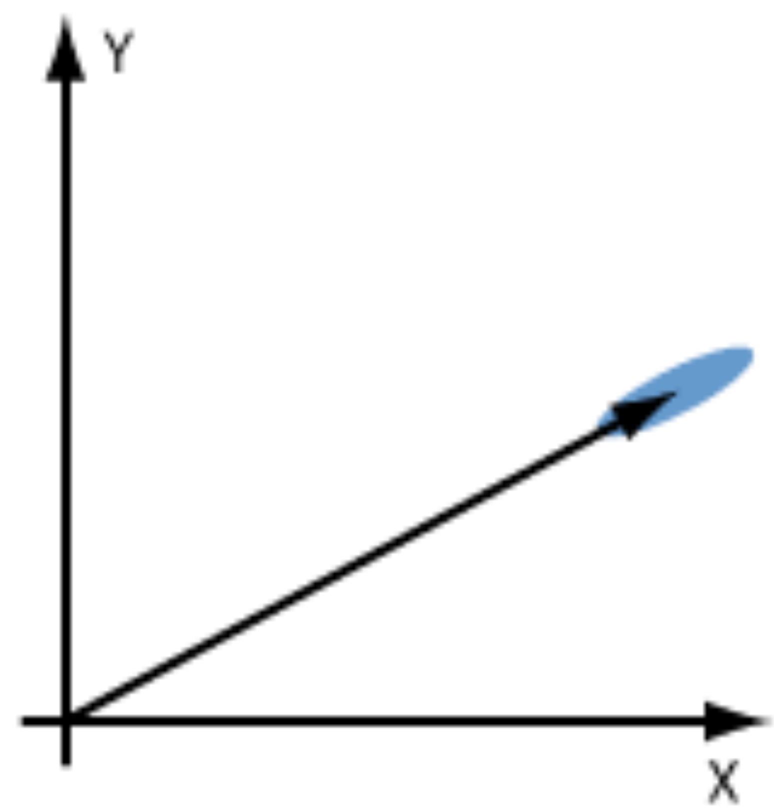
Combining these ideas, we can squeeze amplitude (number) or phase:



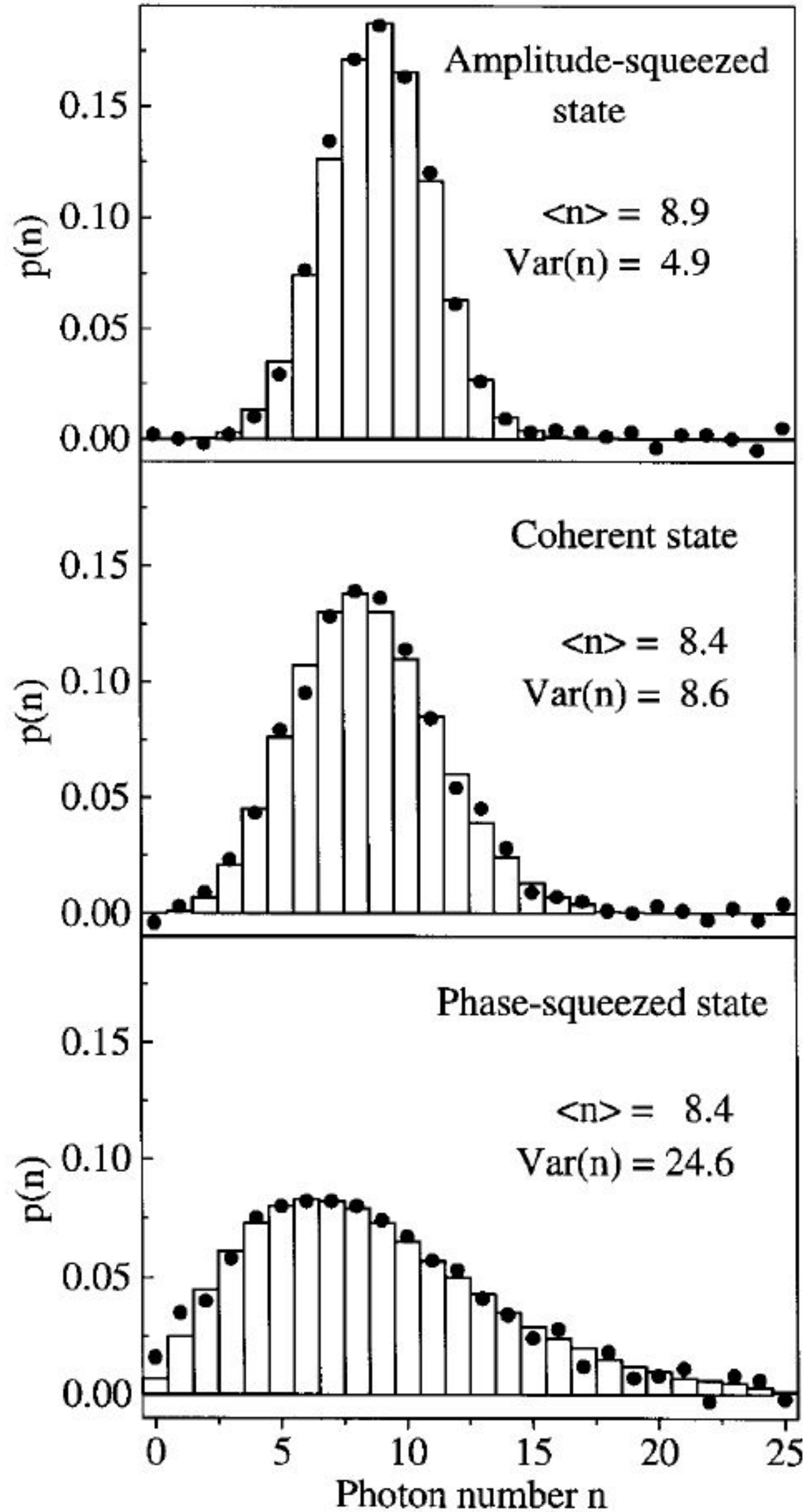
squeezed vacuum



amplitude squeezing



phase squeezing



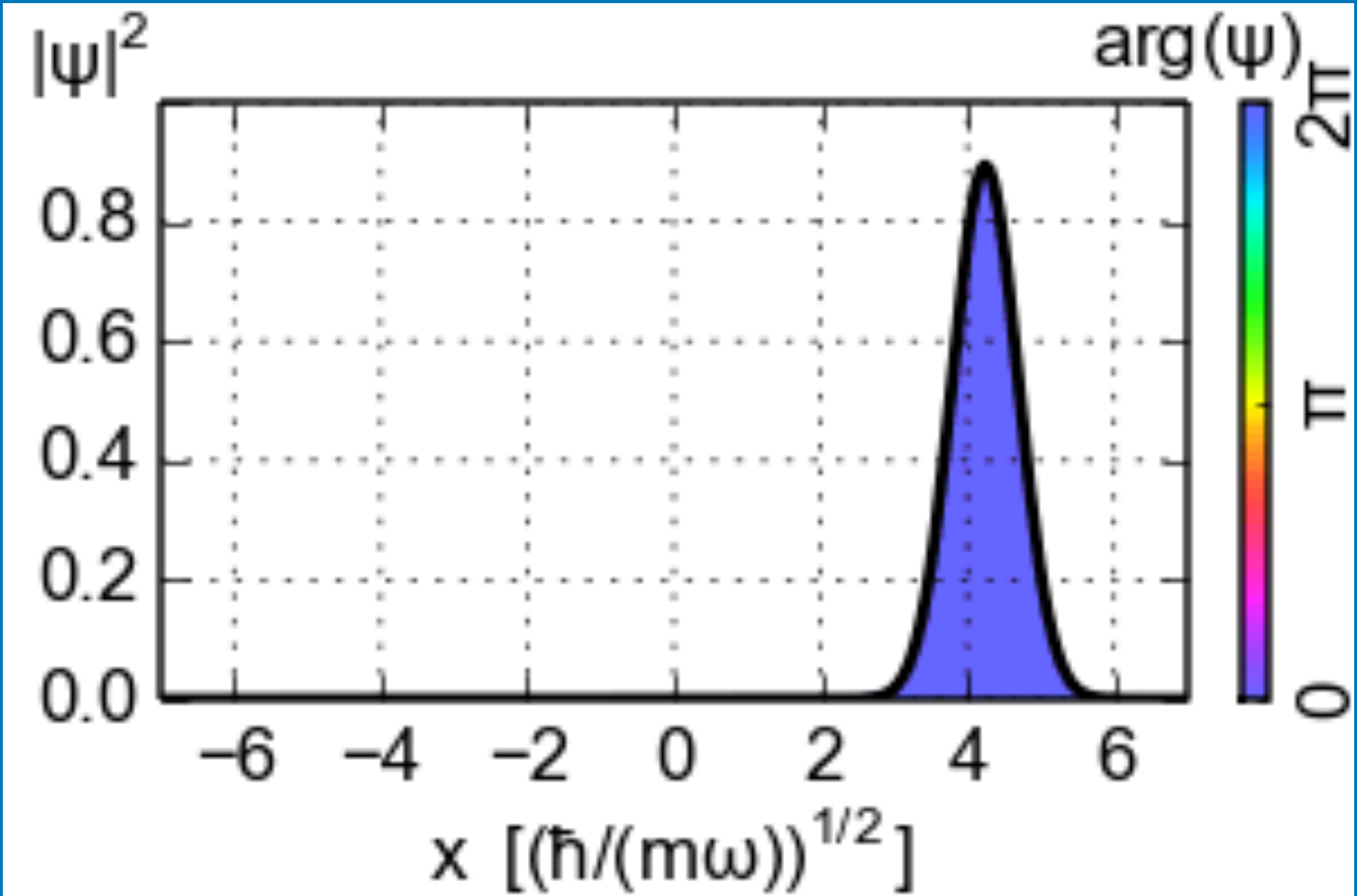
Amplitude-squeezed state
<n> = 8.9
Var(n) = 4.9
Coherent state
<n> = 8.4
Var(n) = 8.6
Phase-squeezed state
<n> = 8.4
Var(n) = 24.6
p(n)
0.15
0.10
0.05
0.00
0
5
10
15
20
25
Photon number n

wave function

Squeezed states allow for $w \neq 1$ in
$\psi(q) \;=\; \exp\left(-(q-q_0)^2/2w^2 + ip_0 q\right)$

This wave function describes an eigenstate of $q + ipw^2$ (or
$(1+w^2)a + (1-w^2)a^\dagger$) .
It is not, of course, an energy eigenstate.

But for the unitary operators that combine displacement and squeezing, we *do* map eigenstates of H into eigenstates of UHU^{-1}. So we get eigenstates of a different harmonic oscillator. For displacement operators, the transformed Hamiltonian is simply translated; under squeezing operators, its effective mass and spring constant generally change.



|ψ|²
arg(ψ)
2π
π
0
0.8
0.6
0.4
0.2
0.0
−6
−4
−2
0
2
4
6
x [(ħ/(mω))^1/2]

“Non-classicality” of squeezing

In the P representation

$$\langle s\,|\,\tilde{q}^2\,|\,s\rangle \;=\; \frac{1}{2}\big(1+\int d^2\alpha\, P(s,\alpha)((\alpha+\alpha^*)^2\big),$$

so squeezing requires that $P(s,\alpha)$ goes negative.

Two-mode squeezing

$$S(\xi) \;=\; \exp\left(\xi^* ab - \xi a^\dagger b^\dagger\right)$$

$$S(\xi)^\dagger a S(\xi) \;=\; \cosh r\; a - e^{i\theta}\sinh r\; b^\dagger$$

$$S(\xi)^\dagger b S(\xi) \;=\; \cosh r\; b - e^{i\theta}\sinh r\; a^\dagger$$

$$|\,r,\theta\rangle \;=\; S(\xi)\,|\,0{,}0\rangle$$

$$|\,r,\theta\rangle \;=\; \frac{1}{\cosh r}\sum_n (-1)^n\, e^{in\theta}\,(\tanh r)^n\,|\,n,n\rangle$$

If we observe only one mode, tracing over the other, we get a thermal distribution (!) with effective temperature $1/T_{\text{eff.}} \;=\; 2\ln\coth r$.

Generating Squeezed States

Paradigms

- Nonlinearity and driving field
    - Electro-optics
    - General relativity; BH quasinormal modes
- Pair creation
    - Hawking radiation
    - Cosmological
- Theoretical (?!)
    - Sudden change of Hamiltonian
    - Resonant source-detector entanglement

Radiation from Afar

Elementary Decay Amplitudes

It is a famous, fundamental result that despite being allowed by naive angular momentum conservation, one-photon transitions between 0 spin states ("0-0 transitions") are forbidden.

This is because the candidate amplitude - linear in the photon polarization and using one unit of orbital angular momentum to make a scalar - vanishes by gauge invariance:

$\epsilon \cdot k \; = \; 0$ (no longitudinal photon).

0-0 transitions can occur through two photon emission. Two famous cases are Higgs decay and π^0 decay.

The relevant amplitudes are:

$\epsilon_1 \cdot \epsilon_2$, scalar

$(\epsilon_1 \times \epsilon_2) \cdot k$, pseudo-scalar (Note: consistent with Bose statistics.).

Although d wave is allowed by angular momentum, in the scalar case, it is forbidden by transversality - we can't soak up two powers of k .

1x1 → 0 + 1 + $\cancel{2}$

More explicitly, the 2-photon final states are described by wave functions of the general form

$$\mathbf{Re}\,(\epsilon_1 \cdot \epsilon_2^*)\; e^{ik(r_1 - r_2)},$$

superposed over k .

We have a similar situation in gravity. Spherically symmetric configurations, even if time-dependent, give no gravitational radiation classically.

What about two-graviton radiation?

Let’s look at the amplitudes:

The polarization of gravitons is described by a two-index (spin 2) tensor with additional conditions:

$$h_{ij}\,,\;\; h_{ij} \;=\; h_{ji}\,,\;\; h_{ii} \;=\; 0.\,,\;\; k_i h_{ij} \;=\; 0\,.$$

As with photons, here too gauge invariance forbids $0 \to 0$ by single graviton emission.

Very similar to the photon case, there are two allowed amplitudes for two-graviton emission:

Scalar, s-wave: $h_{ij}h'_{ij}$

Pseudo-scalar, p-wave: $h_{ij}h'_{ik}\epsilon_{jkl}k_l$

In addition, $1 \to 0$ transitions can't occur through one graviton emission:

$$\pi_i h_{ij} k_j = 0$$

$$\pi_p h_{qr} \epsilon_{pqs} k_r k_s = 0$$

$$\pi_p h_{qr} k_p k_q k_r = 0$$

What about two graviton emission?

$h_{rs}h'_{rs}\,\pi\cdot k$ forbidden by Bose statistics

$h_{ps}h'_{qs}\,\epsilon_{pqu}\,\pi_u$ forbidden, B

$h_{ps}h'_{qs}\,\epsilon_{pqu}\,\pi_d\,\epsilon_{udv}k_v$ forbidden by transversality

$h_{ps}h'_{qs}\,\epsilon_{pqu}\,\pi_d\,k_u k_d$ forbidden, B

$h_{ps}h'_{qs}\;\pi_p\,k_q$ forbidden, t

$h_{ps}h'_{qs}\,\pi_r\epsilon_{qru}k_p k_u$ forbidden, t

$\ldots$

$\Rightarrow$ No $1 \rightarrow 0$ through 2 graviton emission, either!

In the ring-down phase following a black hole merger, the geometry vibrates at characteristic (complex) frequencies in ways that can be read off from linear perturbation theory.

These quasi-normal modes imply observable, and probably observed, forms of gravitational radiation.

The wavelength of this radiation is typically much larger than the black hole size, so a multipole analysis should be appropriate.

In that framework, the absence of single graviton radiation from a spherically symmetrical oscillation (Birkhoff's theorem) is a reflection of the absence of 0-0 transitions.

But two-graviton - and two-photon - processes can proceed. Appropriate vertices can be read off from the microscopic Lagrangian.

$$gFF \rightarrow \sim f(t,r)a^{\dagger}a^{\dagger}$$

$$gR \rightarrow \sim f(t,r)a^{\dagger}a^{\dagger}$$

The frequency of the emitted pairs is half the frequency of the basic oscillation.

This process is distinct from the generation of combination tones, which is also very interesting.

The same sort of analysis applies to ring-down of neutron stars, …

Wigner Functions

Transcending Uncertainty,
Improbably

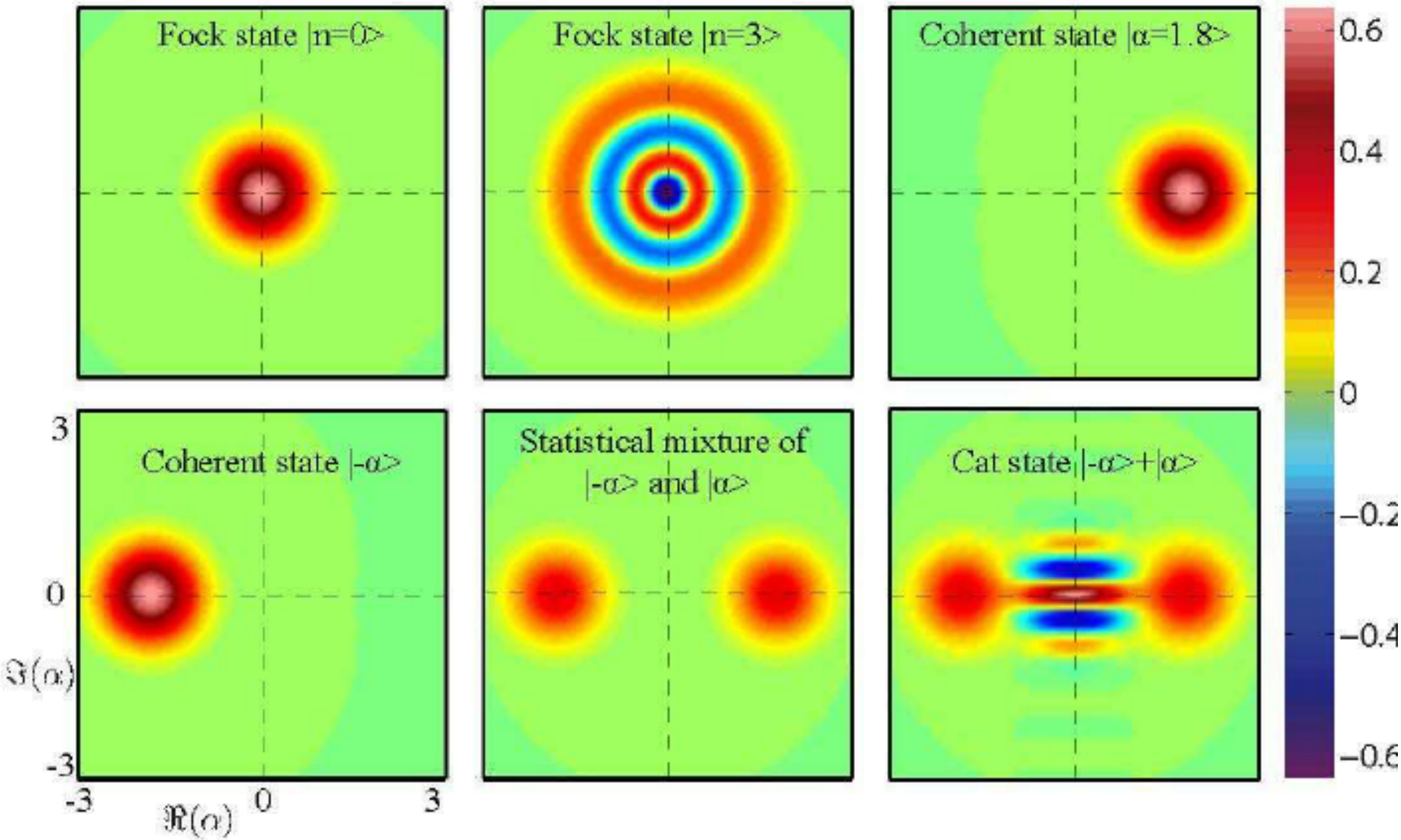
An older but useful and widely used phase space extension of a quantum wave function is the Wigner function

$$W_\psi(q,p) \;=\; \frac{1}{2\pi}\int dx\, \psi^*(q-\frac{x}{2})\, \psi(q+\frac{x}{2})\, e^{ipx}$$

It is real, but not necessarily positive. It has the nice properties

$$\int dq\, W_\psi(q,p) \;=\; |\tilde\psi(p)|^2 \;\;;\;\; \int dp\, W_\psi(q,p) \;=\; |\psi(q)|^2$$

Here are some instructive examples:


Fock state |n=0>
Fock state |n=3>
Coherent state |α=1.8>
Coherent state |-α>
Statistical mixture of |-α> and |α>
Cat state |-α>+|α>
3
0
-3
-3
0
3
ℜ(α)
ℑ(α)
0.6
0.4
0.2
0
−0.2
−0.4
−0.6