Julia Sets in Quantum Evolution

A Complex Dynamics Approach to Dynamical Quantum Phase Transitions

Manmeet Kaur

Ashoka University

July 26, 2025

Long-Range Interactions and Dynamics in Complex Quantum Systems

Nordita, Stockholm, Sweden

Phase Transitions: classical vs quantum

Phase transitions as a parameter is changed

Phase Transitions: classical vs quantum

Phase transitions as a parameter is changed

- "Time" is important: **equilibration, dynamics**, in **QM, nonequilibrium** . . .
  But otherwise "time" has no direct role.

- "Time" is not a tunable parameter in experiments or simulations.
  Tuning Parameters: temperature, pressure, external fields, concentrations . . .

Phase Transitions: classical vs quantum

Phase transitions as a parameter is changed

- "Time" is important: **equilibration, dynamics**, in **QM, nonequilibrium** ...
  But otherwise "time" has no direct role.

- "Time" is not a tunable parameter in experiments or simulations.
  Tuning Parameters: temperature, pressure, external fields, concentrations ...

Dynamical Quantum Phase Transitions (DQPTs)?
(Heyl, Polkovnikov, and Kehrein, PRL 2013)

Dynamical Quantum Phase Transitions

A many-body quantum system at zero temperature during its time evolution may undergo one or more phase transitions in **time.** These transitions are called DQPT.

Dynamical Quantum Phase Transitions

A many-body quantum system at zero temperature during its time evolution may undergo one or more phase transitions in **time.** These transitions are called DQPT.

- No parameter of the Hamiltonian is changed.
- Not related to any quantum phase transitions of the system.

Dynamical Quantum Phase Transitions

A many-body quantum system at zero temperature during its time evolution may undergo one or more phase transitions in **time.** These transitions are called DQPT.

- No parameter of the Hamiltonian is changed.
- Not related to any quantum phase transitions of the system.

Questions:

Can there be new phases and/or new phase transitions?

New? : Not the standard phases or criticality.

The Nature of DQPTs: Real Space Renormalization Group

- Exact analysis
- Scale invariance
  (Lattices built recursively)
- RG in the complex plane
- Model system
  - **Ising (2-state system)**

Dynamical Quantum Phase Transitions

- N-particle quantum system, Hamiltonian H. $H|n\rangle = E_n|n\rangle$
- Start in state: $|\psi_0\rangle = \sqrt{\frac{1}{\mathcal{W}}}\sum_n |n\rangle$ <u>(Not an eigenstate of H)</u>
  ($\mathcal{W}$ is the number of states)

Dynamical Quantum Phase Transitions

- N-particle quantum system, Hamiltonian H. $\quad H|n\rangle = E_n |n\rangle$
- Start in state: $\quad |\psi_0\rangle = \sqrt{\frac{1}{\mathcal{W}}} \sum_n |n\rangle$ (Not an eigenstate of H)
  ($\mathcal{W}$ is the number of states)
- Time evolution: At time t, $\quad |\psi_t\rangle = e^{-itH} |\psi_0\rangle \quad$ (setting $\hbar = 1$)
- **Loschmidt echo:**
  Probability of returning to the initial state:

$$P(t) = |L(t)|^2, \quad \text{where} \quad L(t) = \langle \psi_0 | \psi_t \rangle$$

Loschmidt amplitude

$$L(t) = \frac{1}{\mathcal{W}} \sum_n e^{-itE_n} \sim e^{-N\boldsymbol{f}(t)}$$

$\boldsymbol{f}(t)$: rate function.

DQPT

$$\boxed{L(t) = \frac{1}{\mathcal{W}} \sum_n e^{-itE_n} \sim e^{-N\boldsymbol{f}(t)}}$$

- $L(t)$ resembles the partition function but with complex Boltzmann factors.
  Partition function: $\quad Z = \sum_n e^{-\beta E_n}, \quad (\beta\text{: inverse temperature})$

DQPT

$$L(t) = \frac{1}{\mathcal{W}} \sum_n e^{-itE_n} \sim e^{-N\boldsymbol{f}(t)}$$

- $L(t)$ resembles the partition function but with complex Boltzmann factors.
  Partition function: $Z = \sum_n e^{-\beta E_n}$, ($\beta$: inverse temperature)
- Quantity of interest is the **rate function**: (analogous to free energy)

$$\text{Re}\, \boldsymbol{f}(t) = -\lim_{N\to\infty} \frac{1}{N} \ln |L(t)|$$

More is different!

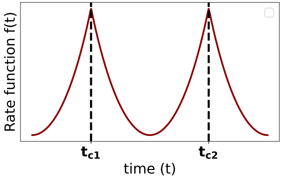
DQPT

$$L(t) = \frac{1}{\mathcal{W}} \sum_n e^{-itE_n} \sim e^{-N\boldsymbol{f}(t)}$$

- $L(t)$ resembles the partition function but with complex Boltzmann factors.
  Partition function: $Z = \sum_n e^{-\beta E_n}$, ($\beta$: inverse temperature)
- Quantity of interest is the **rate function**: (analogous to free energy)

$$\text{Re } \boldsymbol{f}(t) = -\lim_{N\to\infty} \frac{1}{N} \ln |L(t)|$$

More is different!



This is **DQPT**!

- Singular behaviour of $\boldsymbol{f}(t)$ at specific times:
  transition between two macroscopic phases.
- No parameter changed for the transition.
- Phases are characterized by the **probability distributions of the states.**

Transverse-field Ising Model

The Hamiltonian $\mathcal{H} = H + H_F$ has two parts:

(i) H tries to order the spins, and

(ii) H_F disrupts ordering by flipping the spins.

(*the 'transverse-field' term*)

Eigenstates, eigenvalues of H for N spins, $(\alpha = 1, 2, 3, \dots)$

$$H\,|n\rangle = E_n\,|n\rangle\,,$$

$$|n\rangle = \bigotimes_j |\alpha\rangle_j \equiv |\alpha_1\alpha_2\cdots\alpha_N\rangle\,,$$

$$E_n = -J\sum_{\langle jk\rangle}\delta_{\alpha_j,\alpha_k}.$$

Relevant eigenfunction of H_F

$$|\psi_0\rangle = \bigotimes_j \left[\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\right]_j.$$

The Quench: Protocol

$$\mathcal{H} = H + H_F, \text{ where } \begin{cases} H\text{: nearest-neighbour ferromagnetic interaction} \\ H_F\text{: Transverse field, disrupts ordering} \end{cases}$$

- **Start with large** H_F, Initial state: $|\psi_0\rangle$

The Quench: Protocol

$$\mathcal{H} = H + H_F, \text{ where } \begin{cases} H\text{: nearest-neighbour ferromagnetic interaction} \\ H_F\text{: Transverse field, disrupts ordering} \end{cases}$$

- **Start with large H_F**, Initial state: $|\psi_0\rangle$
- At time $t = 0$, set $H_F = 0$. The system then evolves with H.

The Quench: Protocol

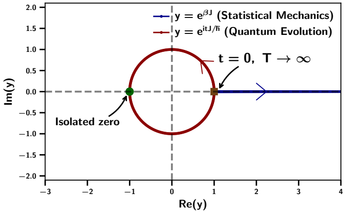
$$\mathcal{H} = H + H_F, \text{ where } \begin{cases} H\text{: nearest-neighbour ferromagnetic interaction} \\ H_F\text{: Transverse field, disrupts ordering} \end{cases}$$

- **Start with large H_F**, Initial state: $|\psi_0\rangle$
- At time $t = 0$, set $H_F = 0$. The system then evolves with H.
- **Loschmidt echo:** $L(t) = \left\langle \psi_0 \middle| e^{-itH} \middle| \psi_0 \right\rangle, \ (\hbar = 1)$
- The natural variable is $\boxed{y = e^{itJ}}$

The Quench: Protocol

$$\mathcal{H} = H + H_F, \text{ where } \begin{cases} H\text{: nearest-neighbour ferromagnetic interaction} \\ H_F\text{: Transverse field, disrupts ordering} \end{cases}$$

- **Start with large H_F**, Initial state: $|\psi_0\rangle$
- At time $t = 0$, set $H_F = 0$. The system then evolves with H.
- **Loschmidt echo:** $L(t) = \langle \psi_0 | e^{-itH} | \psi_0 \rangle$, $(\hbar = 1)$
- The natural variable is $\boxed{y = e^{itJ}}$



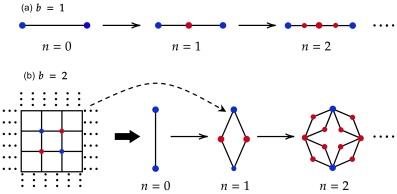
The quantum dynamics is along the unit circle $|y| = 1$ in the complex y-plane.

Thermal problem: $y = e^{\beta J}$: real axis from 0 to ∞.

Hierarchical lattices

Griffiths-Kaufman(1982), Derrida etal(1983)

Consider lattices constructed iteratively:



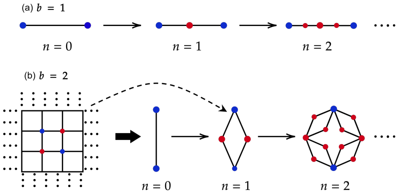
From one bond to larger lattices.

From a large lattice, on coarse graining one gets a smaller lattice: **self-similarity.**

Hierarchical lattices

Griffiths-Kaufman(1982), Derrida etal(1983)

Consider lattices constructed iteratively:



From one bond to larger lattices.

From a large lattice, on coarse graining one gets a smaller lattice: **self-similarity.**

Self-similarity and Renormalization group methods
$\Rightarrow$ Exact determination of

- Loschmidt amplitude: $L(y)$.
- Zeros in the complex y-plane.

Renormalization Group (RG) Approach

- Combine micro d.f. (S) together to define an effective d.f. (S')

  Interactions among S $\Rightarrow$ interactions among S'.

  e.g., $\boxed{y' = R(y)}$. Macro properties remain invariant.

Renormalization Group (RG) Approach

- Combine micro d.f. (S) together to define an effective d.f. (S')

  Interactions among S $\Rightarrow$ interactions among S'.

  e.g., $\boxed{y' = R(y)}$. Macro properties remain invariant.

- This coarse-graining process can be repeated indefinitely.
  - **Stable fixed points:** determine the phases of the system.
  - **Unstable fixed points:** the phase transitions or critical points.

  Stability analysis yields the results of interest.

Complex Dynamics: RG on the complex plane

$$\text{RG: } y' = R(y) = \left(\frac{y^2+1}{2y}\right)^b, \quad y = \begin{cases} e^{itJ}, & \text{(quantum)} \\ e^{\beta J}, & \text{(thermal)} \end{cases}$$

- Ising: $b = 1$ (one-d) or $b = 2$ (two-d).

Complex Dynamics: RG on the complex plane

$$\text{RG: } y' = R(y) = \left(\frac{y^2+1}{2y}\right)^b, \quad y = \begin{cases} e^{itJ}, & \text{(quantum)} \\ e^{\beta J}, & \text{(thermal)} \end{cases}$$

- Ising: $b = 1$ (one-d) or $b = 2$ (two-d).
- Complex dynamics:
  Flow: Track successive y': $R(y), R(R(y)), \ldots, R^{(n)}(y), \ldots$
  A flow is generally to one of the stable fixed points $\boxed{y^* = R(y^*)}$.

Complex Dynamics: RG on the complex plane

$$\text{RG: } y' = R(y) = \left(\frac{y^2+1}{2y}\right)^b, \quad y = \begin{cases} e^{itJ}, & \text{(quantum)} \\ e^{\beta J}, & \text{(thermal)} \end{cases}$$

- Ising: $b = 1$ (one-d) or $b = 2$ (two-d).
- Complex dynamics:
  Flow: Track successive y': $R(y), R(R(y)), \ldots, R^{(n)}(y), \ldots$
  A flow is generally to one of the stable fixed points $\boxed{y^* = R(y^*)}$.
- If the trajectories have sensitive dependence on the initial point, these points form a special set, called the **Julia set.** (Zeros of $L(y)$)
  The boundary of the basins of attraction of the stable fixed points.

Julia set and DQPT

- The RG transformation $y' = R(y)$ has a characteristic Julia set.
- Those are points that do not flow to stable fixed points.
- Those are the zeros of the partition function.

- The quantum dynamics is along the unit circle $|y| = 1$.

Julia set and DQPT

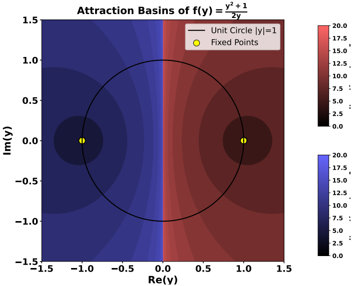
- The RG transformation $y' = R(y)$ has a characteristic Julia set.
- Those are points that do not flow to stable fixed points.
- Those are the zeros of the partition function.

- The quantum dynamics is along the unit circle $|y| = 1$.

The DQPTs correspond to the intersection of the unit circle with the Julia set.

Surprises in 1D!

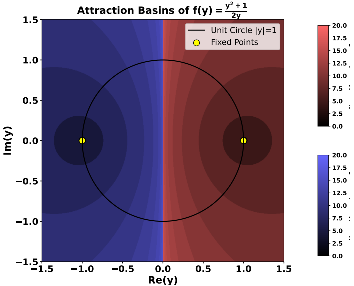
Unitary evolution of $|\psi_0\rangle$



Julia Set in 1D: Fractal structure from iterative dynamics.

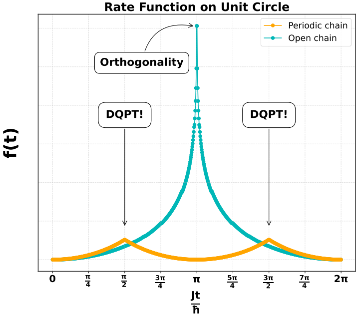
Surprises in 1D!

Unitary evolution of $|\psi_0\rangle$



Julia Set in 1D: Fractal structure from iterative dynamics.

- Julia: the imaginary axis.
- Intersections: Ordered states (transition points)



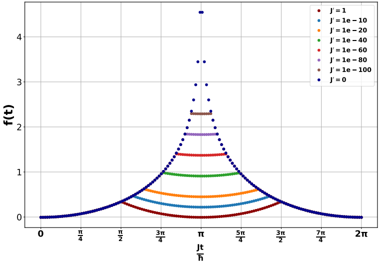
RG results for different boundary conditions for 1D TFIM.

- No thermal counterpart of the middle branch in one-dimension.
- Suppression of phase transition by changing the boundary conditions.

Intermediate Phase and Transfer Matrix

Boundary sensitivity for DQPT

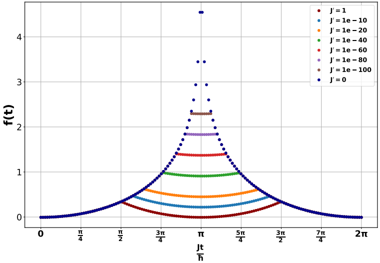
The influence of the boundary coupling J' (open: $J' = 0$, closed: $J' = 1$) reveals how deeply the boundary affects the bulk dynamics and transitions.



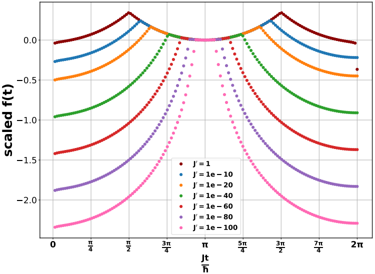
Free energy f for finite N, $N = 100$.

Boundary sensitivity for DQPT

The influence of the boundary coupling J' (open: $J' = 0$, closed: $J' = 1$) reveals how deeply the boundary affects the bulk dynamics and transitions.



Free energy f for finite N, $N = 100$.



scaled: $f - \frac{\ln\left(y'-1\right)}{N}$ for $N = 100$.

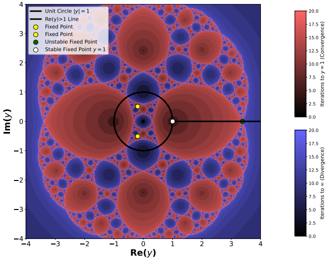
Conclusion

- A quantum system during time evolution explores
  the whole Hilbert space $\Longrightarrow$ **Origin of the transition**.
- Phases transitions: need not be thermal type.
- Complex dynamics, Julia set provide the framework for the phenomena.

Acknowledgment/Ref:
SMB, Phys Rev B 109, 035130 (2024)

$b = 2$ Ising Case

Zeros form a self-similar fractal pattern (complex-y plane):



- There are **4 intersections**.
- Time takes us around the unit circle.
- These critical points occur repeatedly.

$\underbrace{\text{critical points}}_{\text{classical Curie points}}$

There are oscillations from para to ferro phases.

The critical points are like Curie points.