Anomalous transport in long-ranged open quantum systems

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Part-A

Anomalous transport in long-ranged open quantum systems

PRB Letter (2024)

Abhinav Dhawan
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Katha Ganguly
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Part-B

Quantum injection of effectively or inherently interacting particles

Ongoing (2025)

Tamoghna Ray
(ICTS)

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Bijay Agarwalla
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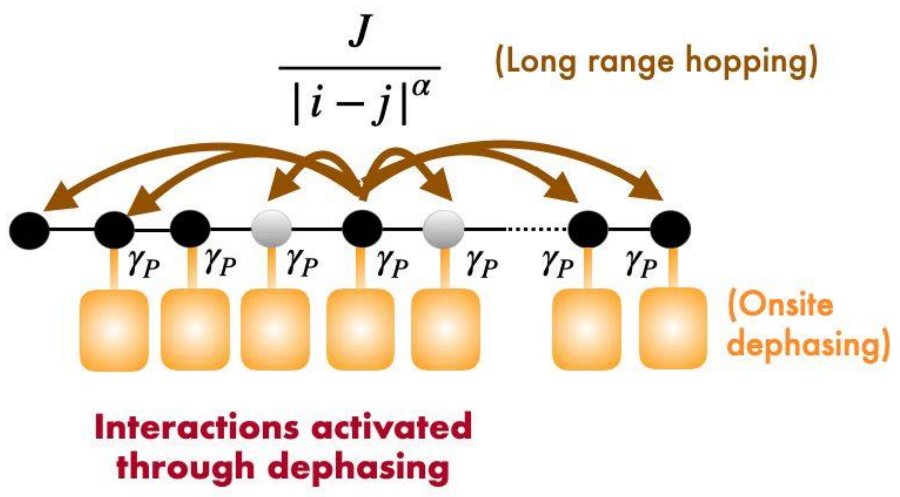
Other completely unrelated works on classical long-ranged (power law) models

J. Kethepalli, **MK**, A. Kundu, H. Spohn, J. Stat. Mech. (2025)
S. Santra, J. Kethepalli, S. Agarwal, A. Dhar, **MK**, A. Kundu, PRL (2022)
J. Kethepalli, **MK**, A. Kundu, S. N. Majumdar, D. Mukamel, G. Schehr, J. Stat. Mech, (2021)
B. Kiran, D. A. Huse, **MK**, PRE (2021),
S. Agarwal, A. Dhar, **MK**, A. Kundu, S. N. Majumdar, D. Mukamel, G. Schehr, PRL (2019), Editors' Suggestion

PART-A

Setup



- one-dimensional fermionic lattice system with long-ranged power-law decaying hopping with exponent α.

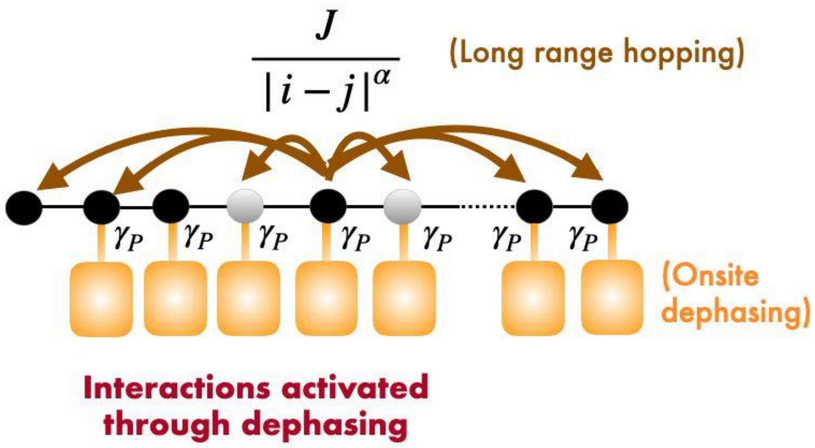
- system is further subjected to dephasing noise in the bulk.

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^{\alpha}} \left[\sum_{r=1}^{N-m} \hat{c}_r^{\dagger} \hat{c}_{r+m} + \hat{c}_{r+m}^{\dagger} \hat{c}_r \right]$$

- To understand the steady-state transport properties, the lattice chain is further connected to a source and a drain reservoir at its two ends, and these reservoirs are maintained at chemical potentials μ_S and μ_D, respectively.

- In addition to the boundary reservoirs at each lattice site we attach Büttiker voltage probes with uniform coupling strength denoted by γ_p. This is done to mimic processes where the phase coherence of particles built during Hamiltonian evolution is lost due to inevitable surroundings

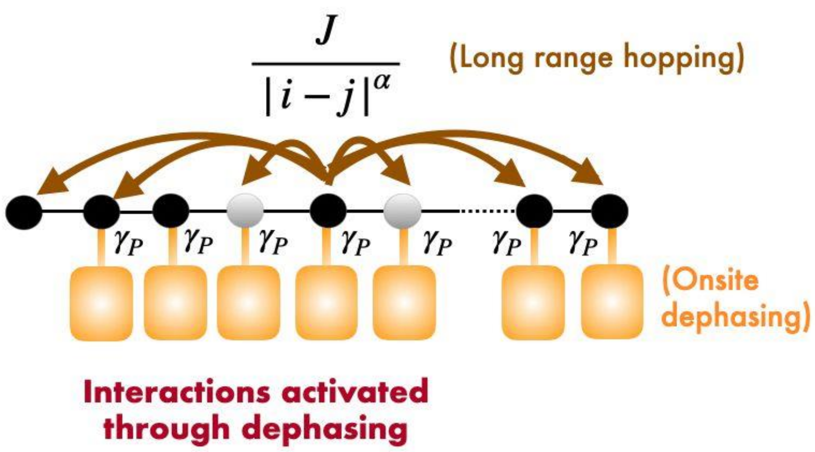
- Approach is widely employed to understand effective many-body transport



Non-equilibrium steady state transport

- We are interested in studying the NESS electronic conductance to characterize transport.
- We focus here in the linear response regime and set for boundary reservoirs $\mu_D = \mu$, $\mu_S = \mu + \delta\mu$, and for the probes $\mu_i = \mu + \delta\mu_i$, $i = 1,2, . . . ,N$.
- At zero temperature, the conductance corresponding to the left to right charge current can be exactly obtained as

Non-equilibrium steady state transport



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- At zero temperature, the conductance corresponding to the left to right charge current can be exactly obtained as

$$\boxed{\begin{aligned}\mathcal{G}(\mu) &= \gamma^2|G_{1N}(\mu)|^2 + \gamma^2\gamma_P \sum_{n,j=1}^{N} |G_{Nn}(\mu)|^2\, \mathcal{W}_{nj}^{-1}(\mu)\\ &\quad \times |G_{j1}(\mu)|^2,\end{aligned}}$$

where $G(\omega) = [\omega I - h_S - \boldsymbol{\Sigma}_L(\omega) - \boldsymbol{\Sigma}_R(\omega) - \boldsymbol{\Sigma}_P(\omega)]^{-1}$

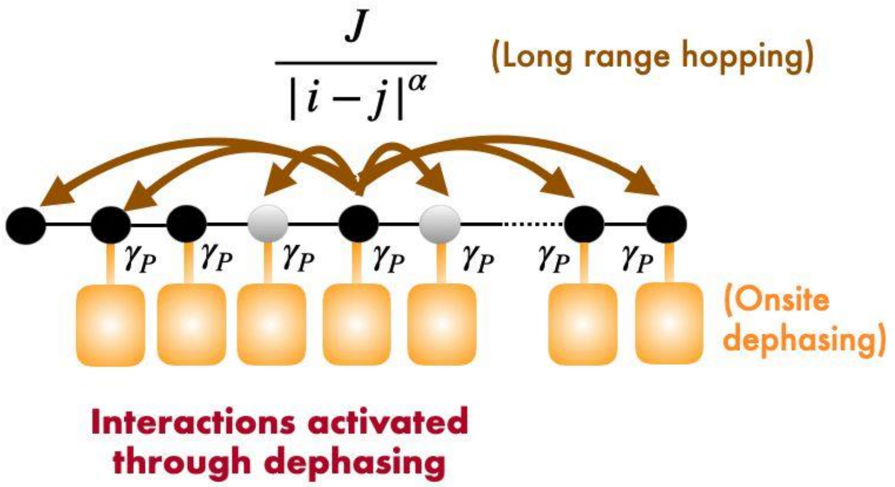
$$\boldsymbol{\Sigma}_L(\omega)|_{11} = -i\gamma/2,\ \ \boldsymbol{\Sigma}_R(\omega)|_{NN} = -i\gamma/2,\ \text{and}\ \ \boldsymbol{\Sigma}_P(\omega)|_{jj} = -i\gamma_p/2$$

$$\mathcal{W}_{ij} = -\gamma_p|G_{ij}|^2\ \forall\ i \neq j$$

$$\mathcal{W}_{ii} = \gamma(|G_{i1}|^2 + |G_{iN}|^2) + \gamma_p \sum_{j\neq i}^{N} |G_{ij}|^2$$

With this expression, we can compute the NESS conductance via extensive numerics

Non-equilibrium steady state transport



In the the context of long-range systems, a natural question is the behavior of system size scaling exponent δ with respect to long-range hopping exponent α.

$$\mathcal{G} \sim \frac{1}{N^{\delta}}$$

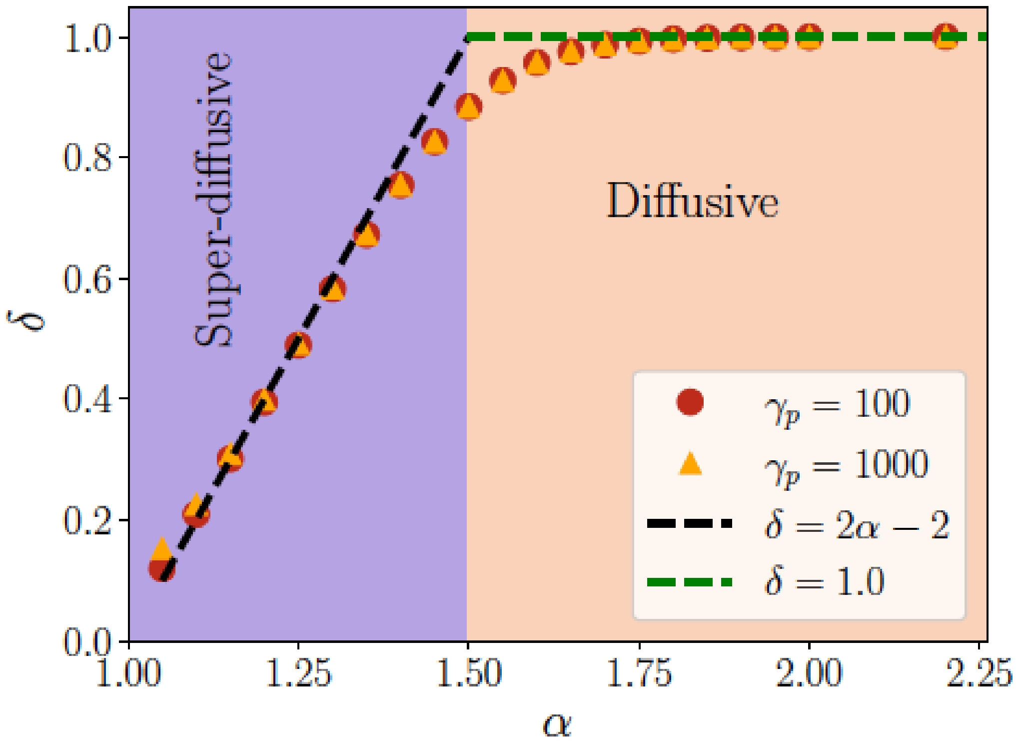
Different universality classes of transport

$\delta = 0$	Ballistic
$0 < \delta < 1$	Super-diffusive
$\delta = 1$	Diffusive
$\delta > 1$	Sub-diffusive

Central question: How does δ depend on α ?

Sample reviews of anomalous behaviour:
Dhar, Adv. Phys (2008)
Landi *et al*, RMP (2022)

Non-equilibrium steady state transport



Recall

$$\mathcal{G}(\mu) = \gamma^2 |G_{1N}(\mu)|^2 + \gamma^2 \gamma_p \sum_{n,j=1}^{N} |G_{Nn}(\mu)|^2 \, \mathcal{W}_{nj}^{-1}(\mu) \times |G_{j1}(\mu)|^2,$$

Recall $\mathcal{G} \sim \dfrac{1}{N^{\delta}}$

$$\delta(\alpha) = \begin{cases} 2\alpha - 2 & \text{for} \quad 1 < \alpha < 1.5 \\ 1 & \text{for} \quad \alpha > 1.5 \end{cases}$$

(relatively long-ranged)

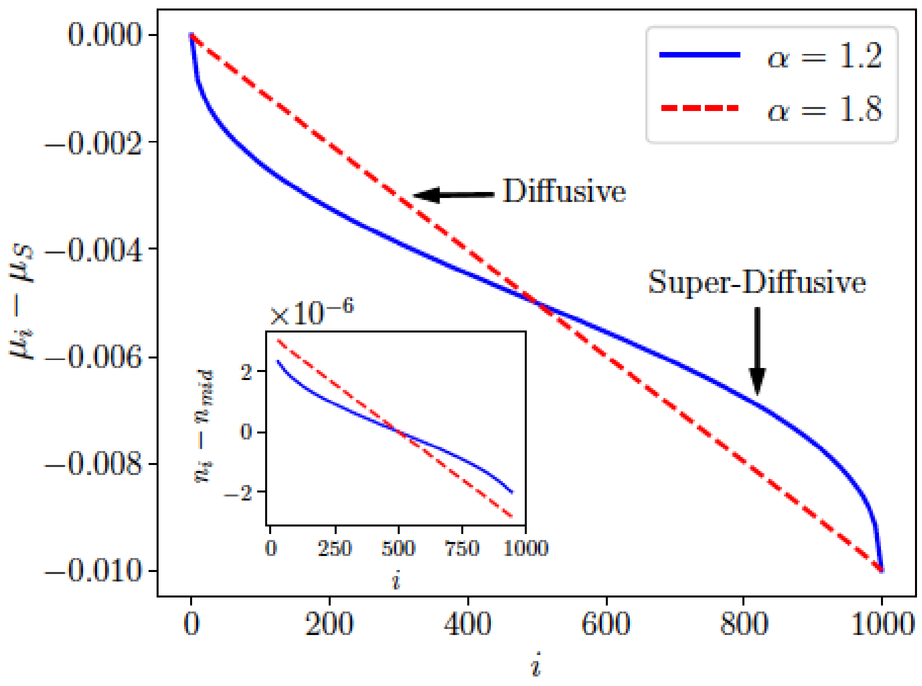
(relatively short-ranged)

What about local chemical potential profiles ?

The zero-particle NESS current from each of the probes ensures a unique chemical potential value at each lattice site and is given by

Purkayastha, Sanyal, Dhar, **MK** PRB (2018)

$$\mu_i = \mu_S + \gamma\, \gamma_p\, (\mu_D - \mu_S) \sum_{j=1}^{N} \mathcal{W}_{ij}^{-1}\, |G_{jN}|^2$$



- We show the local chemical potential profile for two different values of α, one within the superdiffusive regime and one within the diffusive regime.
- For α > 1.5, we notice a linear shape, which is a hallmark of conventional diffusive transport.
- For α < 1.5, the shape is nonlinear, which is a fingerprint of anomalous transport.

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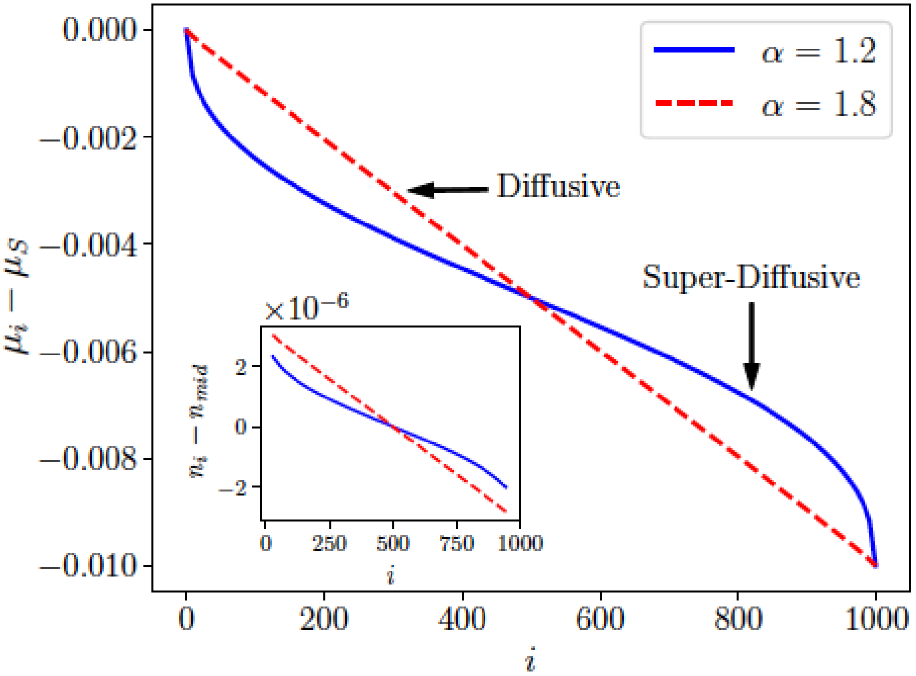

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With this detailed understanding of transport regimes when boundary reservoirs are attached, it is natural to explore the possible relation with the density profile evolution in the absence of reservoirs (but retaining the dephasing mechanism).

Time dynamics of single-particle density profile

- We study the quantum dynamics of single-particle excitation for the long-range lattice setup in the absence of the boundary reservoirs while keeping the dephasing mechanism intact.
- We model the lattice and this dephasing mechanism by a Lindblad quantum master equation

$$\frac{d\hat{\rho}}{dt} = -i[\hat{\mathcal{H}}_S, \hat{\rho}] + \kappa \sum_{i=1}^{N} \left[\hat{n}_i \,\hat{\rho}\, \hat{n}_i - \frac{1}{2}\{\hat{n}_i^2, \hat{\rho}\} \right]$$

Dolgirev et al, PRB 2020

where recall

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^\alpha} \left[\sum_{r=1}^{N-m} \hat{c}_r^\dagger \hat{c}_{r+m} + \hat{c}_{r+m}^\dagger \hat{c}_r \right]$$

Special feature:
evolution of any n-point correlator can be expressed through operators whose order is n or less

- κ represents the effective coupling strength characterising dephasing
- For such a setup, we are interested in studying the time dynamics of a single-particle density profile P(x, t), which is initially localized at the middle site of the lattice.
- One can obtain P(x, t) directly following the Lindblad where one has to deal with N x N matrices (due to special feature).
- An alternative route is to follow unitary quantum unraveling which records the information of wave function in each quantum trajectory.
- The unraveling is carried out by introducing classical δ-correlated Gaussian noise at each lattice

Time dynamics of single-particle density profile

$$\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_S + \sum_{l=1}^{N} \xi_l(t)\,\hat{n}_l$$

where recall

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^\alpha}\left[\sum_{r=1}^{N-m} \hat{c}_r^\dagger \hat{c}_{r+m} + \hat{c}_{r+m}^\dagger \hat{c}_r\right]$$

$$\langle \xi_l(t)\rangle = 0, \ \text{and} \ \langle \xi_l(t)\,\xi_p(t')\rangle = \kappa\,\delta_{lp}\,\delta(t-t')$$

$$P_\xi(x,t) = |\psi_x^\xi(t)|^2$$

(single-particle density profile for a single noise realization)

$$\psi_x^\xi(t) = G_\xi\left(\frac{N+1}{2}+x,t\,\middle|\,\frac{N+1}{2},0\right) = \langle x_1|\mathcal{T}e^{-i\int_{t_0}^{t_1} h(t')dt'}|x_0\rangle$$

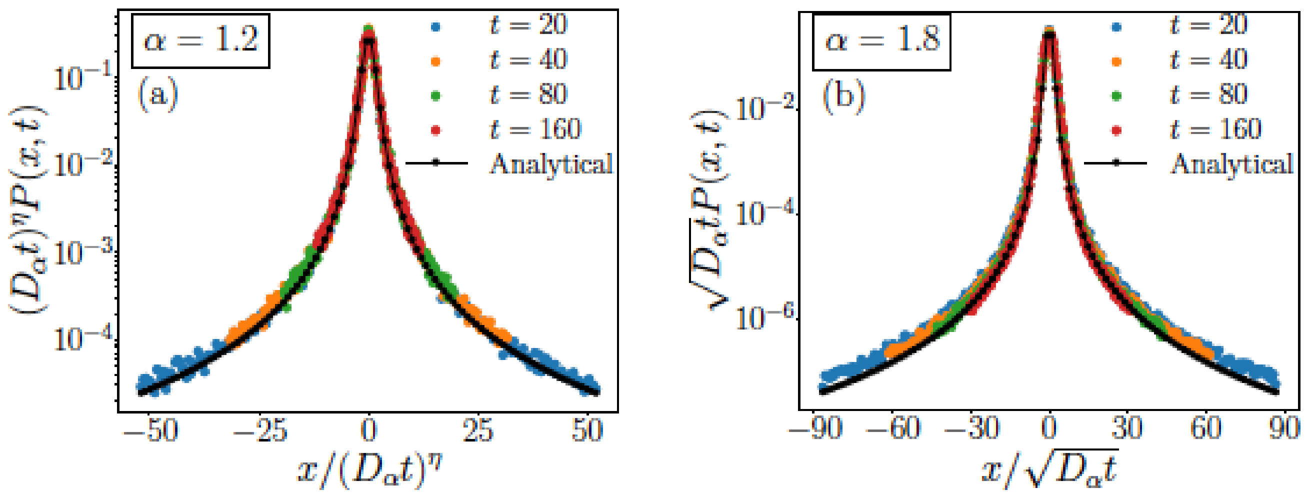
$$\mathcal{T}e^{-i\int_{t_i}^{t_i+dt} h(t')dt'} \approx e^{-i\left(h_S dt+\sqrt{dt}\,\mathcal{M}(t_i)\right)}$$

diagonal noise matrix

$$P(x,t) = \overline{P_\xi(x,t)}$$

average over all quantum trajectories

Time dynamics of single-particle density profile



Relatively long ranged

Relatively short ranged

We can show that $$P(x,t) = \frac{1}{(D_\alpha t)^\eta} f\left(\frac{x}{(D_\alpha t)^\eta}\right)$$

$$\eta(\alpha) = \begin{cases} \frac{1}{2\alpha - 1} & \text{for} \quad 1 < \alpha < 1.5 \\ \frac{1}{2} & \text{for} \quad \alpha > 1.5 \end{cases}$$

(relatively long-ranged)

(relatively short-ranged)

We can analytically calculate the scaling function also. For this we derive a fractional diffusion equation

Derivation of a fractional diffusion equation

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{\mathcal{H}}_S, \hat{\rho}(t)] + \kappa \sum_{i=(-N+1)/2}^{(N-1)/2} \left(\hat{n}_i \rho(t) \hat{n}_i - \frac{1}{2}\{\hat{n}_i^2, \hat{\rho}(t)\} \right)$$

where recall

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^\alpha} \left[\sum_{r=1}^{N-m} \hat{c}_r^\dagger \hat{c}_{r+m} + \hat{c}_{r+m}^\dagger \hat{c}_r \right]$$

Let us define $D_{m,n}(t) \equiv \langle \hat{c}_m^\dagger(t) \hat{c}_n(t) \rangle$

Then

$$\frac{d}{dt} D_{m,n}(t) = iJ \sum_{l \neq 0} \left(\frac{D_{m,n+l}(t) - D_{m+l,n}(t)}{|l|^\alpha} \right) + \kappa\,(\delta_{m,n} - 1) D_{m,n}(t)$$

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- change the variables to $\tau = \kappa\, t$, and work in the strong dephasing limit i.e., $\kappa \gg J$
- introduce a parameter $\epsilon_l = \frac{J\kappa^{-1}}{l^\alpha}$

We then get,

$$\frac{d}{d\tau} D_{m,n}(\tau) = i \sum_{l \neq 0} \epsilon_{|l|} \left(D_{m,n+l}(\tau) - D_{m+l,n}(\tau) \right) + \lambda_{m,n} D_{m,n}(\tau) \qquad \lambda_{m,n} = \delta_{m,n} - 1$$

- We will expand the solution $D_{m,n}$ in terms of a small parameter

$$\epsilon = \sum_{l \geq 1} \epsilon_l = \frac{J}{\kappa} \xi(\alpha),$$

Riemann-Zeta function

Derivation of a fractional diffusion equation

We first seek for a convergent solution for $D_{m,n}(\tau)$ by expanding it in powers of ϵ as

$$D_{m,n}(\tau) = D^{(0)}_{m,n}(\tau) + \epsilon\, D^{(1)}_{m,n}(\tau) + \epsilon^2\, D^{(2)}_{m,n}(\tau) + \dots$$

A multiple scale analysis is employed to finally arrive at
[details in supplementary material of A. Dhawan, K. Ganguly, **M. K**, B. K. Agarwalla, PRB(2024)]

$$\frac{d}{dt} D_{m,m}(t) = \frac{2J^2}{\kappa} \sum_{l \neq 0} \left(\frac{D_{m+l,m+l}(t) - D_{m,m}(t)}{|l|^{2\alpha}} \right)$$

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This is the central equation which describes that a classical master equation for the population that satisfies a fractional diffusion equation.

nearest neighbor case $(\alpha \to \infty)$ $\Longrightarrow$ $l = \pm 1$ becomes conventional diffusion equation with diffusion constant $\Delta = 2J^2/\kappa$

Remarkably, it is possible to find explicit scaling forms,

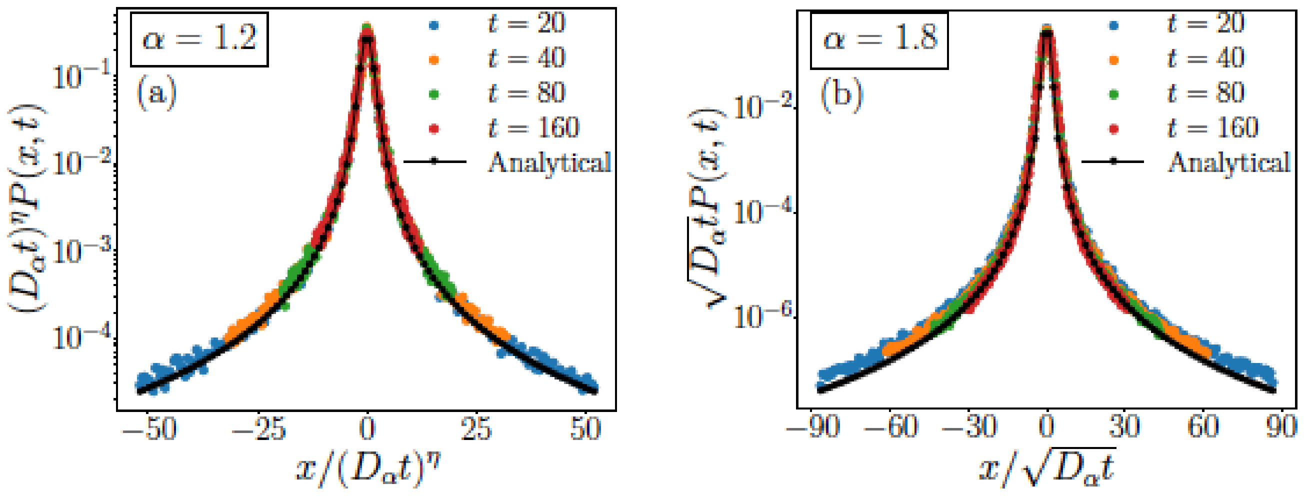
$$f\left(z = \frac{x}{(D_\alpha t)^\eta}\right) \approx \begin{cases} F_\alpha(z), & 1 < \alpha < 1.5 \\ G(z), & \alpha > 1.5 \end{cases}$$

where

$$\eta(\alpha) = \begin{cases} \frac{1}{2\alpha - 1} & \text{for} \quad 1 < \alpha < 1.5 \\ \frac{1}{2} & \text{for} \quad \alpha > 1.5 \end{cases}$$

Schuckert et al, PRB 2020

Fractional diffusion equation and anamolous profile



Relatively long ranged

Relatively short ranged

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$$F_\alpha(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-|k|^{2\alpha-1}} e^{izk}, \quad G(z) = \frac{e^{-z^2/4}}{2\sqrt{\pi}}$$

Generalized diffusion coefficient

$$D_\alpha \approx \begin{cases} -2\Delta\Gamma(1-2\alpha)\sin(\pi\alpha) & \text{for} \quad 1 < \alpha < 1.5 \\ \frac{\Delta}{2\alpha-3} & \text{for} \quad \alpha > 1.5 \end{cases}$$

Relation between the system-size scaling exponent of steady-state conductance (δ) and the exponent η associated with the space-time collapse.

Dhar, Saito, Derrida, PRE (2013)
Zaburdaev, Denisov, and Klafter, RMP (2015)

- We note that the anomalous superdiffusive transport in long-range systems is often governed by Lévy flights
- In our case, we find an intriguing connection with a well-known random walk model in low-dimensional systems, i.e., Lévy walk
- <u>Specific Lévy walk case:</u> for a single particle, each step of the walk consists in (i) choosing a time of flight τ from a given distribution $\varphi(\tau)$ and (ii) moving it at speed v over a distance $x = v\tau$ in either direction, with equal probability.
- The typical space-time scaling in the central region of a pulse dictated by Lévy walker is $x \sim t^{1/\beta}$ where β is the exponent associated with the time of flight distribution of a Lévy walker
- If such a system is connected to boundary reservoirs, then the system size scaling of conductance is given by $N^{1-\beta}$
- For our setup, following the time dynamics of single-particle density profile, we find that $\beta = 2\alpha - 1$ which immediately gives a relation between the exponents δ and α as $\delta = 2\alpha - 2$.
-

Conclusions of Part A

A. Dhawan, K. Ganguly, **M. K**, B. K. Agarwalla, PRB(2024)

- Studied quantum transport properties in one-dimensional long-range fermionic system subjected to dephasing noise.
- Interesting interplay between the incoherent dephasing mechanism and the coherent long-range hopping results in an anomalous behavior.
- Clear departure from conventional diffusive transport is manifested both in NESS transport and in density profile dynamics.
- Density profile dynamics was shown to emerge from a fractional diffusion equation, which was derived following the multiple scale analysis technique.
- This aided in further cementing the relationship between conductance scaling exponent δ and the long-range hopping exponent α.
- Recent experiments [Joshi *et al*, Science 2022] in interacting quantum spin chains have reported such anomalous transport by studying unequal space-time spin-spin correlators. Our work reveals a possible intriguing connection between such interacting quantum systems and systems subjected to dephasing noise mechanisms

Part B

Quantum Injection of particles

Central question: What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting

Main quantities of interest in this part of the talk are spatial density profile and the total number of particles.

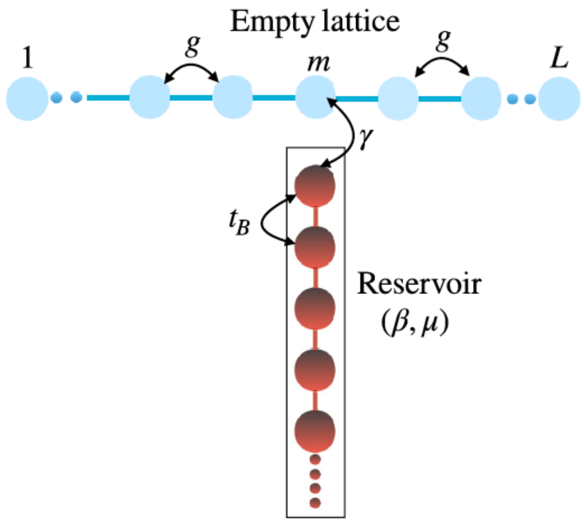
Part B

Quantum Injection of particles

Central question: What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting

Main quantities of interest in this part of the talk are spatial density profile and the total number of particles.

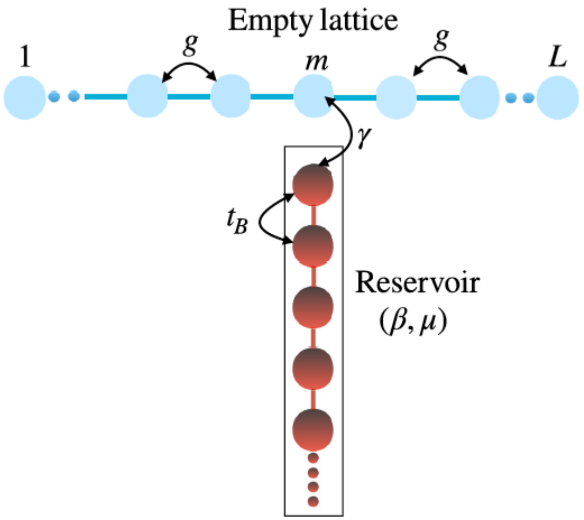
Before proceeding to the case (i) and (ii) mentioned above, we will discuss an earlier result, Trivedi, Gupta, Agarwalla, Dhar, Mk, Kundu, Sabhapandit (PRA 2023) on "Filling an empty lattice by local injection of quantum particles"



- Setup to study quantum dynamics of filling an empty lattice of size L by connecting it locally with an equilibrium bath that injects noninteracting bosons or fermions.
- We will mainly discuss the Lindblad approach

Krapivsky, Mallick, Sels (2019,2020)
Butz, Spohn (2010)

Many past and recent literature on "localized loss"



$$H_S = g\sum_{i=1}^{L-1}(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i),$$

System

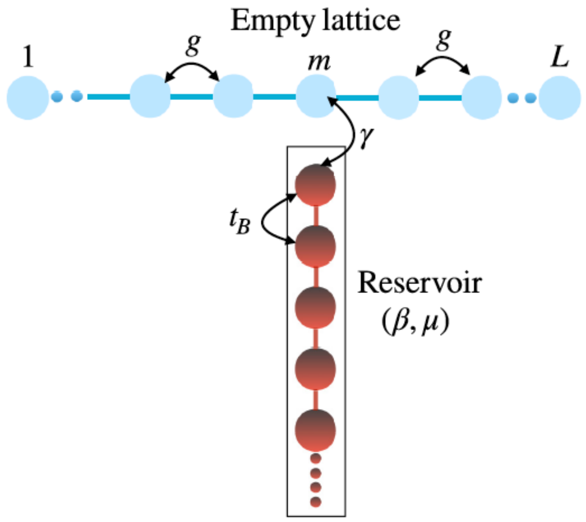
Gain of particles

$$\dot{\rho}_{SS} = i[\rho_{SS}, H_S] + \Gamma_G[2a_m^\dagger \rho_{SS} a_m - \{a_m a_m^\dagger, \rho_{SS}\}]$$
$$+ \Gamma_L[2a_m \rho_{SS} a_m^\dagger - \{a_m^\dagger a_m, \rho_{SS}\}],$$

Reduced density matrix of system

Loss of particles

m stands for
middle injection
point



$$H_S = g\sum_{i=1}^{L-1}(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i),$$

System

Gain of particles

$$\dot{\rho}_{SS} = i[\rho_{SS}, H_S] + \Gamma_G[2a_m^\dagger \rho_{SS} a_m - \{a_m a_m^\dagger, \rho_{SS}\}] + \Gamma_L[2a_m \rho_{SS} a_m^\dagger - \{a_m^\dagger a_m, \rho_{SS}\}],$$

Reduced density matrix of system

Loss of particles

m stands for middle injection point

We will write down equation for the correlation matrix $C_{i,j} = \langle a_i^\dagger a_j \rangle$

$$\frac{dC_{i,j}}{dt} = i\,g\,(C_{i-1,j} - C_{i,j+1} + C_{i+1,j} - C_{i,j-1}) - \Gamma'(\delta_{im} + \delta_{jm})\,C_{i,j} + 2\,\Gamma_G \delta_{mi}\delta_{mj}$$

$\Gamma' = \Gamma_L \mp \Gamma_G$ plus/minus stands for bosons/fermions respectively

Spatial density profile $n_i(t) = 2\,\Gamma_G \int_0^t d\tau\, |\tilde{S}_i(\tau)|^2$

$$\tilde{S}_i(\tau) = J_i(2\,g\,\tau) - \Gamma' \int_0^\tau d\bar{t}\, e^{-\Gamma' \bar{t}} \times \left(\frac{\tau - \bar{t}}{\tau + \bar{t}}\right)^{i/2} J_i[2\,g\sqrt{\tau^2 - \bar{t}^2}]$$

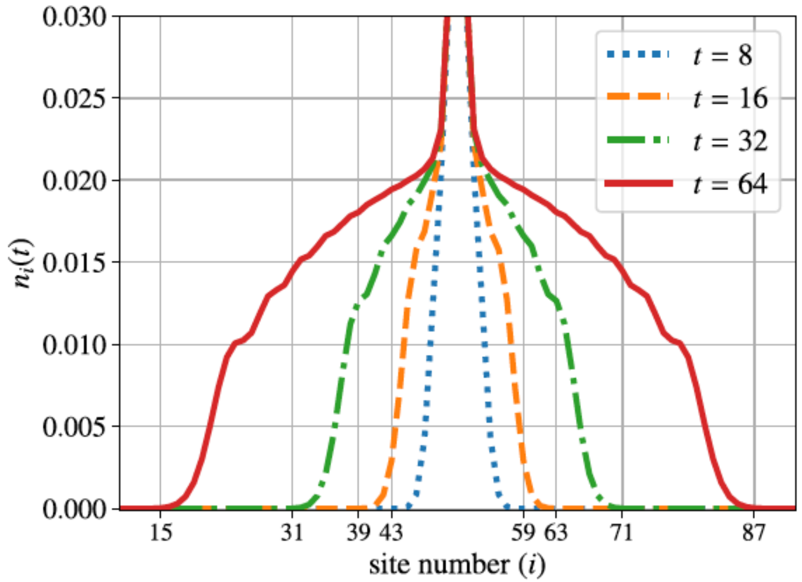
Interestingly, it turns out that $n_i(t)$ can admit an interesting scaling form. To do so, let us take the limits

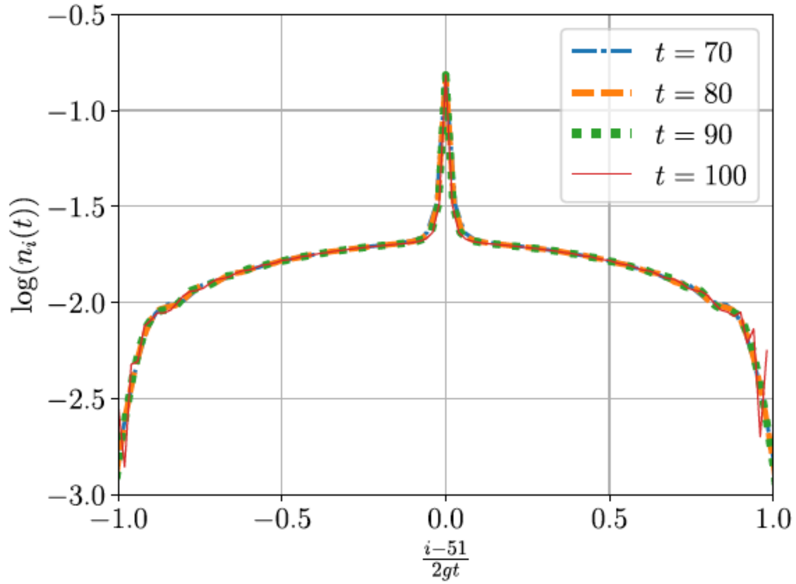
$$i \to \infty, \quad t \to \infty, \quad \nu = \frac{i}{2\,g\,t} \sim O(1)$$

$$n_i(t) = \Phi\left(\frac{i}{2gt}\right), \quad \text{where} \quad \Phi(\nu) = \frac{\tilde{g}\,(1+\nu\,\tilde{g})[\ln(1+\nu\,\tilde{g}) - \ln(\tilde{g}+\nu-\sqrt{(\tilde{g}^2-1)(1-\nu^2)})] - \sqrt{(\tilde{g}^2-1)(1-\nu^2)}}{(\tilde{g}^2-1)^{3/2}(\nu\,\tilde{g}+1)}, \quad 0<\nu<1$$

$$\tilde{g} = \frac{2g}{\Gamma'} = \frac{4g}{J(0)}$$

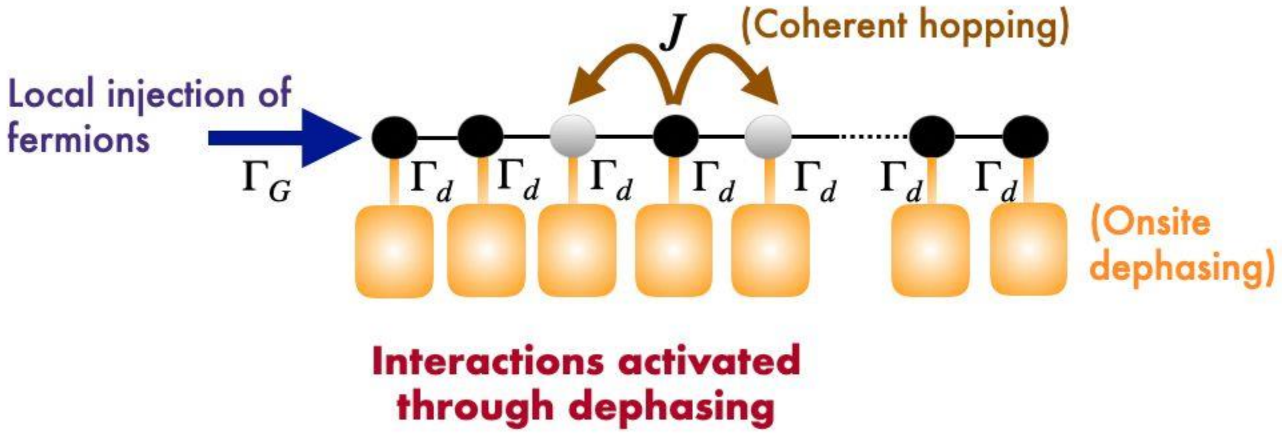
Total occupation $\quad N(t) = -\dfrac{2\,\Gamma_G\,t}{\pi\,(1-\tilde{g}^2)}\left[2\tilde{g} - \pi\,(1-\tilde{g}^2) + 2(1-2\tilde{g}^2)\dfrac{\cos^{-1}(\tilde{g})}{\sqrt{1-\tilde{g}^2}}\right]$





What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting.

We will first discuss case (i)



$$H_S = -J \sum_{i=1}^{L} c_i^\dagger c_{i+1} + h.c.$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S, \rho] + \Gamma_G \left[2c_1^\dagger \rho c_1 - \{c_1 c_1^\dagger, \rho\} \right] - \frac{\Gamma_d}{2} \sum_{i=1}^{L} [n_i, [n_i, \rho]]$$

What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting.

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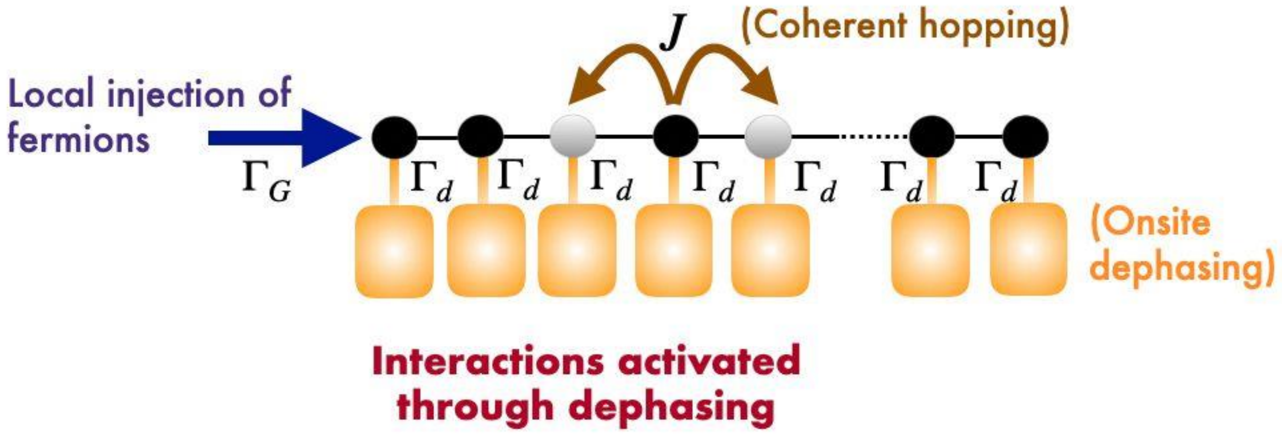

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$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m}+\delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n}-1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

where $\quad C_{m,n}(t) = \langle c_m^\dagger(t) c_n(t)\rangle$

This equation is difficult to solve analytically.

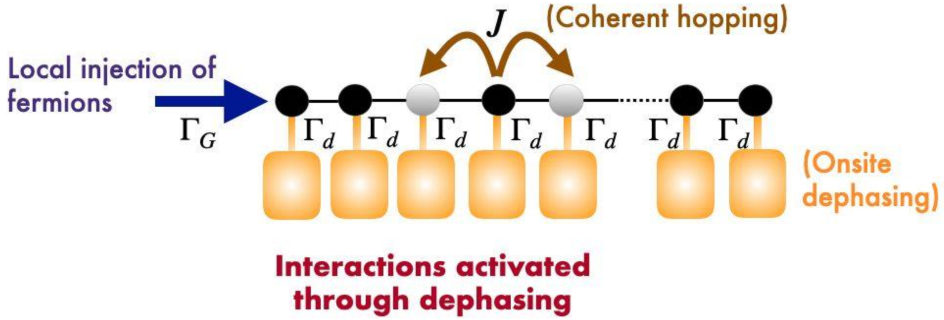
[without gain]
Ishiyama, Fujimoto, Sasamoto
J. Stat. Mech. (2025)

Recall

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

where $C_{m,n}(t) = \langle c_m^\dagger(t) c_n(t)\rangle$

Before attempting an analytical solution, we will present below the numerical findings.



Recall

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

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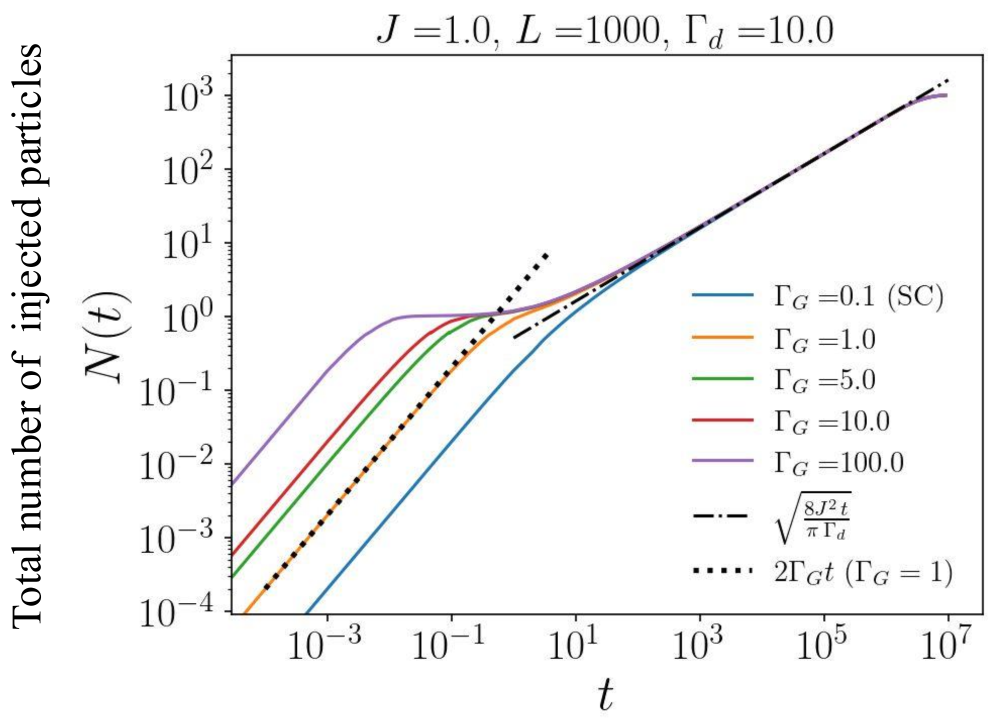
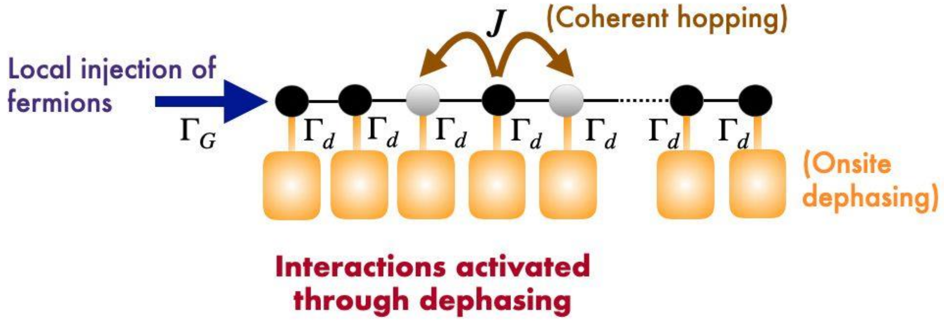


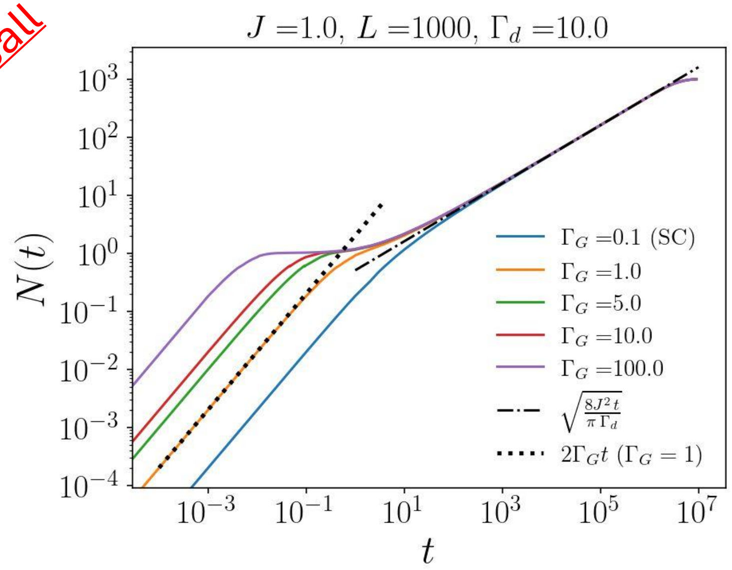


Key findings from numerics

- There is a linear early time behaviour that depends on injection rate.
- There is a square-root behaviour at later times with diffusion constant that depends on dephasing rate but is independent of injection rate
- Between these two time-scales the system goes trough a “congestion” which almost takes the shape of a plateau.

- We will now discuss some analytical results both at early and late times.
- For late-times, we will use the fact that the behaviour is independent of injection rate thereby enabling us to use a special value of injection rate that makes analytics more feasible.

Recall



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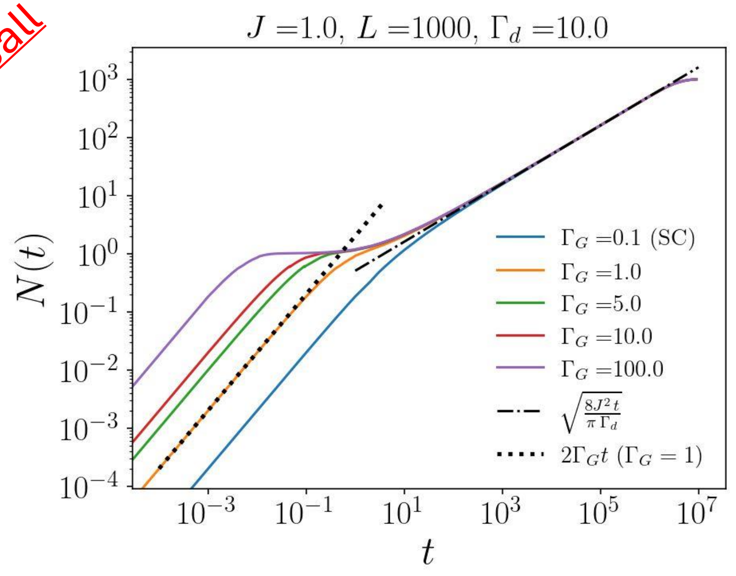


<u>Very early times:</u>

- This is a trivial regime where just at most one particle enters the system.
- During this time-scale even the hopping rate J does not play a role. Hence, coherences also donot develop
- Only the below equation for the first site matters

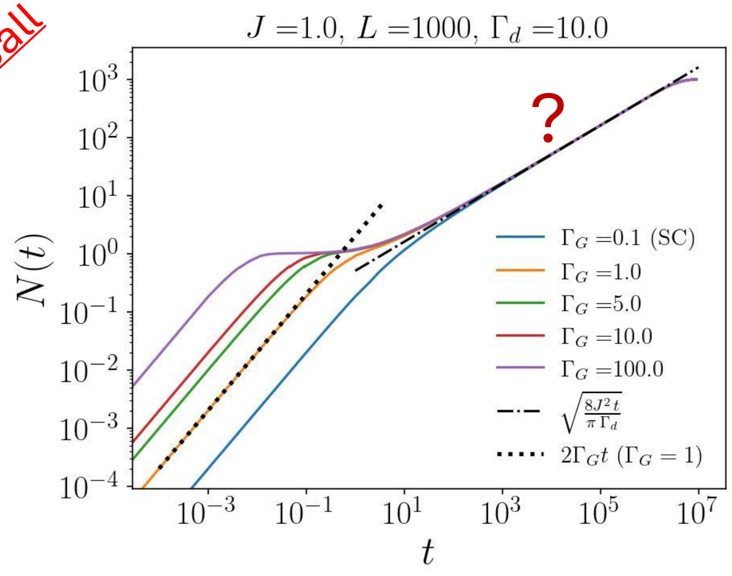
$$\frac{dC_{1,1}}{dt} = 2\Gamma_G(1 - C_{1,1}) \Longrightarrow C_{1,1}(t) = 1 - e^{-2\Gamma_G t} \Longrightarrow C_{1,1}(t) = 2\Gamma_G t$$ (short time expansion)

- This linear growth has been verified with exact numerics as seen in plot above

The late time square-root behavour is analytically more tricky which we will discuss next

Analytical understanding of the diffusive behaviour

Recall



We start with the equations for the correlation matrix

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

We will do an "adiabatic approximation". This involves taking a large dephasing limit $\Gamma_d >> J$

The adiabatic approximation is about a separation of time-scales. The time scale at which the coherences relax is assumed to be much shorter than the times-scales of the population. This leads to

$$C_{m,n} = -\frac{\mathrm{i}J}{\Gamma_G(\delta_{1,m} + \delta_{1,n}) + \Gamma_d}\left(C_{m-1,n} + C_{m+1,n} - C_{m,n-1} - C_{m,n+1}\right) \qquad (m \neq n)$$

The EOM for the diagonal terms of the correlation matrix (densities) is given by

$$\dot{C}_{m,m} = -\mathrm{i}J\left(C_{m-1,m} + C_{m+1,m} - C_{m,m-1} - C_{m,m+1}\right) - 2\Gamma_G\delta_{1,m}C_{m,m} + 2\Gamma_G\delta_{1,m}$$

We will simplify this equation by using the equation above it and ignore second-neighbour correlations

We finally get

Define

$C_m := C_{m,m}$

$$\dot{C}_{m,m} = \begin{cases} \frac{2J^2}{\Gamma_d}\left(C_{m-1} - 2C_m + C_{m+1}\right), & 3 \le m \le L-1 \\ \frac{2J^2}{\Gamma_G+\Gamma_d}\left(-C_m + C_{m+1}\right) - 2\Gamma_G(C_m - 1), & m = 1 \\ \frac{2J^2}{\Gamma_d}\left[\frac{1}{1+\Gamma_G/\Gamma_d}(C_{m-1} - C_m) + C_{m+1} - C_m\right], & m = 2 \\ \frac{2J^2}{\Gamma_d}\left(C_{m-1} - C_m\right), & m = L \end{cases}$$

$\boxed{2J^2/(\Gamma_G+\Gamma_d)}$

$$\frac{\mathrm{d}C_{\mathrm{diag}}}{\mathrm{d}t} = \mathbb{A}C_{\mathrm{diag}} + \mathbb{P}$$

where

$$\mathbb{A} = \begin{pmatrix} -\alpha_2 - 2\Gamma_G & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2 & -(\alpha_1+\alpha_2) & \alpha_1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_1 & -2\alpha_1 & \alpha_1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \cdots & & \alpha_1 & -2\alpha_1 & \alpha_1 \\ 0 & & \cdots & & & \alpha_1 & -\alpha_1 \end{pmatrix}$$

$$\mathbb{P} = [2\Gamma_G, 0, \cdots, 0],$$

$\boxed{2J^2/\Gamma_d}$

which gives $C_{\mathrm{diag}}(t) = \left(e^{\mathbb{A}t} - \mathbb{I}\right)\mathbb{A}^{-1}\mathbb{P}$

The main task in now to analyse the matrix A

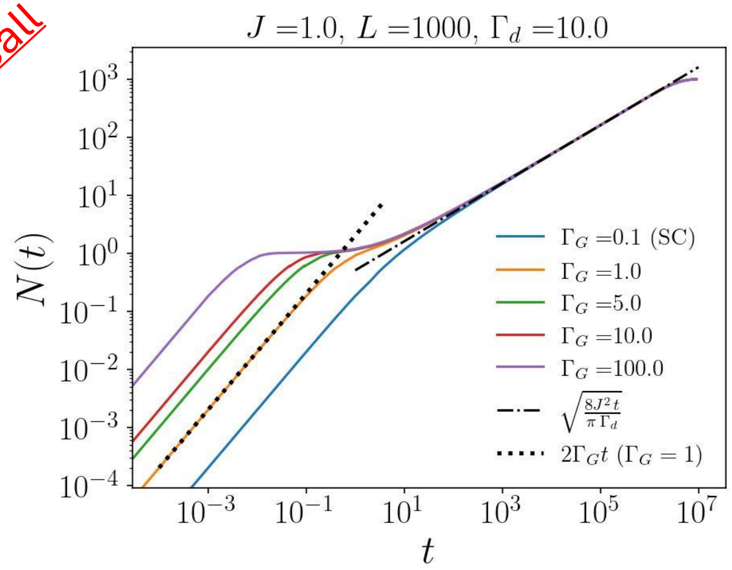
In the weak gain limit $\alpha_2 \approx \alpha_1$

Even after this the matrix is not easy to analyse. So we go to a special case $\Gamma_G = \alpha_1/2$

Special case (SC) $\Gamma_G = \alpha_1/2$

$$\mathbb{A} = \alpha_1 \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \cdots & & 1 & -2 & 1 \\ 0 & & \cdots & & & 1 & -1 \end{pmatrix}$$

Recall



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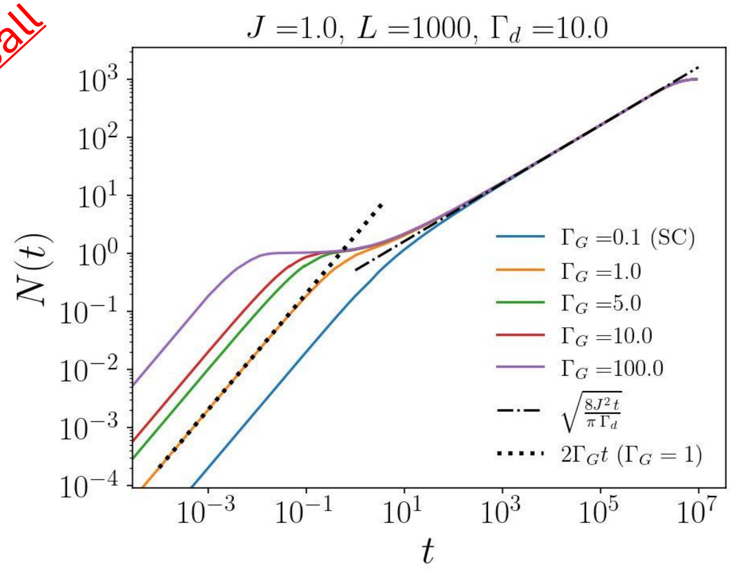
Recall



$$C_{\text{diag}}(t) = \left(e^{\mathbb{A}t} - \mathbb{I}\right)\mathbb{A}^{-1}\mathbb{P}$$

$$n_i(t) = -\frac{8\Gamma_G}{2L+1}\sum_{k=1}^{L} \frac{e^{-4\alpha_1 t \sin^2\left[\frac{(2k-1)\pi}{2(2L+1)}\right]} - 1}{4\alpha_1 \sin^2\left[\frac{(2k-1)\pi}{2(2L+1)}\right]} \sin\left[\frac{(2k-1)\pi}{2L+1}\right] \sin\left[\frac{(2k-1)i\pi}{2L+1}\right]$$

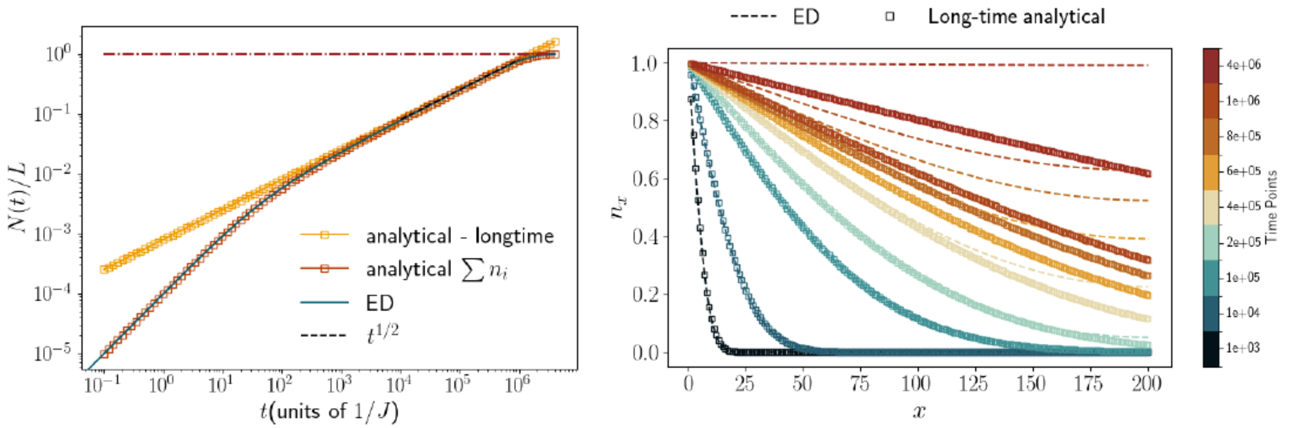
Converting summation to integration

$$n_x(t) = \frac{8\Gamma_G}{\pi}\int_0^{\pi/2} \mathrm{d}\tilde{k}\, \sin\left(2\tilde{k}\right) \sin\left(2\tilde{k}x\right) \frac{1 - e^{-4\alpha_1 t \sin^2 \tilde{k}}}{4\alpha_1 \sin^2 \tilde{k}}$$

In the limit $t \gg 1$ we can show spatial density profile and the total number of particles

$$\boxed{n_x(t) = 1 - Erf\left(\frac{x}{2\sqrt{\alpha_1 t}}\right) \qquad N(t) = 2\sqrt{\frac{\alpha_1 t}{\pi}}}$$

At special case



Direct numerics and analytics match

What happens when we inject into an interacting quantum system ?

We will study two types of interacting systems

Quasi-periodic XXZ chain

$$H_{\mathrm{QP-XXZ}} = J \sum_{i=1}^{L-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + W \sum_{i=1}^{L} \cos(2\pi b\, i + \phi) S_i^z$$

irrational number

$$\dot{\rho} = -i[H_{\mathrm{QP-XXZ}}, \rho] + \Gamma_G \underbrace{\left[S_1^+ \rho S_1^- - \frac{1}{2} \left\{ S_1^- S_1^+, \rho \right\} \right]}_{\text{Injection}}$$

$$|\rho(t)\rangle = e^{\mathscr{L} t} |\rho(0)\rangle$$

$$\mathscr{L} = -i \left(\mathbb{I} \otimes H_{\mathrm{QP-XXZ}} - H_{\mathrm{XXZ}}^T \otimes \mathbb{I} \right) + \Gamma_G \left[(S_1^-)^T \otimes S_1^+ - \frac{1}{2} \left[\mathbb{I} \otimes (S_1^- S_1^+) + (S_1^- S_1^+)^T \otimes \mathbb{I} \right] \right]$$

Next nearest neighbour XXZ spin chain

$$H_{\mathrm{NNN-XXZ}} = J \sum_{i=1}^{L-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + J' \sum_{i=1}^{L-2} S_i^z S_{i+2}^z$$

Next-nearest neighbor
integrability breaking term

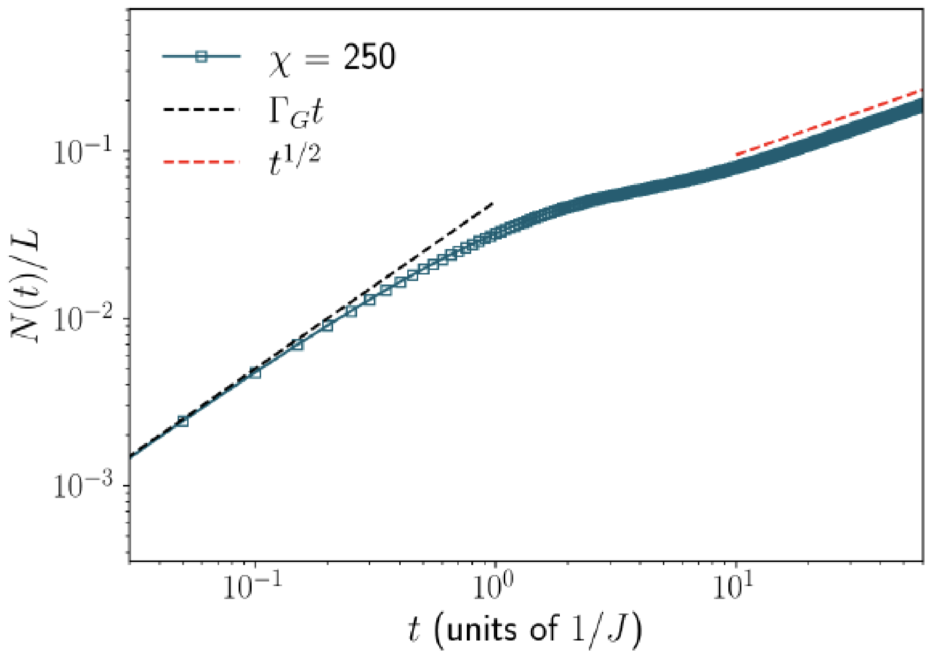
Since these are chaotic / non-integrable quantum systems, we would expect to see diffusive behaviour.
We now numerically look for evidence for it using TEBD algorithm.

What happens when we inject into an interacting quantum system ?

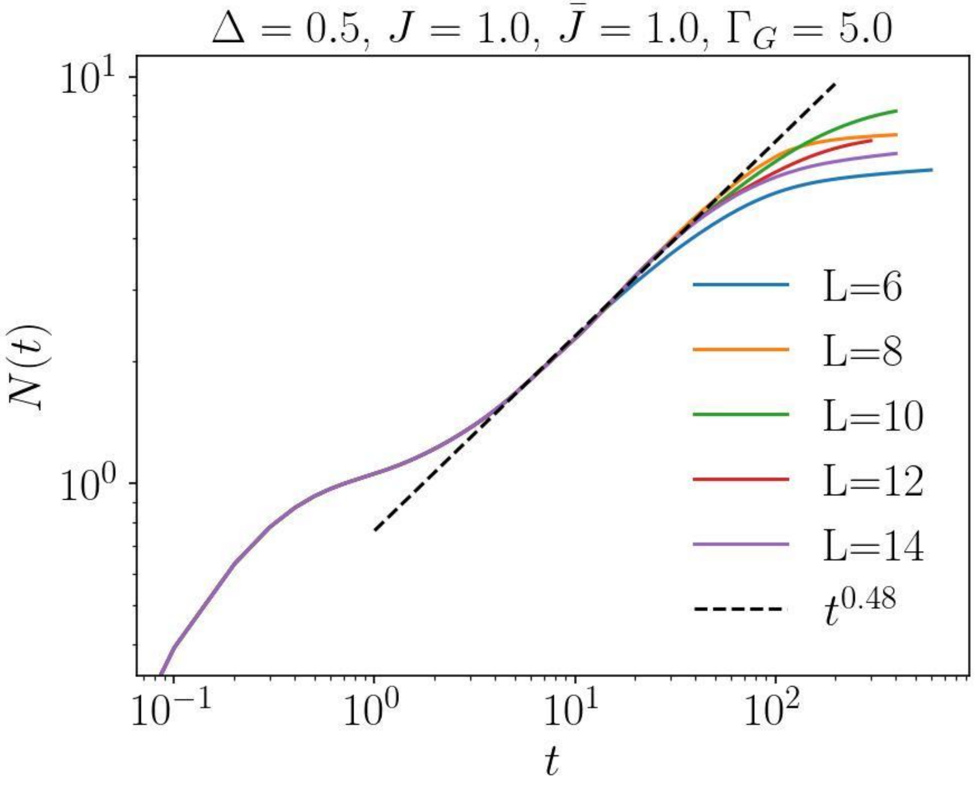
Quasi-periodic model

$$H_{\mathrm{QP-XXZ}} = J\sum_{i=1}^{L-1}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) + W\sum_{i=1}^{L}\cos(2\pi b i + \phi) S_i^z$$

$$H_{\mathrm{NNN-XXZ}} = J\sum_{i=1}^{L-1}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) + J'\sum_{i=1}^{L-2} S_i^z S_{i+2}^z$$



$L = 20, \Gamma_G = 1.0J, W = 1.0J$

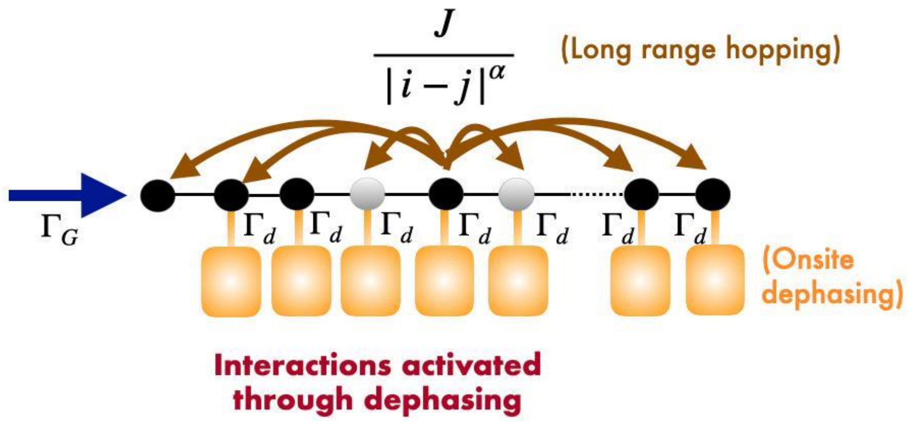


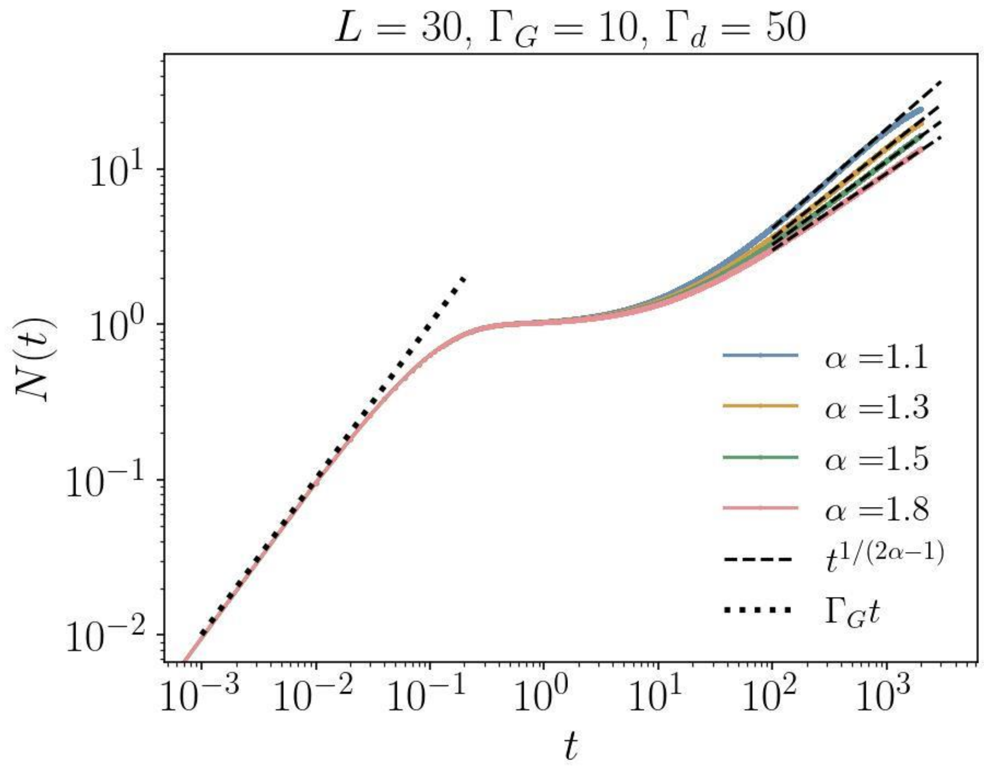
We see squareroot time behaviour but we are still awaiting better data

What happens when we inject long-ranged systems that are subject to dephasing ?

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^\alpha}\left[\sum_{r=1}^{N-m} \hat{c}_r^\dagger \hat{c}_{r+m} + \hat{c}_{r+m}^\dagger \hat{c}_r\right]$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S,\rho] + \Gamma_G\left[2c_1^\dagger \rho c_1 - \{c_1 c_1^\dagger, \rho\}\right] - \frac{\Gamma_d}{2}\sum_{i=1}^{L}[n_i,[n_i,\rho]]$$





See,
Schuckert , Lovas, Knap (PRB 2020)
Dhawan, Ganguly, MK, Agarwalla (PRB 2024)
Nishikawa, Saito (2025)
Catalano et al, PRL (2025)

Key findings

$$N(t) \propto \begin{cases} t^{\frac{1}{2\alpha-1}} & \text{for} \quad \alpha \leq \frac{3}{2} \\ t^{\frac{1}{2}} & \text{for} \quad \alpha > \frac{3}{2} \end{cases}$$

These exponents of the injection problem seem to be the same as several long-ranged papers in slightly different contexts

Conclusions to Part B

- We studied injection of particles on lattices
- Two cases (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting.
- For the dephasing case, we provided numerical and analytical results.
- For the interacting case, we provided TEBD results showing square-root behaviour.
- We discussed long-ranged case subject to dephasing.

Thank you