Disordering a permutation symmetric system: revivals, thermalisation and chaos

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"Long range interactions and dynamics in complex quantum systems", Nordita

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1. *PhysRevE.62.6366, Emergence of quantum chaos in the quantum computer core and how to manage it, Georgeot, B. and Shepelyansky, D. L, 2000.*
2 *Transmon platform for quantum computing challenged by chaotic fluctuations, Berke etal., Nature communications,2022.*

Disorder in kicked all-to-all interacting spin chain

The Hamiltonian of the system is given by:

$$H = \frac{k}{2N\tau} \sum_{\ell<\ell'=1}^{N} (1 + \epsilon_{ll'})\sigma_\ell^x \sigma_{\ell'}^x + \frac{p}{2} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{N} \sigma_\ell^y \delta(t - n\tau), \tag{1}$$

- $k \Rightarrow$ strength of interaction

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- $k \Rightarrow$ strength of interaction
- $\tau \Rightarrow$ is the period of delta kicks
- ϵ is a random number from a Normal distribution with standard deviation w.
- For $\epsilon_{ll'} = 0$,
- $[J^2, J_i] = 0$, where $J_\alpha = \sum_{i=1}^{N} \frac{\sigma_{\alpha_i}}{2}$, $(\alpha = x, y, z)$, J^2 is a constant of motion

Kicked top - Classical limit

Classical Phase Space

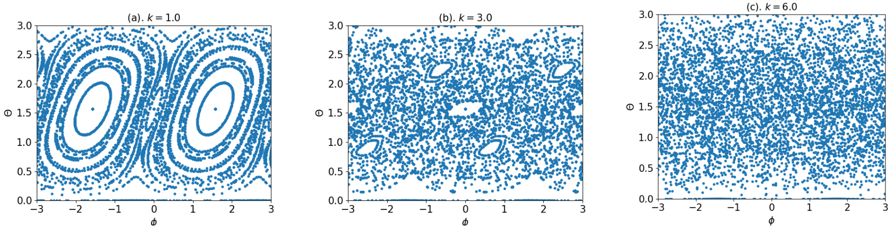

Figure 1: [1] The classical phase space for different chaos parameter k. (a). $k = 1$ predominantly regular region,(b). $k = 3$ mixed phase space and (c). $k = 6$ fully chaotic.

[1]Shohini Ghose et al. "Chaos, entanglement, and decoherence in the quantum kicked top". Physical Review A 78.4 (2008), p. 042318.

- The initial state of our multi qubit system :

$$|\theta\phi\rangle^{\otimes N} = \left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right)^{\otimes N} \tag{2}$$

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Upon evolution ($\epsilon_{ll'} = 0$) $\longrightarrow$ **Dynamics resides in the permutation symmetric subspace of dimension** $N + 1$

- $\epsilon_{ll'} \neq 0$ $\rightarrow$ **Permutation symmetry is broken** $\rightarrow$ **Dynamics traverse the full Hilbert space (FHS)**

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- The Floquet operator given by:

$$U' = \exp(-\frac{ik}{2N}\sum_{l<l'=1}^{N}(1+\epsilon_{ll'})\sigma_l^x\sigma_{l'}^x) \quad \exp(-\frac{ip}{2}\sum_{l=1}^{N}\sigma_l^y) \tag{3}$$

Growth of linear entropy

- The single qubit linear entropy:

$$S_1 = 1 - Tr(\rho_1^2) \tag{4}$$

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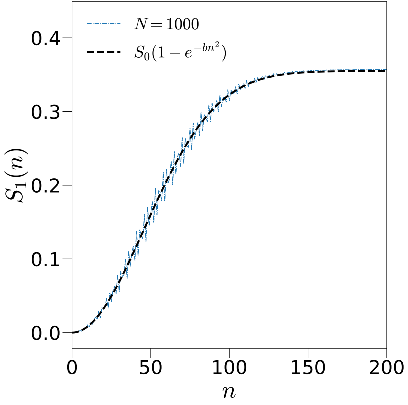
- For kicked top, in terms of collective angular momentum operators:

$$S_1(n) = \frac{1}{2}\left[1 - \left(\frac{\langle J_x(n)\rangle^2 + \langle J_y(n)\rangle^2 + \langle J_z(n)\rangle^2}{j^2}\right)\right], \tag{5}$$

Disorder vs No Disorder

- Disorder free case

$$S = S_0(1 - e^{-bn^2})$$



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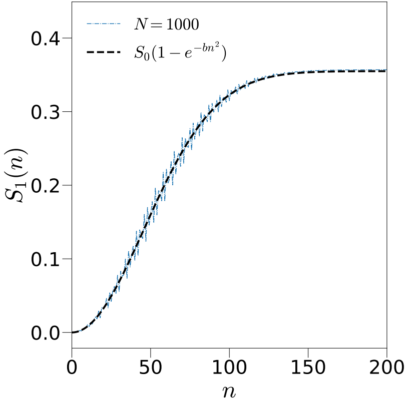

- Disordered case

$$S = S_0^{'}(1 - e^{-Dn}e^{-cn^2})$$

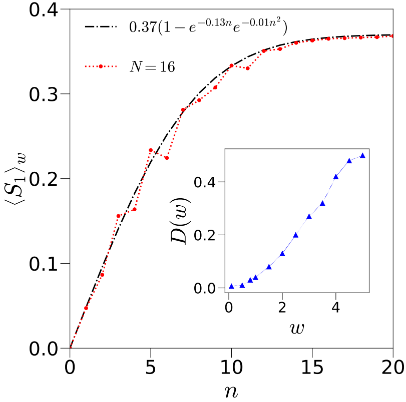

Figure 2: Plots showing the growth of linear entropy $S_1(n)$ for the disorder free and disordered cases respectively at $k = 1$.

Classical Dynamics of Near-Integrable Systems

Disorder Free Case

$$\theta(t) = \theta(0) + \omega(I)\, t$$

Disordered Case

For an observable $f(\theta)$, the average $\langle f \rangle(t)$

$$= \sum_{k=-\infty}^{\infty} f_k\, e^{-\frac{k^2\sigma^2}{2}(\omega'(I_0)^2 t^2 + 1)}\, e^{-ik\omega(I_0)t}$$

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$S_1(n)$ has a classical limit,

$$S_{cl}(n) = \frac{1}{2}\left(1 - [\langle X_n \rangle^2 + \langle Y_n \rangle^2 + \langle Z_n \rangle^2]\right)$$

where X, Y, Z are the classical variables.

Disordered Case

$$\theta(t) = \theta(0) + \omega(I)\, t + \eta(t)$$

$$P(\eta) = \frac{1}{\sqrt{2\pi D t}} e^{-\frac{\eta^2}{2Dt}}$$

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Disordered Case

$$\theta(t) = \theta(0) + \omega(I)\, t + \eta(t)$$

$P(\eta) = \frac{1}{\sqrt{2\pi D t}} e^{-\frac{\eta^2}{2Dt}}$

The average $\langle f \rangle(t)$

$$= \sum_{k=-\infty}^{\infty} f_k e^{-ik\omega(I_0)t} e^{-\frac{k^2\sigma^2}{2}(1+\omega'(I_0)^2 t^2)}$$

$$e^{-\frac{k^2 D t}{2}}$$

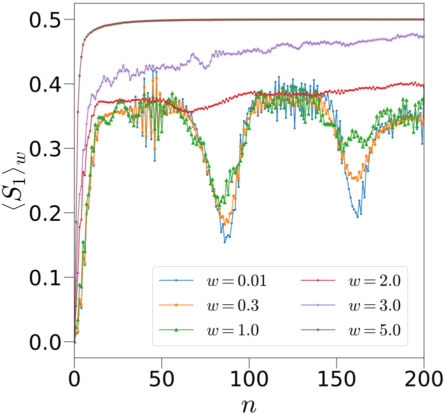

Figure 3: Plot showing disorder averaged S_1 with respect to time n for $N = 14$ at $k = 1$ for different disorder strengths w.

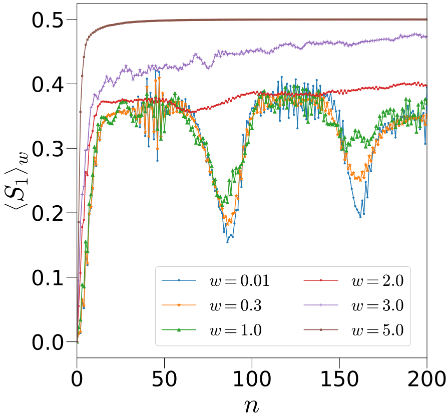

Figure 3: Plot showing disorder averaged S_1 with respect to time n for $N = 14$ at $k = 1$ for different disorder strengths w.

Increase in Disorder $\rightarrow$ Absence of Revivals

Overlap with permutation symmetric basis

$$|\Psi^{n'}_{2^{2j}}\rangle = a_0\,|000\rangle + a'_1\,|001\rangle + a''_1\,|010\rangle + a'_2\,|011\rangle + a'''_1\,|100\rangle + a''_2\,|101\rangle + a'''_2\,|110\rangle + a_3\,|111\rangle$$

- The permutation symmetric basis set:

$$|000\rangle = |0_3\rangle$$

$$\frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}} = |1_3\rangle$$

$$\frac{|011\rangle + |110\rangle + |101\rangle}{\sqrt{3}} = |2_3\rangle$$

$$|111\rangle = |3_3\rangle$$

- The overlap of $|\Psi^{n'}_{2^{2j}}\rangle$ onto each of the permutation symmetric basis is given by:

$$\langle 0_3|\Psi^{n'}_{2^{2j}}\rangle = a_0 = \alpha_1$$

$$\langle 1_3|\Psi^{n'}_{2^{2j}}\rangle = \frac{a_1' + a_1'' + a_1'''}{\sqrt{3}} = \alpha_2$$

$$\langle 2_3|\Psi^{n'}_{2^{2j}}\rangle = \frac{a_2' + a_2'' + a_2'''}{\sqrt{3}} = \alpha_3$$

$$\langle 3_3|\Psi^{n'}_{2^{2j}}\rangle = a_3 = \alpha_4$$

The sum of the absolute value of the square of the overlap given by:

$$\chi = |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 \tag{6}$$

Disorder free case : $\chi = 1$
Disordered case : $\chi < 1$

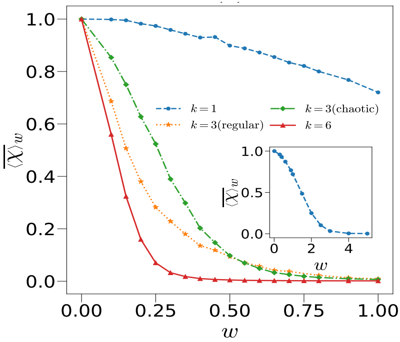

Figure 4: The plot of overlap χ with respect to disorder strength w for different values of k for $N = 14$.

Effective dimension D_{eff}

- Initial state in terms of Floquet eigenstates:

$$|\theta\phi\rangle = \sum_{i=1}^{2^N} c_i \, |\phi_i\rangle \tag{7}$$

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$$|\theta\phi\rangle = \sum_{i=1}^{2^N} c_i |\phi_i\rangle \tag{7}$$

- Obtain $c_i \rightarrow$ overlap of the initial state with the Floquet eigen states $|\phi_i\rangle$.
- D_{eff} is that value of i for which $\sum_{i=1}^{D_{\text{eff}}} |c_i|^2 = 0.999$, when the coefficients $c_i's$ are arranged in decreasing order.

Disorder free case : $D_{\text{eff}} \sim N+1$
For large disorder : $D_{\text{eff}} \sim 2^N$

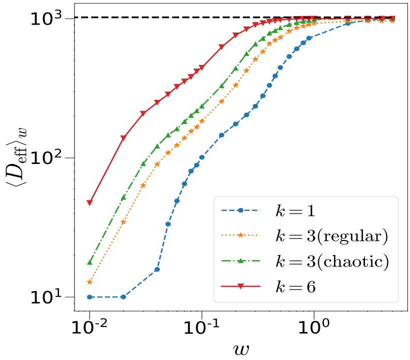


Figure 5: The plot of D_{eff} with respect to disorder strength w for different values of k for $N = 10$ qubits.

Long time averaged linear entropy

$$\langle S_Q \rangle_{W,2^Q} = \frac{(2^Q - 1)(2^{N-Q} - 1)}{2^N + 1}, \tag{8}$$

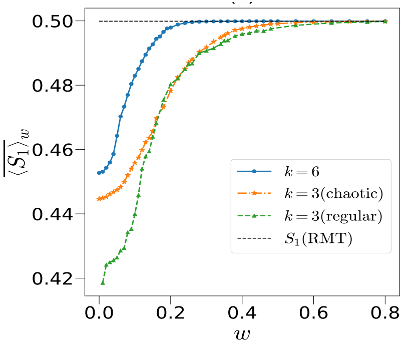
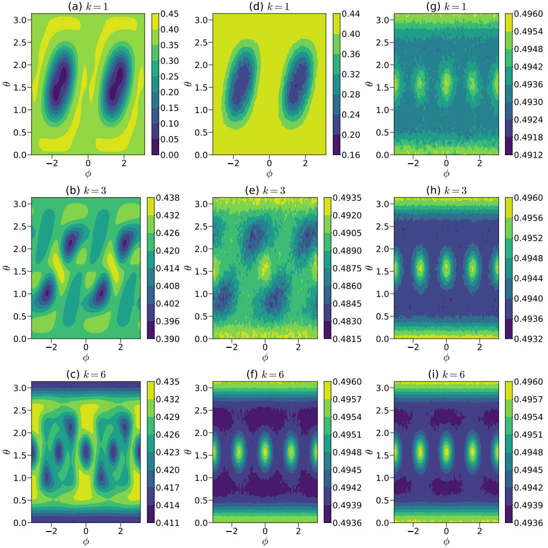


Figure 6: The plot of $\overline{\langle S_1 \rangle}_w$ with respect to disorder strength w for different values of k for $N = 14$ and $Q = 1$ qubits.

Pseudo Phase space



With increase in disorder, different k values exhibit the same behaviour

Pseudo Phase space

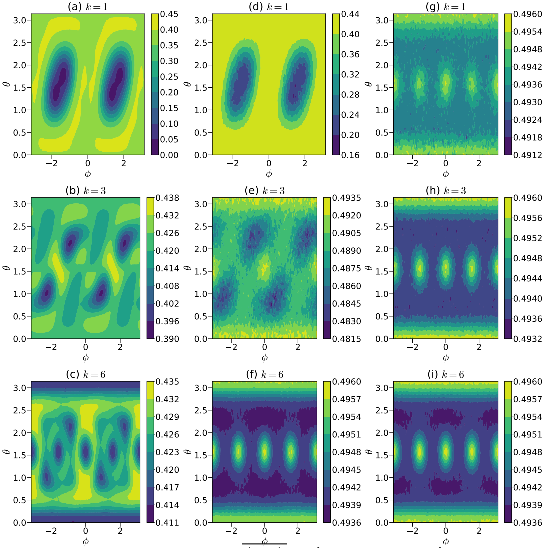


Figure 7: $\overline{\langle S_1 \rangle}_w$ (columnwise) for $w = 0.01, 1.0$ and 6 respectively.

With increase in disorder, different k values exhibit the same behaviour

$p = 4\pi/11$

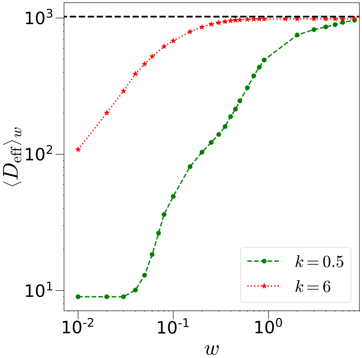
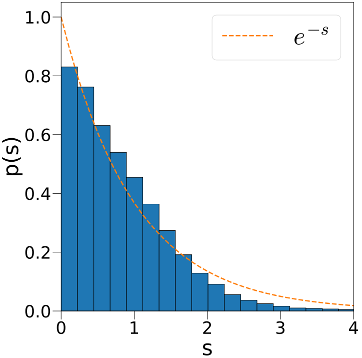


Figure 8: D_{eff} for $k = 0.5$ and $k = 6$ that shows a transition from $N + 1$ to 2^N as the strength of disorder w is increased. The system size $N = 10$ qubits and the initial state is $|\theta, \phi\rangle = |2.25, 1.1\rangle$.

Spectral Statistics



Spectral Statistics

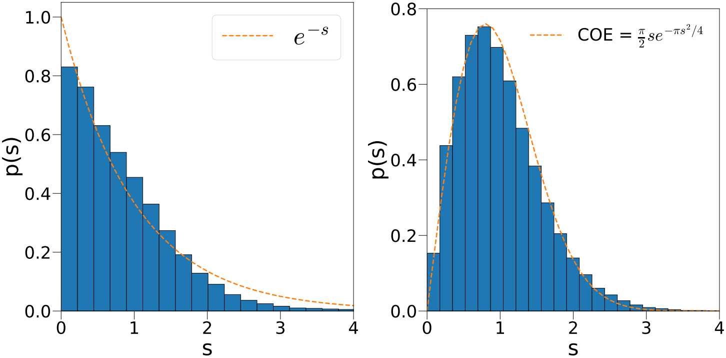


Figure 9: The spacing statistics for $N = 12$ at $k = 0.5$ for (a) small disorder $w = 0.5$ and (b) large disorder $w = 8.0$. The dashed lines shows the spacing statistics for the Poisson and Wigner Dyson (COE) distribution.

- **With increase in disorder system enters a chaotic phase**

Conclusion

✓ Breaking of permutation symmetry with disorder.

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- ✓ Saturation value of linear entropy to RMT values in full Hilbert space.

Conclusion

- ✓ Breaking of permutation symmetry with disorder.
- ✓ Dynamics from PSS to FHS
- ✓ Effect of disorder on entanglement growth.
- ✓ Dynamics taken out of symmetric subspace:
  - ✡ Overlap χ
  - ✡ Effective dimension D_{eff}
- ✓ Saturation value of linear entropy to RMT values in full Hilbert space.
- ✓ Increase in disorder drives the system into a chaotic phase, as reflected in the spectral statistics.

C.Manju, U. Divakaran, A. Lakshminarayan, arXiv:2505.24453 (2025)

Acknowledgements

- ☆ Dr. Uma Divakaran (Supervisor), IIT Palakkad
- ☆ Prof. Arul Lakshminarayan (IIT Madras), our collaborator for his valuable suggestions.
- ☆ HPC facility Chandra at IIT Palakkad.

Thankyou!

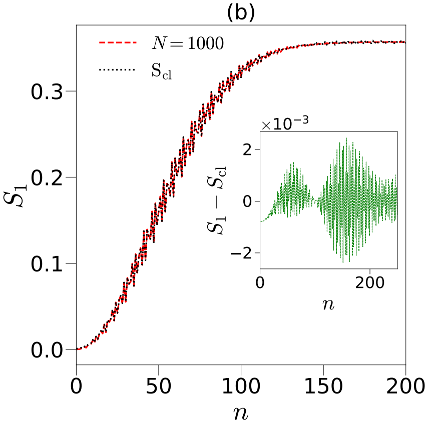


Figure 10: The growth and saturation of linear entropy both for the quantum case and for the classical case.

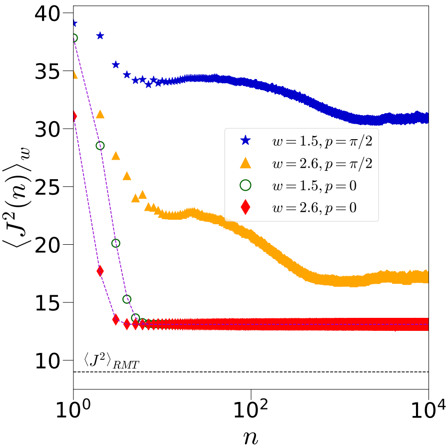
Kicked Top Model

The Hamiltonian of quantum kicked top is given by:

$$H = \frac{\hbar k}{2j\tau} J_x^2 + \hbar p J_y \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{9}$$

where J_x, J_y, J_z are the angular momentum operators obeying $[J_i, J_j] = i\hbar\epsilon_{ijk} J_k$.

- $k \Rightarrow$ strength of the twist, chaos parameter
- $p \Rightarrow$ angle of turn per kick
- $\tau \Rightarrow$ is the time between the kicks
- Since $[J^2, J_i] = 0$, square of the angular momentum operator commutes with the Hamiltonian.
- Thus $J^2 = j(j+1)\hbar^2$ is a constant of motion. We are restricted to 2j+1 dimensional Hilbert space.



40
35
30
25
20
15
10
$\langle J^2(n)\rangle_w$
10^0
10^2
10^4
n
$\langle J^2\rangle_{RMT}$
$w=1.5, p=\pi/2$
$w=2.6, p=\pi/2$
$w=1.5, p=0$
$w=2.6, p=0$

$$\rho(\theta, I, 0) = \frac{1}{2\pi\sigma^2} \sum_{l=-\infty}^{\infty} e^{-(\theta+2\pi l)^2/(2\sigma^2)} e^{-(I-I_0)^2/(2\sigma^2)}, \tag{10}$$

$\rho(\theta, I, 0)$ be the initial ensemble, which evolves to $\rho(\theta, I, t) = \rho(\theta - \omega(I)t, I, 0)$ For an observable $f(\theta)$,

$$\langle f \rangle(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f(\theta)\rho(\theta, I, t)\, d\theta\, dI.$$

Taylor expanding $\omega(I) = \omega(I_0) + \omega'(I_0)(I - I_0)$ to first order, and using Fourier series, we get:

$$\langle f \rangle(t) = \sum_{k=-\infty}^{\infty} f_k e^{-\frac{k^2\sigma^2}{2}(\omega'(I_0)^2 t^2 + 1)} e^{-ik\omega(I_0)t}. \tag{11}$$

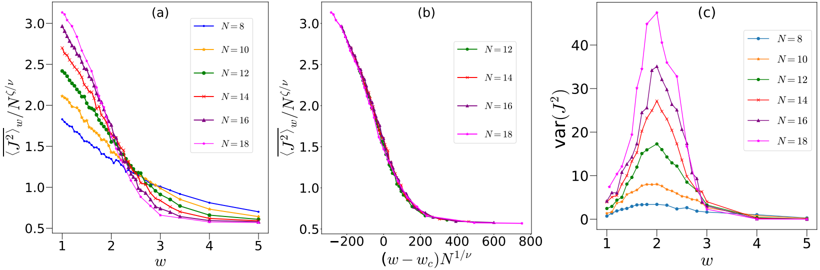


Figure 11: (a): $\overline{\langle J^2 \rangle}_w / N^{\zeta/\nu}$ where (ζ and ν are the critical exponents) as a function of disorder strength w at $k = 1, p = \pi/2$ and initial state $|2.25, 1.1\rangle$ for different system sizes. These curves cross each other at $w = w_c \approx 2.11$. (b): The collapse of the data with $\zeta = 0.57$ and $\nu = 0.52$. (c): var(J^2) with respect to disorder strength w for $k = 1$ that shows a peak around 2.0.

The Floquet operator describing unitary evolution one kick to the next kick is given by:

$$U = e^{\frac{-ik}{2j}J_x^2}e^{-ipJ_y} \tag{12}$$

From Heisenberg evolution equations:

$$\langle J_i \rangle_{n+1} = \left\langle U^\dagger J_i U \right\rangle_n \tag{13}$$

where U is the Floquet operator.

j $\rightarrow \infty \Rightarrow$ classical equations of motion for $p = \pi/2$ is given by:

$$\begin{aligned} X_{n+1} &= Z_n \\ Y_{n+1} &= Y_n \cos(kZ_n) + X_n \sin(kZ_n) \\ Z_{n+1} &= -X_n \cos(kZ_n) + Y_n \sin(kZ_n) \end{aligned} \tag{14}$$

We parametrize these equations with $\phi = \tan^{-1}(\frac{Y}{X})$ and $\theta = \cos^{-1}(Z)$

Classical Phase space

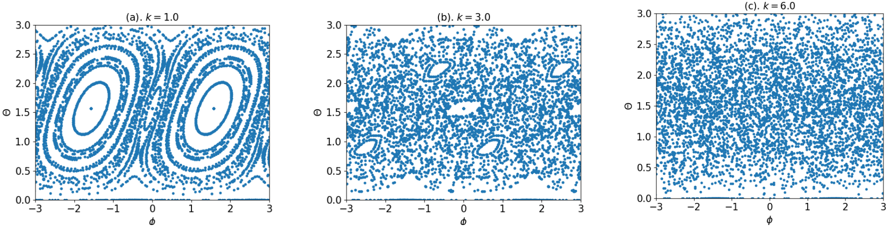


Figure 12: [2] The classical phase space for different chaos parameter k. (a). $k = 1$ predominantly regular region,(b). $k = 3$ mixed phase space and (c). $k = 6$ fully chaotic.

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Long time saturation for large disorder

Using Random matrix theory (RMT):
To find $\langle J^2 \rangle$: If $|\phi_j\rangle$ are the eigenvectors of J_z, we have

$$J_z |\phi_j\rangle = \lambda_j |\phi_j\rangle , \quad (15)$$

$$\langle J_z \rangle = \langle \psi | J_z |\psi\rangle = \sum_j \lambda_j |\langle \psi|\phi_j\rangle|^2 \quad (16)$$

$$\langle J_z \rangle_{RMT} = \sum_j \lambda_j \overline{|\langle \psi|\phi_j\rangle|^2}^{RMT} \quad (17)$$

$$= \frac{1}{2^N} \sum_j \lambda_j \quad (18)$$

$$= \frac{1}{2^N} \mathrm{Tr}\, J_z = \frac{1}{2^N} 2^{N-1} \mathrm{Tr} \sum_i \frac{\sigma_i^z}{2} \quad (19)$$

$$\langle J_z^2 \rangle_{RMT} = \frac{1}{2} \mathrm{Tr} \sum_i \frac{\sigma_i^{z\,2}}{4} = \frac{N}{4} \quad (20)$$

Similarly $\langle J_x^2 \rangle_{\mathrm{RMT}} = \langle J_y^2 \rangle_{\mathrm{RMT}} = N/4$, and hence $\langle J^2 \rangle_{\mathrm{RMT}} = 3N/4$.

k2_new_withbar.pdf

k4_withbar22.pdf

Figure 13: Left: The main figure shows the data corresponding to different system sizes for $k = 2$ crossing each other at $w = w_c \sim 0.41$ whereas the inset shows the collapse of the data with $\nu = 0.40$ and $\zeta = 0.49$. Right: Same as (a) but for $k = 4$ with $w = w_c \sim 0.17$, $\nu = 0.3$ and $\zeta = 0.32$.