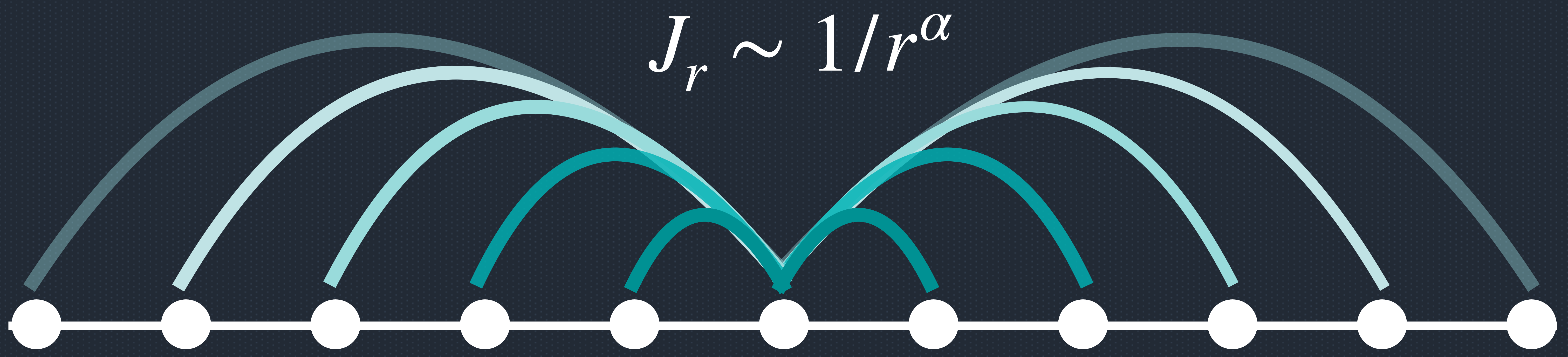
The long-range origin of the black hole entropy

Nordita, 25/07/2025

Nicolò Defenu

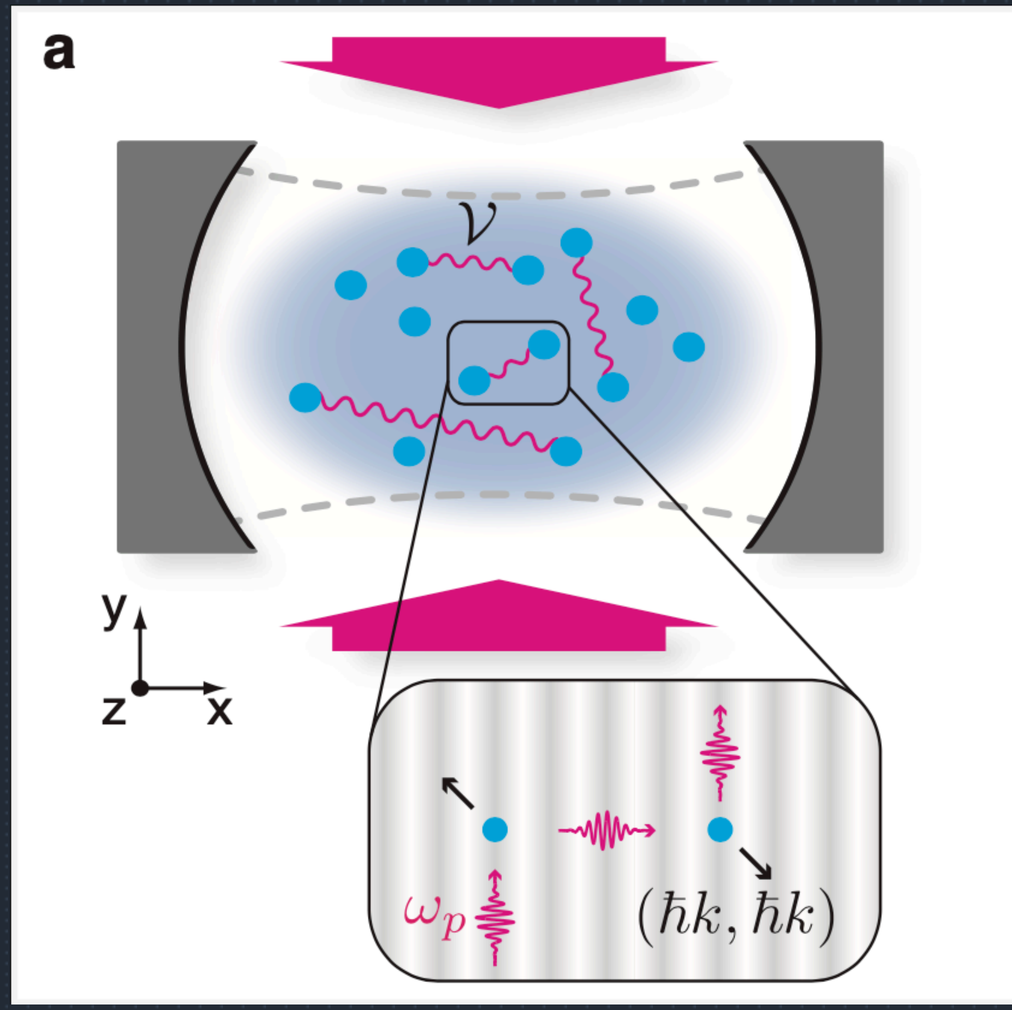
Long-range interacting quantum systems

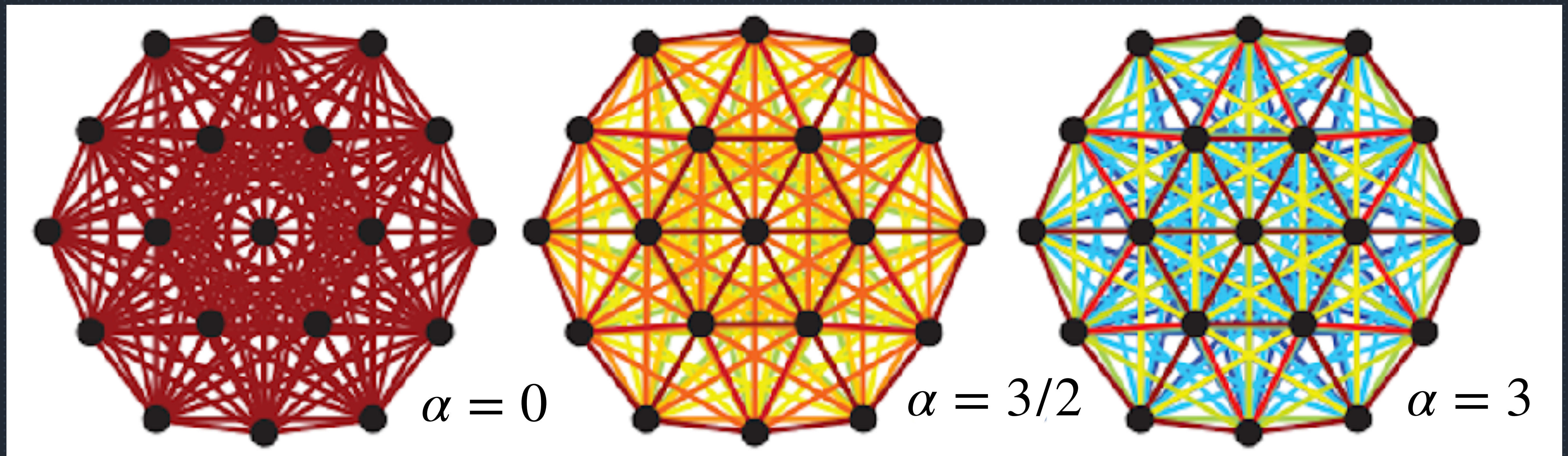


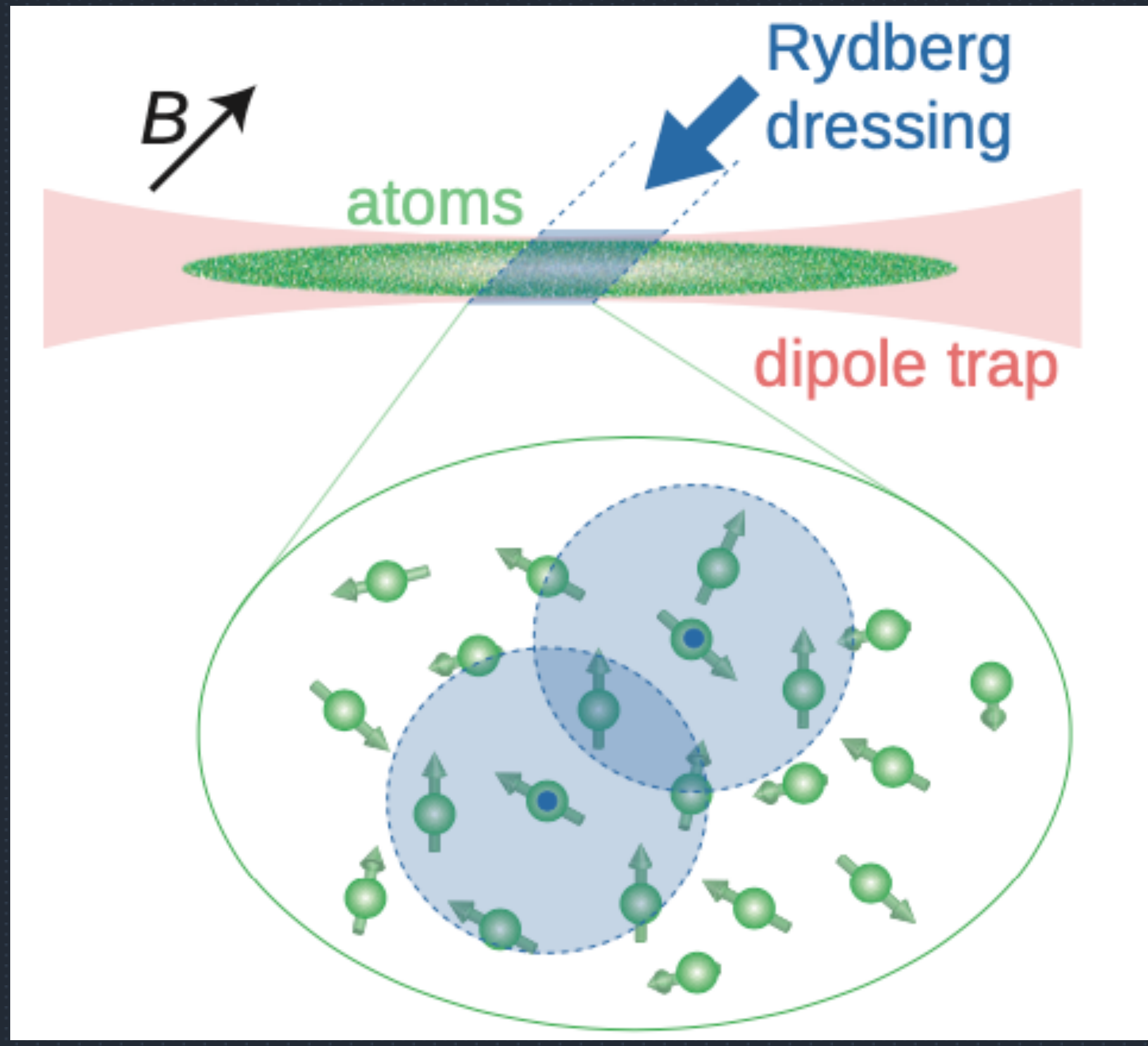
$\alpha < d$	Strong Long-range
$d < \alpha < \alpha^*$	Weak Long-range
$\alpha > \alpha^*$	Short-range physics

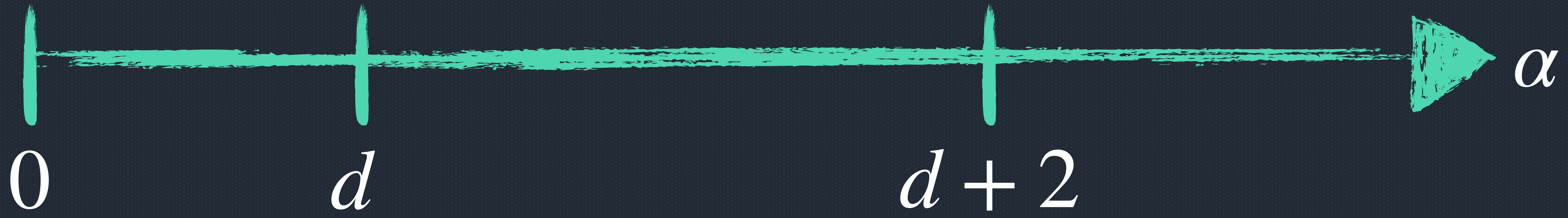
Long-range quantum simulators

$$V(r) \sim r^{-\alpha}$$









"Long-range interacting quantum systems", ND et al. *Rev. Mod. Phys. 95, 035002 (2023).*

"Out-of-equilibrium dynamics of quantum many-body systems with long-range interactions", ND et al. *Phys. Rep. 1074, 1, (2024)*

Sachdev-Ye-Kitaev model

$$H = \sum_{\substack{a_1, \dots a_q \\ b_1, \dots b_q}} J_{a_1,\dots a_q,\dots b_1,\dots,b_q} c^\dagger_{a_1} \dots c^\dagger_{a_q} c_{b_1} \dots c_{b_q}$$

Identically distributed independent, Gaussian random couplings

$$\langle J_{a_1,\dots a_q,b_1\dots,b_q} \rangle = 0$$

$$\langle | J_{a_1,\dots a_q,b_1\dots,b_q} |^2 \rangle = J^2 / q N^{2q-1} (q!)^2$$

N fermonic flavor $a_i, b_i = 1, \dots, N$

All-to-all connected

q-body interactions

Sachdev-Ye-Kitaev model

Interesting porperties:

1. **NO quasiparticle description -** level spacing above the ground state is exponentially small $\sim e^{-S_0 N}$
2. **Residual zero temperature entropy** $\lim_{T\to 0}\lim_{N\to\infty}\frac{S(N,T)}{N} = S_0 > 0$
3. **“Maximally chaotic” - fast scrambler**
4. High energy perspective: toy model for a **quantum black hole**
5. Condensed matter perspective: beyond **Fermi-Liquid** description

Sachdev-Ye-Kitaev model: exact solution

Disorder average of the Euclidean action: replica trick

Saddle point $N \to \infty$

Matsubara Frequencies $\omega_n = \dfrac{(2n+1)\pi}{\beta}$

$$G(i\omega_n) = \frac{1}{i\omega_n - \Sigma(i\omega_n)}$$

$$\Sigma(\tau) = (-1)^{q+1} J^2 G^q(\tau) G^{q-1}(-\tau)$$

$$f(\tau) = \frac{1}{\beta} \sum_n f(i\omega_n) e^{-i\omega_n \tau}, \quad f(i\omega_n) = \int_0^\beta d\tau f(\tau) e^{i\omega_n \tau}.$$

Sachdev-Ye-Kitaev model: exact solution

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Low energy solution a T = 0 : power law ansatz

$$G(z) = C e^{-i(\pi\Delta - \theta)} z^{2\Delta - 1}$$

$\mathrm{Im}(z) > 0, |z| \ll J$

$\Delta = \frac{1}{2q}$ Fermions scaling dimension

Sachdev-Ye-Kitaev model: exact solution

$G(z) = Ce^{-i(\pi\Delta-\theta)}z^{2\Delta-1}$ $\quad \mathrm{Im}(z) > 0, |z| \ll J$ $\quad \Delta = \frac{1}{2q}$ Fermions scaling dimension

$G(\tau) \propto \mathrm{sgn}(\tau)|\tau|^{-2\Delta} = \mathrm{sgn}(\tau)|\tau|^{-1/q}$ $\quad T = 0$

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$$\int d\tau G(\tau,\tau')\Sigma(\tau,\tau') = -\delta(\tau-\tau') \qquad \Sigma(\tau,\tau') = J^2[G(\tau,\tau')]^{(2q-1)}$$

$\tau \to f(\tau)$ Euclidean time reparametrizations (conformal) invariance

$$G(\tau,\tau') \to [f'(\tau)f'(\tau')]^{\Delta}G(f(\tau),f(\tau')) \qquad \Sigma(\tau,\tau') \to [f'(\tau)f'(\tau')]^{\Delta(2q-1)}\Sigma(f(\tau),f(\tau'))$$

Sachdev-Ye-Kitaev model: exact solution

$G(z) = Ce^{-i(\pi\Delta-\theta)}z^{2\Delta-1}$ $\quad$ $\mathrm{Im}(z) > 0, |z| \ll J$ $\quad$ $\Delta = \dfrac{1}{2q}$ Fermions scaling dimension

$G(\tau) = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-2\Delta} = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-1/q}$ $\quad$ $T = 0$

$f(\tau) = \tan(\pi\tau/\beta)$

Sachdev-Ye-Kitaev model: exact solution

$$G(z) = Ce^{-i(\pi\Delta-\theta)}z^{2\Delta-1} \qquad \mathrm{Im}(z) > 0, |z| \ll J \qquad \Delta = \frac{1}{2q}$$

Fermions scaling dimension

$$G(\tau) = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-2\Delta} = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-1/q} \qquad T = 0$$

$$G(\tau) = b\left[\frac{\pi}{\beta\sin(\pi\tau/\beta)}\right]^{2\Delta}\mathrm{sgn}(\tau) \qquad T > 0$$

Sachdev-Ye-Kitaev model: exact solution

$$G(z) = Ce^{-i(\pi\Delta-\theta)}z^{2\Delta-1} \qquad \mathrm{Im}(z) > 0, |z| \ll J \qquad \Delta = \frac{1}{2q}$$

Fermions scaling dimension

$$G(\tau) = b\,\mathrm{sgn}(\tau)\,|\tau|^{-2\Delta} = b\,\mathrm{sgn}(\tau)\,|\tau|^{-1/q} \qquad T = 0$$

$$G(\tau) = b\left[\frac{\pi}{\beta\sin(\pi\tau/\beta)}\right]^{2\Delta}\mathrm{sgn}(\tau) \qquad T > 0$$

$$\beta F = -\sum_n \ln\left(-i\omega_n + \Sigma(i\omega_n)\right) - \frac{J^2}{2q}\int_0^\beta d\tau\left[G^q(\beta-\tau)G^q(\tau) + \Sigma(\tau)G(\beta-\tau)\right]$$

Sachdev-Ye-Kitaev model: exact solution

$$G(z) = Ce^{-i(\pi\Delta-\theta)}z^{2\Delta-1} \qquad \mathrm{Im}(z) > 0, |z| \ll J \qquad \Delta = \frac{1}{2q}$$

Fermions scaling dimension

$$G(\tau) = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-2\Delta} = b\,\mathrm{sgn}(\tau)\,|\,\tau\,|^{-1/q} \qquad T = 0$$

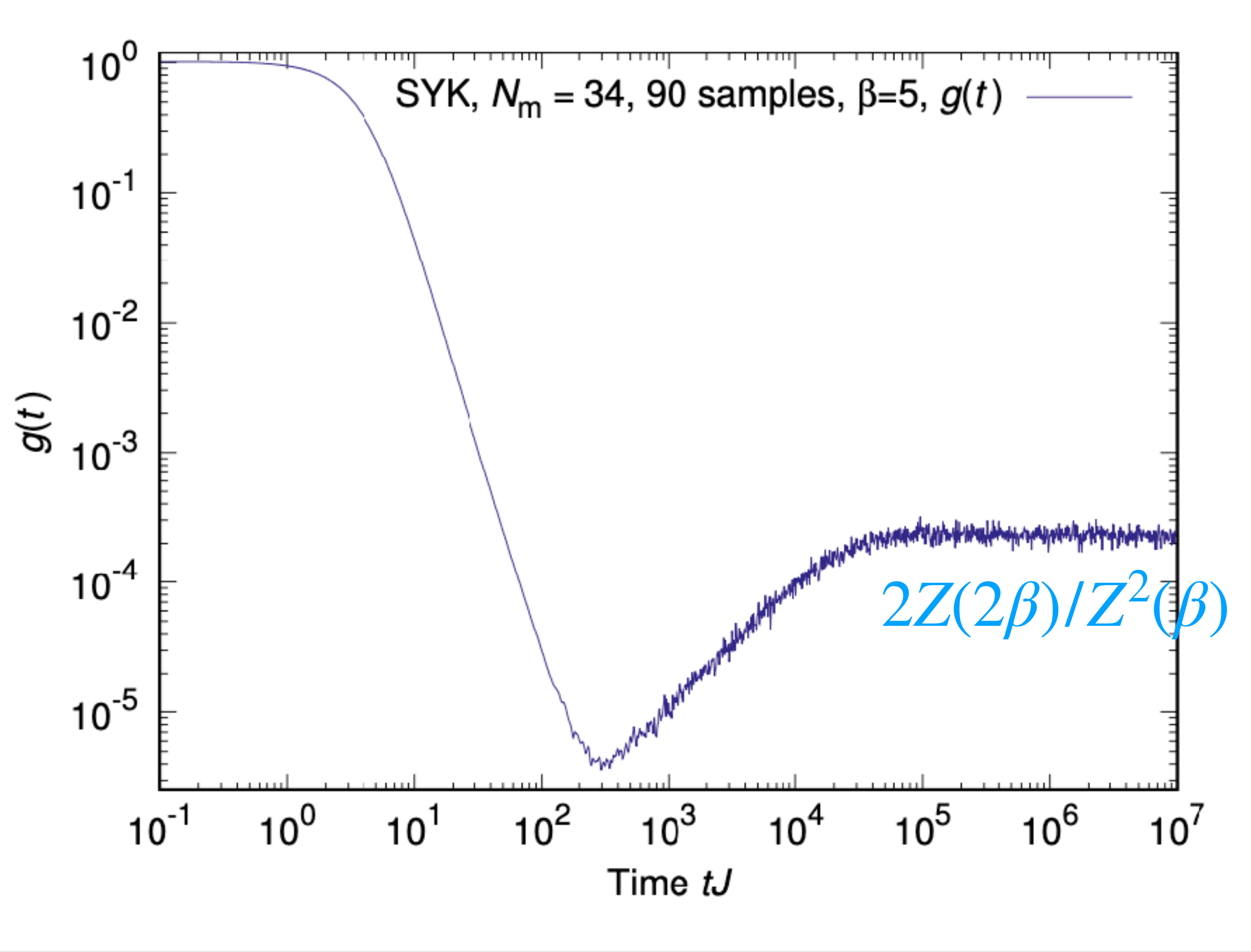
$$G(\tau) = b\left[\frac{\pi}{\beta\sin(\pi\tau/\beta)}\right]^{2\Delta}\mathrm{sgn}(\tau) \qquad T > 0$$

$$S_0 = \lim_{T\to 0}\lim_{N\to\infty}\frac{S(N,T)}{N} = \frac{1}{2}\log(2) - \int_0^{\Delta} dx\pi(1/2 - x)\tan(\pi x) + \mathcal{O}(T)$$

Residual entropy

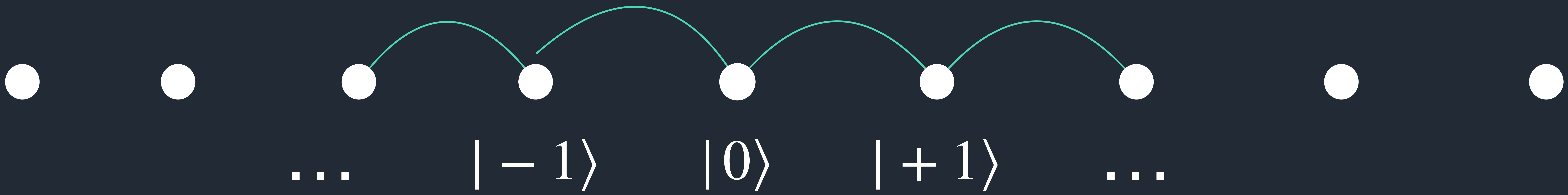
Inspired by

$$g(t;\beta) \equiv \frac{\langle Z(\beta,t)Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2}$$



$$2Z(2\beta)/Z^2(\beta) \sim e^{-aS}$$

An analogy?



Initial State: $|\psi_i\rangle = |i\rangle$ With $i \in \{1,\cdots,N\}$

Final Hamiltonian: $H_f = -J\sum_i |i\rangle\langle i+1| + h.c. = \sum_k E_k\hat{\Pi}_k$

Fidelity: $f(t) = ||\langle\psi|e^{-i\hat{H}t}|\psi\rangle||^2 = |\chi(t)|^2$ And $\chi(t) = \sum_n p_n e^{-itE_n}$ with $p_n = \langle\psi|\hat{\Pi}_n|\psi\rangle$.

The characteristic function

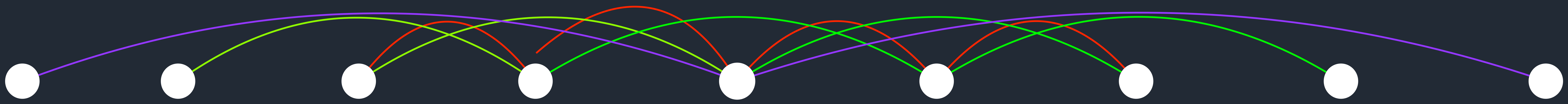
If the initial state only overlaps with eigenstate in the continuous portion of the spectrum.

$$\lim_{t \to \infty} \chi(t) = 0$$

Or more in general

$$\lim_{T \to \infty} \langle f(t) \rangle_T = 0, \quad \text{where} \quad \langle [\ldots] \rangle_T = \frac{1}{T} \int_0^T [\ldots] dt$$

The long-range case



$$\dots \quad |-1\rangle \quad |0\rangle \quad |+1\rangle \quad \dots$$

Initial State: $|\psi_i\rangle = |i\rangle$ With $i \in \{1, \cdots, N\}$

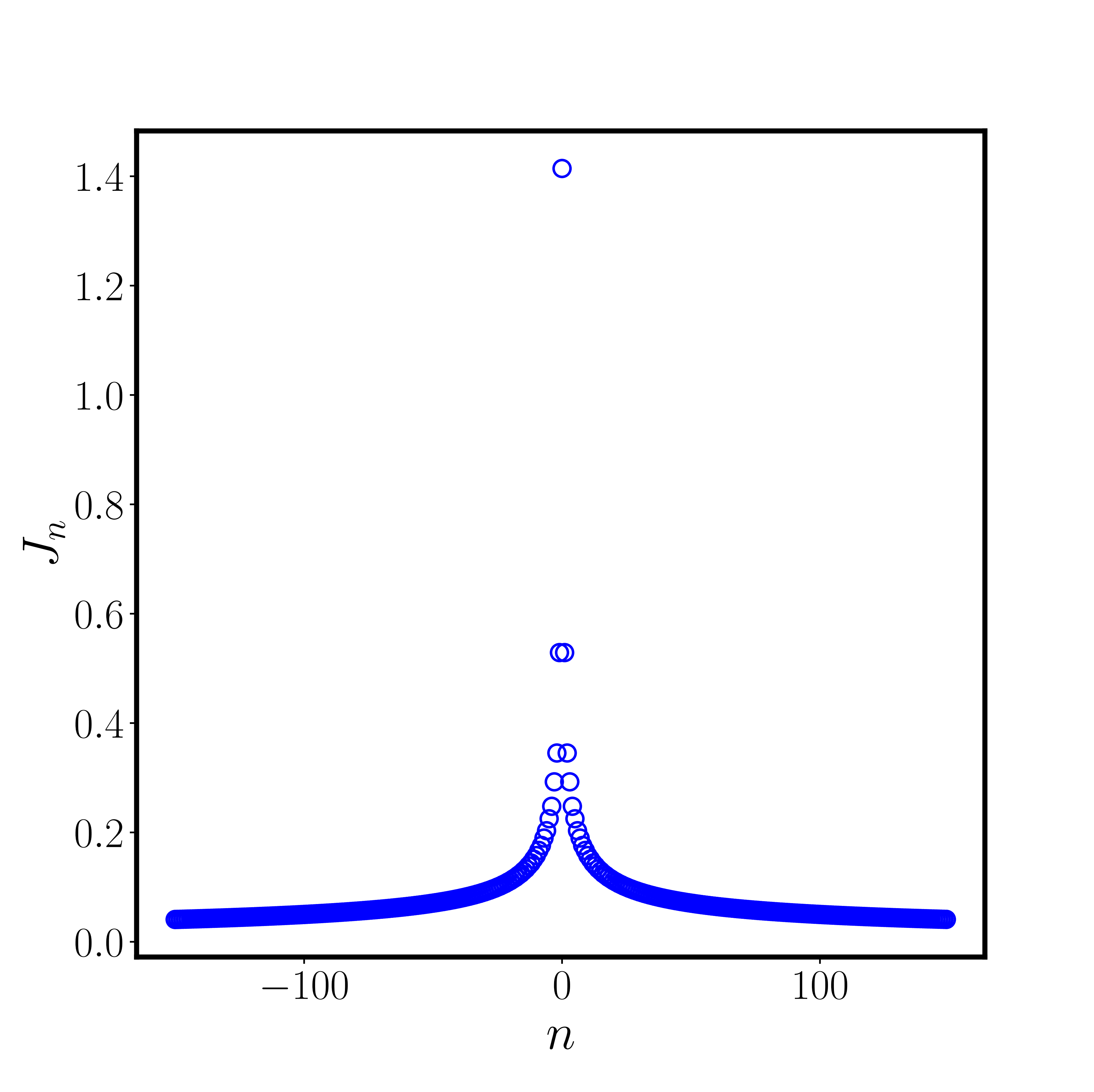
Final Hamiltonian: $H_f = -J\sum_i V_{ij}|i\rangle\langle j| + h.c. = \sum_k E_k \hat{\Pi}_k$

Fidelity: $f(t) = ||\langle\psi|e^{-i\hat{H}t}|\psi\rangle||^2 = |\chi(t)|^2$ And $\chi(t) = \sum_n p_n e^{-itE_n}$ with $p_n = \langle\psi|\hat{\Pi}_n|\psi\rangle$.

Discrete Spectrum

$$\lim_{N\to\infty} J_k = \lim_{N\to\infty} \frac{1}{N_\alpha} \sum_{r=1}^{N/2-1} \frac{\cos(kr)}{r^\alpha} \approx \frac{c_\alpha}{N} \sum_{r=1}^{N/2} \frac{\cos\left(2\pi n \frac{r}{N}\right)}{(r/N)^\alpha} \equiv J_n$$

$$J_n \equiv c_\alpha \int_0^{\frac{1}{2}} \frac{\cos(2\pi n\, s)}{s^\alpha} ds\,.$$



An example involving disorder

Harmonic Oscillator Variables: $[\hat{s}_i, \hat{p}_j] = i\delta_{ij}$ $\quad [\hat{s}_i, \hat{s}_j] = [\hat{p}_i, \hat{p}_j] = 0$

$$H = \frac{g}{2}\sum_i \hat{p}_i^2 - \frac{1}{2}\sum_{i,j} V_{ij}\hat{s}_i\hat{s}_j + \mu\left(\sum_i \hat{s}_i^2 - \frac{N}{4}\right).$$

The chemical potential μ is chosen in such a way that for each values of g one has:

$$A = \left\langle \frac{4\sum_i \hat{s}_i^2}{N} \right\rangle = 1$$

The disordered case

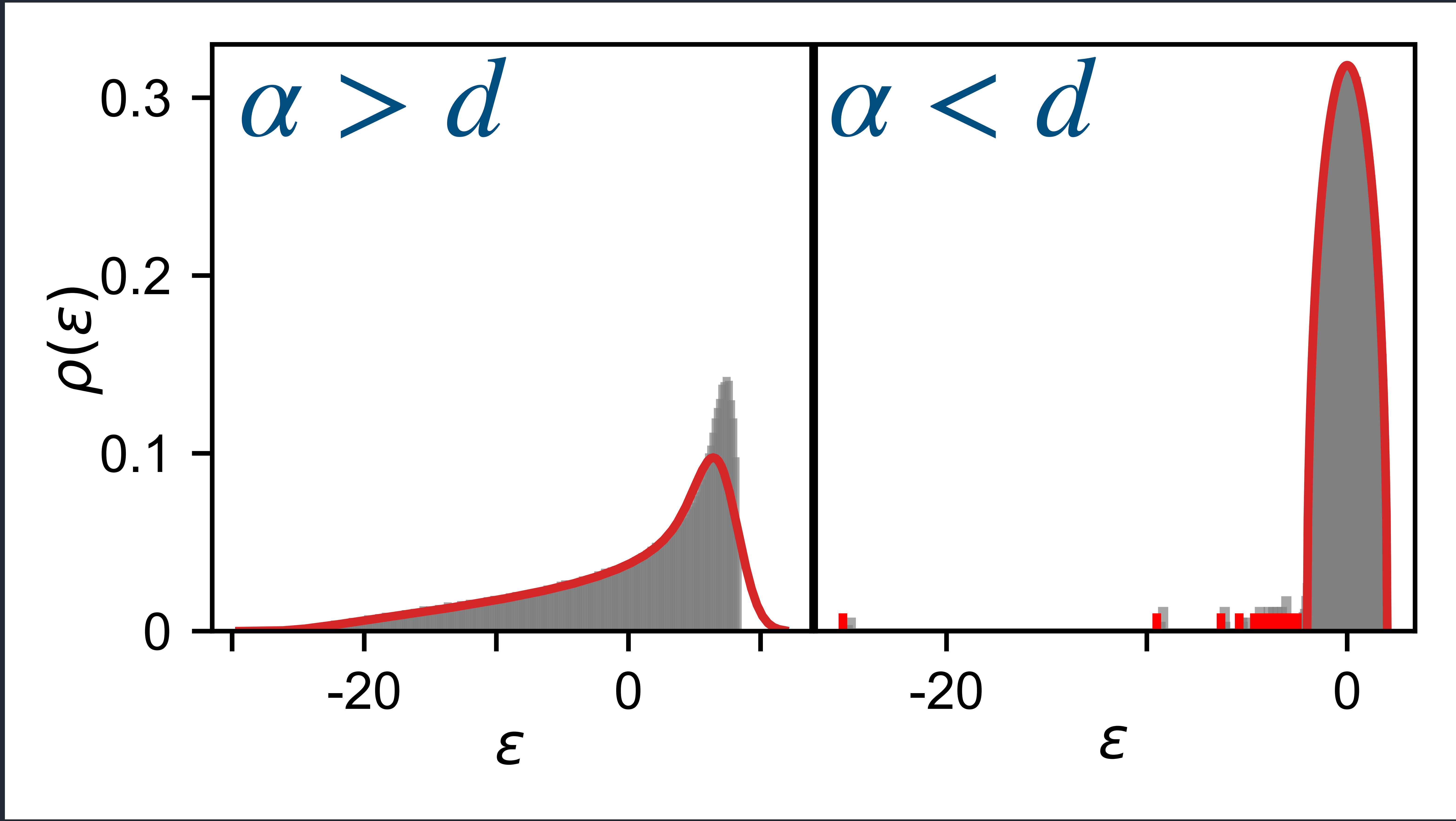
$$V_{ij} = J_{ij} + u_{ij} \quad \text{with} \quad P(u_{ij}) \sim \exp(-Nu_{ij}^2/2J^2)$$

Leading to the density of states

$$\rho(\varepsilon) = \rho_0(\varepsilon) + \sum_{n<n_*} \frac{\delta(\varepsilon - \varepsilon_n)}{N} \quad \text{with} \quad \rho_0(\varepsilon) = \begin{cases} \frac{2}{\pi} \frac{\sqrt{\varepsilon_1^2 - \varepsilon^2}}{\varepsilon_1^2} & \text{if } |\varepsilon| < \varepsilon_1, \\ 0 & \text{if } |\varepsilon| > \varepsilon_1 \end{cases}$$

with $\varepsilon_1 = 2J$

Long-range interactions interplay with disorder



$$\rho(\varepsilon) \approx \varepsilon^{\frac{d_s}{2}-1}$$

$$d_s = \frac{2d}{\alpha - d}$$

Strong long-range regime

$$d_s \to \infty$$

“Freezing and Shielding under Global Quenches for Long-Range Interacting Many-Body Systems”, D. A. Viciente, ND. *arXiv:2407.06072 (202*

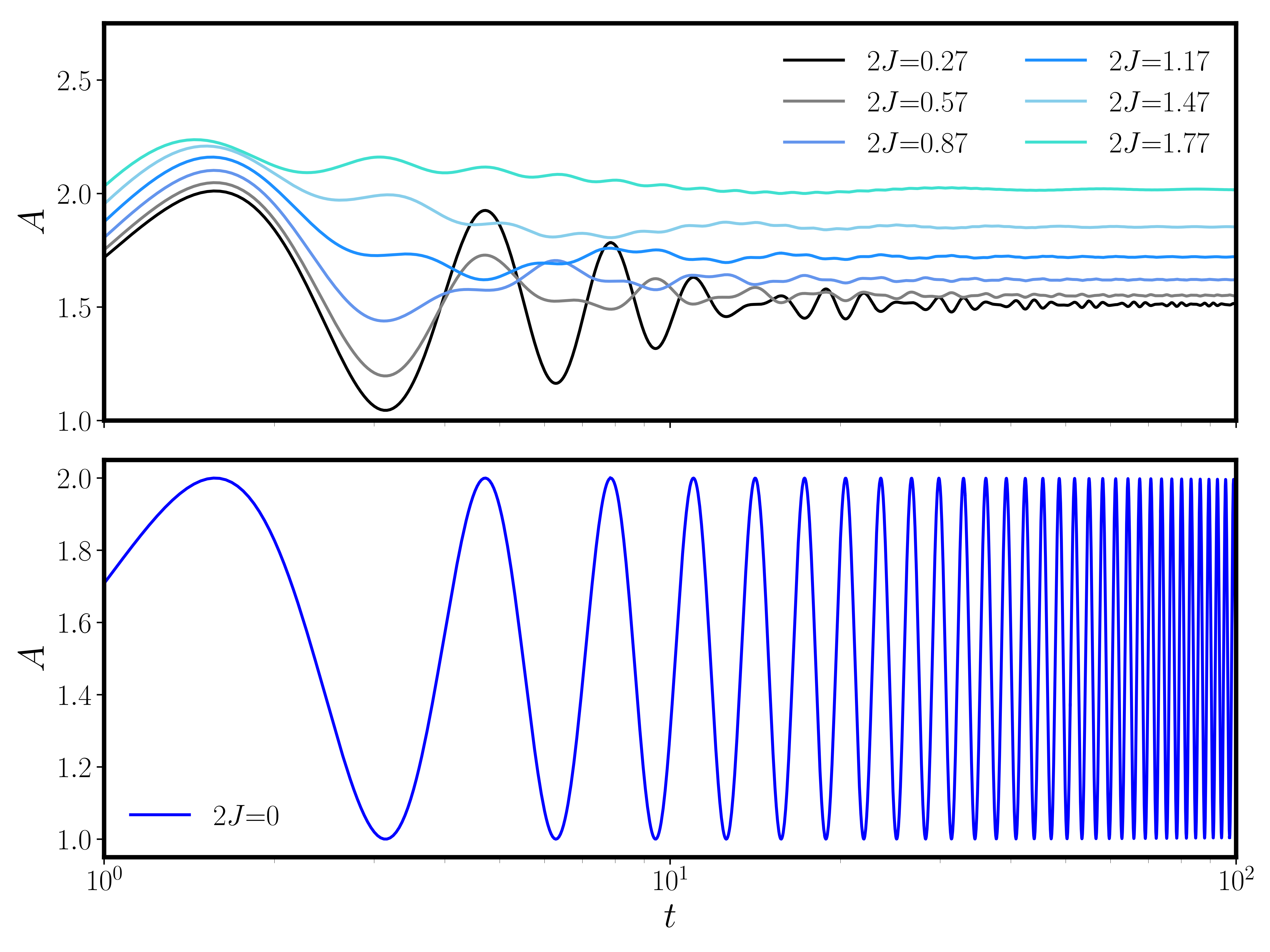
Lifting the constraint

Initial State: $g > g_c$ such that $\mu = 2\mu_c$

At $t = 0$ the constraint is removed:

$$\mu = \mu_c$$

And the system is let free to evolve.



My question

Are long-range interactions the origin of the black-hole entropy?

Maldacena's answer:

Chain of black holes!

A.k.a. a non-fermi liquid…

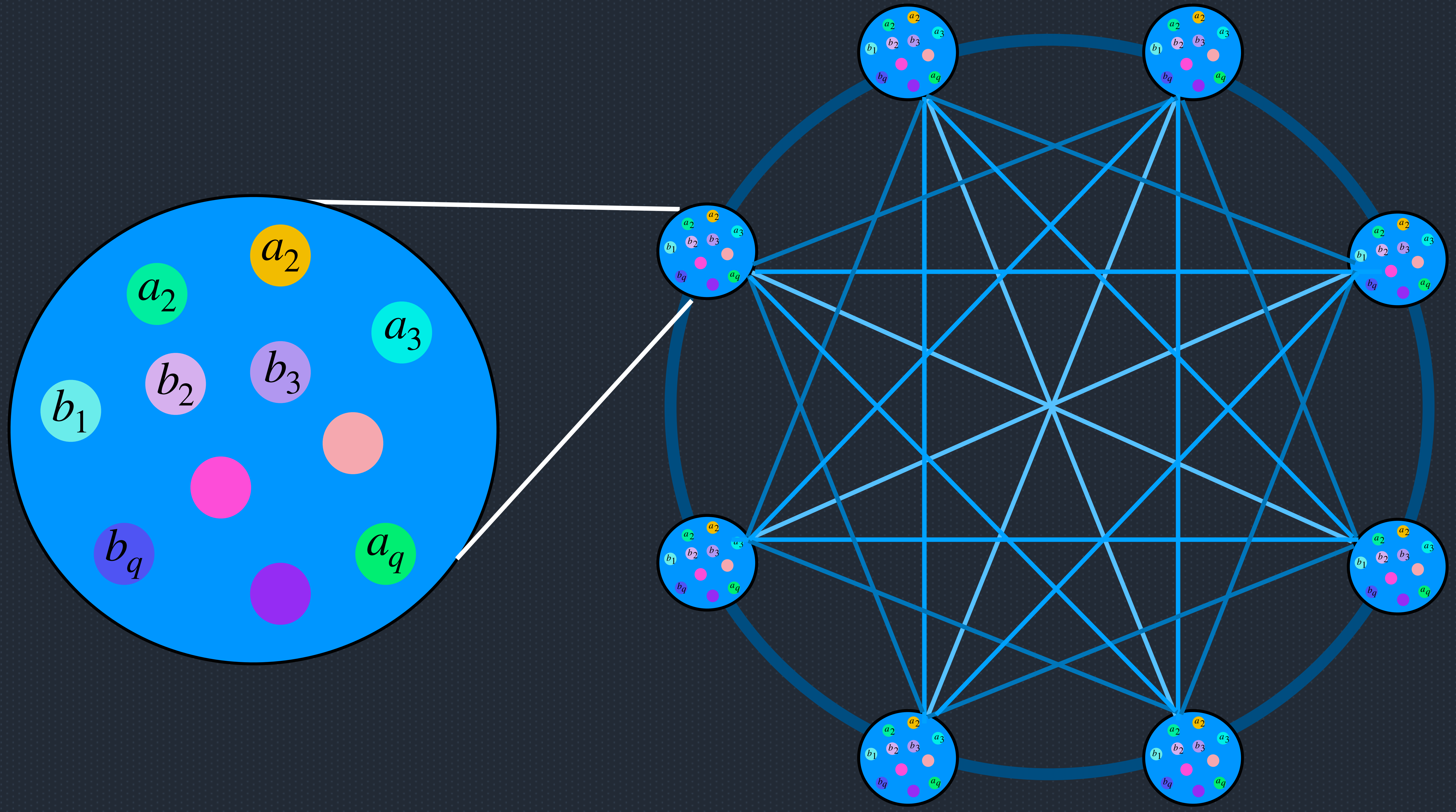
X-Y. Song, C-M. Jian, and L. Balents, Phys. Rev. Lett. 119, 216601 (2017)

The α-SYK model

$$H = \sum_{x=1}^{L} \sum_{\substack{a_1,\ldots a_q \\ b_1,\ldots b_q}} J_{a_1,\ldots a_q,\ldots b_1,\ldots,b_q} c^\dagger_{x,a_1} \ldots c^\dagger_{x,a_q} c_{x,b_1} \ldots c_{x,b_q} - \sum_{x,y=1}^{L} \sum_{a=1}^{N} t_{x,y} c^\dagger_{x,a} c_{y,a} + \mu \sum_{x=1}^{L} \sum_{a=1}^{N} c^\dagger_{x,a} c_{x,a}$$

$$t_{xy} = \frac{1}{\mathcal{N}_\alpha} \frac{1}{r_{xy}^\alpha}$$

$$\mathcal{N}_\alpha = \sum_{r=1}^{L} \frac{1}{r^\alpha}$$



The long-range spectrum

$$H = -\sum_{x,y=1}^{L}\sum_{a=1}^{N} t_{x,y} c^{\dagger}_{x,a} c_{y,a} + \mu \sum_{x=1}^{L}\sum_{a=1}^{N} c^{\dagger}_{x,a} c_{x,a} + H_{\text{int}}$$

Single particle spectrum $\varepsilon_\alpha(k) = \mu - f_\alpha(k)$ $f_\alpha(k_n) = \frac{1}{\mathcal{N}_\alpha}\sum_{r=1}^{L/2}\frac{\cos(k_n r)}{r^\alpha}$

$$\mathcal{N}_\alpha \approx \begin{cases} L^{1-\alpha} & \text{if } \alpha < 1 \\ \ln L & \text{if } \alpha = 1 \\ \zeta(\alpha) & \text{if } \alpha > 1 \end{cases}$$

$$k_n = \frac{2\pi n}{L} \qquad n = -\frac{L}{2}, -\frac{L}{2}+1, \ldots, \frac{L}{2} \quad \text{(PBC)}$$

The long-range spectrum: weak long-range

$\alpha > d = 1$ $\mathcal{N}_\alpha \approx \zeta(\alpha)$ Continuum limit $k_n \to k$

$$f_\alpha(k) \approx \frac{1}{\zeta(\alpha)}\left[\mathrm{Li}_\alpha(e^{ik}) + \mathrm{Li}_\alpha(e^{-ik})\right]$$

$$\mathrm{Li}_x(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^x}$$ polylogarithm

The long-range spectrum: weak long-range

$\alpha > d = 1$ $\quad \mathcal{N}_\alpha \approx \zeta(\alpha)$ $\quad$ Continuum limit $\quad k_n \to k$

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Dispersion relation $\quad k \to 0$

$$f_\alpha(k) = 1 + \sin\left(\frac{\alpha\pi}{2}\right) \frac{\Gamma(1-\alpha)}{\zeta(\alpha)} |k|^{\alpha-1} + \mathcal{O}(k^2) \qquad \text{if } 1 < \alpha < 3,$$

$$f_\alpha(k) = 1 + \frac{2\ln(k) - 3}{4\zeta(3)} k^2 + \mathcal{O}(k^3) \qquad \text{if } \alpha = 3,$$

$$f_\alpha(k) = 1 - \frac{\zeta(\alpha-2)}{2\zeta(\alpha)} k^2 + \mathcal{O}(k^{\alpha-1}) \qquad \text{if } \alpha > 3.$$

The long-range spectrum: weak long-range

$\alpha > d = 1$ $\mathcal{N}_\alpha \approx \zeta(\alpha)$ Continuum limit $k_n \to k$

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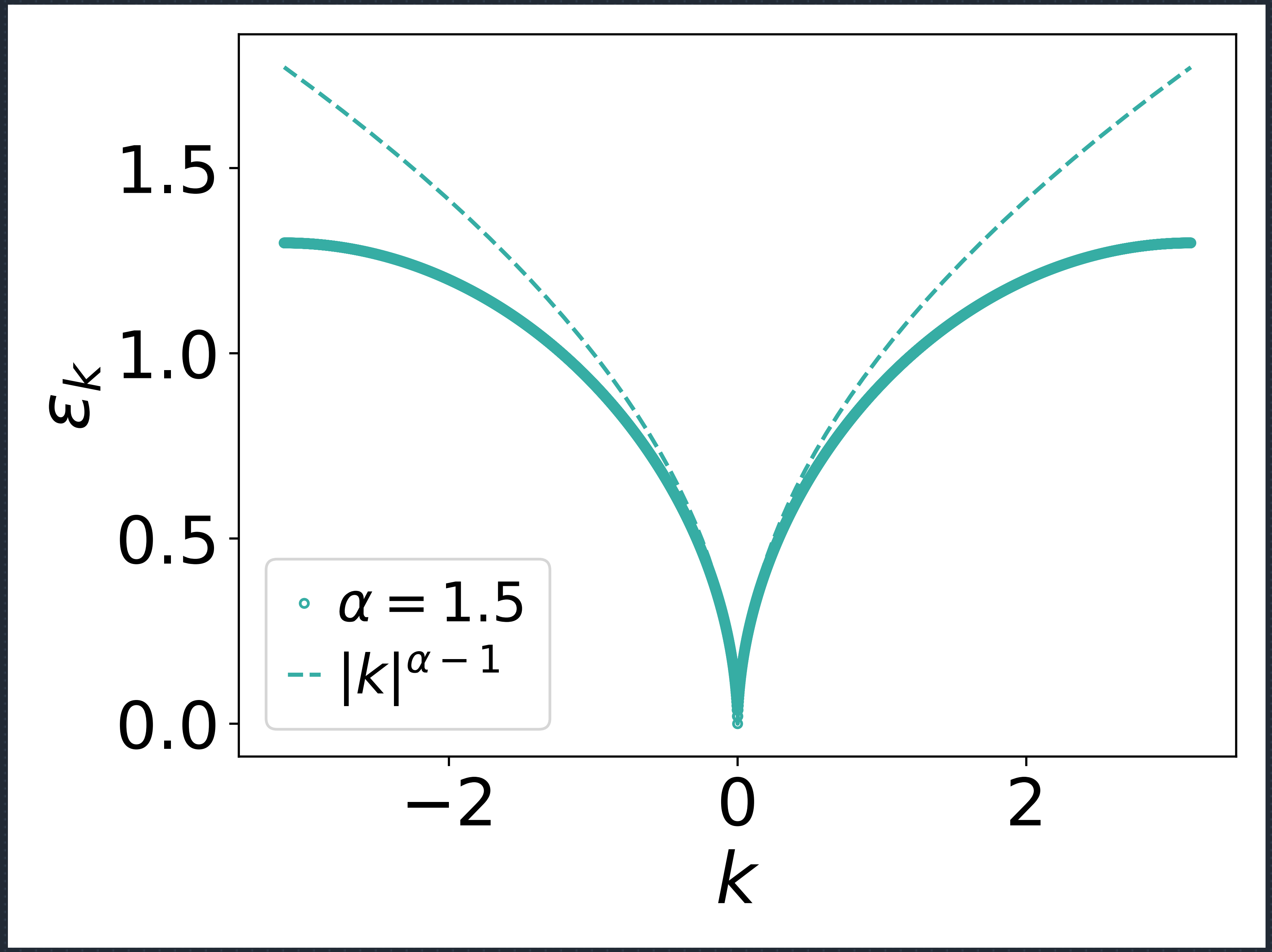
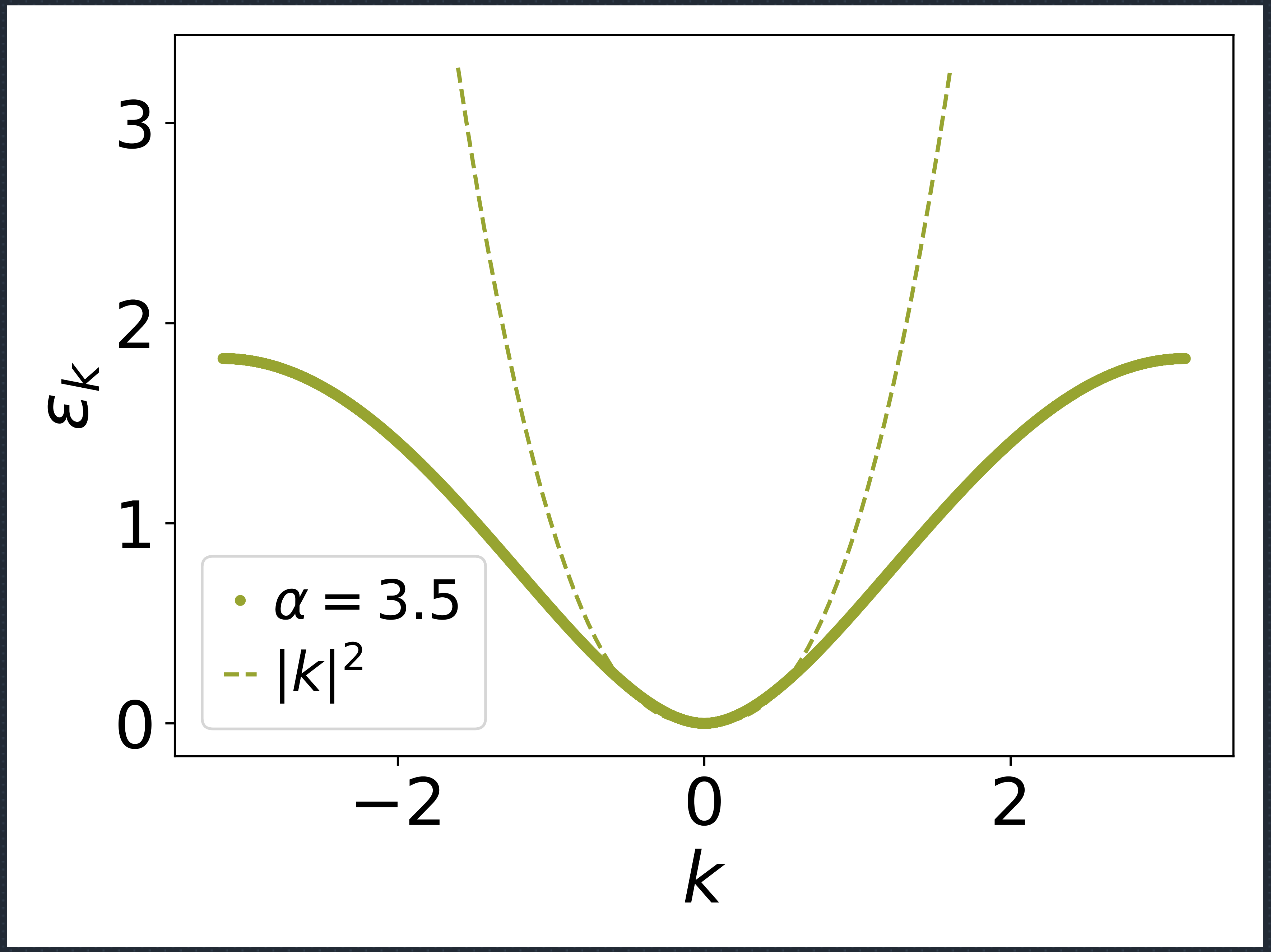
$$\mathrm{Li}_x(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^x}$$ polylogarithm

$1 < \alpha < 3$



$\alpha > 3$



The long-range spectrum: strong long-range

$\alpha < d = 1 \qquad \mathcal{N}_\alpha \approx L^{1-\alpha}$

$$\lim_{L\to\infty} \frac{1}{\mathcal{N}_\alpha} \sum_{r=1}^{L/2} \frac{\cos(kr)}{r^\alpha} \approx \frac{c_\alpha}{L} \sum_{r=1}^{L/2} \frac{\cos(2\pi n \frac{r}{L})}{(r/L)^\alpha}$$

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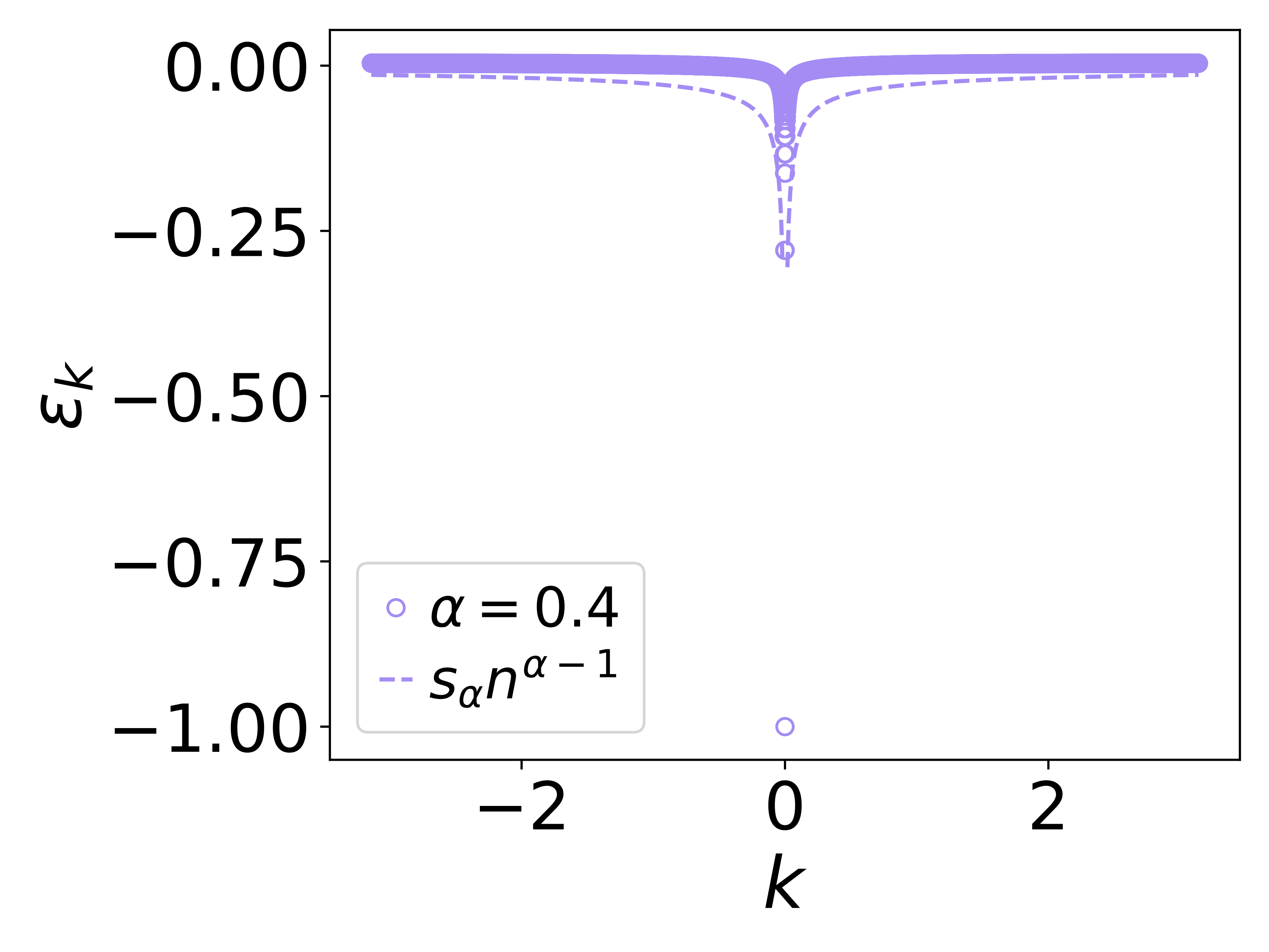
$$f_\alpha(n) = \lim_{N\to\infty} f_\alpha(k) = \int_0^{1/2} ds \frac{\cos(2\pi n s)}{s^\alpha} \qquad \lim_{n\to\infty} f_\alpha(n) = 0 \qquad f_\alpha(n) = s_\alpha n^{\alpha-1} + \mathcal{O}(n^{-2})$$

Discrete spectrum $\quad \varepsilon_\alpha(n) = \mu - f_\alpha(n)$ Accumulation point $\quad \max_n \varepsilon_\alpha(n) = \mu$

The long-range spectrum: strong long-range

$\alpha < d = 1$ $\qquad$ $\mathcal{N}_\alpha \approx L^{1-\alpha}$

Discrete spectrum $\quad \varepsilon_\alpha(n) = \mu - f_\alpha(n)$ $\quad$ Accumulation point $\quad \max_n \varepsilon_\alpha(n) = \mu$



Low energy T = 0 solution

Saddle point equations $N \to \infty$ Analitic continuation $i\omega_n \to \omega + i0^+$

$$G(i\omega_n) = \frac{1}{L}\sum_k \frac{1}{i\omega_n - \varepsilon_k - \Sigma(i\omega_n)} \qquad \Sigma(\tau) = (-1)^{q+1}U^2G^q(\tau)G^{q-1}(-\tau)$$

Ansatz $G(\omega) = Ce^{i\theta}\omega^{2\Delta-1}$ $\Delta = \Delta(\alpha, q)$ $\omega \to 0$

$\omega \ll \Sigma(\omega)$ Interaction-dominated solution: **Non-Fermi-Liquid**

$\Sigma(\omega) \ll \omega$ Interaction becomes irrelevant: **Fermi-Liquid**

Strong long-range $\alpha < 1$

$$G(\omega) \approx \frac{1}{L} \sum_n \frac{1}{z - \varepsilon_n} \qquad z = \omega - \Sigma(\omega) \qquad \varepsilon_\alpha(n) = s_\alpha n^{\alpha-1} + \mathcal{O}(n^{-2})$$

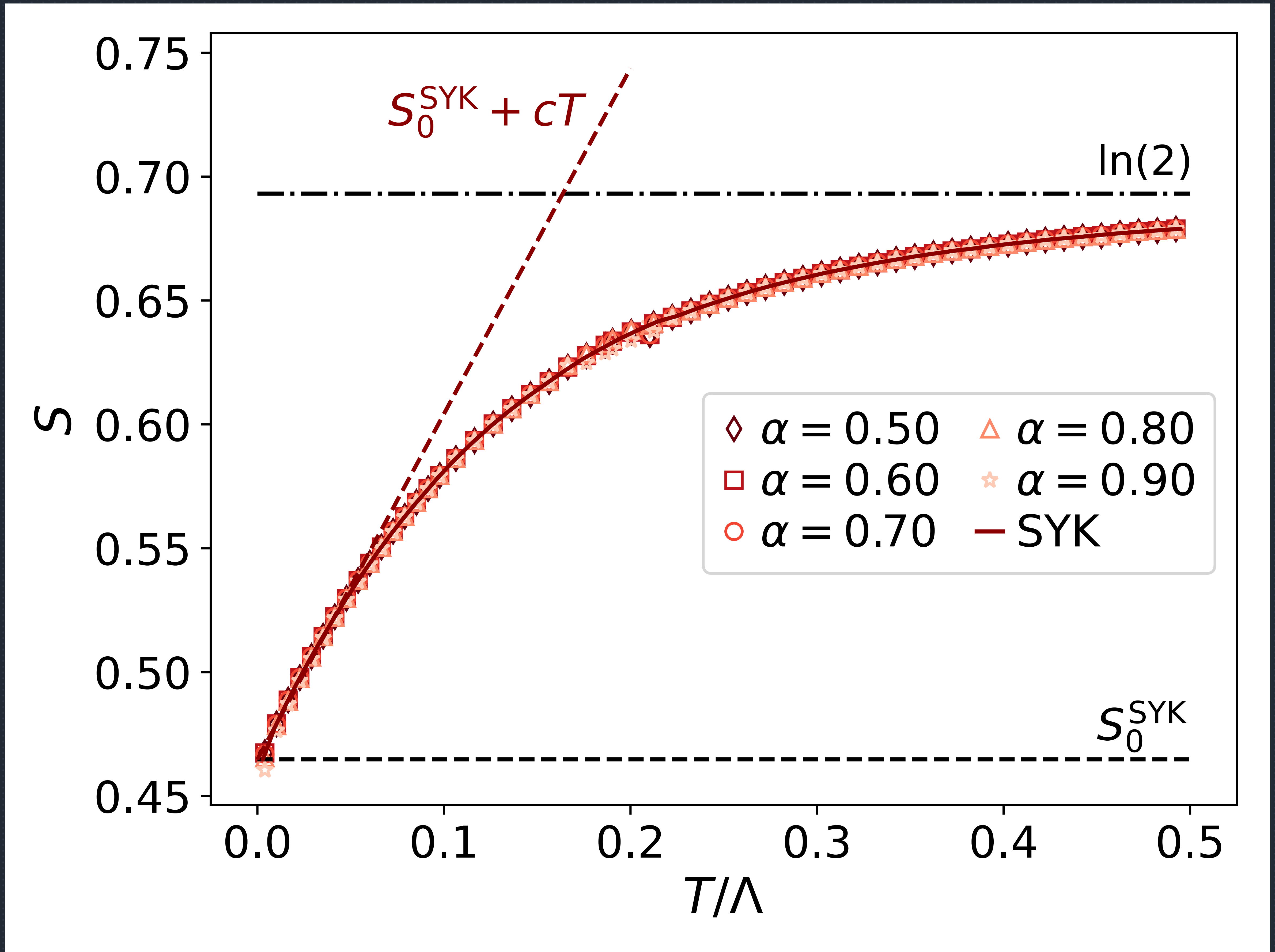
$$G(\omega) \approx \frac{1}{z} \left[1 - \frac{s_\alpha}{zL} \sum_n \frac{1}{n^{1-\alpha}} + \mathcal{O}(L^{-1}) \right] \qquad \sum_n n^{\alpha-1} = \mathcal{O}(L^{\alpha-1})$$

$$G(i\omega_n) = \frac{1}{i\omega_n - \Sigma(i\omega_n)} + \mathcal{O}(L^{\alpha-1})$$

Same equations as for the standard SYK model

Strong long-range $\alpha < 1$

Residual entropy $$S_0^{\alpha-\mathrm{SYK}} = S_0^{\mathrm{SYK}} + \mathcal{O}(L^{\alpha-1})$$



Weak long-range $\alpha > 1$

Saddle point equations $N \to \infty$

$$G(\omega) \approx \int_{-\Lambda}^{\Lambda} d\varepsilon \frac{g(\varepsilon)}{\omega - \Sigma(\omega) - \varepsilon} \qquad \Sigma(\tau) = (-1)^{q+1} U^2 G^q(\tau) G^{q-1}(-\tau)$$

Weak long-range $\alpha > 1$

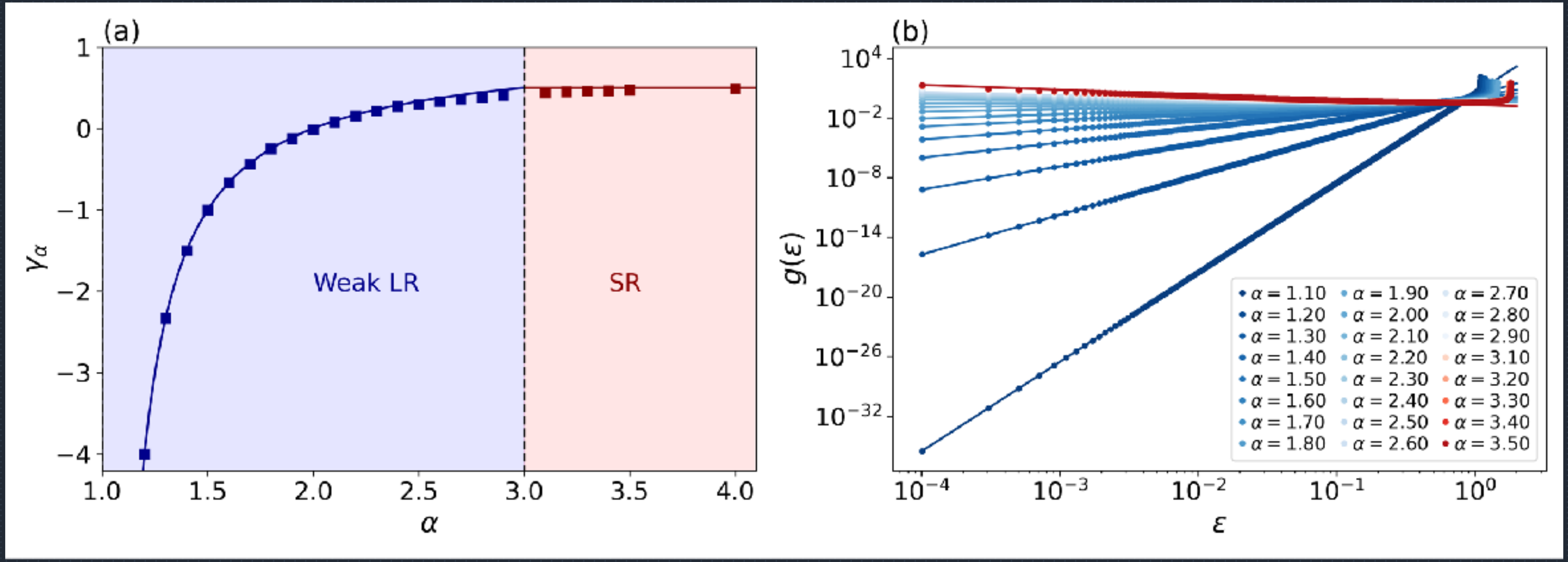
Saddle point equations $N \to \infty$

$$G(\omega) \approx \int_{-\Lambda}^{\Lambda} d\varepsilon \frac{g(\varepsilon)}{\omega - \Sigma(\omega) - \varepsilon} \qquad \Sigma(\tau) = (-1)^{q+1} U^2 G^q(\tau) G^{q-1}(-\tau)$$

Density of states

$$g(\varepsilon) \approx |\varepsilon|^{-\gamma_\alpha}$$

$$\gamma_\alpha = \begin{cases} 1 - \dfrac{1}{\alpha - 1} & 1 < \alpha < 3 \\ \dfrac{1}{2} & \alpha > 3 \end{cases}$$



T=0 solution: weak long-range

$$G(\omega) \approx G(z) = \int_{-\Lambda}^{\Lambda} d\varepsilon \frac{g(\varepsilon)}{z - \varepsilon} \qquad g(\varepsilon) \approx |\varepsilon|^{-\gamma_\alpha} \qquad z \to 0$$

$$G(z) \propto \begin{cases} z^{-\frac{1}{2}} & \alpha \geq 3 \\ z^{-1+\frac{1}{\alpha - 1}} & \frac{3}{2} < \alpha < 3 \\ z & 1 < \alpha \leq \frac{3}{2} \end{cases}$$

$$G(\omega) = Ce^{i\theta}\omega^{2\Delta - 1}$$

T=0 solution: weak long-range

$$G(\omega) = Ce^{i\theta}\omega^{2\Delta-1}$$

$$\Delta = \begin{cases} \dfrac{3}{(2+4q)} & \alpha > 3 \\ \dfrac{2\alpha-3}{2[1+2q(\alpha-2))]} & \alpha_c(q) < \alpha < 3 \end{cases}$$

$$\omega \ll \Sigma(\omega)$$

$$\alpha \le \alpha_c(q) = \frac{1}{2} + q$$

Non-Fermi-Liquid

$$\Delta = \begin{cases} \dfrac{1}{2(\alpha-1)} & \frac{3}{2} < \alpha \le \alpha_c(q) \\ 1 & 1 < \alpha \le \frac{3}{2} \end{cases}$$

$$\Sigma(\omega) \ll \omega$$

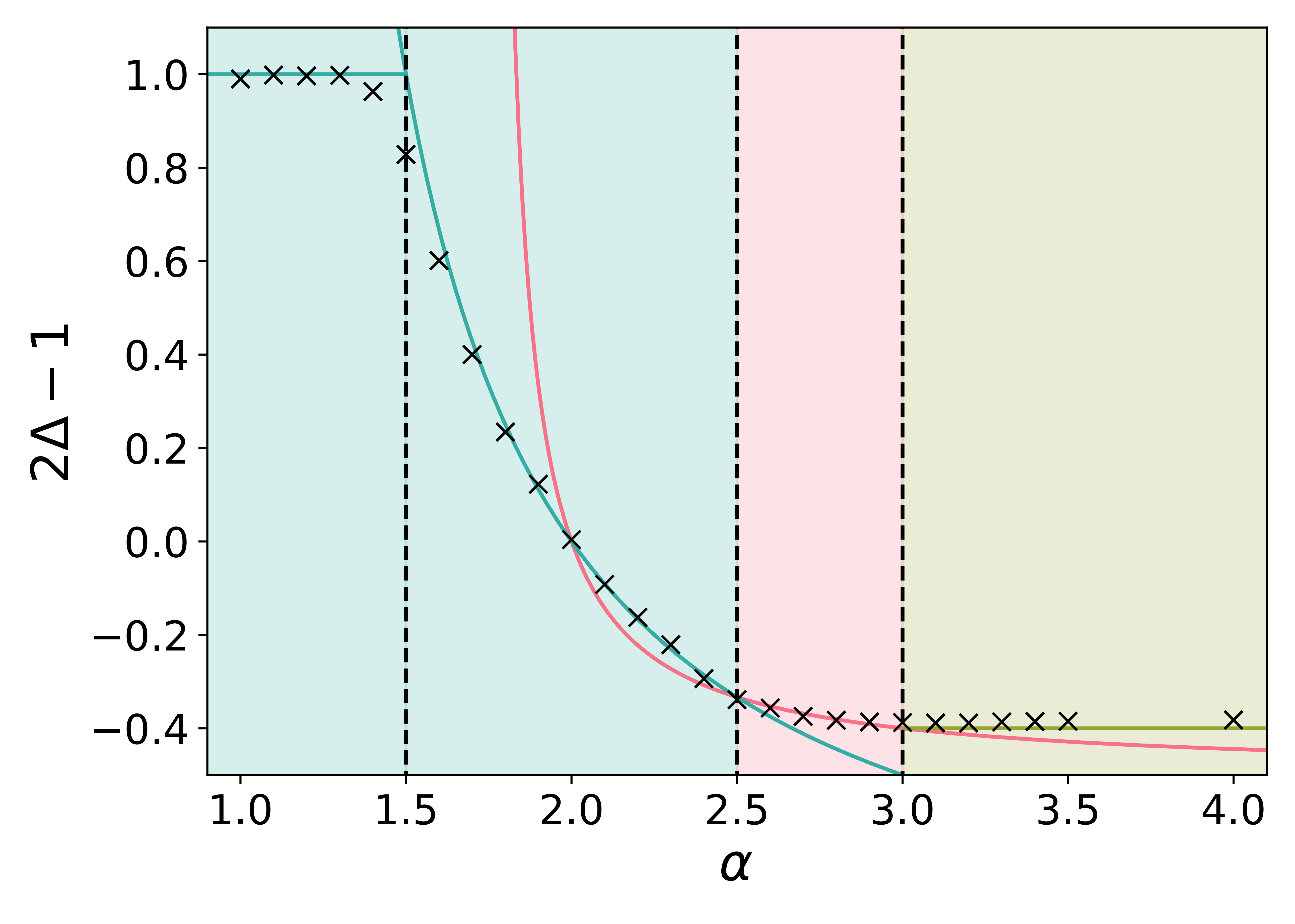
Fermi-Liquid

T=0 solution: weak long-range

$$G(\omega) = Ce^{i\theta}\omega^{2\Delta-1}$$

$$\Delta = \begin{cases} \frac{3}{(2+4q)} & \alpha > 3 \\ \frac{2\alpha-3}{2[1+2q(\alpha-2))]} & \alpha_c(q) < \alpha < 3 \end{cases}$$

$$\Delta = \begin{cases} \frac{1}{2(\alpha-1)} & \frac{3}{2} < \alpha \le \alpha_c(q) \\ 1 & 1 < \alpha \le \frac{3}{2} \end{cases}$$



Residual entropy

$$F = -\frac{1}{\beta L}\sum_n \sum_k \ln\left(-i\omega_n + \varepsilon_k + \Sigma(i\omega_n)\right) - \left(\frac{2q-1}{2q}\right)\frac{1}{\beta}\sum_n \Sigma(i\omega_n)G(i\omega_n)$$

$$F(T) \approx T^{\zeta+1} \qquad S(T) = -\partial F/\partial T \approx T^{\zeta}$$

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$$F(T) \approx T^{\zeta+1} \qquad S(T) = -\partial F/\partial T \approx T^{\zeta}$$

$$\zeta = \frac{(2(2q-1)\Delta - 1)}{\alpha - 1} = \begin{cases} \dfrac{4-q}{1+2q} & \alpha > 3 \\ \dfrac{4q - 10 - 2\alpha(2q-3)}{\left[1 + 2q(\alpha-2)\right](\alpha-1)} & \alpha_c(q) < \alpha < 3 \end{cases}$$

NFL - SR ($\alpha > 3$)

NFL - LR ($\alpha_c(q) < \alpha < 3$)

Residual entropy

$$F \approx -\frac{1}{\beta}\sum_n \int d\varepsilon g(\varepsilon) \ln\left(-i\omega_n + \varepsilon\right) \approx -\frac{1}{\beta}\int d\varepsilon g_\alpha |\varepsilon|^{\frac{1}{\alpha-1}-1} \ln\left(1 + e^{\beta\varepsilon}\right)$$

$F(T) \approx T^{\zeta+1}$ $\quad S(T) = -\,\partial F/\partial T \approx T^{\zeta}$ $\quad \zeta = \dfrac{1}{\alpha - 1}$ **FL - LR**

Residual entropy

1. **SRL**: Strong Long-range

$0 < \alpha < 1$ $\quad S(T=0) \sim S_0 > 0$

2. **FL**:

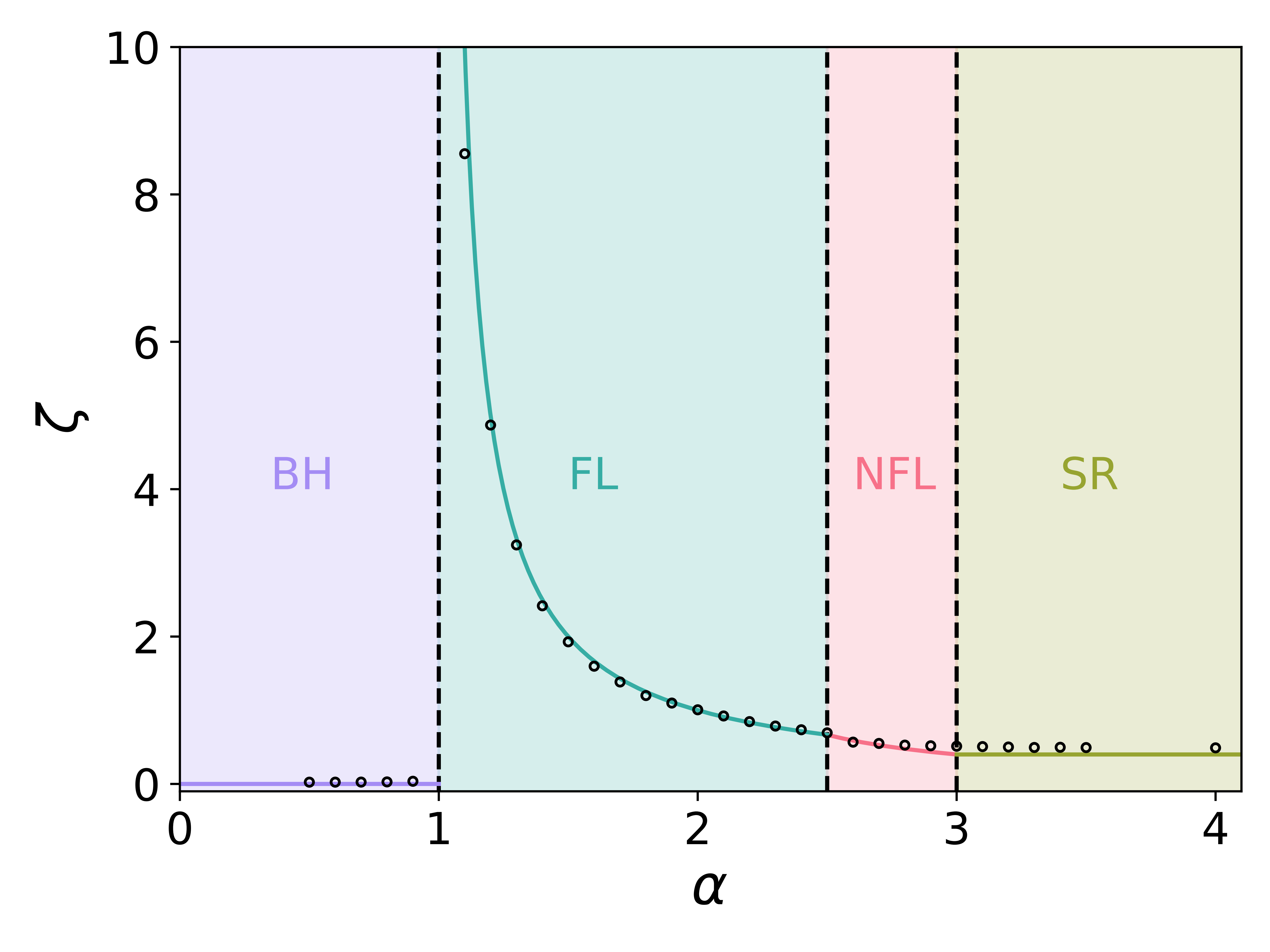
$1 < \alpha < \alpha_c$ $\quad S \sim T^{\frac{1}{\alpha - 1}}$

3. **LR-NFL**:

$\alpha_c < \alpha < \alpha^*$ $\quad S \sim T^{\zeta} \quad \zeta = \zeta(\alpha, q)$

4. **SR-NFL**

$\alpha > \alpha^*$ $\quad S \sim T^{\zeta} \quad \zeta = \zeta(q)$




(a)
Strong LR
Weak LR
SR
Black Hole
$S(T \to 0) \sim S_0 \neq 0$
Fermi Liquid LR
$S(T \to 0) \sim T^{1-\frac{1}{\alpha-1}}$
Non − Fermi Liquid LR
$S(T \to 0) \sim T^{\frac{4q-10-2\alpha(2q-3)}{(1+2q(\alpha-2))(\alpha-1)}}$
Non − Fermi Liquid SR
$S(T \to 0) \sim T^{\frac{4-q}{1+2q}}$
$\alpha = 0.4$
$s_\alpha n^{\alpha-1}$
$\alpha = 1.5$
$|k|^{\alpha-1}$
$\alpha = 3.5$
$|k|^2$
ε_k
k
α
(b)
$2\Delta - 1$
(c)
ζ
BH
FL
NFL
SR

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