- Lecture 1:
  - Introduction
    * Two-body Problem in GR
    * Comparison with different methods
    * Why scattering?
  - Worldline EFT formalism
    * Organizing Principles
    * Conservative sector
    * Dissipation sector
    * Scalar Toy Model
- Lecture 2:
  - Scattering Calculation
    * Waveform & Deflection at leading order (LO)
    * Extracting effective potential
    * Dimensional Regularization
    * Sketch of next-to-leading order (NLO)

△ Additional Reading Materials :

(Far from complete)

- Goldberger , hep-ph/0701129
- Porto , 1601.04914

Research papers

- Goldberger , Ross , 0912.4254
- Goldberger , Ridgway , 1611.03493
- amplitude & EFT , 1908.01493
- Worldline QFT , 2010.02865
- Self-Force EFT , 2406.14770 , 2308.15304

⋮

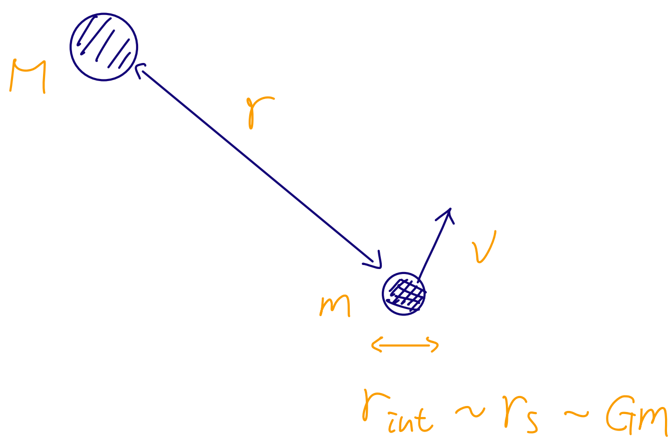
(and references therein)

- Introduction :

  - Given the advances in experiments (O5 of LVK, 3G exp, LISA) we need to improve our theorectical precision on the waveform models.

  - Scales in the two-body problem in GR :

    * Natural units : $\hbar = c = 1$



Nonrelativistic limit : $v/c = v \ll 1$

Inspiral regime (GR correction) : $\frac{r_{int}}{r} \sim \frac{Gm}{r} \ll 1$

Probe limit (SF correction) : $\frac{m}{M} \ll 1$

(Quantum effects) : $\frac{r_{Bohr}}{r_{int}} \sim \frac{\hbar}{m \cdot (Gm)} \sim \left(\frac{m_{Pl}}{m}\right)^2 \ll 1$

($m_{Pl} \sim 10^{-38} M_\odot$; not relevant for astrophysical BH / Neutron stars)

Standard Binary Coalescence

- The two-body problem in GR has multiple scales.

Numerical Relativity (NR) is the only way to cover all stages.

But there are several cases where NR is not so efficient.

* long inspiral phase (Neutron star (NS)
  Extreme Mass Ratio Inspiral (EMRI))
  → LISA

* Extreme Mass Ratio Binaries

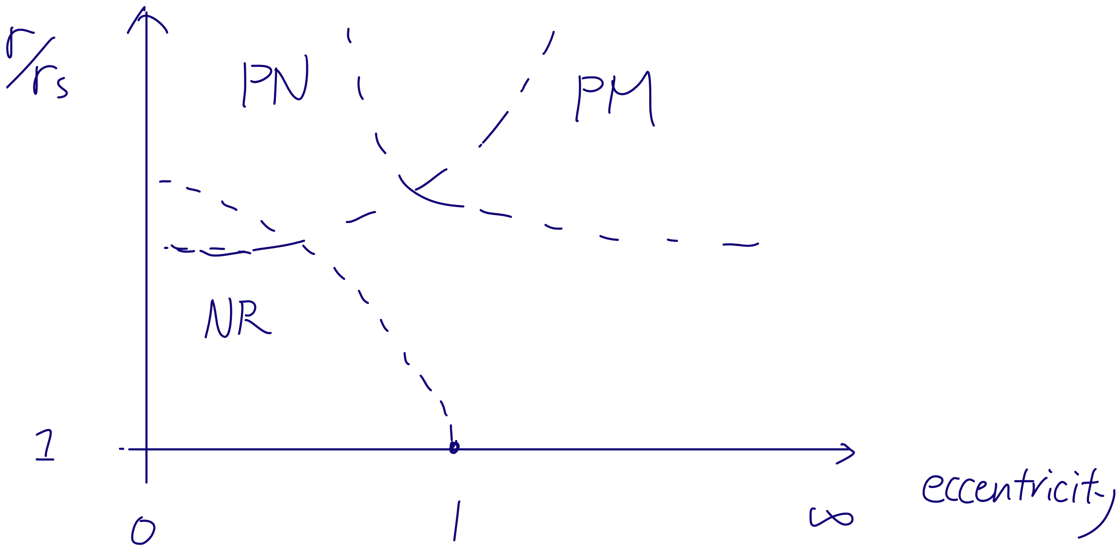
* Large eccentricity / Scattering

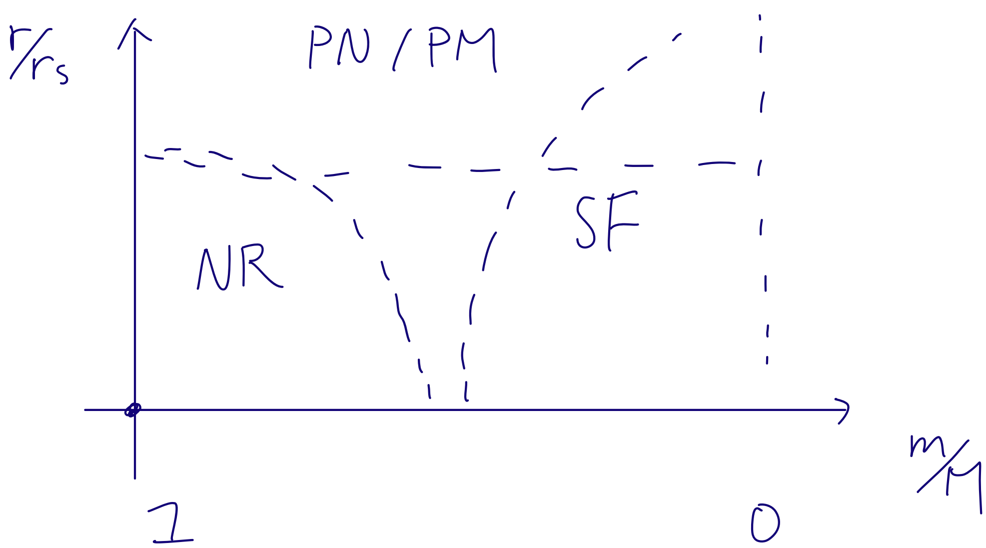
* Large spin

To cover binaries in all scenarios, developing analytic methods are necessary.

△ One can develope analytic methods when some parameters become small.

- Post Newtonian : $r_s/r \ll 1$, $v \ll 1$
- Post Minkowskian : $r_s/r \ll 1$
- Self Force : $m/M \ll 1$





◦ Why Scattering ?

- Many (intermediate) quantities in GR are gauge dependent

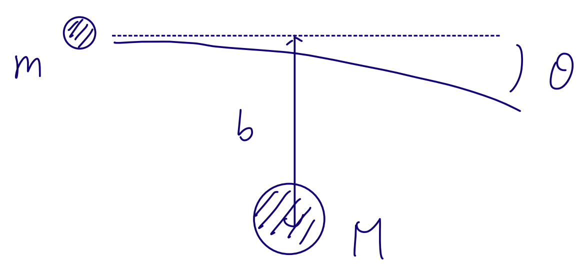
  e.g. effective potential ,

  effective multipole moments

- Scattering experiements measure observables in asymptotic flat spacetime $\longrightarrow$ gauge invariant

- Furthermore , the observables can expose hidden simplicity

- GR example :



1) Standard PN potential

$$V(r, \vec{p}^2, \vec{p}\cdot\hat{r})$$

2) Scattering input

$$\left.\begin{aligned} E &\longrightarrow \vec{p}^2 \\ J &= p\times b \end{aligned}\right\}$$ only two inputs ((s,t) in amplitudes)

$\theta(E, J)$ implies V only depends on two variables $\Rightarrow$ gauge redundancy in $V(r, \vec{p}^2, \vec{p}\cdot\hat{r})$

3) Natural variables :

$$P_1\cdot P_2 = \underset{\uparrow}{\gamma}\, m M$$

relative boost

$\Rightarrow$ Scattering observables expose the underlying Poincare invariance .

" Hidden Simplicity "

△ Worldline EFT Formalism :

We will focus on $r \gg r_{int}$ from now on.

The object can be shrinked to a point

$\begin{cases} r_{int} \sim r_s & : \text{single BH or NS} \\ r_{int} \sim r_{orbit} & : \text{binary system} \end{cases}$

Write down action based on symmetries & locality

① Diffeomorphism (gauge invariance)

② Reparametrization of the worldline (RPI)

③ Additional symmetry

(e.g. no spin $\longrightarrow$ $SO(3)$ symmetry)

Action :

$$S = S_{bulk} + S_{pp}$$

S_{bulk}: S_{EH} or S_{EM}

S_{pp}: point-particle action

- Minimal Coupling :

$$S_{pp} \supset -m\int d\tau = -m\int d\lambda \left(g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2}$$

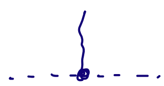
$$d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad ; \quad V^\mu = \frac{dx^\mu}{d\lambda}$$

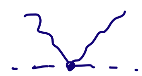
Coupling to gravity :

$$d\tau = d\lambda \left(g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \right)^{1/2} \qquad \left(\bar{V}^2 \equiv \eta_{\mu\nu} V^\mu V^\nu \right)$$

$$= d\lambda \left(\bar{V}^2 + h_{\mu\nu} V^\mu V^\nu \right)^{1/2}$$

$$= d\lambda \cdot |\bar{V}| \cdot \left(1 + \frac{1}{2|\bar{V}|} h_{\mu\nu} V^\mu V^\nu - \frac{1}{8|V|^2} \left(h_{\mu\nu} V^\mu V^\nu \right)^2 + \cdots \right)$$





2nd version of minimally coupling

$$S_{pp} \supset -\int d\lambda \left(\frac{m^2}{2e} + \frac{1}{2} e\, V^2 \right)$$

Reparametrization : $\lambda \to \lambda'(\lambda)$

$$d\lambda \to d\lambda' = d\lambda \cdot \frac{d\lambda'}{d\lambda}$$

$$e(\lambda) \to e(\lambda') = e(\lambda) \cdot \left(\frac{d\lambda'}{d\lambda} \right)$$

EOM for e :
$$\frac{1}{2} V^2 - \frac{1}{2e^2} \cdot m^2 = 0$$

$$V^2 = \frac{1}{e^2} m^2 \Rightarrow e = m (V^2)^{-1/2}$$

$$\Rightarrow S_{pp} = -m \int d\lambda \cdot \left(g_{\mu\nu} V^\mu V^\nu \right)^{1/2}$$

same as before

But we can also use RPI to choose $e(\lambda) = m$

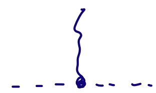
"gauge fixing" : $e(\lambda) = m \quad \Rightarrow \quad V^2 = 1$

$$(P^2 = m^2)$$

$$\Rightarrow \quad S_{pp} \supset -\int d\lambda \left(\tfrac{1}{2} m + \tfrac{1}{2} m V^2 \right)$$

$\Rightarrow$ gravitational coupling

$$= -\int d\lambda \left(\tfrac{1}{2} m + \tfrac{1}{2} m \left(\bar{V}^2 + h_{\mu\nu} V^\mu V^\nu \right) \right)$$



∘ Compare to the 1st version, we use RPI to linearize the coupling of the metric to the worldline.

EOM :

1) metric : $\dfrac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} = 8\pi G \cdot T_{\mu\nu} = 8\pi G \left(\dfrac{-2}{\sqrt{-g}} \dfrac{\delta S_M}{\delta g^{\mu\nu}} \right)$

$$S_M = -\int d\lambda \left(\tfrac{1}{2} m + \tfrac{1}{2} m\, g_{\mu\nu} V^\mu V^\nu \right)$$

$$= \int d^d x \sqrt{-g} \int d\lambda \cdot \frac{1}{\sqrt{-g}} \delta(x - X(\lambda)) \left(-\tfrac{1}{2} m - \tfrac{1}{2} m\, g_{\mu\nu} V^\mu V^\nu \right)$$

$$T_{\mu\nu} = \int d\lambda \cdot \frac{1}{\sqrt{-g}} \delta(x - X(\lambda)) \left(m\, V_\mu(\lambda) V_\nu(\lambda) \right)$$

2) Worldline : $\dfrac{\delta S_{pp}}{\delta X^\mu} = 0 \quad \Rightarrow$ geodesic eq.

$$\Rightarrow \quad \ddot{X}^\mu - \Gamma^\mu{}_{\rho\sigma} V^\rho V^\sigma = 0$$

- EFT at $\mathcal{O}(\partial^2)$:
  - But the power of EFT is that we can systematically go beyond the leading order.

$$S_{pp} \supset \underbrace{C_1}_{\substack{\text{coupling}\\ (\text{Wilson}\\ \text{coefficient})}} \int d\lambda \, \underbrace{\tfrac{1}{e} R}_{\text{operator}} \; + \; \underbrace{C_2}_{\substack{\text{coupling}\\ (\text{Wilson}\\ \text{coefficient})}} \int d\lambda \, \underbrace{e \; R_{\mu\nu} \, v^\mu v^\nu}_{\text{operator}}$$

Since $\underbrace{[S]}_{\text{mass dimension}} = 0$, $[R] = [R_{\mu\nu}] = +2 \quad (\sim 1/r^2)$

$$\begin{cases} [E] = [P] = 1 \\ [L] = -1 \end{cases}$$

$\Rightarrow \quad [C_1] = [C_2] = -1$

$$\Rightarrow \int d\tau \, (-m + C_1 R) \sim \int d\tau \cdot \left(-m + m \cdot \left(\frac{r_{int}}{r} \right)^2 \right)$$

$$C_{1,2} \sim m \times \mathcal{O}(r_{int}^2)$$

$\Rightarrow$ As we go to higher order

$$\underbrace{C}_{m \cdot r_{int}^n} \int d\lambda \cdot \underbrace{\mathcal{O}}_{\partial^n \sim 1/r^n} \quad \Rightarrow$$

We probe the system at $\mathcal{O}(r_{int}^n)$

Same as multipole expansion in EM.

Finite-size effect !

- However, these two operators do not survive since $R_{\mu\nu} = R = 0$ as per Einstein eq. in the vacuum.

- More precisely, when we redefine $\phi \to \phi' = \phi + \delta\phi$

$$S(\phi) \longrightarrow S(\phi) + \int d^4x \cdot \underbrace{\left(\frac{\delta S}{\delta \phi} - \partial_\mu \left(\frac{\delta \phi}{\delta \partial_\mu \phi} \right) \right)}_{\text{EOM}} \delta\phi$$

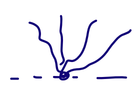
Therefore, we can use field redefinition to remove operators that are proportional to leading order EOM.

$$R_{\mu\nu} = T_{\mu\nu}$$
$$= m^2 \, V_\mu V_\nu \, \delta(x - x(\lambda))$$

$$\int d\lambda \, R_{\mu\nu} V^\mu V^\nu \longrightarrow \int d\lambda \cdot m^2 (V^2)^2 \cdot \delta(0)$$

$\Rightarrow$ absorbed by the minimal coupling (divergence $\delta(0)$ can be removed by redefine m)

$\Rightarrow$ There is NO $\mathcal{O}(\partial^2)$ finite-size effect !

(Similar to the removal of  by RPI)

- **Finite-size Effect** :

The first two non-trivial interactions we can write down are the ones with Weyl tensor $C_{\mu\nu\rho\sigma}$ ("traceless" Riemann)

$$S_{pp} \supset \int d\tau \left\{ \tfrac{1}{2} C_E \underline{E_{\mu\nu} E^{\mu\nu}} + \tfrac{1}{2} C_B \underline{B_{\mu\nu} B^{\mu\nu}} \right\}$$

parity preserving

$$+ \, C_{EB} \underline{E_{\mu\nu} B^{\mu\nu}} \Big\}$$

parity violating

- We use the 3+1 decomposition of Weyl tensor

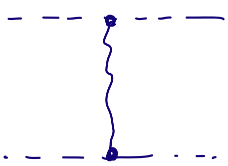
$$\begin{cases} E_{\mu\nu} \equiv C_{\mu\rho\nu\sigma} v^\rho v^\sigma \\ B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} C^{\alpha\beta}{}_{\delta\nu} v^\gamma v^\delta \end{cases} \xrightarrow{\text{rest frame}} \begin{cases} E_{ij} = C_{0i0j} \\ B_{ij} = \epsilon_{ik\ell} C_{k\ell 0j} \end{cases}$$

$$(\epsilon_{0123} = 1)$$

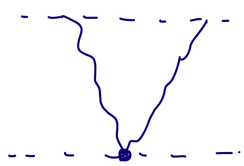
- Since $R \sim C \sim E \sim B \sim \mathcal{O}(\partial^2)$, the same dim'l analysis gives

$$C_E \sim C_B \sim C_{EB} \sim m \cdot r_{int}^4$$

$\triangle$ For BH, this gives the correction to potential as



$\Rightarrow$ Newtonian (0PN)



$$\Rightarrow (C_{E,B}/r^4) \times \underbrace{v^2}_{}$$

from loop (see later)

$$\sim \left(\frac{Gm}{r}\right)^4 \times v^2$$

$$\sim v^{10}$$

$\Rightarrow$ 5PN effect !

△ As we will see, the effect corresponds to tidal deformation

The couplings (C_E, C_B, C_{EB}) are called "Love numbers".
(static)

△ It is easy to generalize to higher orders

1) More derivatives

e.g. In the rest frame

$$S_{tidal} = \int d\tau \left\{ C_{E,\ell,n} \left(\nabla_{\mu_1} \cdots \nabla_{\mu_\ell} E_{\rho\sigma} \right) \underbrace{d_\tau^n}_{n \text{ time-derivatives}} \left(\underbrace{\nabla^{\mu_1} \cdots \nabla^{\mu_\ell}}_{\ell\text{-spatial derivatives}} E^{\rho\sigma} \right) + \cdots \right\}$$

usually projected to Symmetric-trace-free (STF)

e.g.

$$\nabla_\mu \nabla_\nu \rightarrow \nabla_\mu \nabla_\nu - \frac{\delta_{\mu\nu}}{d-1} \nabla^2$$

※ The $n=0$ terms $\Rightarrow$ static

$n \neq 0$ $\Rightarrow$ dynamical

$$\begin{cases} n \in \text{even} \Rightarrow \text{conservative} \\ n \in \text{odd} \Rightarrow \text{dissipative} \end{cases}$$

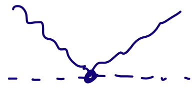
2) More Curvatures :

$$S_{NL-tidal} = \int d\tau \left\{ C \, E_{ij} E_{jk} E_{ki} + \cdots \right\}$$

$\Rightarrow$ "Non-linear" tidal interaction.

△ Dissipative Effect :

So far the finite-size interactions are quadratic in R.

 $\Rightarrow$ No emission

To include emission, we need to add single R terms.

In the rest frame of the particle,

$$S \supset \int d\tau \left\{ I_{ij} E^{ij} + J_{ij} B^{ij} + I_{ijk} \left(\partial^k E^{ij} \right)_{STF} + \cdots \right\}$$

We need to introduce the multipole moments

to get a nontrivial action.

$$\Rightarrow \quad I_{ij} \sim J_{ij} \sim m \cdot r_{int}^2 \qquad \left(\text{c.f.} \quad I_{ij} = \int \rho(x) \left(X_i X_j - \tfrac{1}{3} \delta_{ij} \vec{x}^2 \right) \right)$$

The worldline EFT describes not only stable BH, NS,

but also binary systems

Consider the previous tidal action

$$S_{tidal} = \int d\tau \left(\tfrac{1}{2} C_E \cdot E_{ij} E^{ij} + \cdots \right)$$

If E_{ij} has a background $\quad E_{ij} = \langle E_{ij} \rangle + \delta E_{ij}$

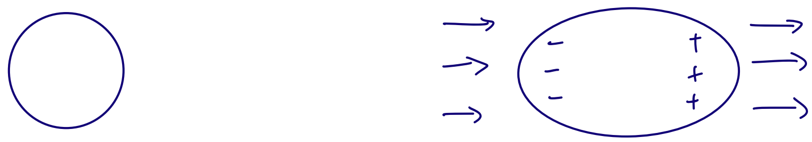
$$S_{tidal} \rightarrow \int d\tau \left(C_E \langle E_{ij} \rangle \delta E^{ij} + \cdots \right)$$

$$\longleftrightarrow \int d\tau \; I_{ij} E^{ij}$$

This induces a multipole moment

$$I'_{ij} = I_{ij} + C_E \langle E_{ij} \rangle$$

$\Rightarrow$ C_E is the analog of "susceptibility" that encodes how the moment is induced by a background.



$\langle E \rangle = 0$ $\qquad\qquad$ $\langle E \rangle \neq 0$

That's why we called them tidal interaction.

Worldline EFT Summary :

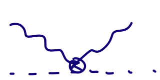
$$S = S_{bulk} + S_{pp}$$

$$S_{bulk} = \frac{1}{16\pi G} \int d^d x \sqrt{-g}\, R$$

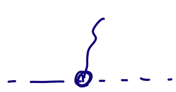
$$S_{pp} = -m \cdot \int d\tau \quad \Leftarrow \text{ minimal}$$

tidal

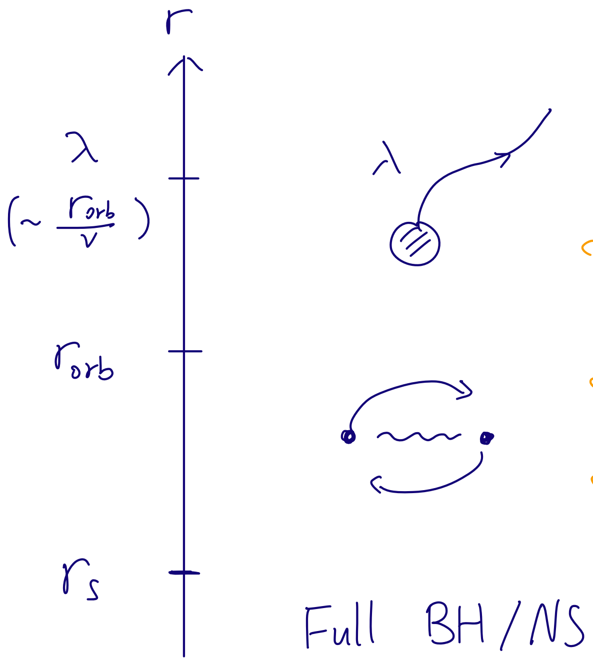
$$+ \int d\tau \left(\tfrac{1}{2} C_E E^2 + \tfrac{1}{2} C_B B^2 + C_{EB} E_{ij} \cdot B^{ij} + \cdots \right)$$



$$+ \int d\tau \left(I_{ij} E^{ij} + J_{ij} B^{ij} + \cdots \right) \Leftarrow \text{ emission}$$



△ Tower of EFTs :



integrate out orbital dynamics $\longrightarrow I_\ell$

integrate out $h_{\mu\nu}$: GR $\longrightarrow V_{eff}$

point particle EFT