Lectures on black hole perturbation theory and self-force

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Nordita program on
Amplitudes, Strong-Field Gravity and Resummation
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Plan

- **Overview**
  - Types of perturbative expansions in GR
  - Applications of black-hole perturbation theory
- **PART I: Basics of perturbation theory in GR**
  - Metric perturbations and gauge freedom
  - Perturbations via the Newman–Penrose formalism
- **PART II: Methods of black Hole perturbation theory**
  - Lorenz-gauge formulation
  - Regge-Wheeler-Zerilli formalism
  - Teukolsky equation & metric reconstruction
- **PART III: EMRIs and self-force theory**
  - EMRIs as sources of gravitational waves
  - Self-force theory
  - Self-force in scattering

A few bits left for you to work out...

Problem 0 (EXAMPLE) ● ○ ○

Draw me a sheep.

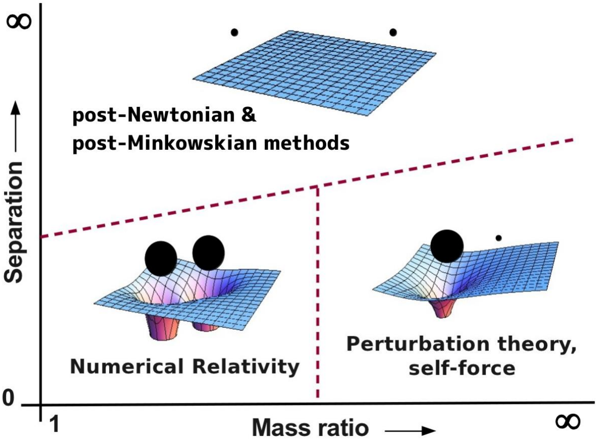
- ● ○ ○ a ~ 1-line calculation
- ● ● ○ a ~ 1-paragraph calculation
- ● ● ● a ~ 1-page calculation

Types of perturbative expansions in GR

4 main systematic perturbative frameworks for solving Einstein's Field Equations:

- **Post-Newtonian theory:** expands about Newtonian gravity, in powers of G & v/c
  - examples: SatNav; large-separation compact-object binaries
- **Post-Minkowskian theory:** expands about Special Relativity, in powers of G
  - examples: radiation at scri; scattering particles
- **Black-hole perturbation theory:** expands about Kerr spacetime, in magnitude of small metric perturbation
  - example: large mass-ratio binary; post-merger ringing
- **FLRW perturbation theory:** expands about FLRW cosmological spacetime, in powers of density fluctuation

Overlapping expansions in the binary problem



Applications of black-hole perturbation theory

- (Historical) Stability of the BH/event horizon
- Stability/development of internal structure; strong cosmic censorship
- Semi-classical BH theory
- Interaction with radiation (QNR, superradiance, power-law decay, universality)
- Post-merger ringing
- Compact object in a tidal environment
- Extreme Mass Ratio Inspirals (EMRIs), self-force

PART I: PERTURBATION THEORY IN GR

- **Metric perturbations and gauge freedom**
- Perturbations via the Newman–Penrose formalism

Metric perturbation equations

- We want to solve

$$G_{\mu\nu}[g_{\alpha\beta}(\epsilon)] = \frac{8\pi G}{c^4} T_{\mu\nu}(\epsilon) \qquad \text{for } \epsilon \ll 1$$

  where $g_{\alpha\beta}$ and $T_{\mu\nu}$ depend smoothly on a dimensionless parameter ϵ (e.g., binary mass ratio), so that $g_{\alpha\beta}(0)$ is a known spacetime [e.g., Kerr, with $T_{\mu\nu}(0) = 0$].
- Think of $g_{\alpha\beta}(\epsilon)$ as a 1-parameter family of spacetimes.

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- Think of $g_{\alpha\beta}(\epsilon)$ as a 1-parameter family of spacetimes.
- Taylor-expand

$$\begin{aligned} g_{\alpha\beta}(\epsilon) &= g_{\alpha\beta}(0) + \epsilon \frac{dg_{\alpha\beta}}{d\epsilon}\Big|_{\epsilon=0} + \frac{1}{2}\epsilon^2 \frac{d^2 g_{\alpha\beta}}{d\epsilon^2}\Big|_{\epsilon=0} + \cdots =: g^{(0)}_{\alpha\beta} + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta} + \cdots \\ T_{\mu\nu}(\epsilon) &= T_{\mu\nu}(0) + \epsilon \frac{dT_{\mu\nu}}{d\epsilon}\Big|_{\epsilon=0} + \frac{1}{2}\epsilon^2 \frac{d^2 T_{\mu\nu}}{d\epsilon^2}\Big|_{\epsilon=0} + \cdots =: T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \cdots \end{aligned}$$

- Regard $h^{(n)}_{\alpha\beta}$ and $T^{(n)}_{\mu\nu}$ as tensor fields on the “background” spacetime $g^{(0)}_{\alpha\beta}$.
- By convention, indices are raised and lowered using $g^{(0)}_{\alpha\beta}$. E.g., $h^{\alpha\beta}_{(1)} = g^{\alpha\mu}_{(0)} g^{\beta\nu}_{(0)} h^{(1)}_{\mu\nu}$.

Metric perturbation equations

Problem 1

Given $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h^{(1)}_{\alpha\beta} + O(\epsilon^2)$ and $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$, show $g^{\alpha\beta} = g^{\alpha\beta}_{(0)} - h^{\alpha\beta}_{(1)} + O(\epsilon^2)$.

Metric perturbation equations

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What equations do the perturbations $h^{(n)}_{\alpha\beta}$ satisfy?

- **At 0-th order** we simply have

$$G^{(0)}_{\mu\nu}[g^{(0)}_{\alpha\beta}] = 8\pi T^{(0)}_{\mu\nu} \quad (= 0 \text{ for Kerr}),$$

where $G^{(0)}$ is the Einstein operator with derivatives $\nabla^{(0)}_\alpha$ compatible with $g^{(0)}_{\alpha\beta}$.

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  where $G^{(0)}$ is the Einstein operator with derivatives $\nabla^{(0)}_{\alpha}$ compatible with $g^{(0)}_{\alpha\beta}$.

- **At orders ≥ 1** we have a little complication: $G_{\mu\nu}$ involves ∇_{α}, not $\nabla^{(0)}_{\alpha}$, acting on tensors defined in $g^{(0)}_{\alpha\beta}$. We need to express ∇_{α} in terms of $\nabla^{(0)}_{\alpha}$.

Derivation of the metric perturbation equations

Problem 2

Follow through the steps below to derive the linearized Einstein Field Equations

- **Step 1:** Show that, for an arbitrary vector field w^α,

$$(\nabla_\beta - \nabla^{(0)}_\beta) w^\alpha = C^\alpha_{\beta\gamma} w^\gamma$$

where

$$C^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}_{(0)} \left(\nabla^{(0)}_\beta h^{(1)}_{\gamma\mu} + \nabla^{(0)}_\gamma h^{(1)}_{\beta\mu} - \nabla^{(0)}_\mu h^{(1)}_{\beta\gamma} \right) + O(\epsilon^2)$$

(Assume here that ∇h is "as small" as h itself in terms of ϵ order counting.)

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  (Assume here that ∇h is "as small" as h itself in terms of ϵ order counting.)

- **Step 2:** Using $R^\alpha{}_{\beta\gamma\delta} w^\beta = (\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) w^\alpha$, next show

$$R^\alpha{}_{\beta\gamma\delta}(g) = R^{\alpha(0)}_{\ \ \beta\gamma\delta}(g^{(0)}) + 2\nabla^{(0)}_{[\gamma} C^\alpha_{\delta]\beta} + 2C^\alpha_{\mu[\gamma} C^\mu_{\delta]\beta}.$$

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- **Step 3:** Obtain The linear perturbation in the Ricci tensor, $\delta R_{\alpha\beta} := R_{\alpha\beta} - R^{(0)}_{\alpha\beta}$ and from it the linear perturbation in the Einstein tensor:

$$\delta G_{\alpha\beta} = \delta \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} R_{\mu\nu} \right).$$

Metric perturbation equations: 1st & 2nd order

- **Step 4:** Hence obtain the **linearized Einstain's equations** (here specialized to vacuum background, $R^{(0)}_{\mu\nu}[g^{(0)}_{\alpha\beta}] = 0$ for simplicity):

$$\delta G_{\alpha\beta} = -\frac{1}{2}\Box^{(0)}\bar{h}^{(1)}_{\alpha\beta} - R^{(0)}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}_{(1)} + \nabla^{(0)}_{(\alpha}\nabla^{\mu}_{(0)}\bar{h}^{(1)}_{\beta)\mu} - \frac{1}{2}g^{(0)}_{\alpha\beta}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}\bar{h}^{\mu\nu}_{(1)} = 8\pi T^{(1)}_{\alpha\beta} \quad (1)$$

where

$\Box^{(0)} \;:=\; g^{\mu\nu}_{(0)}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}$ "D'Alambertian" operator (GWs!)

$\bar{h}^{(1)}_{\alpha\beta} \;:=\; h^{(1)}_{\alpha\beta} - \frac{1}{2}g^{(0)}_{\alpha\beta}h^{(1)}$ "trace-reversed "perturbation (since $\bar{h}^{(1)} = -h^{(1)}$)

$h^{(1)} \;:=\; g^{\alpha\beta}_{(0)}h^{(1)}_{\alpha\beta}$ trace of $h^{(1)}_{\alpha\beta}$

Metric perturbation equations: 1st & 2nd order

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$h^{(1)}$	$:=$	$g^{\alpha\beta}_{(0)}h^{(1)}_{\alpha\beta}$	trace of $h^{(1)}_{\alpha\beta}$

- Derivation of **2nd-order pert. equation** more laborious but follows similarly.

$$\delta G_{\alpha\beta}[h^{(2)}] = 8\pi T^{(2)}_{\alpha\beta} - \delta^2 G_{\alpha\beta}[h^{(1)}]$$

where $\delta^2 G_{\alpha\beta}$ is a sum of quadratic combinations like $h^{(1)}\nabla\nabla h^{(1)}$ and $\nabla h^{(1)}\nabla h^{(1)}$.

Gauge freedom

Complication: $h^{(n)}_{\alpha\beta}$ can be changed arbitrarily by a small coordinate transformation

$$x^\alpha \to x^\alpha - \epsilon\xi^\alpha_{(1)} + \frac{1}{2}\epsilon^2\xi^\alpha_{(2)} + \cdots \qquad (2)$$

Problem 3 ●●○

Show that under the coordinate transformation $x^\alpha \to x^\alpha - \xi^\alpha$, where $\xi^\alpha = O(\epsilon)$, the first-order metric perturbation changes according to

$$h^{(1)}_{\alpha\beta} \to h^{(1)}_{\alpha\beta} + \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha. \qquad (3)$$

Show that this can also be written in terms of a Lie derivative:

$$h^{(1)}_{\alpha\beta} \to h^{(1)}_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta}.$$

- So, different $h^{(1)}_{\alpha\beta}$ can correspond to the same physics. Ergo, $h^{(n)}_{\alpha\beta}$ itself is not physically meaningful without additional information on the gauge (coordinates).

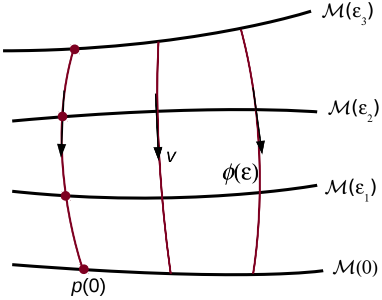
Gauge freedom: an important Clarification

- Even though gauge freedom in perturbation theory originates from the usual coordinate ambiguity in GR, it is **not the same** as the usual freedom to express the components of a tensor in different coordinates.
- Rather, a gauge transformation takes an $O(\epsilon)$ bit of $h^0_{\alpha\beta}$ into $h^1_{\alpha\beta}$, so it changes the split between "background" and "perturbation", and it thus changes the definition of "1st-order metric perturbation".
- Hence, a scalar field can be gauge dependent, while a rank-4 tensor can be gauge invariant—we'll encounter examples of both later on!

Meaning of gauge freedom is made more clear (and a general transformation rule is obtained) using following geometrical picture:

Gauge freedom: geometric interpretation

- 1-parameter family of spacetimes $(\mathcal{M}(\epsilon), g(\epsilon))$ defines a 5D manifold.
- We wish to describe the perturbation of some tensor $T(0,x)$ on $\mathcal{M}(0)$
- For this, we need $T(\epsilon, x)$, with an **identification map** between points of $\mathcal{M}(\epsilon)$ and points of $\mathcal{M}(0)$.
- Introduce a vector field v in the 5D manifold, transverse to each $\mathcal{M}(\epsilon)$, and let $\phi(\epsilon)$ be its integral curves.
- We then say that a point $p(\epsilon)$ of $\mathcal{M}(\epsilon)$ is identified with a point $p(0)$ of $\mathcal{M}(0)$ lying on the same curve.



Gauge freedom: geometric interpretation

- The perturbation of $T(0,x)$ is defined by

$$\delta T(x) = \epsilon \frac{\partial T(\epsilon,x)}{\partial \epsilon}\Big|_{\epsilon=0} = \mathcal{L}_v T\Big|_{\epsilon=0}$$

  for a given choice (v,ϕ) of ident. map.

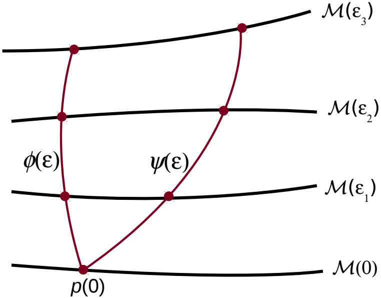
- Another map (w,ψ) would yield another perturbation, $\mathcal{L}_w T\Big|_{\epsilon=0}$. The difference between the two is

$$\left(\mathcal{L}_v T - \mathcal{L}_w T\right)|_{\epsilon=0} = \mathcal{L}_{v-w} T|_{\epsilon=0} = \mathcal{L}_\xi T$$

  where $\xi := (v-w)|_{\epsilon=0}$ is tangent to $\mathcal{M}(0)$.

- We have obtained a general formula for the gauge transformation of a tensor T under $x \to x - \xi$:

$$\delta T \to \delta T + \mathcal{L}_\xi T.$$



However, under $x^\alpha \to x^\alpha - \xi^\alpha$ we have $\delta G_{\alpha\beta} \to \delta G_{\alpha\beta}$, i.e. $\delta G_{\alpha\beta}$ is "**gauge invariant**".
This makes sense, since $\delta G_{\alpha\beta} = 8\pi T^{(1)}_{\alpha\beta}$, and $T^{(1)}_{\alpha\beta}$ is "physics".

Problem 4

By direct substitution of (3) in (1), show that $\delta G_{\alpha\beta}$ is gauge invariant.

$$h^{(1)}_{\alpha\beta} \to h^{(1)}_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \tag{3}$$

$$\delta G_{\alpha\beta} = -\frac{1}{2}\Box^{(0)}\bar{h}^{(1)}_{\alpha\beta} - R^{(0)}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}_{(1)} + \nabla^{(0)}_{(\alpha}\nabla^{\mu}_{(0)}\bar{h}^{(1)}_{\beta)\mu} - \frac{1}{2}g^{(0)}_{\alpha\beta}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}\bar{h}^{\mu\nu}_{(1)} = 8\pi T^{(1)}_{\alpha\beta} \tag{1}$$

This means Eq. (1) applies in **any** gauge.

Gauge freedom: some general results

- In particular, a perturbation in a scalar field Φ transforms as

$$\delta\Phi \to \delta\Phi + \mathcal{L}_\xi \Phi = \delta\Phi + \xi^\alpha \nabla^{(0)}_\alpha \Phi.$$

  A scalar field need not be invariant under a gauge transformation!

- **Problem 5**

  Prove the following **Theorem**: If a tensor T vanishes on the background $g^{(0)}$, then its linear perturbation δT about $g^{(0)}$ is gauge-invariant.

- In particular, the linear perturbation of the Riemann tensor, $\delta R_{\alpha\beta\gamma\delta}$, is gauge invariant **on flat space** (while, of course, its components still transform under a coordinate transformation the way tensor components do!)

Gauge freedom: Lorenz-gauge example

- Recall general form of the 1st-order perturbed Einstein's equations:

$$-\frac{1}{2}\Box^{(0)}\bar{h}^{(1)}_{\alpha\beta} - R^{(0)}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}_{(1)} + \nabla^{(0)}_{(\alpha}\nabla^{\mu}_{(0)}\bar{h}^{(1)}_{\beta)\mu} - \frac{1}{2}g^{(0)}_{\alpha\beta}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}\bar{h}^{\mu\nu}_{(1)} = 8\pi T^{(1)}_{\alpha\beta}$$

- Note last two terms on the left involve the divergence $Z^{\mu} := \nabla^{(0)}_{\nu}\bar{h}^{\mu\nu}_{(1)}$.
- Impose

$Z^{\mu} = 0$ Lorenz gauge condition

(like Lorenz gauge of electromagnetism, $\partial_{\mu}A^{\mu} = 0$).

- Obtain **Lorenz-gauge form** of the 1st-order perturbed Einstein's equations:

$$\Box^{(0)}\bar{h}^{(1)}_{\alpha\beta} + 2R^{(0)}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}_{(1)} = -16\pi T^{(1)}_{\alpha\beta} \qquad \text{field equation}$$

$$\nabla^{(0)}_{\nu}\bar{h}^{\mu\nu} = 0 \qquad \text{gauge condition}$$

Note the field equations **must** be supplemented with the gauge condition. (A solution of the former that is not a solution of the latter is **not** a solution of the Einstein equations!)

Gauge freedom: Lorenz-gauge example

How do we know that $Z^\mu = 0$ is a valid gauge condition, i.e. that it corresponds to a valid choice of coordinates?

Starting with a metric perturbation $h^{\rm old}_{\alpha\beta}$ in some gauge, can we always find a gauge transformation $x^\alpha \to x^\alpha - \xi^\alpha$ such that $h^{\rm new}_{\alpha\beta} = h^{\rm old}_{\alpha\beta} + \nabla^{(0)}_\alpha \xi_\beta + \nabla^{(0)}_\beta \xi_\alpha$ satisfies the Lorenz-gauge condition?

Problem 6 ●●○

Show that

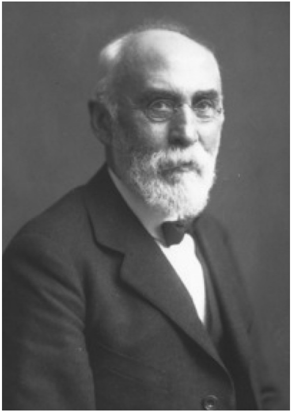
$$\nabla^{(0)}_\nu \bar{h}^{\mu\nu}_{\rm new} = 0 \quad \Leftrightarrow \quad \Box^{(0)}\xi^\alpha = -\nabla^{(0)}_\nu \bar{h}^{\mu\nu}_{\rm old}.$$

This is a hyperbolic equation for ξ^α, which can always be solved. Thus, we can construct a Lorenz-gauge perturbation.

Moreover, from the above we see that there are infinitely many distinct Lorenz-gauge perturbations, all related via gauge transformations whose generators satisfy

$$\Box^{(0)}\xi^\alpha = 0.$$

Lorentz or Lorenz?



Lorentz, Hendrik (Dutch, 1853-1928)
Nobel Prize 1902 (Zeeman effect)
Lorentz transformations
Lorentz force
Lorentzian metric

Lorenz, Ludvig (Dane, 1829-1891)
Lorenz gauge condition
Light propagation in media
. . . Lorentz-Lorenz equation

PART I: PERTURBATION THEORY IN GR

- Metric perturbations and gauge freedom
- **Perturbations via the Newman–Penrose formalism**

Newman–Penrose formulation of GR

- An example of a **tetrad formalism**, in which relevant tensors of the theory are expressed in terms of their projections onto a chosen vector basis ("tetrad")
- If tetard is chosen to reflect symmetries of spacetime, certain components may vanish, leading to simplification of field equations.
- In the case of the NP formalism (1962) the tetrad is complex and **null**:

$$\{e_a^\alpha\} = \{\ell^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha\} \qquad (a = 1, \ldots, 4),$$

with $e_a^\alpha e_{b\alpha} = 0$ for all a, b except $\ell^\alpha n_\alpha = -1$ and $m^\alpha \bar{m}_\alpha = 1$.

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- **In Kerr geometry**, ℓ^α and n^α are chosen to coincide with the two principal null directions (i.e., $C_{\alpha\beta\gamma\delta}\ell^\beta\ell^\delta = \lambda_1 \ell_\alpha \ell_\gamma$ and $C_{\alpha\beta\gamma\delta}n^\beta n^\delta = \lambda_2 n_\alpha n_\gamma$; read about the Petrov classification!)
- Hence particularly suited for describing outgoing and incoming radiation in the asymptotic region of Kerr geometry.

Newman–Penrose formulation of GR

Problem 7

Show that the metric can be written in terms of the tetrad legs as

$$g_{\alpha\beta} = -\ell_\alpha n_\beta - \ell_\beta n_\alpha + m_\alpha \bar{m}_\beta + m_\beta \bar{m}_\alpha.$$

Newman–Penrose formulation of GR

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- Instead of connections, one works with **spin coefficients**,

$$\gamma_{abc} := g_{\mu\lambda} e^\mu_a e^\nu_c \nabla_\nu e^\lambda_b,$$

  which in the NP formalism are given the extremely non-descriptive symbols $\kappa(= -\gamma_{311}), \tau, \sigma, \rho, \varpi, \nu, \mu, \lambda, \epsilon, \gamma, \beta, \alpha$.

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- Instead of Riemann components (**20** DoF), one works with
  - 5 complex **Weyl scalars** Ψ_n, encoding the **10** non-trace DoF of Riemann (i.e. the Weyl tensor): $\Psi_0 = C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \ell^\gamma m^\delta$, etc.
  - 4 real scalars $\{\Phi_{00}, \Phi_{11}, \Phi_{12}, \Lambda\}$ and 3 complex scalars $\{\Phi_{20}, \Phi_{21}, \Phi_{22}\}$, encoding the **10** trace DoF of Riemann (i.e. the Ricci tensor): $\Lambda = \frac{1}{24} R, \quad \Phi_{00} = \frac{1}{2} R_{\alpha\beta} \ell^\alpha \ell^\beta$, etc.

Newman–Penrose formulation of GR

- Then the field equations (now called **NP equations**) relate between derivatives of spin coefficients and the Ψ_n's and Φ_n's. For example:

$$D\tau - \Delta\kappa = (\tau + \bar{\varpi})\rho + (\bar{\tau} + \varpi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}$$

  where $D := \ell^\alpha \nabla_\alpha$ and $\Delta := n^\alpha \nabla_\alpha$.

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where $D := \ell^\alpha \nabla_\alpha$ and $\Delta := n^\alpha \nabla_\alpha$.

- **In Kerr**, only nonvanishing curvature scalar is

$$\Psi_2 = M\rho^3 = -\frac{M}{(r - ia\cos\theta)^3}.$$

That's 2 nontrivial DoF out of 20!

- Shows how special Kerr geometry is — but to have made that manifest required the use of a specially adapted tetrad.

Grav. Perturbations of Kerr in the NP formalism

- Since all curvature scalars but Ψ_2 vanish on the Kerr background, the linear perturbations in all curvature scalars but Ψ_2 are **gauge invariant**.
- There is a residual arbitrariness in Ψ_n, associated with the freedom to perform **infinitesimal rotations** of the tetrad basis. It can be shown that
  - The perturbations in Ψ_1 and Ψ_3 can be made to vanish via a tetrad rotation.
  - The perturbations in Ψ_0, Ψ_2 and Ψ_4 are invariant under such rotations.
- Thus Ψ_0 and Ψ_4 are true gauge-invariant fields in Kerr.
- In fact, the two DoF of **either** Ψ_0 or Ψ_4 encode the entire physics of gravitational waves (2 polarizations!) far from any sources.

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- In fact, the two DoF of **either** Ψ_0 or Ψ_4 encode the entire physics of gravitational waves (2 polarizations!) far from any sources.

- Teukolsky (1973) showed that, remarkably, Ψ_0 and Ψ_4 each satisfies **decoupled**, wave-like field equations
- Moreover, the equations admit a full separation into Fourier-harmonic modes and thus reduce to **Ordinary** Differential Equations—even on Kerr (which lacks spherical symmetry).

PART II:
Methods of BH perturbation theory

- **Lorenz-gauge formalism**
- Regge-Wheeler-Zerilli formalism
- Teukolsky equation & metric reconstruction

Direct Lorenz-gauge treatment

Recall the Lorenz-gauge form of the 1st-order perturbation equations:

$$\Box^{(0)}\bar{h}^{(1)}_{\alpha\beta} + 2R^{(0)}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}_{(1)} = -16\pi T^{(1)} \qquad \text{field equation}$$

$$\nabla^{(0)}_{\nu}\bar{h}^{\mu\nu} = 0 \qquad \text{gauge condition}$$

The field equation is **hyperbolic**, so one can attempt to solve it numerically as an initial/boundary-value problem. This is a **linear** equation, so in principle much simpler than the full Einstein's equations tackled by Numerical Relativists!

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Still, there are difficult issues to address, including —

- Recall field equation equivalent to Einstein's only if gauge condition is satisfied. How do we ensure the numerical evolution picks out a Lorenz-gauge solution from among all the (infinitely many) solutions that are not Lorenz gauge?
- There exist pure (Lorenz)-gauge vacuum solutions that grow $\propto t$ at late time (they encode CoM drift). Left unchecked, they dominate any numerical evolution.
- The field equation is consistent only if the source is conserved, $\nabla^{\alpha}T_{\alpha\beta} = 0$ (because of the Bianchi identities). This can be a problem, e.g., if we want $T_{\alpha\beta}$ to represent the (nongeodesic) inspiral trajectory of a point particle.

Multipole decomposition in spherical symmetry

In **spherical symmetry** (e.g., on Schwarzschild) we can reduce the problem from 3+1D to 1+1D, or even to ODEs, using a multipole decomposition.

Scalar field:

$$\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(t,r)\, Y^{\ell m}(\theta,\varphi)$$

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Vector field: On S^2, $\{v_t, v_r\}$ transform like scalars, but $\{v_\theta, v_\varphi\}$ transform like components of a vector. Use basis of **vector harmonics**:

$$\begin{aligned} v_a &= \sum_{\ell,m} a_a^{\ell m} Y^{\ell m} \qquad (a \in \{t,r\}) \\ v_A &= \sum_{\ell,m} \left(b^{\ell m} Y_A^{\ell m} + c^{\ell m} X_A^{\ell m} \right) \qquad (A \in \{\theta,\varphi\}) \end{aligned}$$

where

$$\begin{aligned} Y_A^{\ell m} &= \partial_A Y^{\ell m} \quad \text{(transforms like vector, } \nabla Y^{\ell m}\text{)} \\ X_A^{\ell m} &= \epsilon_A{}^B \partial_B Y^{\ell m} \quad \text{(transforms like axial vector, } \hat{\mathbf{r}} \times \nabla Y^{\ell m}\text{)} \end{aligned}$$

Multipole decomposition in spherical symmetry

Rank-2 symmetric tensor field:

On S^2, h_{ab} transform like scalars, h_{aA} transform like components of a vector, and h_{AB} transform like components of a tensor. Use basis of **tensor harmonics**:

$$h_{ab} = \sum_{\ell,m} a_{ab}^{\ell m} Y^{\ell m} \qquad \text{(3 components)}$$

$$h_{aA} = \sum_{\ell,m} \left(b_a^{\ell m} Y_A^{\ell m} + c_a^{\ell m} X_A^{\ell m} \right) \qquad \text{(4 components)}$$

$$h_{AB} = \sum_{\ell,m} \left(d^{\ell m} \Omega_{AB} Y^{\ell m} + e^{\ell m} Y_{AB}^{\ell m} + f^{\ell m} X_{AB}^{\ell m} \right) \qquad \text{(3 components)}$$

where

$$Y_{AB}^{\ell m} = D_A D_B Y^{\ell m} + \frac{1}{2}\ell(\ell+1)\Omega_{AB} Y^{\ell m} \qquad \text{(tensor)}$$

$$X_{AB}^{\ell m} = D_A X_B^{\ell m} + D_B X_A^{\ell m} \qquad \text{(axial tensor)}$$

and Ω_{AB} and D_A are the metric and covariant derivative on the unit 2-sphere.

*Note that $\mathrm{tr}(h_{AB}) = \Omega^{AB} h_{AB}$ transforms like a scalar.

** Note there are 3 "axial" modes $\{c_a, f\}$, and 7 "polar" modes $\{a_{ab}, b_a, d, e\}$. These are also called **odd-parity** and **even parity** modes.

Multipole decomposition in spherical symmetry

Compact notation:

$$h_{\alpha\beta} = \sum_{\ell,m} \sum_{i=0}^{10} h^{(i)\ell m}(t,r) Y^{(i)\ell m}_{\alpha\beta}(\theta,\varphi)$$

$Y^{(i)\ell m}_{\alpha\beta}$ are 10 orthonormal tensor-harmonic basis functions whose components are linear combinations of $Y^{\ell m}$, $Y^{\ell m}_A$, $X^{\ell m}_A$, $Y^{\ell m}_{AB}$ and $X^{\ell m}_{AB}$.

The basis functions are organized such that $Y^{(1)\ell m}_{\alpha\beta}, \ldots, Y^{(7)\ell m}_{\alpha\beta}$ coincide with the 7 even-parity modes, and $Y^{(8)\ell m}_{\alpha\beta}, \ldots, Y^{(10)\ell m}_{\alpha\beta}$ coincide with the 3 odd-parity modes.

Multipole decomposition in spherical symmetry

The tensor-harmonic expansion **separates** the Lorenz-gauge field equations with respect to ℓ, m. For each given ℓ, m we obtain 2 coupled sets for the functions $h^{(i)\ell m}(t,r)$: 7 coupled equations for the 7 even-parity modes, and 3 coupled equations for the 3 odd-parity modes.

The equations are hyperbolic, and, conveniently, decouple in the principal part:

$$\Box^{(2)} h^{(i)\ell m} + D^i{}_j h^{(j)\ell m} = T^{(i)\ell m}$$

where $\Box^{(2)}$ is the 2D wave operator, and $D^{(i)}_{(j)}$ are certain 1st-order differential operators that couple between different i-modes.

The expansion also separates the supplementary gauge conditions, which, in modal form, look like $\tilde{D}^i{}_j h^{(j)\ell m} = 0$ with $\tilde{D}^i{}_j$ another 1st-order operator. There are 3 coupled gauge equations for the even-parity sector, and a single equation for the odd-parity sector.

Multipole decomposition in spherical symmetry

Frequency-domain decomposition:

One can further reduce the dimensionality of the problem using a Fourier transform:

$$h_{\alpha\beta} = \sum_{\ell,m} \sum_{i=0}^{10} \int_{-\infty}^{\infty} dt\, h^{(i)\ell m\omega}(r) Y^{(i)\ell m}_{\alpha\beta}(\theta,\varphi) e^{-i\omega t},$$

which converts the field and gauge equations into **ordinary** DEs for the radial (frequency-dependent) functions $h^{(i)\ell m\omega}(r)$.

Useful especially when spectrum of $h_{\alpha\beta}$ is **discrete**, as in problem of calculating the perturbation from a particle on a bound orbit.

Multipole decomposition in spherical symmetry

Frequency-domain decomposition:

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Useful especially when spectrum of $h_{\alpha\beta}$ is **discrete**, as in problem of calculating the perturbation from a particle on a bound orbit.

Numerical implementation:

- The Lorenz-gauge field equations have been tackled numerically since ~ 2005 in both their time-domain 1+1D form and their frequency-domain form.
- Gauge conditions imposed by inserting "gauge damping" terms $\propto Z_\alpha$ into the equation (similar to Z4 technique of Numerical Relativity). CoM-drift modes controlled using special filters.
- Main framework for self-force calculations in Schwarzschild

Lorenz-gauge perturbations on Kerr

There is no known way (yet!) to decompose $h_{\alpha\beta}$ into multipole modes on Kerr

There are several options:

- Solve the Lorenz-gauge field equation directly in 3+1D.
- Since Kerr is **axially symmetric**, a separation into azimuthal modes is still possible:

$$h_{\alpha\beta} = \sum_{m=0}^{\infty} h^m_{\alpha\beta}(t, r, \theta) e^{im\varphi},$$

  and tackle the resulting hyperbolic equations for $h^m_{\alpha\beta}(t, r, \theta)$ in 2+1D.
- Separate into azimuthal and frequency modes:

$$h_{\alpha\beta} = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dt\, h^{m\omega}_{\alpha\beta}(r, \theta) e^{i(m\varphi - \omega t)},$$

  and tackle the resulting **elliptic** equations for $h^{m\omega}_{\alpha\beta}(r, \theta)$ in 2D.
- Decompose into spherical tensor harmonics as in Schwarzschild, and deal with resulting **mode-coupling**

PART II:
Methods of BH perturbation theory

- Lorenz-gauge formalism
- **Regge-Wheeler-Zerilli formalism**
- Teukolsky equation & metric reconstruction

Regge-Wheeler perturbation formalism for Schwarzschild

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

JOHN A. WHEELER, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

The Regge-Wheeler gauge

- The RW formalism is based on a choice of gauge (different from Lorenz) that reflects the spherical symmetry of the background, leading to much simplified field equations.
- **The 4 Regge-Wheeler gauge conditions:**

$$b_a^{\ell m} = 0, \qquad e^{\ell m} = 0, \qquad f^{\ell m} = 0$$

  i.e., we set to zero the even-parity part of h_{aA} and the entire tensorial (trace-free) part of h_{AB}.

- In this gauge, the perturbation has a simple structure:
  - Even-parity sector with 4 scalar modes (3 in h_{ab} and 1 in h_{AB})
  - Odd-parity sector with 2 vector modes (in h_{aA})
- Only applicable in spherical symmetry!

The Regge-Wheeler-Zerilli master equations

Defining the scalar-like "master" variables

$$\Psi^{\ell m}_{\text{even}} := \frac{2r}{\ell(\ell+1)}\left[d_{\ell m} + \frac{2f}{k}\left(f\, a^{\ell m}_{rr} - r\partial_r d_{\ell m}\right)\right]$$

$$\Psi^{\ell m}_{\text{odd}} := \frac{2r}{(\ell-1)(\ell+2)}\left(\partial_r c^{\ell m}_t - \partial_t c^{\ell m}_r - \frac{2}{r} c^{\ell m}_t\right)$$

where $f := 1 - 2M/r$ and $k = (\ell-1)(\ell+2) + 6M/r$

one obtains

Zerilli's master equation:

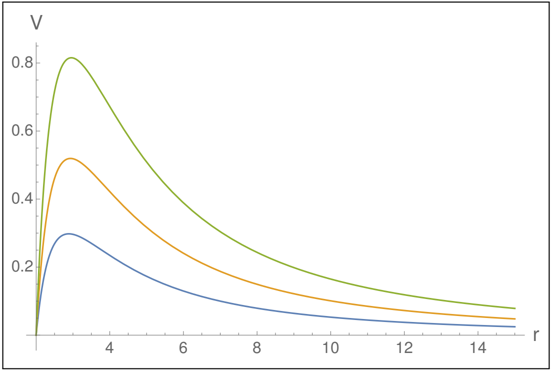
$$\left(\partial_{tt} - \partial_{r_* r_*} + V_{\text{even}}\right)\Psi^{\ell m}_{\text{even}} = S_{\text{even}}[T_{\alpha\beta}]$$

Regge-Wheeler's master equation:

$$\left(\partial_{tt} - \partial_{r_* r_*} + V_{\text{odd}}\right)\Psi^{\ell m}_{\text{odd}} = S_{\text{odd}}[T_{\alpha\beta}]$$

where V_{even} and V_{odd} are certain (ℓ-dependent) **effective potentials**.

The Regge-Wheeler effective potential



- V_{even} is similar
- Intuitive explanation for mechanism by which BH "sheds its hair"

The Regge-Wheeler formalism

- There are formulas for reconstructing the metric perturbation out of $\Psi^{\ell m}_{\rm even}$, $\Psi^{\ell m}_{\rm odd}$
- The two GW polarizations and the GW flux of energy and angular momentum can be expressed in a simple form in terms of $\Psi^{\ell m}_{\rm even}$ and $\Psi^{\ell m}_{\rm odd}$ (summed over modes).
- Also simple relations with Ψ_0 and Ψ_4.
- Extremely popular formulation, widely used.

Problems/issues:

- The $\ell = 0, 1$ modes have to be calculated separately
- Regge–Wheeler gauge pathological when source of perturbation is a point particle
- Can't do Kerr!

PART II: Methods of BH perturbation theory

- Lorenz-gauge formalism
- Regge-Wheeler-Zerilli formalism
- **Teukolsky equation & metric reconstruction**

Teukolsky's master equation

$$
\begin{aligned}
&-\left(\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right)\frac{\partial^2\phi_s}{\partial t^2}-\frac{4Mar}{\Delta}\frac{\partial^2\phi_s}{\partial t\partial\varphi}-\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\frac{\partial^2\phi_s}{\partial\varphi^2}\\
&+\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\phi_s}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi_s}{\partial\theta}\right)+2s\left(\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right)\frac{\partial\phi_s}{\partial t}\\
&+2s\left(\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial\phi_s}{\partial\varphi}-(s^2\cot^2\theta-s)\phi_s=T_s=\hat{\mathcal{S}}_s[T_{\mu\nu}]
\end{aligned}
$$

where $\phi_{+2}=\Psi_0,\quad \phi_{-2}=\rho^{-4}\Psi_4,\quad \Delta:=r^2-2Mr+a^2.$

$\hat{\mathcal{S}}_s$ is a certain 2nd-order differential operator that gives the source of the Teukolsky equation out of $T_{\mu\nu}$.

In operator form: $\hat{\mathcal{O}}_s[\phi_s]=\hat{\mathcal{S}}_s[T_{\mu\nu}]$

Separation of Teukolsky's equation

Remarkably, the master equation can be fully separated even on Kerr ($a \neq 0$), using

$$\phi_s = \int d\omega \sum_{\ell m} {}_sR_{\ell m\omega}(r)\, {}_sS_{\ell m\omega}(\theta) e^{i(m\varphi - \omega t)},$$

${}_sS_{\ell m\omega}(\theta)$: A set of orthogonal functions called "spin-weighted spheroidal harmonics".

- For $\omega = 0$, ${}_sS_{\ell m\omega}(\theta)e^{im\varphi}$ reduce to spin-weighted spherical harmonics ${}_sY^{\ell m}(\theta, \varphi)$ (related to tensorial harmonics).
- For $s = 0$ & $\omega = 0$, ${}_sS_{\ell m\omega}(\theta)e^{im\varphi}$ reduce to spherical harmonics $Y^{\ell m}(\theta, \varphi)$.

Note: Since the angular functions depend on ω, separation is only possible in the frequency domain; there is no 1+1D separation as in Schwarzschild.

The radial Teukolsky equation

The modal radial functions ${}_sR_{\ell m\omega}(r)$ satisfy the the ODE

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right){}_sR_{\ell m\omega} - \left[\lambda_{s\ell m\omega} - 4is\omega r - \frac{K^2 - 2is(r-M)K}{\Delta}\right]{}_sR_{\ell m\omega} = T_{s\ell m\omega},$$

where $K_{m\omega} := (r^2 + a^2)\omega - am$, and $\lambda_{s\ell m\omega}$ is the eigenvalue of the angular equation.

- Can be solved **numerically** with boundary conditions (say, outgoing waves at infinity, ingoing waves at the even horizon)
- Can also be solved **semi-analytically** using the Mano-Suzuki-Takasugi (MST) method: Two homogeneous solutions with appropriate asymptotic behavior are constructed from infinite sums of special (hypergeometric-type) functions, with the condition that they join smoothly translating to a continued-fraction equation. Highly accurate solutions can be computed very efficiently.

From ${}_sR_{\ell m\omega}(r)$ one readily constructs the two GW polarizations and the radiative fluxes.

Almost all work on perturbation of—and radiation from—a Kerr black hole is based on the Teukolsky equation.

Metric reconstruction

An important question: Can one reconstruct the physical metric perturbation $h_{\alpha\beta}$ from Ψ_0 and/or Ψ_4 alone? Is there enough information ? (Restricting to vacuum, say)

Consider the operation that constructs ϕ_s out of $h_{\alpha\beta}$:

$$\hat{\mathcal{T}}_s[h_{\alpha\beta}] = \phi_s$$

[E.g., $\hat{\mathcal{T}}_2(h_{\alpha\beta}) = \ell^\alpha m^\beta \ell^\gamma m^\delta C_{\alpha\beta\gamma\delta}(h_{\alpha\beta}) = \Psi_0$].

Another way to phrase above question: Are there perturbations for which

$$\hat{\mathcal{T}}_s[h^{\mathrm{kern}}_{\alpha\beta}] = 0?$$

Or: What is the kernel of the operator $\hat{\mathcal{T}}_s$?

If it's empty, then either Ψ_0 or Ψ_4 uniquely determine $h_{\alpha\beta}$. If it's not empty, then there is associated more than one $h_{\alpha\beta}$ to a given Ψ_0 or Ψ_4.

Metric reconstruction

Answer given by Wald (1973): **In vacuum,**

$$h_{\alpha\beta}^{\text{kern}} = h_{\alpha\beta}^{\text{gauge}} \oplus \delta M_{\alpha\beta} \oplus \delta J_{\alpha\beta} \oplus \delta N_{\alpha\beta} \oplus \delta C_{\alpha\beta}$$

where

$$\begin{aligned}
h_{\alpha\beta}^{\text{gauge}} &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha && \text{gauge perturbation} \\
\delta M_{\alpha\beta} &= \delta_M(g_{\alpha\beta}^{\text{Kerr}}) && \text{mass perturbation} \\
\delta J_{\alpha\beta} &= \delta_J(g_{\alpha\beta}^{\text{Kerr}}) && \text{angular-momentum perturbation} \\
\delta N_{\alpha\beta} &= \delta_N(g_{\alpha\beta}^{\text{Kerr}}) && \text{perturbation into the Kerr-NUT family} \\
\delta C_{\alpha\beta} &= \delta_C(g_{\alpha\beta}^{\text{Kerr}}) && \text{perturbation into the C-metric family}
\end{aligned}$$

In typical BH-perturbation scenarios, $\delta N_{\alpha\beta}$ and $\delta C_{\alpha\beta}$ excluded by regularity condition, and $\delta M_{\alpha\beta}$ and $\delta J_{\alpha\beta}$ are "trivial".

Thus, in vacuum, either Ψ_0 or Ψ_4 contain "almost" all the physical (non-gauge) metric information.

Metric reconstruction

So, how do we obtain $h_{\alpha\beta}$ from $\Psi_{0/4}$ in practice?
Relevant equations in operator form:

$$\begin{aligned}
\hat{\mathcal{E}}[h_{\alpha\beta}] &= T_{\mu\nu} && \text{Einstein's equation} \\
\hat{\mathcal{O}}_s[\phi_s] &= T_s = \hat{\mathcal{S}}_s[T_{\mu\nu}] && \text{Teukolsy's equation} \\
\hat{\mathcal{T}}_s[h_{\alpha\beta}] &= \phi_s
\end{aligned}$$

Note the operator equality

$$\hat{\mathcal{S}}_s\hat{\mathcal{E}} = \hat{\mathcal{O}}_s\hat{\mathcal{T}}_s$$

Metric reconstruction

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Note the operator equality

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Recall the notion of **adjoint** operator: If $\hat{\mathcal{L}} : \phi \to \psi$, then $\hat{\mathcal{L}}^\dagger : \psi \to \phi$, such that

$$\psi(\hat{\mathcal{L}}\phi) = (\hat{\mathcal{L}}^\dagger\psi)\phi \quad \text{up to a divergence}$$

The adjoint has the property $(\hat{\mathcal{A}}\hat{\mathcal{B}})^\dagger = \hat{\mathcal{B}}^\dagger\hat{\mathcal{A}}^\dagger$.

Metric reconstruction

Theorem (Wald 1978):

If $\hat{\mathcal{O}}_s^\dagger \Psi = 0$, then $h^{\text{rec}}_{\alpha\beta} = \mathcal{S}_s^\dagger[\Psi]_{\alpha\beta}$ is a solution of the vacuum Einstein's equation.

The proof is very simple, and based on the following observation:

Problem 8

Show that $\hat{\mathcal{E}}$ is self-adjoint, i.e. $\hat{\mathcal{E}} = \hat{\mathcal{E}}^\dagger$ (hint: integration by parts!)

Metric reconstruction

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Show that $\hat{\mathcal{E}}$ is self-adjoint, i.e. $\hat{\mathcal{E}} = \hat{\mathcal{E}}^\dagger$ (hint: integration by parts!)

Proof of theorem:

$$\begin{aligned}
\hat{\mathcal{E}}[h^{\rm rec}_{\alpha\beta}] = \hat{\mathcal{E}}\hat{\mathcal{S}}_s^\dagger[\Psi] &= (\hat{\mathcal{S}}_s\hat{\mathcal{E}}^\dagger)^\dagger[\Psi] \\
&= (\hat{\mathcal{S}}_s\hat{\mathcal{E}})^\dagger[\Psi] \quad (\hat{\mathcal{E}}\ \text{self-adjoint}) \\
&= (\hat{\mathcal{O}}_s\hat{\mathcal{T}}_s)^\dagger[\Psi] \quad (\text{operator identity}) \\
&= \hat{\mathcal{T}}_s^\dagger\hat{\mathcal{O}}_s^\dagger[\Psi] \\
&= 0 \quad (\text{by theorem's assumption})
\end{aligned}$$

Note $\hat{\mathcal{O}}_s^\dagger = \hat{\mathcal{O}}_{-s}$, so we have a prescription for constructing a vacuum perturbation from a solution to the Teukolsky equation

Metric reconstruction

To ensure $h^{\rm rec}_{\alpha\beta}$ corresponds to a given $\Psi_0 = \phi_{+2}$ or $\Psi_4 = \rho^4\phi_{-2}$, we must also demand

$$\hat{\mathcal{T}}_s[h^{\rm rec}_{\alpha\beta}] = \phi_s \quad \Rightarrow \quad \hat{\mathcal{T}}_s\hat{\mathcal{S}}_s^\dagger\Psi = \phi_s$$

Metric reconstruction

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Summary of metric reconstruction procedure (for vacuum perturbations):

1. Given $\Psi_0 = \phi_{+2}$ or $\Psi_4 = \rho^4\phi_{-2}$, find a **Hertz potential** Ψ satisfying both

$$\hat{\mathcal{O}}^\dagger_s\Psi = 0 \quad \text{and} \quad \hat{\mathcal{T}}_s\hat{\mathcal{S}}^\dagger_s\Psi = \phi_s.$$

   (One can prove there is a unique simultaneous solution)

2. Construct the metric perturbation via

$$h^{\text{rec}}_{\alpha\beta} = \mathcal{S}^\dagger_s[\Psi]_{\alpha\beta} + \delta M_{\alpha\beta} + \delta J_{\alpha\beta}$$

   The mass and angular-momentum perturbations need to be determined separately (e.g., from conditions on the ADM integrals of spacetime)

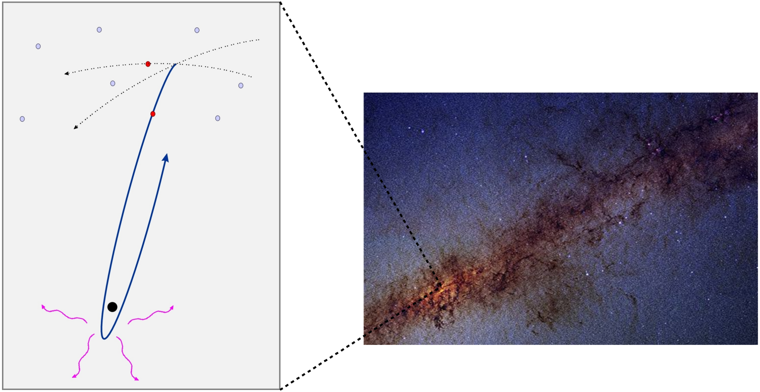
- Reconstruction in **non-vacuum** spacetimes (e.g.: point particle, 2nd-order perturbation theory) much harder. Recent formulations: GHZ 22, WKD 24.

PART III:
EMRIs and self-force theory

- **EMRIs as sources of GWs**
- Self-force theory
- Some self-force results

Nonexpert review:
Barack & Pound, *Self-force and radiation reaction in general relativity*, 2019 Rep. Prog. Phys. **82** 016904 [arXiv:1805.10385]

Extreme Mass Ratio Inspirals (EMRIs) in Nature



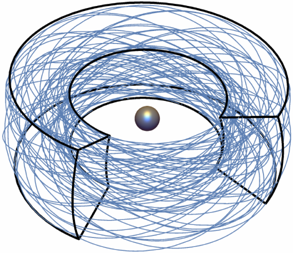
- LISA sensitive to $M_{\rm MBH} \sim 10^{5.5}$-$10^{7.5} M_\odot \Rightarrow$ mass ratio $\epsilon \sim 1 : 10^4$-10^7.
- LISA sees 10s-1000s(?) EMRIs out to $z \sim$ a few.
- $(T_{\rm orb} \sim \text{hour}) \ll (T_{\rm RR} \sim T_{\rm orb}/\epsilon \sim \text{yrs})$

EMRIs as probes of strong-field geometry

Assuming central object is a Kerr BH:

- Orbit tri-periodic (1 rotation + 2 librations)
- Orbit ergodic (space-filling) in general
- Principal elements drift in time → radiation
- Positional elements drift in time → precession

[movie]



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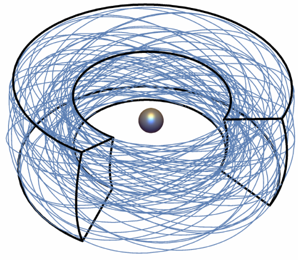
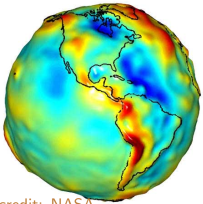




credit: NASA

- Excellent probe of strong-field geometry:
  - Precision "black-hole geodesy"
  - Tests of GR
- Need accurate templates for matched filtering!

“Self-force Programme”

- Calculate EMRI orbits and waveforms, phase-accurate over $T_{\rm RR}$
- Strong field (no resort to PN)
- Generic eccentricity, inclination, spins
- Accuracy requirement for local radiation-reaction force:

  $\Phi = \Phi_0 + \Omega\Delta t + \dot{\Omega}\Delta t^2 + \ldots$

  To keep $\delta(\dot{\Omega}\Delta t^2) \lesssim 1$ over $\Delta t = T_{\rm RR}$ need $\delta(\dot{\Omega}) \lesssim T_{\rm RR}^{-2} = O(\epsilon^2)$

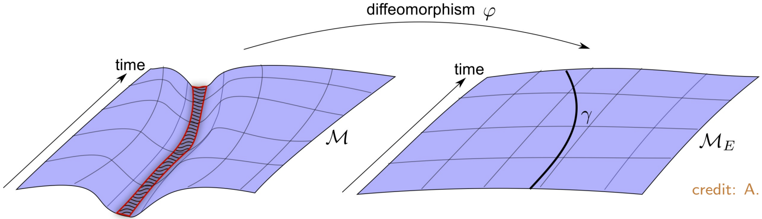
  $\Rightarrow$ Second-order self-force

PART III:
EMRIs and self-force theory

- EMRIs as sources of GWs
- **Self-force theory**
- Self-force in scattering

"Problem of motion"

FIELD degrees of freedom $\to$ PARTICLE degrees of freedom



credit: A. Pound

Body in exact spacetime

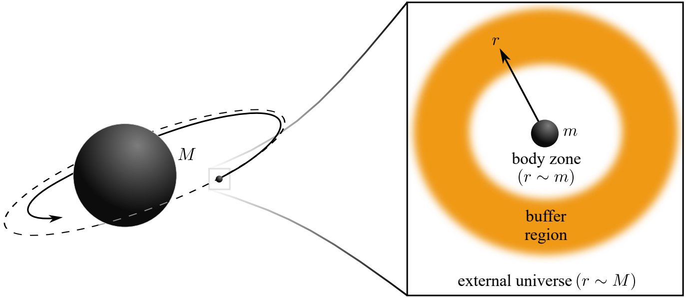
Effective representation of motion in background spacetime

Guiding principle:
"point particles" don't make sense as fundamental objects in GR,
but "point particle equation of motion" does — in a certain effective way.

Matched Asymptotic Expansions

Mino, Sasaki & Tanaka (1997), Poisson (2003)

building on early works by Burke, d'Eath, Kates, Thorne & Hartle,. . .



- Trajectory defined on background spacetime using a suitable far-zone limit; constrained by matching near & far expansions of the metric in the matching zone.
- No resort to "point particles": notion *derived* rather than assumed
- More rigorous derivation by Gralla & Wald (2008) using a 1-parameter metric family (extending work by Geroch & Ehlers on geodesic motion).

Equation of Motion at 1st post-geodesic order

Result at $\mathbf{O}(\epsilon^{\mathbf{0}})$: trajectory is a geodesic of the background spacetime:

$$\ddot{z}^\alpha = 0 + O(\epsilon)$$

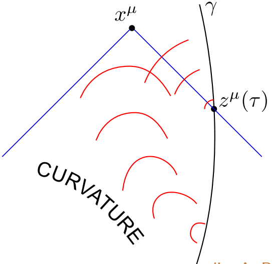
Metric perturbation $h^{(1)}_{\alpha\beta}$ is as from a δ-function source moving on the geodesic.

Result at $\mathbf{O}(\eta)$: "Self-force" exerted by the **tail** part of the $h^{(1)}_{\alpha\beta}$:

$$h^{(1)}_{\alpha\beta} = h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$

$$\ddot{z}^\alpha = -u^\beta u^\gamma C^\alpha_{\beta\gamma}(h^{\text{tail}})_{\perp u} =: F^\alpha_{\text{self}}/m$$

where u^α is 4-velocity along geodesic.



credit: A. Pound

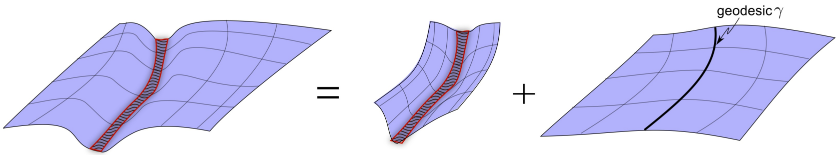
Explicitly (for a nonspinning particle):

$$\ddot{z}^\alpha = -\frac{1}{2}(g^{\alpha\beta}_{(0)} + u^\alpha u^\beta)\, u^\gamma u^\delta \left(2\nabla^{(0)}_\delta h^{\text{tail}}_{\beta\gamma} - \nabla^{(0)}_\beta h^{\text{tail}}_{\gamma\delta}\right)\Big|_{z(\tau)} =: \nabla^{\alpha\beta\gamma} h^{\text{tail}}_{\beta\gamma}$$

“R field” reformulation (Detweiler & Whiting 2003)

- $h^{\rm tail}_{\alpha\beta}$ is **not a vacuum solution** of the linearized Einstein equations
- But one can construct a vacuum solution $h^{R}_{\alpha\beta}$ [associated with a certain (a-causal) Green function in the Hadamard representation] such that

$$
\begin{aligned}
F^{\alpha}_{\rm self} &= m\nabla^{\alpha\beta\gamma} h^{\rm R}_{\beta\gamma} \\
&= m\nabla^{\alpha\beta\gamma}\left(h^{(1)}_{\beta\gamma} - h^{\rm S}_{\beta\gamma}\right)
\end{aligned}
$$

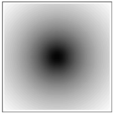


full metric $g^{\rm full}_{\alpha\beta}$ “self-field” $h^{\rm S}_{\alpha\beta}$ effective metric $g^{(0)}_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$

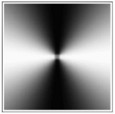
- **Interpretation:** Self-accelerated trajectory is a geodesic in the effective metric $g^{(0)}_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$. (This is not the physical metric. The latter is $g^{(0)}_{\alpha\beta} + h^{(1)}_{\alpha\beta}$.)

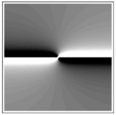
Self-force and gauge

- Self-force is gauge-dependent,
  but $\{F^\alpha_{\rm self}, h_{\alpha\beta}\}$ contains invariant information
- EoM originally formulated in Lorenz gauge, $\nabla^\beta \bar{h}_{\alpha\beta} = 0$.



- **Generalizations:**
  - Continuous deformations of Lorenz
    (LB & Ori 2001)
  - Direction-dependent (bounded) deformations of Lorenz
    (Gralla & Wald 2008)
  - Parity-regular gauges (Gralla 2011)
  - Radiation gauges (Pound, Merlin & LB 2014)





Last generalization allows convenient calculation via Teukolsky's formalism.

Practical schemes in black-hole spacetimes:

I. Mode-sum method (LB & Ori 2000)

- Subtraction of $h^S_{\alpha\beta}$ done mode-by-mode in a multipole expansion about large BH:

$$
\begin{aligned}
F^\alpha_{\rm self}/m &= \sum_{\ell=0}^{\infty}\left[(\nabla^{\alpha\beta\gamma}h^{(1)}_{\beta\gamma})^\ell - (\nabla^{\alpha\beta\gamma}h^S_{\beta\gamma})^\ell\right] \\
&= \sum_{\ell=0}^{\infty}\left[(\nabla^{\alpha\beta\gamma}h^{(1)}_{\beta\gamma})^\ell - A^\alpha\ell - B^\alpha - C^\alpha/\ell\right] - D^\alpha
\end{aligned}
$$

where $D^\alpha := \sum_{\ell=0}^{\infty}\left[(\nabla^{\alpha\beta\gamma}h^S_{\beta\gamma})^\ell - A^\alpha\ell - B^\alpha - C^\alpha/\ell\right]$

- Regularization parameters A, B, C, D derived analytically (as functions along the orbit) from local form of $h^S_{\alpha\beta}$; known for generic orbits in Kerr (LB & Ori 2000-03)

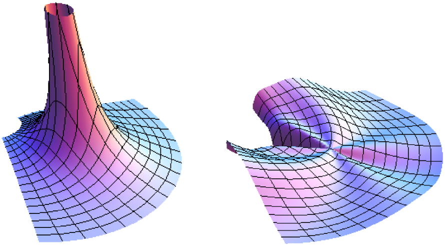
- Numerical input: Modes of $h^{(1)}_{\beta\gamma}$ obtained by solving metric perturbation equations with a particle (delta function) source and retarded boundary conditions.

Practical schemes in black-hole spacetimes:
II. Puncture (or "effective source") method

- Analytically construct Puncture field $h^P_{\alpha\beta} \approx h^S_{\alpha\beta}$ so that $\nabla h^P = \nabla h^S$ at particle.
- Write linearized field equation $\delta G_{\mu\nu}(h) = T_{\mu\nu}$ in "punctured" form

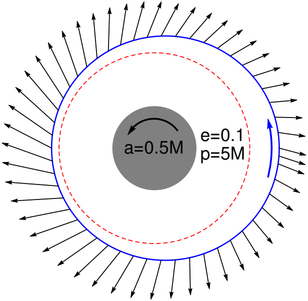
$$\delta G_{\mu\nu}(h - h^P) = T_{\mu\nu} - \delta G_{\mu\nu}(h^P) =: S^{\text{eff}}_{\mu\nu}$$

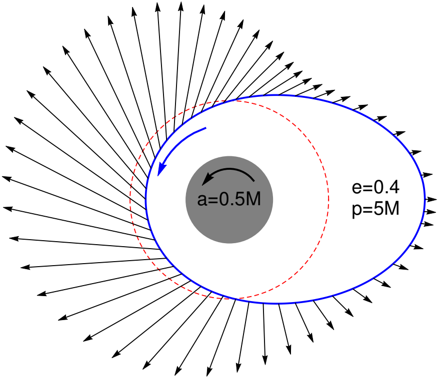
- Numerically solve for Residual field $h^{\text{Res}} := h - h^P$. Then $F_{\text{self}} = m\nabla h^{\text{Res}}$.



credit: J. Thornburg & B. Wardell

sample results for equatorial orbits in Kerr $(a = 0.5M)$

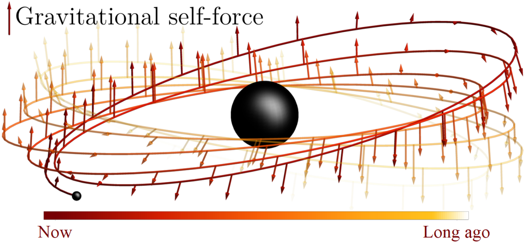




Maarten van de Meent (2016)
using numerical implementation of Mano-Suzuki-Takasugi method
+ metric reconstruction + mode-sum regularization.

Self-force along fixed geodesic orbits

sample results for an inclined eccentric orbit in Kerr



$$a = 0.5M, \quad p = 10, \quad e = 0.1, \quad \cos\theta_{\min} = 0.3$$

[M. Van de Meent]

Conservative/dissipative split (1st order)

$$m\ddot{z}^\alpha = F^\alpha_{\rm cons} + F^\alpha_{\rm diss} := \frac{1}{2}\left[F^\alpha_{\rm self}(h^{\rm ret}) + F^\alpha_{\rm self}(h^{\rm adv})\right] + \frac{1}{2}\left[F^\alpha_{\rm self}(h^{\rm ret}) - F^\alpha_{\rm self}(h^{\rm adv})\right]$$

Kerr geodesics have a symmetry that implies, at each point,

$$F^\alpha_{\rm adv}(u_t, u_r, u_\theta, u_\varphi) = \epsilon_{(\alpha)} F^\alpha_{\rm ret}(u_t, -u_r, -u_\theta, u_\varphi)$$

$$\epsilon_{(\alpha)} = (-1, 1, 1, -1)$$

For equatorial geodesics, this allows to split cons/diss using $F^\alpha_{\rm ret}$ alone:

$$F^\alpha_{\rm cons}(r,\dot{r}) = \frac{1}{2}\left(F^\alpha_{\rm ret}(r,\dot{r}) + \epsilon_{(\alpha)} F^\alpha_{\rm ret}(r,-\dot{r})\right)$$

$$F^\alpha_{\rm diss}(r,\dot{r}) = \frac{1}{2}\left(F^\alpha_{\rm ret}(r,\dot{r}) - \epsilon_{(\alpha)} F^\alpha_{\rm ret}(r,-\dot{r})\right)$$

Self-force and adiabatic expansion 3-layer structure

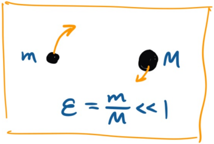
$$\boxed{G_{\mu\nu}(g_{\alpha\beta})=0 \;\; /. \quad g_{\alpha\beta} = g^{\mathrm{KERR}}_{\alpha\beta} + \varepsilon h^{(1)}_{\alpha\beta} + \varepsilon^2 h^{(2)}_{\alpha\beta} + \ldots}$$

SELF-FORCE THEORY

from PDEs to
point-particle orbits

$$\boxed{\ddot{x}^\alpha = \varepsilon F^\alpha_{(1)} + \varepsilon^2 F^\alpha_{(2)} + \ldots}$$

TWO-TIMESCALE/ADIABATIC
EXPANSION



$$\boxed{\varphi = \varepsilon^{-1}\varphi_{0PA}(\varepsilon t) + \varepsilon^0 \varphi_{1PA}(\varepsilon t) + O(\varepsilon)}$$

Should suffice for
parameter extraction
if ε sufficiently small

Self-force and adiabatic expansion

$$G_{\mu\nu}(g_{\alpha\beta})=0 \quad /. \quad g_{\alpha\beta}=g^{\text{KERR}}_{\alpha\beta}+\varepsilon h^{(1)}_{\alpha\beta}+\varepsilon^2 h^{(2)}_{\alpha\beta}+\dots$$

Equation of Motion

$$\ddot{x}^{\alpha}=\varepsilon F^{\alpha}_{(1)}+\varepsilon^2 F^{\alpha}_{(2)}+\dots$$

$\langle F_{(1)}\rangle$ $\langle F_{(2)}\rangle$

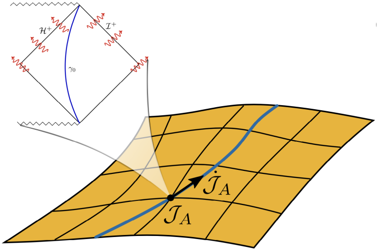
phase evolution (Radiation-reaction timescale)

$$\varphi=\varepsilon^{-1}\varphi_{0PA}(\varepsilon t)+\varepsilon^{0}\varphi_{1PA}(\varepsilon t)+O(\varepsilon)$$

- Treat $h^{(n)}_{\alpha\beta}$ as functions on extended manifold:

$$h_{\alpha\beta}(t, x^i) \to \sum_{n=1}^{\infty} \epsilon^n h^{(n)}_{\alpha\beta}(x^i; \mathcal{J}_A, \varphi_A)$$

where $\mathcal{J}_A = (J_A, M_{\rm BH}, J_{\rm BH}, \ldots)$.



- With a suitable choice of (J_A, φ_A):

$$h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n)\Omega_k}_{\alpha\beta}(x^i; \mathcal{J}_A) e^{-ik^A\varphi_A}$$

with $\Omega_k := k^A \dot{\varphi}_A$.

Offline step: Solve field equations for amplitudes $h^{(n)\Omega_k}_{\alpha\beta}$ on grid of $\mathcal{J}_{\mathcal{A}}$ values.

Online step (FEW): Rapidly evolve through parameter space

$$\begin{aligned} \dot{\varphi}_A &= \Omega_A(\mathcal{J}_B) \\ \dot{\mathcal{J}}_A &= \epsilon \tilde{F}^{(0)}(\mathcal{J}_B) + \epsilon^2 \tilde{F}^{(1)}(\mathcal{J}_{\mathcal{B}}) + \cdots \end{aligned}$$

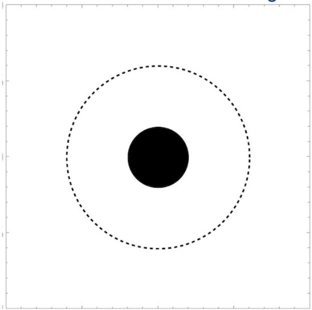
PART III:
EMRIs and self-force theory

- EMRIs as sources of GWs
- Self-force theory
- **Self-force in scattering**

Why scattering?

Main idea: scattering as an efficient probe of strong interaction

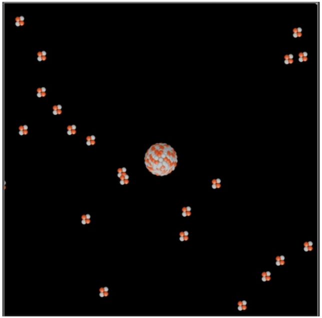
EMRI "zoom-whirl" scattering



credit: Oliver Long

10^{+11} m

Rutherford scattering



credit: PhET, university of Colorado Boulder

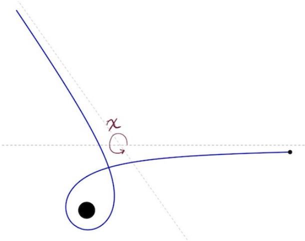
10^{-13} m

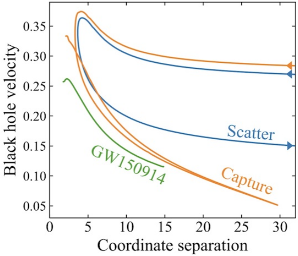
Why scattering?

- Diagnostic "observables" (e.g. scattering angle χ) defined unambiguously from $r \to \infty$ asymptotics.
- Handle on fuller binary parameter space
- $\chi(E, b) \Rightarrow$ full Hamiltonian dynamics
- New way of calibrating EOB theory using post-Minkowskian χ information (Damour 2016)
- "Boundary to bound" maps (Goldberger & Rothstein 2006; Kalin & Porto 2019+)
- Intense cross-disciplinary interest, new participants: EFT, QCD Amplitudes (Bern et al 2019+).

Self-force calculations for scattering:

- ``Easier'' than bound inspiral: no two timescales!
- ``Easy'' access to full high-order PM theory (next slide)
- Access to strong-field dynamics: no weak-field approx.





credit: Oliver Long

Self-force and post-Minkowskian theory

[Damour 2019:] From mass-exchange symmetry and polynomial structure $\Rightarrow$ nSF determines the **full** conservative dynamics to $(2n+2)$PM order (arbitrary ϵ).

$$
\begin{aligned}
\chi_{\rm cm} = \frac{E^*_{\rm cm}}{b}\Big[\ & a_0(v) && \textbf{1PM}\\
&+ \frac{a_1(v)(M+m)}{b} && \textbf{2PM}\\
&+ \frac{a_2(v)(M^2+m^2) + a_{11}(v)Mm}{b^2} && \textbf{3PM}\\
&+ \frac{a_3(v)(M^3+m^3) + a_{21}(v)(M^2m+Mm^2)}{b^3} && \textbf{4PM}\\
&+ \frac{\underset{\substack{\uparrow\\ \text{from 0SF!}}}{a_4(v)(M^4+m^4)} + \underset{\substack{\uparrow\\ \text{needs 1SF}}}{a_{31}(v)(M^3m+Mm^3)} + \underset{\substack{\uparrow\\ \text{needs 2SF}}}{a_{22}(v)M^2m^2}}{b^4} + \cdots\Big] && \textbf{5PM}
\end{aligned}
$$

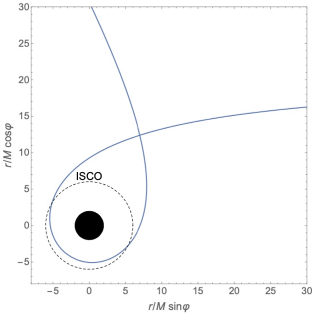
$^*E_{\rm cm} = \sqrt{M^2+m^2+2Mm\gamma}$ is initial total energy in initial CoM frame.

Scattering geodesics preliminaries

parametrisation (Schwarzschild)

- $E := g_{\alpha\beta} t^\alpha u^\beta \Big|_{t\to-\infty} = (1 - v_\infty^2)^{-1/2} = \gamma_\infty$ (specific) Energy
- $L := g_{\alpha\beta} \varphi^\alpha u^\beta \Big|_{t\to-\infty} > L_{\rm crit}(E)$ (specific) Angular Momentum
- $b := r \sin \Big| \varphi(t) - \varphi(-\infty) \Big|_{t\to-\infty} = \dfrac{L}{\sqrt{E^2 - 1}}$ impact parameter
- $e > 1$ eccentricity
- $p > 6 + e$ semilatus rectum
- $r_{\rm min} = \dfrac{pM}{1+e} > 3M$ periastron distance

... any two of these!



Scattering geodesics preliminaries

“observables”

- Scattering angle:

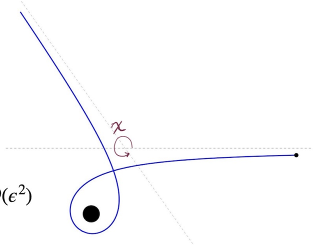
$$\chi = \int_{-\infty}^{\infty} \frac{d\varphi}{d\tau}(\tau; e, p, F^{\alpha}_{\rm self})\, d\tau - \pi + (\delta\chi)_{\rm frame} = \chi_{\rm 0SF} + \chi_{\rm 1SF} + O(\epsilon^2)$$

- Time delay:

$$\Delta t = \int_{-\infty}^{\infty} \left(\frac{dt}{d\tau}(\tau; e, p, F^{\alpha}_{\rm self}) - \frac{dt}{d\tau}(\tau; e, p, 0) \right) d\tau + (\delta\Delta t)_{\rm frame} = \Delta t_{\rm 1SF} + O(\epsilon^2)$$

- Radiated energy and angular momentum:

$$E_{\rm rad} = -\int_{-\infty}^{\infty} F_t^{\rm self}(\tau; e, p)\, d\tau + (\delta E)_{\rm frame} \qquad J_{\rm rad} = \int_{-\infty}^{\infty} F_\varphi^{\rm self}(\tau; e, p)\, d\tau + (\delta J)_{\rm frame}$$



$$\chi_{1SF}^{\rm diss} = \alpha(p, e) E_{\rm rad} + \beta(p, e) J_{\rm rad}$$ (LB & Long 2022)

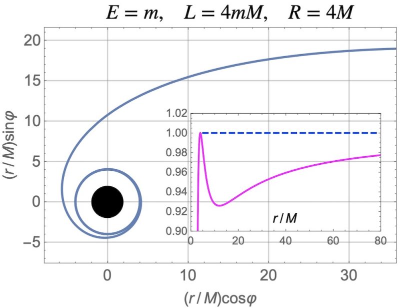
Practice problem: $F_{\rm cons}$ effects on the zero-energy zoom-whirl orbit (ZEZO) (LB, Colleoni, Damour, Isoyama & Sago 2019)

- **Heteroclinic orbit** connects circular orbit to infinity, allowing identification of circular orbit's "binding energy" as a Bondi-type quantity through $O(m^2)$.
- 1SF corrections to $\Omega_{\rm circ}$ and to $L_{\rm crit}$ obtained by integrating self-force along the geodesic orbit:

$$\Omega = (8M)^{-1}\left(1 + 8\epsilon F^r(4M) - 3\epsilon\int_{-\infty}^{\infty} F_t\, d\tau\right)$$

$$L = Mm\left(4 + 4\epsilon - 2\epsilon + \epsilon\int_{-\infty}^{\infty}(F_\varphi - 8F_t)d\tau\right)$$

0SF recoil gauge



$\Omega = (8M)^{-1}[1 + 0.5536(2)\epsilon]$
$L = 4Mm[1 - 0.304(2)\epsilon]$

Final workout (with a moral)

Problem 10 ●●○

For a scattering geodesic in the equatorial plane of a Schwarzschild black hole, parametrized by proper time τ, the impact parameters is defined as

$$b = \lim_{\tau \to -\infty} r(\tau) \sin |\varphi(\tau) - \varphi(-\infty)| \,.$$

Show that

$$b = \frac{L}{\sqrt{E^2 - 1}},$$

where $E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$ and $L = r^2 \frac{d\varphi}{d\tau}$ are the conserved energy and angular momentum (per unit mass).

Hence the ZEZO (with $E = 1$ and $L = 4M$) has $b = \infty$, even though its periastron is at $r = 4M$. Large b does not imply a weak-field orbit!

Scalar-charge toy model

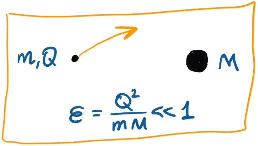
- Charge Sources Klein-Gordon field Φ:

$$\nabla^\alpha \nabla_\alpha \Phi = -4\pi Q \int_{-\infty}^{\infty} (-g)^{-1/2}\, \delta^4(x - z(\tau))d\tau$$

- Treat Φ as linear perturbation on Kerr, ignore gravitational self-force, consider only back-reaction from Φ:

$$F^\alpha_{\text{self}} = Q\,\nabla^\alpha \tilde{\Phi} \propto Q^2$$

- Deviation from geodesic remains small $[O(\epsilon)]$ during scattering, so at leading order can evaluate scattering observables by integrating F_{self} along limiting **geodesic** trajectory.

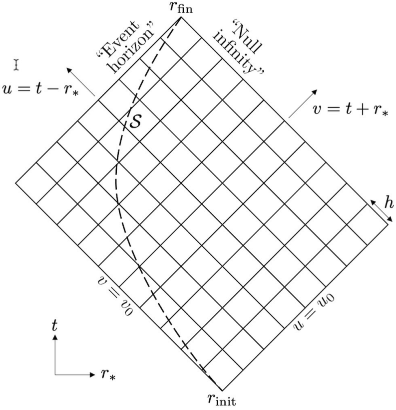


Advantage: Similar mathematical structure, simpler field equation, no frame ambiguities

t-domain numerical method

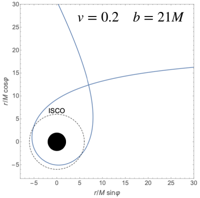
(LB & Long 2022)

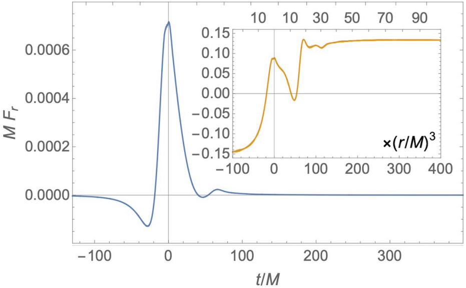
- $\Phi = \dfrac{Q}{r} \sum_{\ell,m} \phi_{lm}(r,t) Y_{lm}(\theta,\varphi)$
- 1+1D field equation for $\phi_{lm}(r,t)$ discretised and solved in double-null coords.
- zero initial conditions, transient junk discarded
- $F^{\alpha}_{\text{self}}(\tau)$ constructed from ϕ_{lm} using mode-sum regularisation.

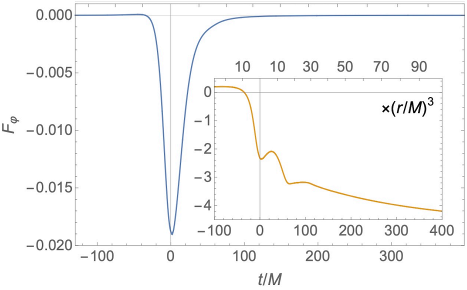


t-domain calculation sample results

(LB & Long 2022)

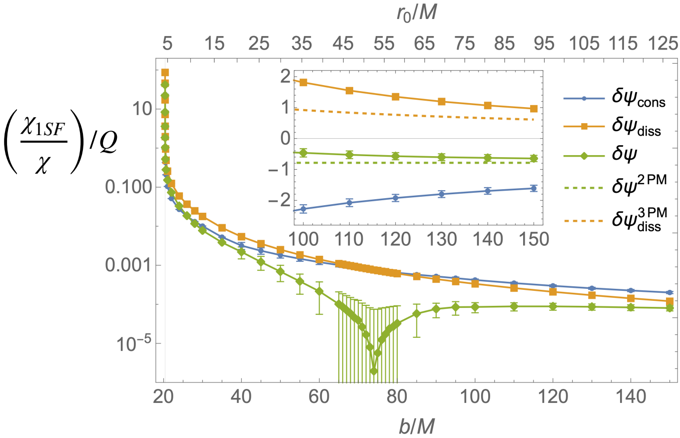






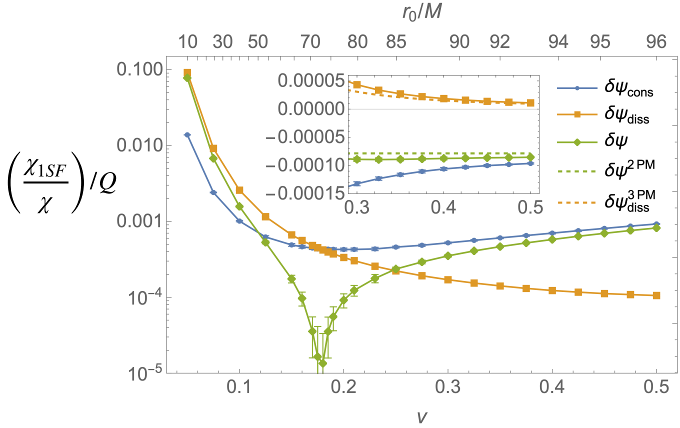
t-domain calculation sample results ($\nu = 0.2$)

(LB & Long 2022)



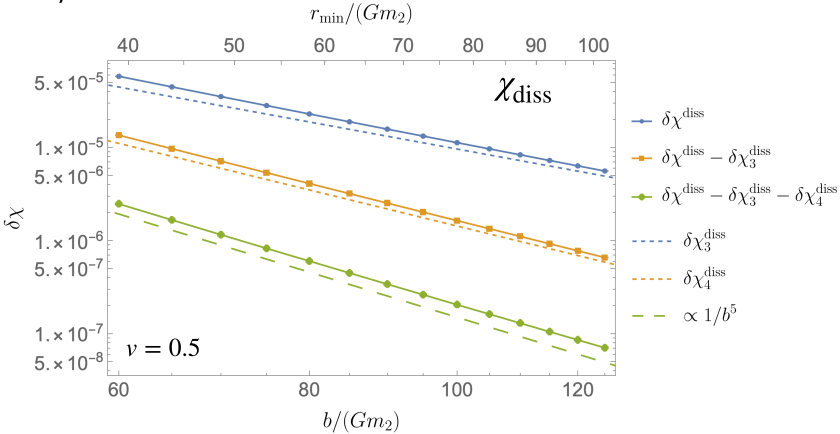
t-domain calculation sample results ($b = 100M$)

(LB & Long 2022)



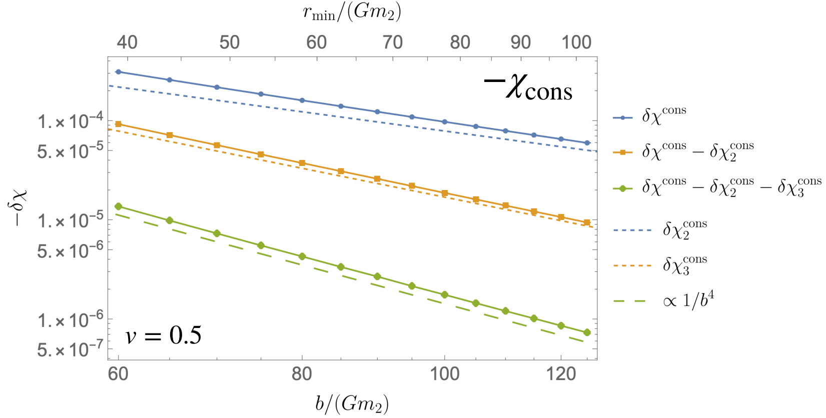
Comparison with PM results from Amplitudes

(LB, Bern et al 2023)



Comparison with PM results from Amplitudes

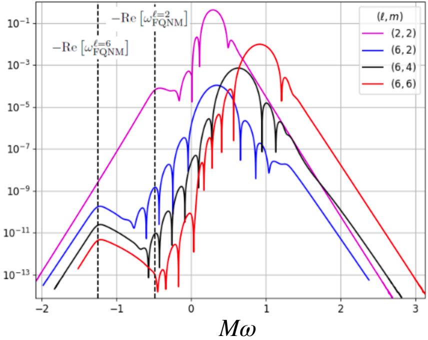
(LB, Bern et al 2023)



f-domain numerical method

(Whittall & LB 2023)

- $\Phi = \frac{Q}{r} \sum_{\ell,m} \int d\omega\, \phi_{lm\omega}(r) Y_{lm}(\theta, \varphi) e^{-i\omega t}$
- $\phi_{lm\omega}(r)$ obtained by solving ODEs with BCs.
- **Much** more precise than TD method in strong field.
- $\phi_{lm}(r, t)$ reconstructed using the method of extended homogeneous solutions

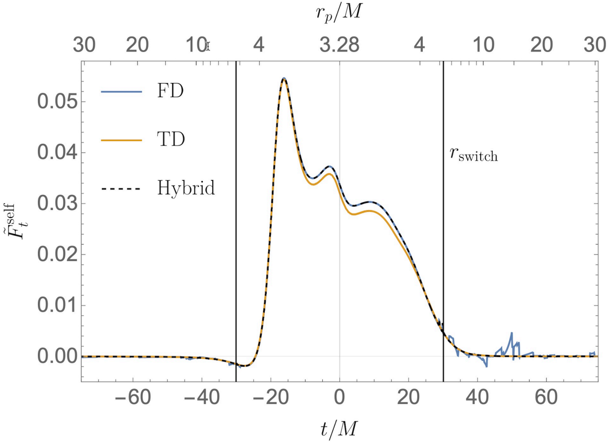


f/t-domain hybridisation

(Long, Whittall & LB 2024)

$v = 0.7 \qquad b - b_{crit} = 0.001M$

(In this case, strong beaming sends power to large l modes, where TD method struggles.)



Application: resummation of $\chi_{\rm PM}$ using separatrix info

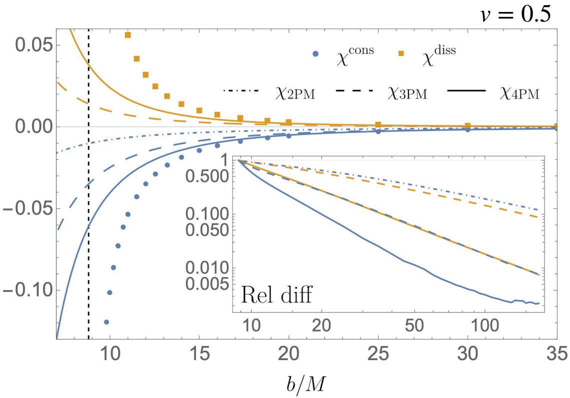
(Long, Whittall & LB 2024)

Scattering angle diverges at separatrix:

$$\chi_{0SF} \sim A_0(v)\,\log\big(b - b_{\rm crit}(v)\big)$$

$$\chi_{1SF} \sim \frac{A_1(v)}{b - b_{\rm crit}(v)}$$

$$A_1 = \int_{-\infty}^{\infty} \big(c_t(v)F_t^{\rm self} + c_\varphi(v)F_\varphi^{\rm self}\big)\,d\tau$$

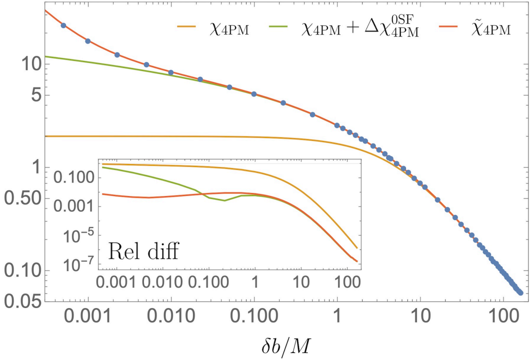


Application: resummation of $\chi_{\rm PM}$ using separatrix info

(Long, Whittall & LB 2024)

Resummation formula:

$$\tilde{\chi} = \chi_{4PM} + A_0 \log\left(1 - \frac{1-\epsilon A_1/A_0}{b/b_{\rm crit}}\right) + \sum_{k=1}^{4} \frac{A_0}{k}\left(\frac{1-\epsilon A_1/A_0}{b/b_{\rm crit}}\right)^k$$



Application: resummation of $E_{\rm rad}$ using separatrix info

(LB, Gonzo, Leather, Long & Warburton (2026))

Near separatrix $E_{\rm rad}$ dominated by whirl $\Rightarrow$

$$\begin{aligned} E_{\rm rad} &\sim \dot{E}_{\rm whirl} \times T_{\rm whirl} \times N_{\rm whirl} \\ &\sim \dot{E}(R) \times T(R) \times (\chi + \pi)/(2\pi) \\ &= \dot{E}(R) \times \frac{R^2/M}{\sqrt{6 - R/M}} \log(b - b_{\rm crit}) \end{aligned}$$

Circular-orbit flux $\dot{E}(R)$ known numerically, with accurate analytical fit over $3M < R \leq 6M$