NR, EOB & DATA ANALYSIS
LECTURE 1: NUMERICAL RELATIVITY

AMPLITUDES, RESUMMATION & STRONG GRAVITY @ NORDITA, APRIL 2026

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UNIVERSITY OF BIRMINGHAM | GRAVITATIONAL WAVE ASTRONOMY

NR, EOB & DATA ANALYSIS

- Lecture 1: Introduction & Numerical Relativity
  - Tuesday, April 7 @ 14.30
- Lecture 2: Effective-One-Body (Geraint Pratten)
  - Wednesday, April 8 @ 9.30
- Lecture 3: GW Data Analysis
  - Thursday, April 9 @ 11.30
- Hands-on session:
  - Friday, April 10 @ 14.30

LECTURE 1

NUMERICAL RELATIVITY

Literature:

Miguel Alcubierre: Introduction to 3+1 Numerical Relativity

Thomas Baumgarte & Stuart Shapiro: Numerical Relativity

Eric Gourgoulhon: “3+1 Formalism and the Bases of Numerical Relativity” (available at https://arxiv.org/pdf/gr-qc/0703035.pdf)

WHY NUMERICAL RELATIVITY (NR)?

- Examples include:
  - **Relativistic two-body problem** & gravitational waves ("holy grail")

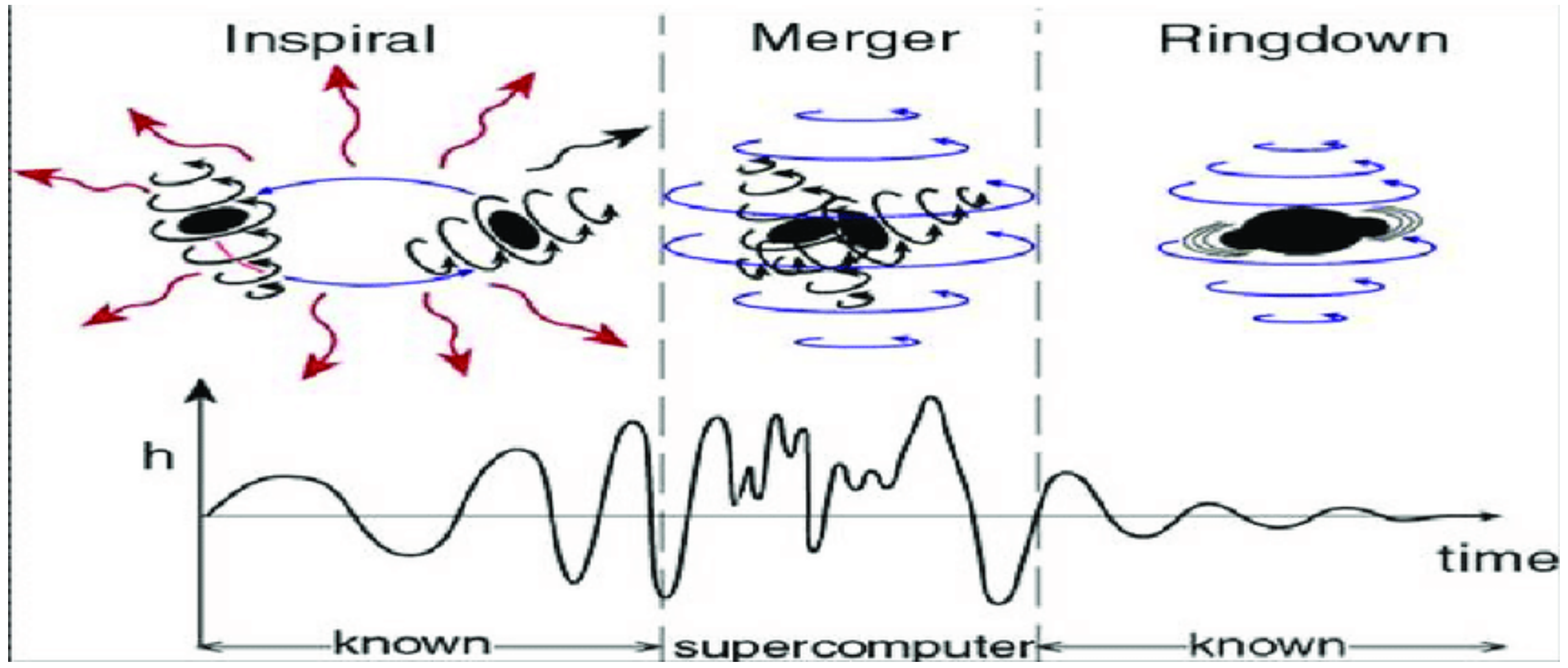
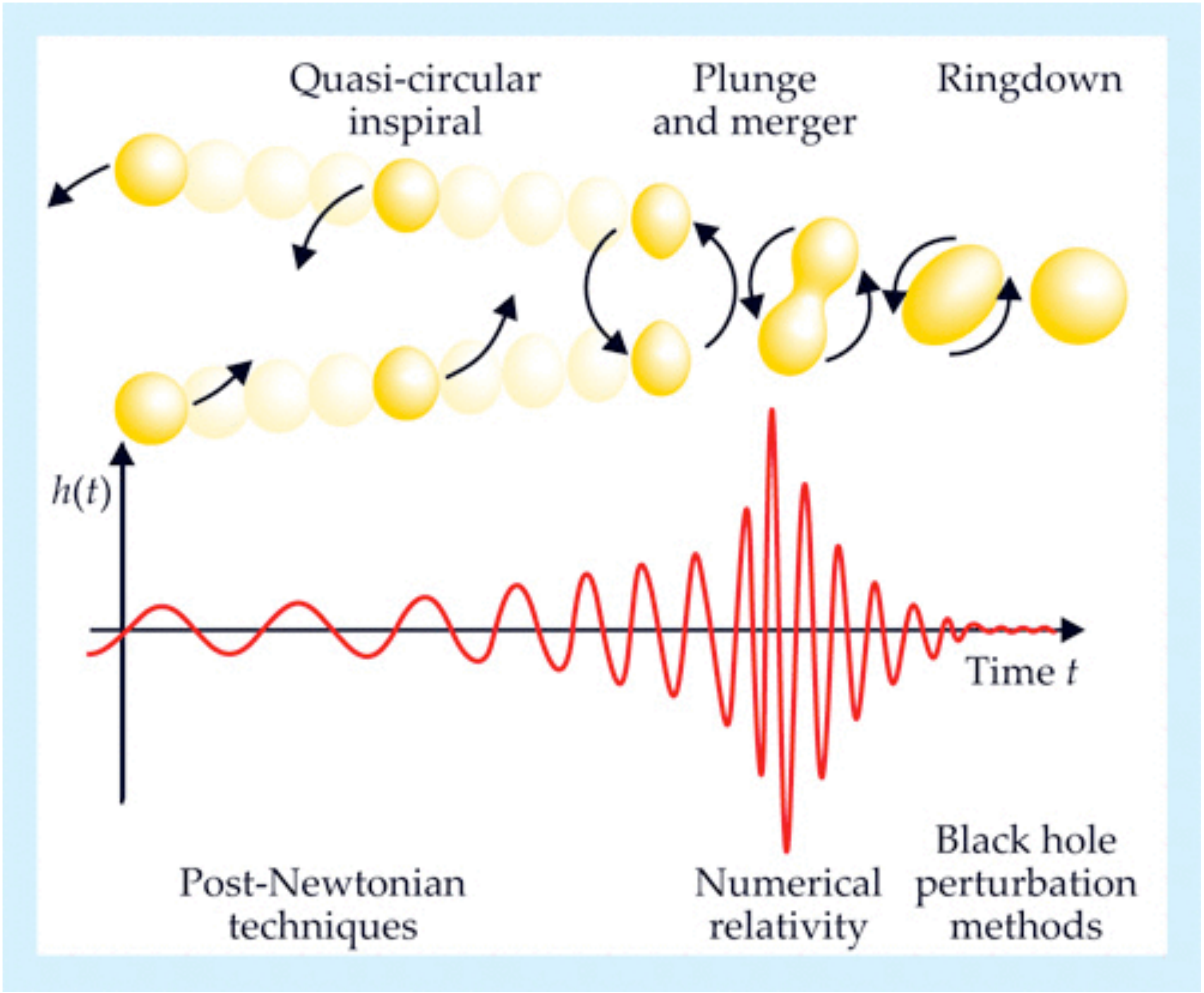
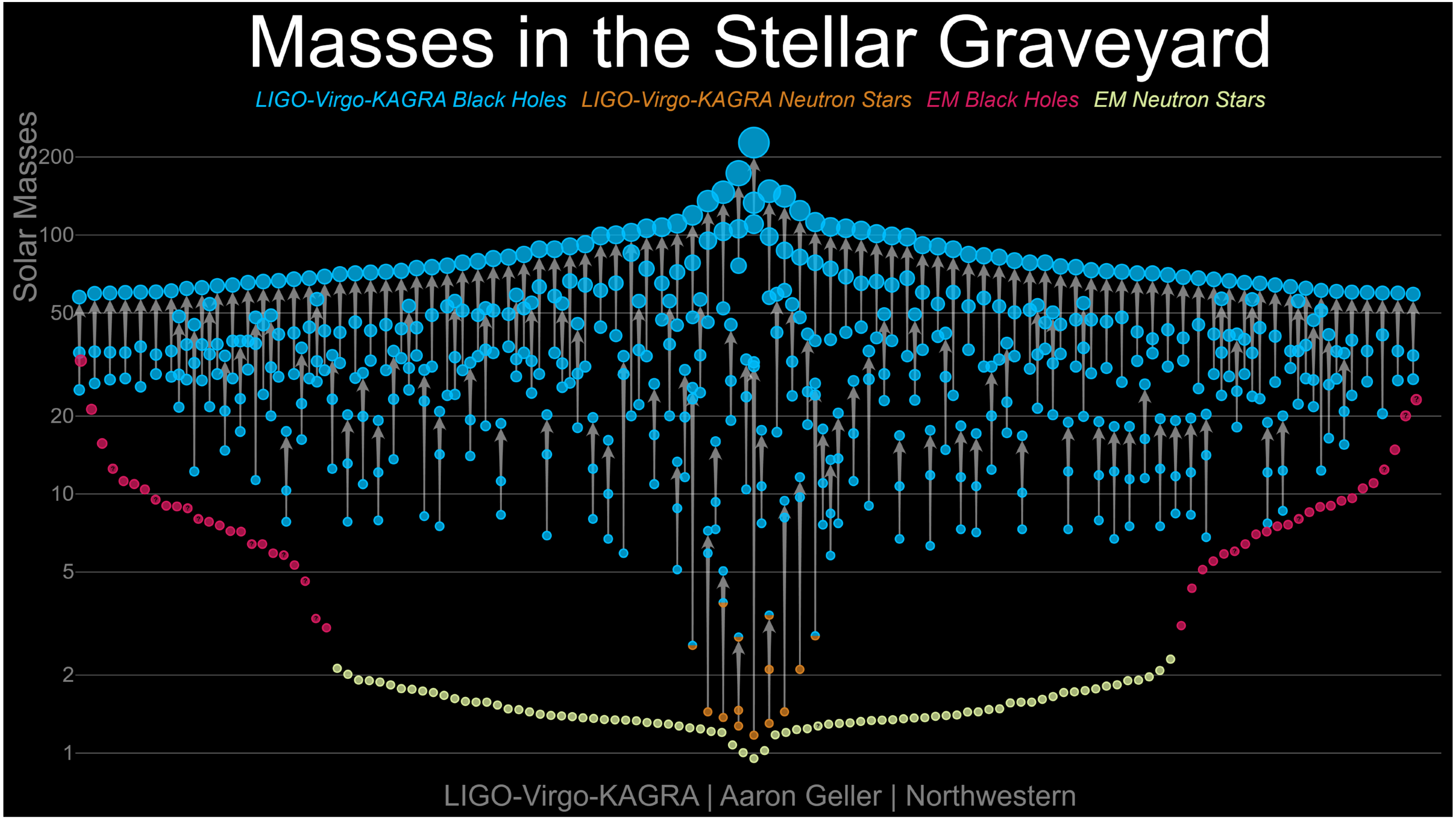


Image: K. Thorne

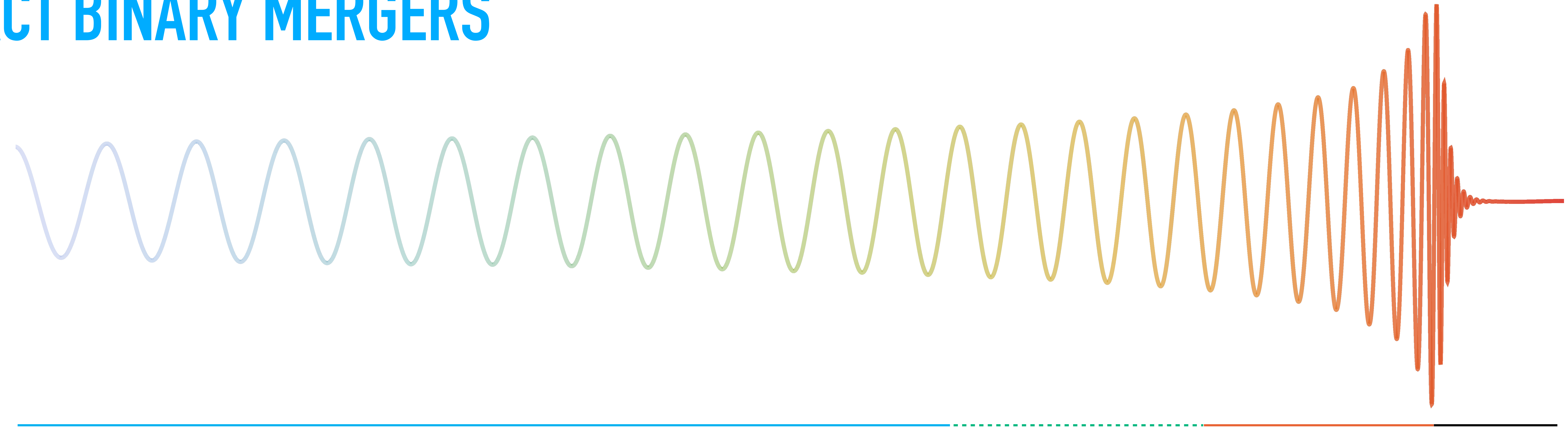
  - Supernovae explosions
  - Perturbations of isolated stars or BHs
  - Cosmological simulations beyond Newton gravity
  - Critical collapse phenomena (e.g. primordial BH formation)
  - Black hole shadows and jet formation
- … and of course most things beyond GR (not discussed here)



[Image: Baumgarte & Shapiro]


Masses in the Stellar Graveyard
LIGO-Virgo-KAGRA Black Holes
LIGO-Virgo-KAGRA Neutron Stars
EM Black Holes
EM Neutron Stars
Solar Masses
200
100
50
20
10
5
2
1
LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

COMPACT BINARY MERGERS

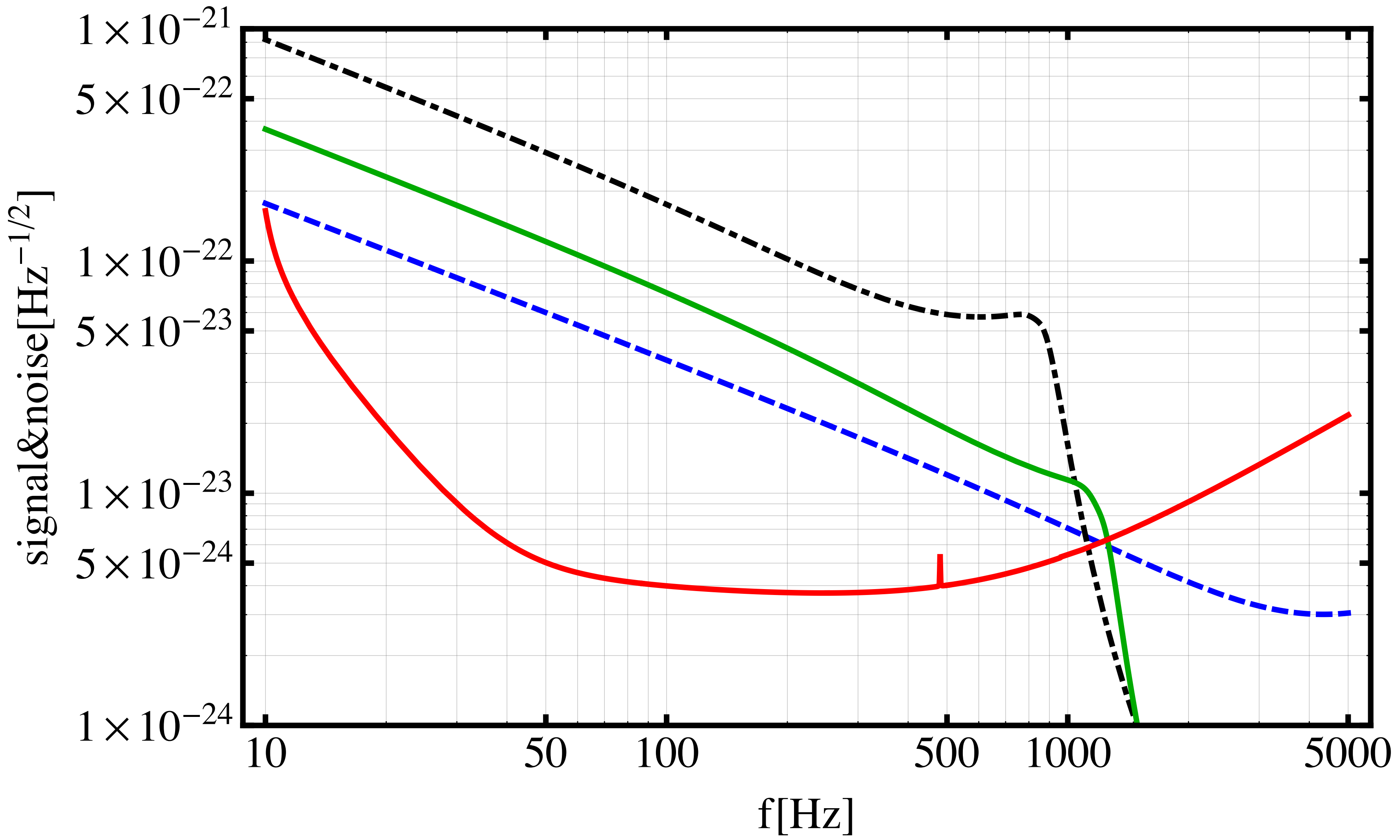


Inspiral

Analytical approximations begin to break down

Merger: Numerical Relativity

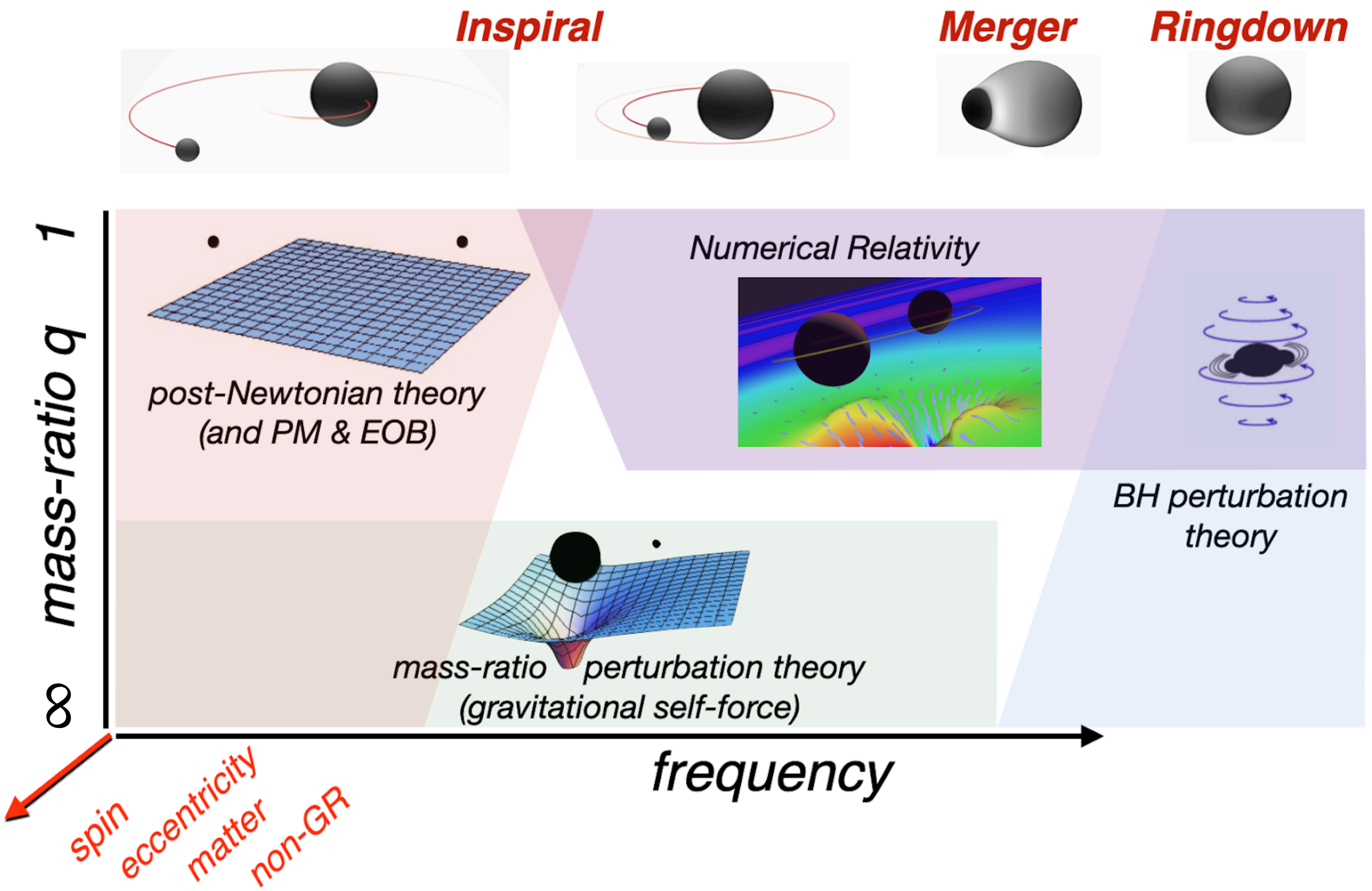
Ringdown: Black Hole Perturbation Theory and Numerical Relativity



[Image credit: G. Pratten]

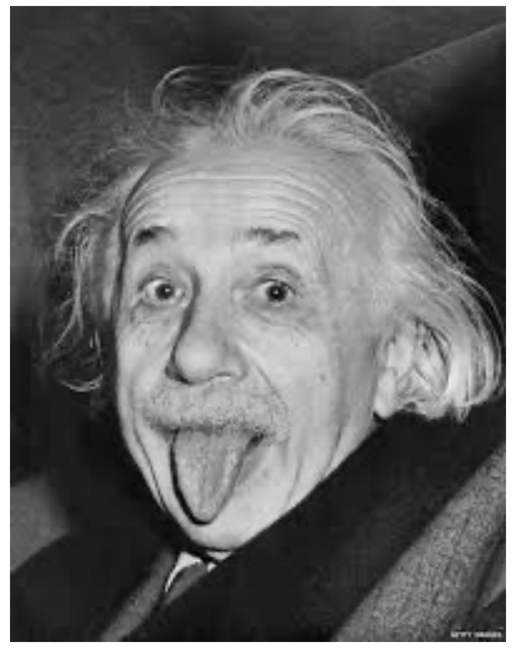
SOLVING THE TWO-BODY PROBLEM IN GR

$$q \equiv \frac{m_1}{m_2}$$



[Image credit: H. Pfeiffer]

GOAL: SOLVING THE TWO-BODY PROBLEM IN GR



- Need to solve the **Einstein field equations** (EFEs):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Mass/energy curve spacetime

Metric tensor

- GR is a **non-linear** theory
  - 10 coupled non-linear PDEs
- Analytic solutions only exist for a handful of special cases, .e.g. Schwarzschild, Kerr, TOV
  - No (known) analytic solutions for more **general spacetimes**, e.g. multiple black holes
- The EFEs can be solved:
  - "Exactly" with **Numerical Relativity (NR)**
  - Approximately using perturbative methods (weak-field, slow-velocity, etc.)

- In practice, we **require both** analytical and numerical solutions to build accurate waveform models for compact binaries.

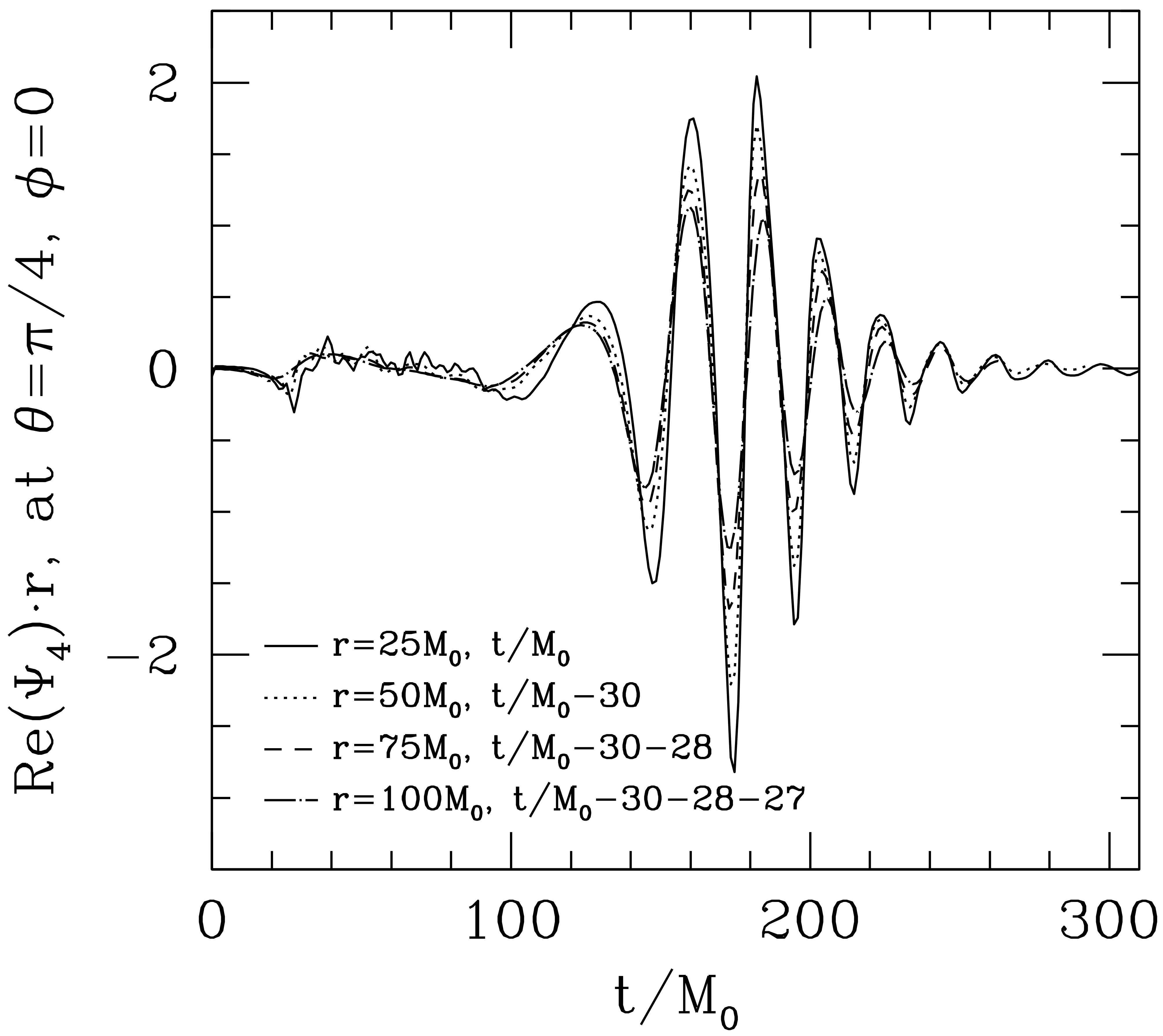
NUMERICAL RELATIVITY (NR)

- In the strong-field regime, the perturbative approach breaks down.
  - We need to solve the EFEs numerically.

- Breakthrough in 2005 [Pretorius; Baker+; Campanelli+]: First simulation of the final inspiral, merger & ringdown plus extraction of the gravitational-signal.

- Today, thousands of NR waveforms are publicly available, see e.g. the SXS database: https://data.black-holes.org/simulations/index.html

[Pretorius 2005]



- Many decades of research preceded the 2005 breakthrough

1952 Choquet-Bruhat IVP

1962 ADM formulation

1964 Hahn-Linquist 2 wormholes

1984 Unruh excision

1992,3 Choptiuk Abrahams, Evans critical phenomena

1997 Brandt&Brügmann puncture data

1994-98 BBH grand challenge

1999-00 AEI/PSU Grazing Collisions

2000-04 AEI/UTB/NASA Lazarus

~2000 Choptuik, Brügmann, Schnetter mesh refinement

2005 Gundlach+ constraint damping

2005 Pretorius IMR w harmonic

2005-06 Campanelli+, Baker+; IMR w BSSN & moving punctures

2006-08 SXS IMR w spectral

2011 Lousto+ q=100

2011 Lovelace+ a=0.97

2015 Szilagyi+ 175 orbits

2014-17 Ruchlin+ high spin puncture ID

1950s — 2005 — 2015+

1975-77 Smarr-Eppley head on collisions

1979 York Kinematics & dynamics of GR

1989-95 Bona-Masso Modified ADM

1994 Cook Bowen-York ID

94-95 NCSA/WSU improved head-on

1999-2005 JW York, Cornell, Caltech, LSU hyperbolic formulations

1999 BSSN

1999 York CTS ID

2000-02 Alcubierre gauge conditions

2004 Brügmann+ one orbit

2003-08 Cook, Pfeiffer ea improved ID

2000 Ashtekar isolated horizons

2006,07 Baker+; Gonzalez+ non-spinning BBH kicks

2007-11 RIT; Jena; AEI;... BBH superkicks

2008 NINJA

2009-11 Bishop+ CCE

2010 Bernuzzi+ C4z

Adapted from slides by C. Lousto & H. Pfeiffer

NR INGREDIENTS LIST (FOR BBH)

- (Well-posed) Initial value problem formulation of the EFEs
  - Yvonne Choquet-Bruhat (1952)
- Numerically stable formulation
  - Well-posedness & hyperbolicity
  - ADM, BSSN, Z4C
- Initial data
  - Bowen-York, conformal think sandwich (CTS), Kerr-Schild
- "Good" coordinates (gauge conditions)
  - Slicing & shift conditions
- Handle singularities
  - Moving punctures, excision
- "Find" and "track" black holes (dynamics)
  - Apparent horizons
- Extract gravitational waves
  - Newman-Penrose formalism, Regge-Wheeler-Zerilli formalism

3+1 DECOMPOSITION AND FORMULATIONS

3+1 FORMALISM

- Goal: Determine the dynamical evolution of a physical system governed by the Einstein field equations (EFE)
    - **Initial value problem** (Cauchy problem) formulation of GR:

> Given a set of adequate initial (and boundary) conditions, the fundamental equations must predict the future (past) evolution of the system.

-
    - To rewrite the EFE as an IVP, we first separate spacetime into space *and* time (3+1 formalism)

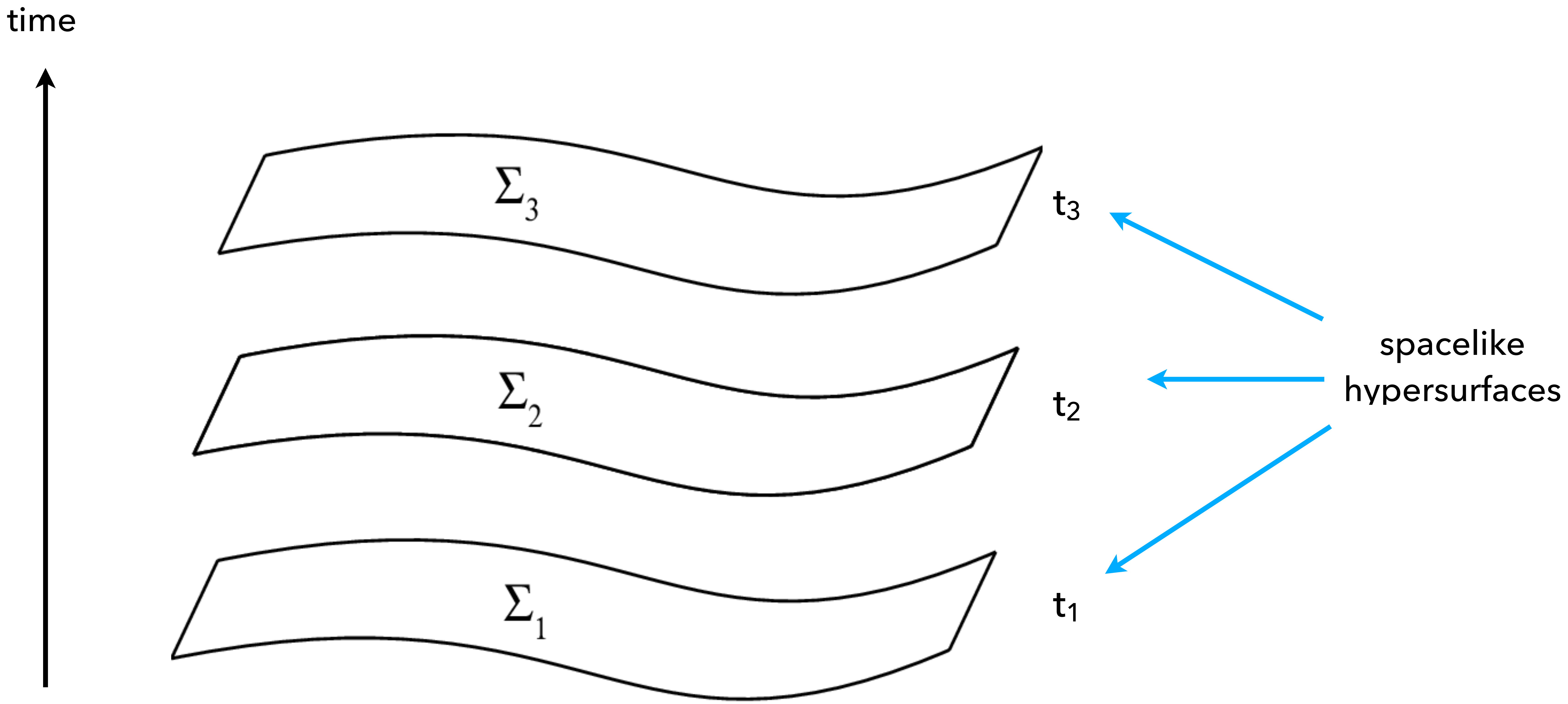
- Assumption: The considered spacetime $(M, g_{\mu\nu})$ is ***globally hyperbolic***, i.e. the spacetime admits a Cauchy surface.
- Definition: A **Cauchy surface** is a spacelike hypersurface in M such that each causal curve intersects it once and only once.

3+1 FORMALISM

- Any globally hyperbolic spacetime M can be completely **foliated** by space-like hypersurfaces (foliation, slicing):

$$M = \bigcup_t \Sigma_t$$

- The foliation can be identified as the **level sets** of a global, smooth, regular scalar function (e.g. time).



SPACELIKE HYPERSURFACES Σ_t

- We define spacelike hypersurfaces Σ_t as the level set of the scalar field t on M. From t, we define the (unnormalised) 1-form:

$$\Omega_\mu = \nabla_\mu t$$

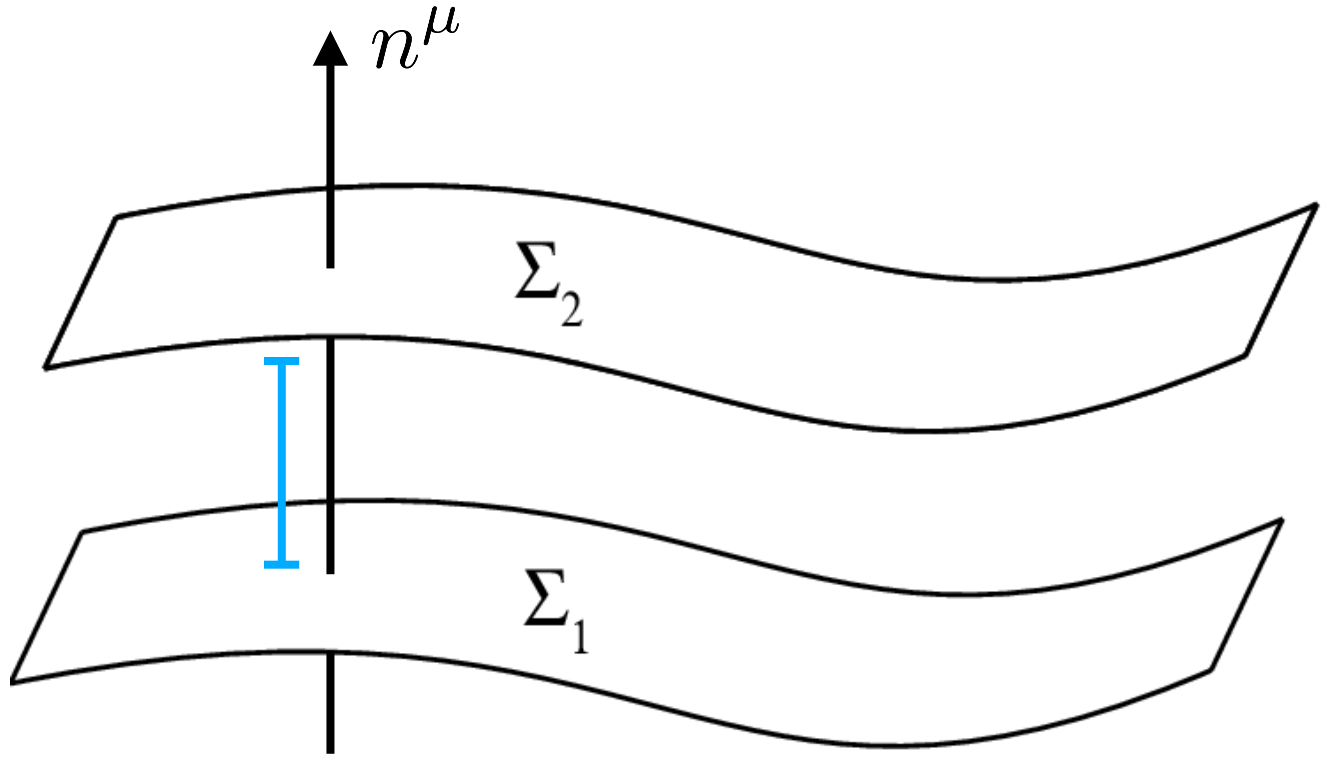
- From the 4-metric $g_{\mu\nu}$, we can compute its square norm:

$$g^{\mu\nu}\Omega_\mu\Omega_\nu = g^{\mu\nu}\nabla_\mu t \nabla_\nu t \equiv -\frac{1}{\alpha^2}$$

  - We assume $\alpha > 0$, then Σ_t is space like and Ω_μ is timelike.

- The unit normal vector to Σ_t is then given by:

$$n^\mu = -\alpha g^{\mu\nu}\nabla_\nu t = -\alpha \nabla^\mu t$$



- α denotes the lapse and n^μ can be thought of as the 4-velocity of a normal (Eulerian) observer
- Note: Since Σ_t is spacelike, $g_{\mu\nu}n^\mu n^\nu = -1$.
- Convention: $\mathrm{sign}(g_{\mu\nu}) = (-+++)$.

SPACELIKE HYPERSURFACES

- With the definition of a hypersurface normal n^μ, we can now construct the spatial metric γ_{ij} on the hypersurface induced by the spacetime metric $g_{\mu\nu}$:

$$\mathcal{T}_p(M) = \mathcal{T}_p(\Sigma) \oplus \mathrm{span}(n) \quad \forall p \in \Sigma$$
$$\gamma : \mathcal{T}_p(M) \mapsto \mathcal{T}_p(\Sigma)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- To "break up" 4D objects into their components parallel and orthogonal to the hypersurface, we require a projection operator:

$$\gamma^\mu{}_\nu := \delta^\mu{}_\nu + n^\mu n_\nu$$

- The projection operator allows us to construct purely spatial objects. Together with n^μ we have the tools to relate 4D objects in M to 3D objects on Σ_t.

SPACELIKE HYPERSURFACES

- 3D covariant derivative: Let $\gamma_{\mu\nu}$ be the induced non-degenerate metric on Σ_t. Then there exists a unique connection (covariant derivative) D:

$$D_\mu f = \gamma_\mu{}^\nu \nabla_\nu f$$

- The Riemann tensor associated with this connection defines the intrinsic curvature of the hypersurface:

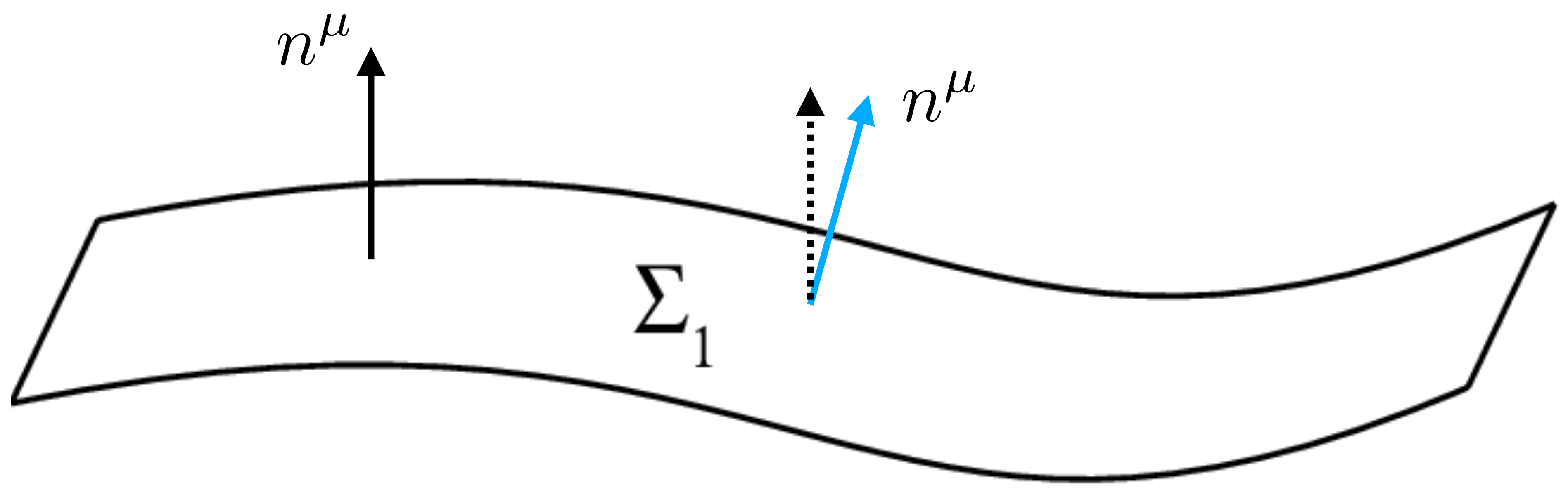
$$(D_\mu D_\nu - D_\nu D_\mu) v^\sigma = {}^{(3)}R^\sigma{}_{\mu\nu\rho} v^\rho \quad \forall v \in \mathcal{T}(\Sigma)$$

- The intrinsic (scalar) curvature of the hypersurface is given by:

$$^{(3)}R = \gamma^{\mu\nu}\,{}^{(3)}R_{\mu\nu}$$

SPACELIKE HYPERSURFACES

- Extrinsic curvature: Describes the "bending" of (3D) Σ_t in (4D) M, and is measured by the change of direction of n^μ as one moves along the hypersurface via parallel transport.



- The extrinsic curvature tensor (second fundamental form) is defined as:

$$K_{\mu\nu} := \gamma_\mu{}^\alpha \gamma_\nu{}^\beta \nabla_\alpha n_\beta = -\nabla_\mu n_\nu - n_\mu n^\alpha \nabla_\alpha n_\nu \equiv -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$$

Lie derivative

- $K_{\mu\nu}$ is symmetric & purely spatial
- $(\gamma_{\mu\nu}, K_{\mu\nu})$ are the fundamental variables in the 3+1 formulation.

GAUSS, CODAZZI & RICCI EQUATIONS

- $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ cannot be chosen arbitrarily - they are related to our 4D manifold $(M, g_{\mu\nu})$. In particular, we need to relate the 3D and the 4D Riemann tensor. Obtained via contractions with n^μ and $\gamma^\mu{}_\nu$.
- Complete spatial projection of ${}^{(4)}R^\mu{}_{\nu\sigma\rho}$ => **Gauss equation:**

$$\gamma_\mu{}^\alpha \gamma_\nu{}^\beta \gamma_\sigma{}^\gamma \gamma_\rho{}^{\delta}\,{}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\mu\nu\sigma\rho} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}$$

- 3 spatial + 1 normal projection of ${}^{(4)}R^\mu{}_{\nu\sigma\rho}$ => **Codazzi equation:**

$$\gamma_\mu{}^\alpha \gamma_\nu{}^\beta \gamma_\sigma{}^\gamma n^{\delta}\,{}^{(4)}R_{\alpha\beta\gamma\delta} = D_\nu K_{\mu\sigma} - D_\mu K_{\nu\sigma}$$

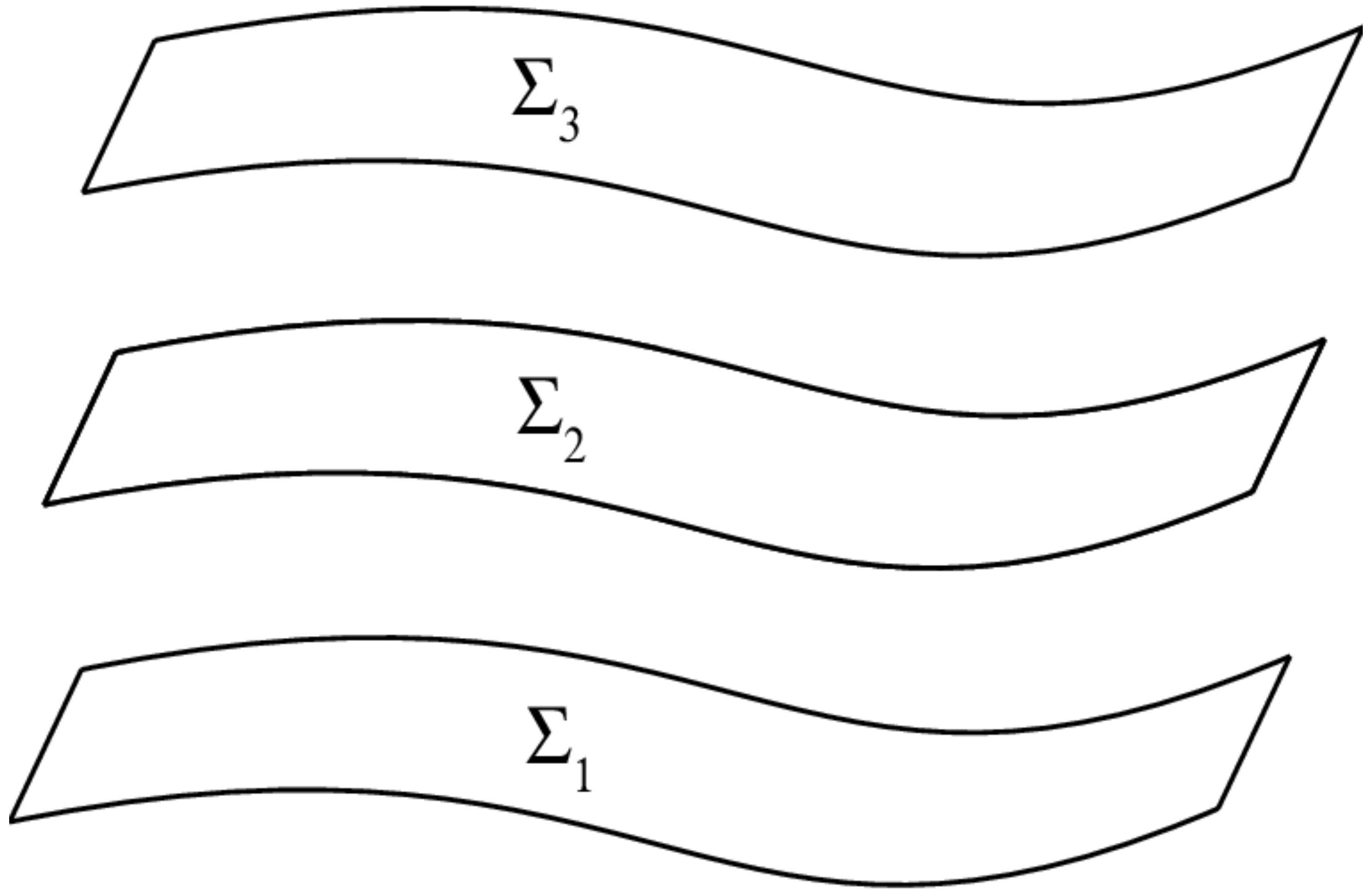
- Note: The Gauss and Codazzi equations depend only on the spatial metric, the extrinsic curvature and their spatial derivatives.
- 2 spatial + 2 normal projections of ${}^{(4)}R^\mu{}_{\nu\sigma\rho}$=> **Ricci equation**:

$$\gamma_\mu{}^\alpha \gamma_\nu{}^\beta n^\gamma n^{\delta}\,{}^{(4)}R_{\alpha\beta\gamma\delta} = \mathcal{L}_n K_{\mu\nu} + \frac{1}{\alpha} D_\mu D_\nu \alpha + K^\gamma{}_\nu K_{\mu\gamma}$$

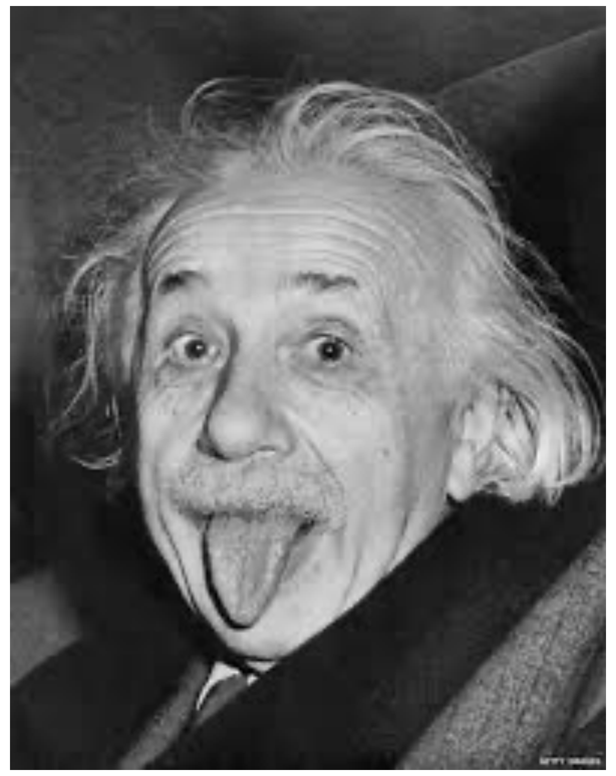
- Note: All other projections vanish due to the symmetries of ${}^{(4)}R^\mu{}_{\nu\sigma\rho}$.

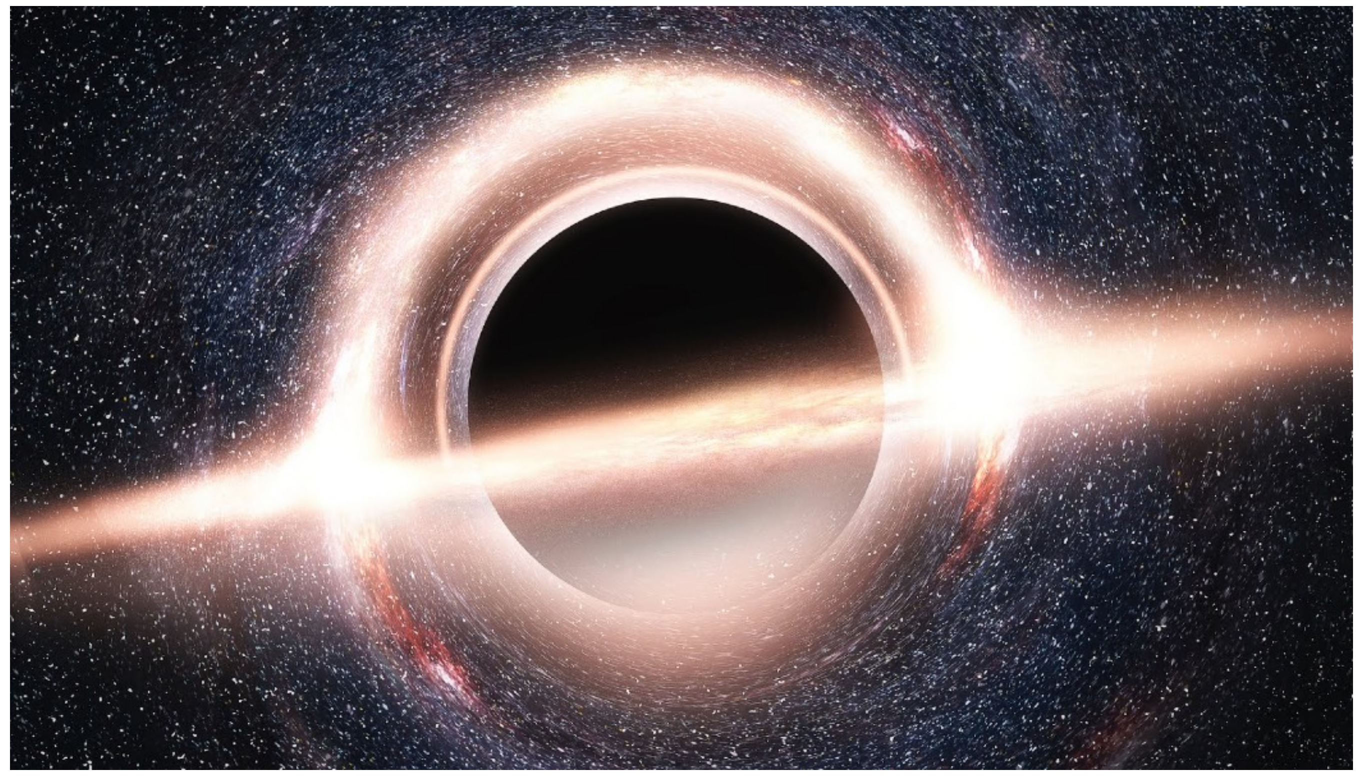
THE EINSTEIN CONSTRAINTS

- So far we have only considered the *kinematics of hypersurfaces*.
- The true gravitational DOF are contained in the Einstein field equations. We now need to link our geometric objects to the physics to obtain a system of constraint & evolution equations.



Geometry





Gravity

CONSTRAINT EQUATIONS

- Contracting the Gauss equation twice and using the EFEs, one finds:

$$^{(3)}R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi\rho$$

Hamiltonian constraint

- $\rho \equiv n_\mu n_\nu T^{\mu\nu}$ is the total energy density measured by a normal observer.
- Contracting the Codazzi equation once yields:

$$D_\alpha K_\mu{}^\alpha - D_\mu K = 8\pi S_\mu$$

Momentum constraint

- $S_\mu \equiv -\gamma_\mu{}^\alpha n^\beta T_{\alpha\beta}$ is the momentum density measured by a normal observer.
- K is the trace of the extrinsic curvature (mean curvature)
- The constraint equations only involve the spatial metric, the extrinsic curvature and their spatial derivatives. $(\gamma_{\mu\nu}, K_{\mu\nu})$ are imposed on a timeslice Σ_t and have to satisfy the constraint equations.
- We need to solve the constraint equations to find suitable initial data.

EVOLUTION EQUATIONS

- We already have "evolution equations" for $(\gamma_{\mu\nu}, K_{\mu\nu})$ from the definition of the extrinsic curvature and the Ricci equation.
  - The evolution equations to evolve the data forward in time are given by the Lie derivative of the spatial metric and the extrinsic curvature along the hypersurface normal n^μ.

- However, $\mathscr{L}_n$ is not a natural time derivative since n^μ is not the dual of Ω_μ:

$$n^\mu \Omega_\nu = -\alpha g^{\mu\nu} \nabla_\mu t \nabla_\nu t = \frac{1}{\alpha} \neq 1$$

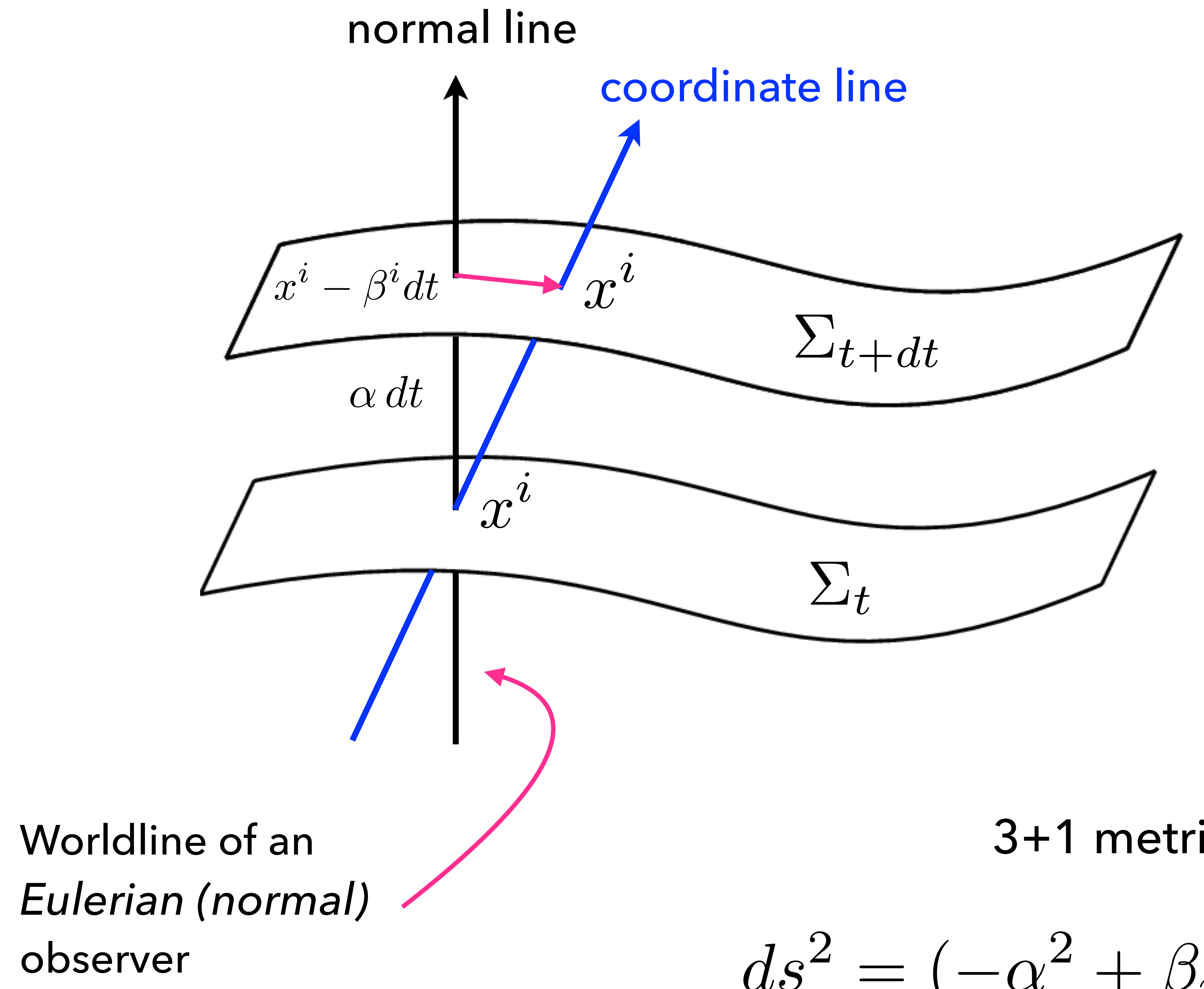
- Instead, consider the following vector:

$$t^\mu = \alpha n^\mu + \beta^\mu$$

- For any purely spatial vector β^μ - the **shift vector** - t^μ is the dual of Ω_μ, i.e. $t^\mu \Omega_\mu = 1$.
- t^μ provides a natural congruence along which to propagate the spatial coordinates from one slice to the next.
- (α, β^μ) are arbitrary (gauge variable) and encode how coordinates evolve in time.

FOLIATION ADAPTED COORDINATES

- Let us now consider coordinates adapted to this 3+1 split:



$$t^\mu = (1,0,0,0)$$
$$\beta^\mu = (0,\beta^i)$$
$$\Rightarrow n^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right)$$
$$n_\mu = (-\alpha,0,0,0)$$
$$\gamma_{ij} = g_{ij}$$

3+1 metric in adapted coordinates:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j$$

ADM EQUATIONS (IN YORK FORM)

- The entire content of the 3+1 decomposed EFE is contained the spatial components.
- In foliation-adapted coordinates $\mathscr{L}_t = \partial_t$.
- **Constraint equations:**

$$
\begin{aligned}
{}^{(3)}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\
D_j(K^{ij} - \gamma^{ij}K) &= 8\pi S^i
\end{aligned}
$$

- **Evolution equations:**

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\partial_t K_{ij} &= -D_iD_j\alpha + \alpha\Big(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j + 4\pi[(S-\rho)\gamma_{ij} - 2S_{ij}]\Big) \\
&\quad + \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k
\end{aligned}
$$

- The ADM equations are quasi-linear.
- Note: The 3+1 evolution equations are non-unique. We can always add arbitrary multiples of the constraints.

WELL-POSEDNESS

- **Well-posedness** of the system of evolution equations is essential for stable numerical evolution, but it is not enough.
  - Definition (Hadamard): A system of PDEs is well-posed if 1) there exists a solution for every admissible set of initial data; 2) the solution is unique and 3) the solution depends continuously on the initial data.
  - A system of linear, first-order PDEs is well-posed if and only if it is strongly hyperbolic.

- Original ADM equations are *mathematically* are weakly hyperbolic, and hence ill-posed.
- The formulation after York is well-posed but still not numerically robust.
  - NB: Mathematical well-posedness does not guarantee numerical stability.

- Due to the non-uniqueness of the evolution equations, we can derive "new" evolution equations, that are mathematically well-posed AND numerically robust.
  - **Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima (BSSNOK) formulation**
  - Conformal rescaling of the spatial metric + auxiliary variable Γ^i
  - Obtain a *strongly hyperbolic system of evolution equations (with the appropriate gauge choice).*

BSSNOK FORMULATION - SUMMARY

- Goal: Modify the ADM equations in York form to obtain a numerically stable evolution.
- It achieves this by doing the following:
  - Introduce a conformal rescaling of the spatial metric: $\bar{\gamma}_{ij} := \psi^{-4}\gamma_{ij}$
  - Choose the conformal factor such that $\det(\bar{\gamma}_{ij}) = 1\ \forall t$. Therefore, $\psi^4 = \gamma^{1/3}$.
  - Introduce $\phi = \ln\psi$ (or $\chi = \exp(-4\phi)$).
  - Separate the extrinsic curvature into its trace K and a trace-free part A_{ij} and conformally rescale:
$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K \quad \& \quad \bar{A}_{ij} = \psi^{-4}A_{ij} = e^{-4\phi}A_{ij}$$
  - Introduce auxiliary variables known as the conformal connection functions:
$$\bar{\Gamma}^i := \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\bar{\gamma}^{ij}$$
  - The BBSNOK variables are: $(\phi, K, \bar{\gamma}_{ij}, \bar{A}_{ij}, \bar{\Gamma}^i)$
  - Addition of multiples of the constraint equations to manipulate the evolution equations and make them numerically stable.

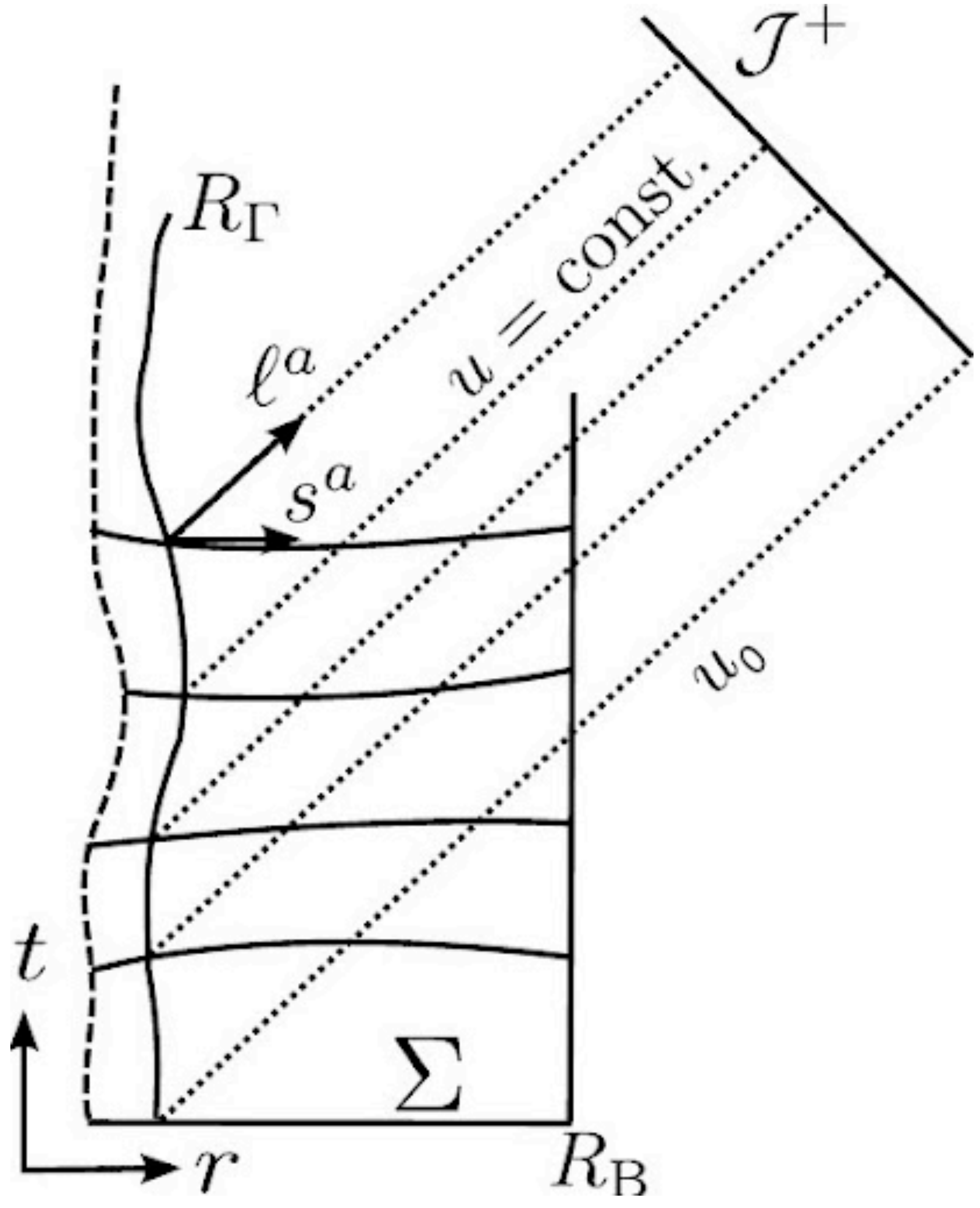
REMARK: OTHER FORMULATIONS

- **Characteristic formalism:** 2+2 formulation, where ingoing (outgoing) null hypersurfaces emanate from a timelike world tube.

- **Conformal formalism:** Hyperboloidal slicing

- **Generalised harmonic formulation:**
  - Evolve the full 4D spacetime
  - Used in SpEC

- **Z4c:** constraint-damping, conformal formulation of EFEs
  - Promotes constraints to dynamical fields
  - Hence any constraint violations can propagate and be damped away



Credit: C. Reisswig

GENERALISED HARMONIC (GH) FORMULATION – SUMMARY

- The EFEs become manifestly hyperbolic of spacetime coordinates satisfy the generalised harmonic gauge condition:

$$\Box x^\mu \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta x^\mu = H^\mu = \Gamma^\mu$$

  - $H^\mu(x, g, \partial g, \dots)$ are freely specifiable gauge source functions.

- Introduce auxiliary variables to replace second order derivatives.
- The EFEs are then rewritten as a system of coupled first-order PDEs in the metric and its derivatives with a symmetric hyperbolic principal part guaranteeing well-posedness (wave equations for the metric!)
  - Constraint:

$$\mathcal{C}^\mu := H^\mu - \Gamma^\mu = 0$$

  - Adds constraint damping terms proportional to the constraints to ensure numerical errors (constrain violations) decay exponentially.
- Note: In a numerical implementation, coordinates are still evolved in a 3+1 split, i.e. H^μ becomes a function of α and β^i

INITIAL DATA

INITIAL DATA

- The spatial metric, the extrinsic curvature (and any matter fields) have to satisfy the constraint equations on every hypersurface $\Sigma_t \forall t$:

$$
\begin{aligned}
{}^{(3)}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\
D_j(K^{ij} - \gamma^{ij}K) &= 8\pi j^i
\end{aligned}
$$

4 coupled elliptic PDEs!

- We first need to specify (γ_{ij}, K_{ij}) on some initial slice. We start with 12 dynamical components
  - 4 constraint equations - not all 12 can be chosen freely
  - 3 components are related to the spatial coordinate freedom
  - 3 components are related to the choice of slicing and embedding
  - Seek a split between the constrained and free components of the field (no natural way!)

- 2 main approaches: Conformal decomposition & conformal thin-sandwich approach

CONFORMAL TRANSVERSE-TRACELESS (CTT) DECOMPOSITION

- Consider a conformal transformation of the spatial metric:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

- The Hamiltonian constraint the becomes:

$$8\bar{D}^2\psi - \psi\bar{R} + \psi^5(K_{ij}K^{ij} - K^2) + 16\pi\psi^5\rho = 0$$

Elliptic PDE for the conformal factor.

- Separate the extrinsic curvature into its trace and trace-free part: $K^{ij} = A^{ij} + \frac{1}{3}\gamma^{ij}K$
    - Then the momentum constraint becomes: $D_j A^{ij} - \frac{2}{3}D^i K - 8\pi j^i = 0$
- Result: Any symmetric trace-free (STF) tensor can be split into a transverse (divergence-free!) and longitudinal part:

$$X^{ij} = X^{ij}_{\mathrm{TT}} + (\mathbf{L}W)^{ij}$$

    - The operator $(\mathbf{L}W)^{ij} := D^iW^j + D^jW^i - \frac{2}{3}\gamma^{ij}D_kW^k$ is the conformal Killing form of the vector W^i.
    - If $(\mathbf{L}W)^{ij} = 0$, then W^i is a conformal Killing vector.

CONFORMAL TRANSVERSE-TRACELESS (CTT) DECOMPOSITION

- Now consider a conformal transformation of the extrinsic curvature:

$$\bar{A}^{ij} = \psi^{10} A^{ij}$$

- And then apply the transverse decomposition:

$$\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{TT}} + (\bar{\mathbf{L}}\bar{W})^{ij}$$

- Then the momentum constraint becomes:

$$\bar{\Delta}_{\bar{\mathbf{L}}}\bar{W}^i - \frac{2}{3}\psi^6 \bar{D}^i K - 8\pi\psi^{10}\bar{j}^i = 0$$
$$\bar{\Delta}_{\bar{\mathbf{L}}}\bar{W}^i := \bar{D}_j(\bar{\mathbf{L}}\bar{W})^{ij}$$

- Given $(\bar{\gamma}_{ij}, K, \bar{A}^{ij}_{\mathrm{TT}})$ we can solve for $\psi, \bar{W}^i$ and reconstruct the physical metric and extrinsic curvature.
- BUT: It is difficult to construct a TT tensor as transversality is a differential condition. Guessing is hard.
- Instead, consider an arbitrary STF tensor $\bar{M}^{ij}$ and perform the analogous decomposition into its TT and longitudinal parts. Assuming $\bar{A}^{ij}_{\mathrm{TT}} = \bar{M}^{ij}_{\mathrm{TT}}$ then

$$\bar{A}^{ij} = \bar{M}^{ij}_{\mathrm{TT}} + (\bar{\mathbf{L}}\bar{W})^{ij} = \bar{M}^{ij} + (\bar{\mathbf{L}}\bar{V})^{ij}$$

- With $\bar{V}^i := \bar{W}^i - \bar{Y}^i$

CONFORMAL TRANSVERSE-TRACELESS (CTT) DECOMPOSITION

- Putting it all together, the constraint equations then have the following final from (York&Lichnerowicz):

$$8\bar{D}^2\psi - \bar{R}\psi + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \frac{2}{3}\psi^5 K^2 + 16\pi\psi^5\rho = 0$$

$$\bar{\Delta}_{\bar{\mathbf{L}}}\bar{V}^i + \bar{D}_j\bar{M}^{ij} - \frac{2}{3}\psi^6\bar{D}^i K - 8\pi\psi^{10}\bar{j}^i = 0$$

- Solve the constraint equations for the vector potential $\bar{V}^i$ and the conformal factor ψ.
- We see that we can freely choose: $\bar{\gamma}_{ij}, K, \bar{M}^{ij}$, and the energy and momentum densities.
- Then construct the physical solutions for γ_{ij}, K_{ij}
- Choices for background fields $\bar{\gamma}_{ij}, K, \bar{M}^{ij}$ need to be motivated.

CONFORMAL THIN-SANDWICH (CTS) DECOMPOSITION

- If we want to have (quasi-)equilibrium solutions, we require initial data that have a certain time evolution.
- CTS: Instead of providing data on one slice Σ, data are provided on two slices with infinitesimal separation.
- Define the time derivative of the conformal metric:

$$\bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij} \quad \Rightarrow \quad \bar{\gamma}^{ij}\bar{u}_{ij} = 0$$

- Consider the trace-free part of the spatial metric and define:

$$u_{ij} := \partial_t \gamma_{ij} - \frac{1}{3}\gamma_{ij}(\gamma^{mn}\partial_t \gamma_{mn}) = -2\alpha A_{ij} + (\mathbf{L}\beta)_{ij}$$

- Adopt the same conformal decomposition but do not split A_{ij} into its TT and longitudinal part, one finds:

$$\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{\mathbf{L}}\beta)^{ij} - \bar{u}^{ij}\right)$$

CONFORMAL THIN-SANDWICH (CTS) DECOMPOSITION

- Applying the conformation transformation, we find:

$$\bar{A}_{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{L}\beta)^{ij} - \bar{u}^{ij}\right) \quad \text{where} \quad \bar{\alpha} = \psi^6\alpha$$

- The momentum constraint then becomes an equation for the shift vector β^i:

$$(\bar{\Delta}_L\beta)^i - (\bar{L}\beta)^{ij}\bar{D}_j \ln(\bar{\alpha}) = \bar{\alpha}\bar{D}_j(\bar{\alpha}^{-1}\bar{u}^{ij}) + \frac{3}{4}\bar{\alpha}\psi^6\bar{D}^iK + 16\pi\bar{\alpha}\psi^{10}S^i$$

- After solving the Hamilton constraint for ψ, the physical solutions are constructed from:

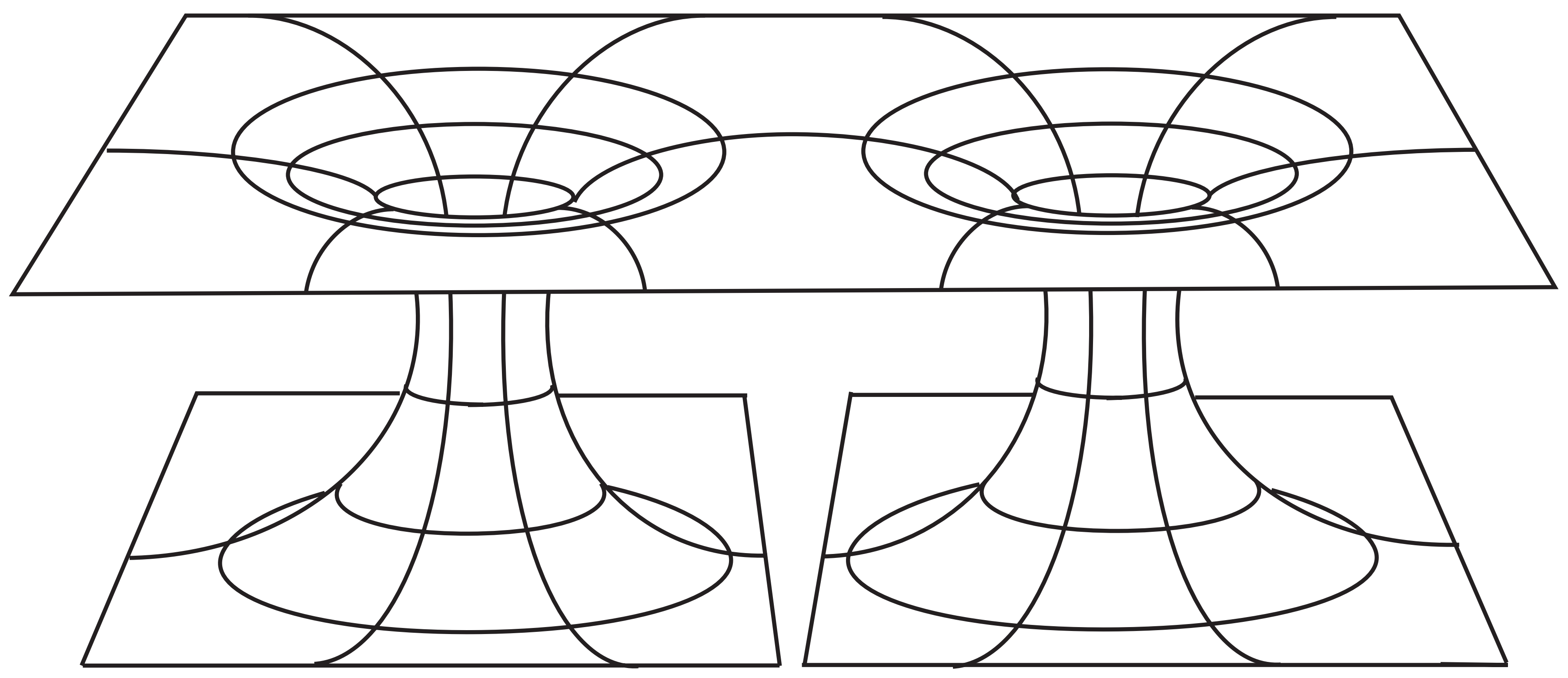
$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}, \quad K^{ij} = \psi^{-10}\bar{A}^{ij} + \frac{1}{3}\gamma^{ij}K, \quad \alpha = \psi^6\bar{\alpha}$$

- The freely specifiable variables are $\bar{\gamma}_{ij}, \bar{u}_{ij}, K, \bar{\alpha}$
- NB: The lapse & shift appear in the initial data construction, i.e. they cannot be specified freely but are solved for instead.
  - Instead of explicitly choosing a gauge one chooses how the metric changes in time.

EXAMPLE: BRILL-LINDQUIST INITIAL DATA FOR MULTIPLE BLACK HOLES

- Assume time symmetry & vacuum = static black holes: $K_{ij} = 0$
    - Momentum constraint is trivially fulfilled.
- Conformal flatness: $\bar{\gamma}_{ij} = \delta_{ij}$
- Hamiltonian constraint becomes flat-space Laplace's equation for the conformal factor:
- Need to impose appropriate boundary conditions to solve Laplace's equation: $\Delta\psi = 0$
    - Asymptotic flatness: $\psi \to 1 \text{ for } r \to \infty$
- Possible solutions:
    - Minkowski space: $\psi = 1$
    - Schwarzschild in isotropic coordinates if $k = M/2$: $\psi = 1 + \frac{k}{r}$
- Linearity allows superposition of solutions: $\psi = 1 + \sum_{i=1}^{N} \frac{m_i}{2|\vec{r} - \vec{r}_i|}$
- Note: The points $r = r_i$ are singularities, i.e. the solution is defined on $\mathbb{R}^3 \backslash \{r_i\}$. The removed points are referred to as **punctures**.

Brill-Lindquist type data



[Image: Alcubierre]

EXAMPLE: BOWEN-YORK INITIAL DATA

- We want moving (boosted) and spinning black holes. Need to solve the momentum constraint.
    - There exists an analytic solution with the desired properties!
- Assume: Vacuum, conformal flatness, maximal slicing ($K = 0$).
- Using the CTT decomposition and choosing $\bar{M}_{ij} = 0$, one finds:

$$\bar{\Delta}_{\bar{\mathbf{L}}} \bar{V}^i = \bar{D}^2 \bar{V}^i + \frac{1}{3} \bar{D}^i \bar{D}^j \bar{V}^j = 0 \quad \Rightarrow \quad \bar{V}^i = -\frac{1}{4r}\left(7P^i + n^i n_j P^j\right) + \frac{1}{r^2}\epsilon^{ijk} n_j S^k$$

- P^i, S^i are constant vectors corresponding to the linear and angular momentum.
- From this, we find the conformal, trace-free extrinsic curvature (as $K_{ij} = \psi^{-2}\bar{A}_{ij}$ for maximal slicing):

$$\bar{A}_{ij} = (\bar{\mathbf{L}}\bar{V})_{ij} = \frac{3}{2r^2}\left[n_i P_j + n_j P_i + n_k P^k (n_i n_j - \delta_{ij})\right] - \frac{3}{r^3}(\epsilon_{ilk} n_j + \epsilon_{jlk} n_i) n^l S^k$$

- Brandt & Brügmann showed that the Brill-Lindquist ID can be generalised to

$$\psi = \sum_{i=1}^{N} \frac{m_i}{2|\vec{r} - \vec{r}_i|} + u \equiv \psi_{\mathrm{BL}} + u$$

EXAMPLE: BOWEN-YORK INITIAL DATA

- Hamiltonian constraint becomes a PDE for u:

$$\Delta u + \frac{1}{8\psi_{\mathrm{BL}}^{7}} \bar{A}_{ij}\bar{A}^{ij}\left(1+\frac{u}{\psi_{\mathrm{BL}}}\right)^{-7} = 0$$

- Regular near puncture, $\Delta u = 0$. Therefore, u is C^2 in all of $\mathbb{R}^3$! Punctures can be ignored!
  - No special boundary conditions for puncture needed.
- Limitations:
  - $P^i = 0,\ S^i \neq 0$: Does not reduce to Kerr but to Kerr + gravitational radiation
  - $P^i \neq 0,\ S^i = 0$: Does not reduce to Schwarzschild but to Schwarzschild + gravitational radiation
  - This "junk radiation" is the result of the the specific choices made to solve the constraint equations.
  - Bowen-York initial data can only approximate Kerr BHs with spins up to 0.93 due to conformal flatness.
    - Alternative: (Superposed) Kerr-Schild initial data

GAUGE CHOICES

GAUGE CHOICES: SLICING & SHIFT CONDITIONS

- In the 3+1 formulation, there are 4 gauge functions: Lapse α and shift vector β^i.
  - We need to impose coordinate conditions (our choice).
- Finding "good" gauge conditions is not trivial but geometric insight combined with numerical experimentation (trial and error) can lead to success.

- Some desired features for BH spacetimes are:
  - Adapted to the underlying geometry/symmetries
  - Should avoid the formation of coordinate singularity
  - Should penetrate the BH horizon
  - Should lead to minimal distortion (keep topology simple)
  - Be mathematically well-behaved and easy to implement numerically

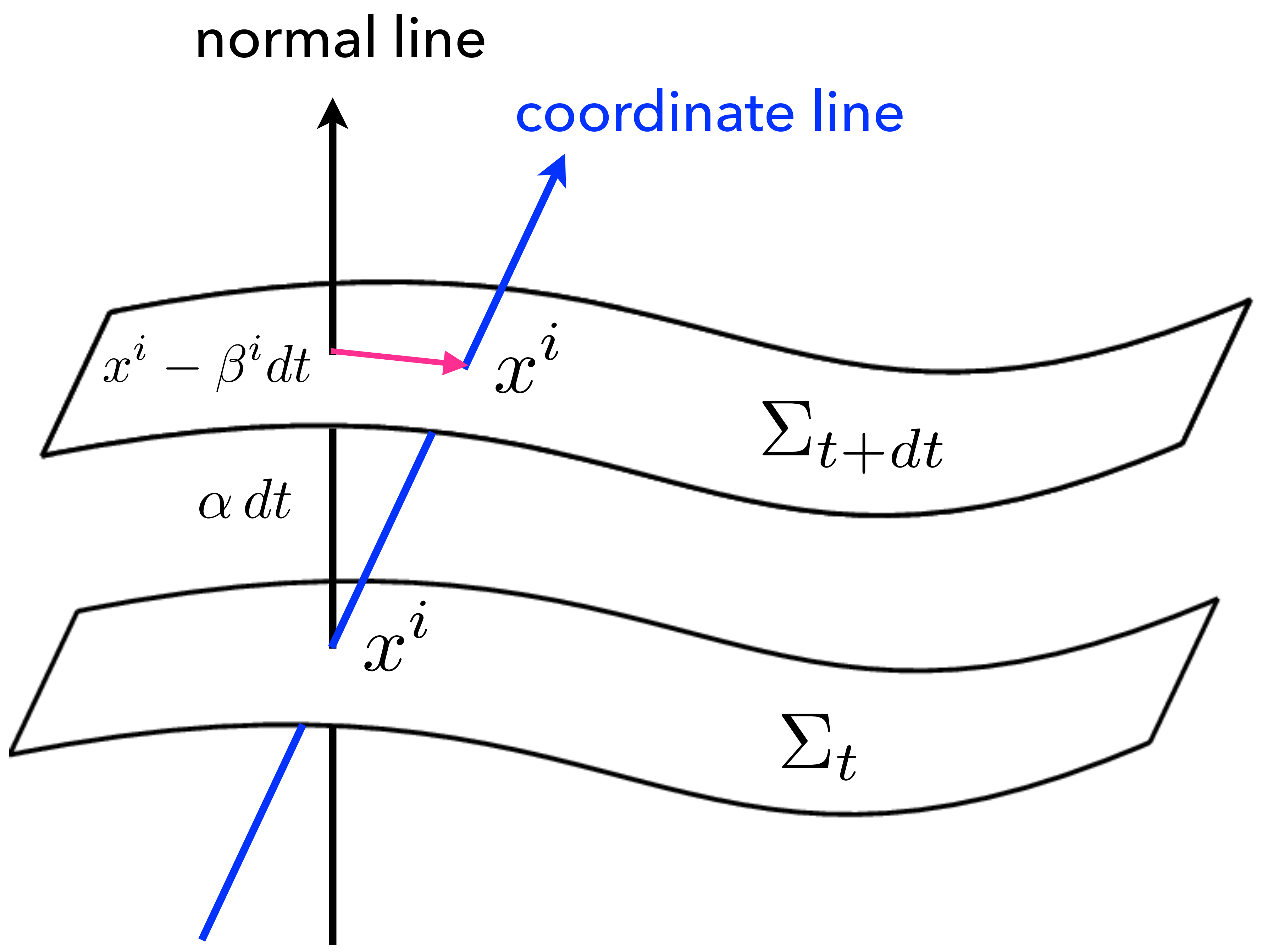
- Slicing condition to determine the lapse, i.e. how proper time advances between two slices
- Shift conditions to determine how the spatial coordinates move within a slice

EXAMPLE 1: GEODESIC SLICING

- Simplest possible choice:

$$\alpha = 1, \quad \beta^i = 0$$

- Coordinate observers coincide with normal observers.
- Normal observers are freely falling ($a_\mu = 0$) and hence follow timelike geodesics.
- Unfortunately, coordinate singularities develop very quickly as geodesics focus near gravitating sources, which can be seen from the expansion (rate of change of volume elements): $\nabla_\mu n^\mu = -K$



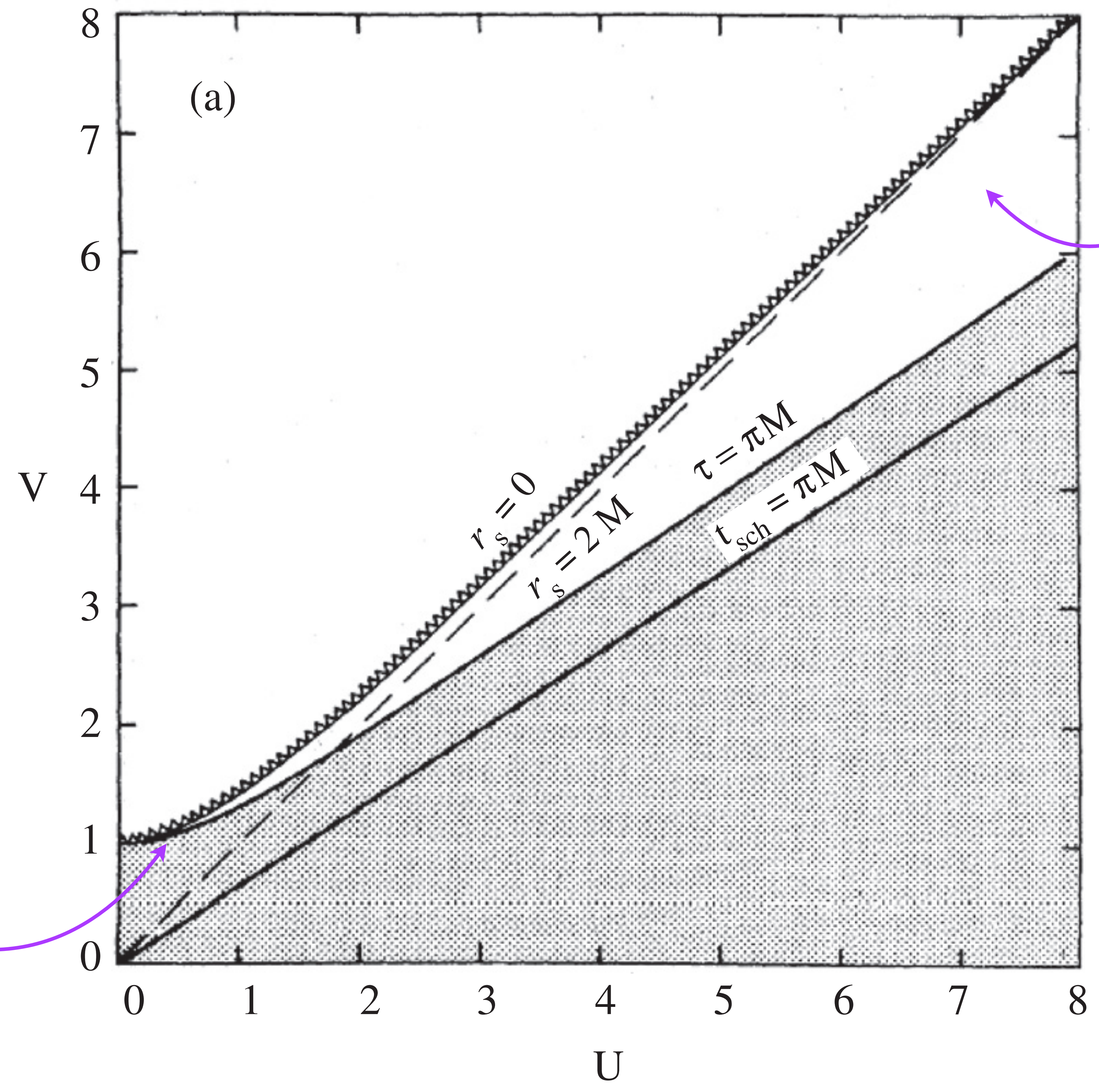
The coordinates become singular.

K grows monotonically in time, the volume element will collapse:

$$\partial_t K \geq 0$$

EXAMPLE 1: GEODESIC SLICING

- Example: Schwarzschild spacetime (in Kruskal coordinates U,V)



Does not fully cover the BH exterior!

Slice "hits" the singularity at coordinate time t = πM (free fall time)

[Taken from Baumgarte & Shapiro, Numerical Relativity]

EXAMPLE 2: MAXIMAL SLICING

- To control the divergence of normal observers, we need to impose a suitable condition on K.
- A common choice is "maximal" slicing (for all times):

$$K = 0 = \partial_t K$$

  - Maximal slicing is volume preserving along congruences of n^μ and the 3-volume is maximal.

- The contraction of the evolution equation for K_{ij} yields an elliptic PDE for the lapse:

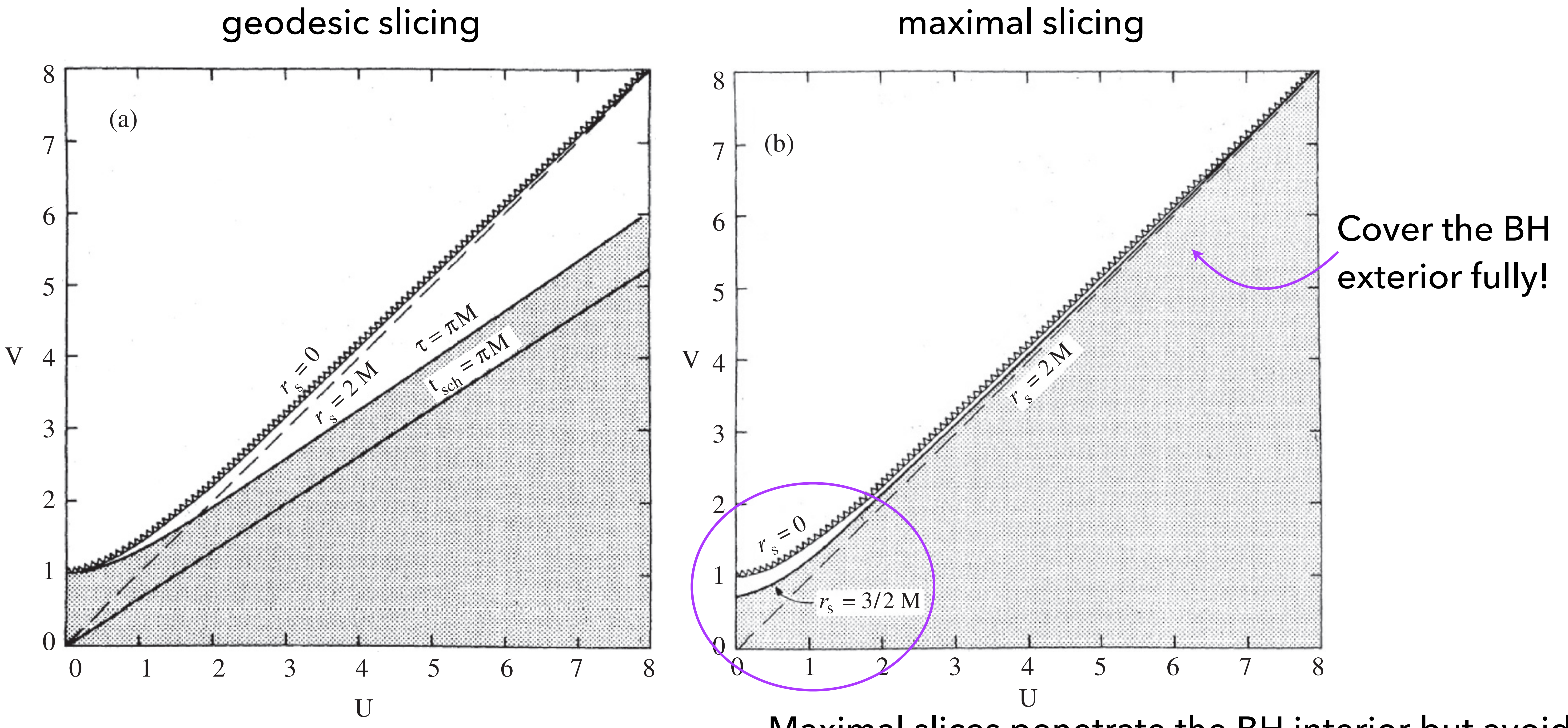
$$D^2\alpha = \alpha(K_{ij}K^{ij} + 4\pi(\rho + S))$$

  - Note: Elliptic equations are costly. We can recast the equation into a parabolic one via "approximate" maximal slicing, i.e. $\partial_t K = -cK$. Also need appropriate boundary conditions, which can be tricky.

- Maximal slicing chooses hypersurfaces that avoid the focusing of normal congruences and that "spread out" as much as possible in spacetime.

GEODESIC VS MAXIMAL SLICES

- Example: Schwarzschild spacetime (in Kruskal coordinates)



Maximal slices penetrate the BH interior but avoid the singularity asymptoting to the limiting surface at $r_S = 3M/2$.

"Collapse of the lapse" when approaching the singularity [Smarr & York] but slice stretching.

[Taken from Baumgarte & Shapiro, Numerical Relativity]

SOME OTHER SLICING CONDITIONS

- Hyperbolic slicing conditions: avoid the costly elliptic PDE but share the good properties of maximal slicing.
  - Harmonic slicing condition:

$$\frac{d}{dt}\alpha \equiv (\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 K \quad \Rightarrow \quad \alpha = \sqrt{\gamma}$$

  - 1+log slicing: Popular (algebraic) slicing condition

$$\alpha = 1 + \log\gamma$$

  - Bona-Masso family of slicing conditions, which encompass both of the above"

$$\frac{d}{dt}\alpha = -\alpha^2 f(\alpha) K$$

  - Taking a second time derivative, one finds that the lapse obeys a wave equation.
  - It can be shown that hyperbolic slicing conditions are singularity avoiding!

SHIFT CONDITIONS

- Many scenarios work well for $\beta^i = 0$, but not for BH spacetimes.
  - Zero shift means eventually all spatial coordinates end up inside inside the BH horizon. :-(
  - Frame-dragging for rotating BHs causes large shears in the spatial metric.

- Similar to slicing conditions, we have elliptic-type and hyperbolic-type shift conditions:
  - Minimal distortion shift condition
  - Gamma freezing shift condition
  - (Hyperbolic) Gamma-driver shift condition: counteracts slice stretching & allows punctures to move
  - Generalised harmonic shift condition

- For orbiting BHs: Use shift vector to transform to a corotating coordinate frame.

FORMULATION COMPARISON

Feature	BSSNOK	GH	Z4c
Hyperbolicity	Strong	Symmetric	Strong
Constraint propagation	No (zero-speed modes)	Yes	Yes
Constraint damping	None	Yes	Yes
Gauge condition	1+log & Gamma-driver	GH gauge	1+log & Gamma-driver
Punctures	Yes	No	Yes

EVOLVING BLACK HOLE SPACETIMES

HOW TO HANDLE SINGULARITIES: MOVING PUNCTURES

- Idea: We keep the physical singularity but make sure that the it never coincides with a grid point, where the calculations are performed. This is controlled via appropriate gauge conditions.
  - 1+log slicing & gamma-driver shift condition
  - Recall: Bowen-York initial data for BHs

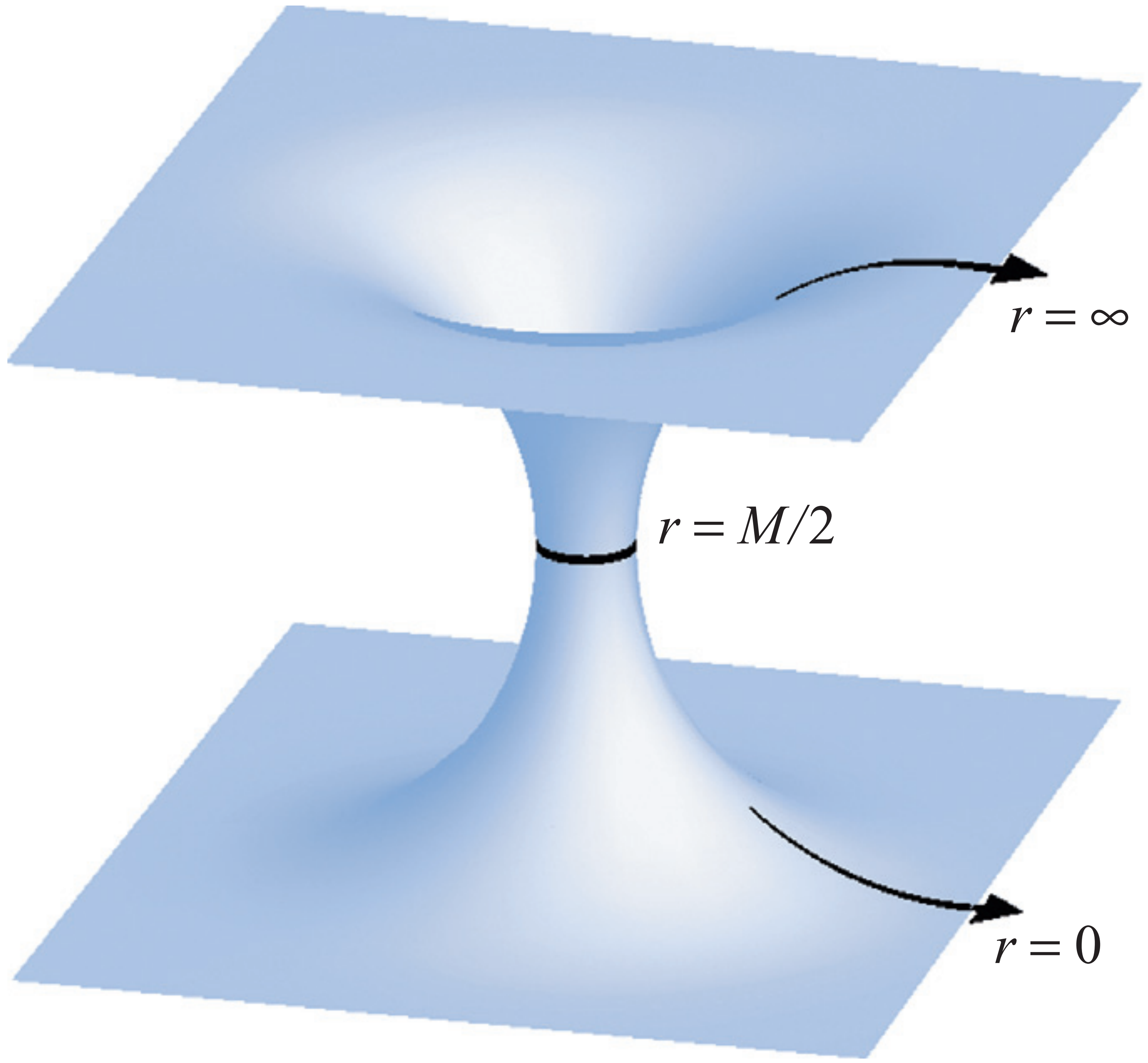
$$\psi = \sum_{i=1}^{N} \frac{m_i}{2r_i} + u$$

Singular part is never evolved

Regular part evolved numerically

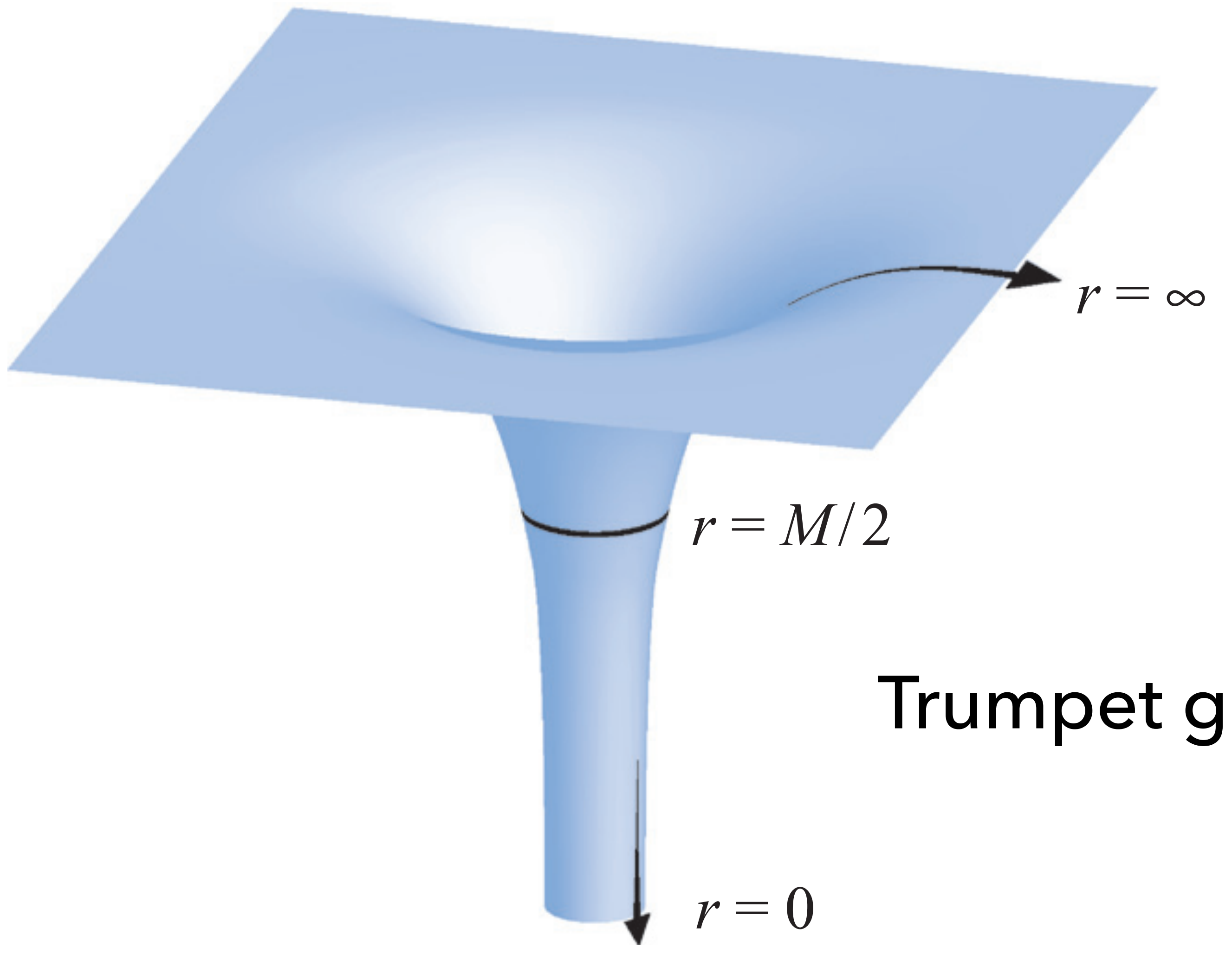
Topology of the initial slice is a wormhole connecting two asymptotically flat copies.

Singularity can be identified with spatial infinity (regular).

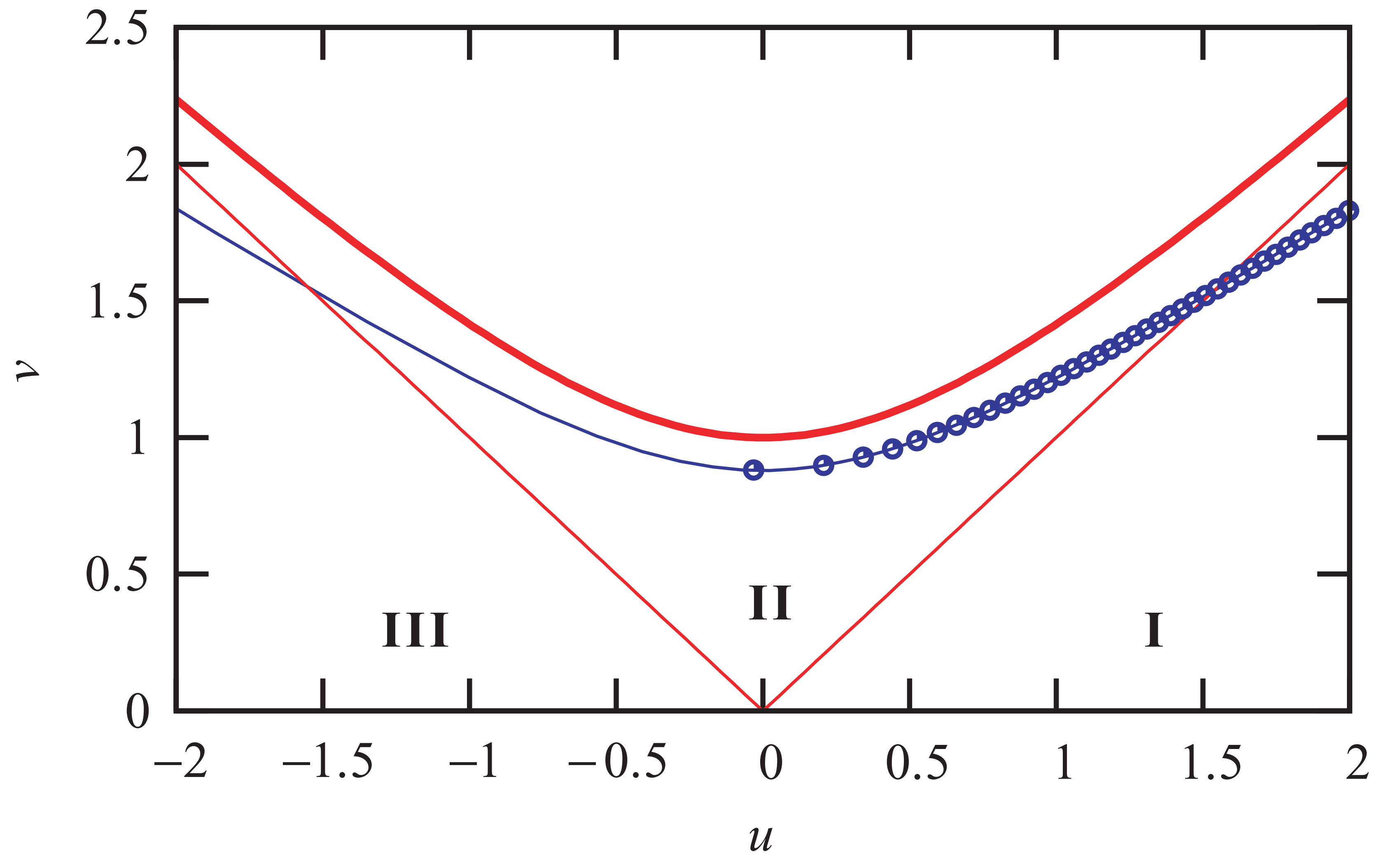


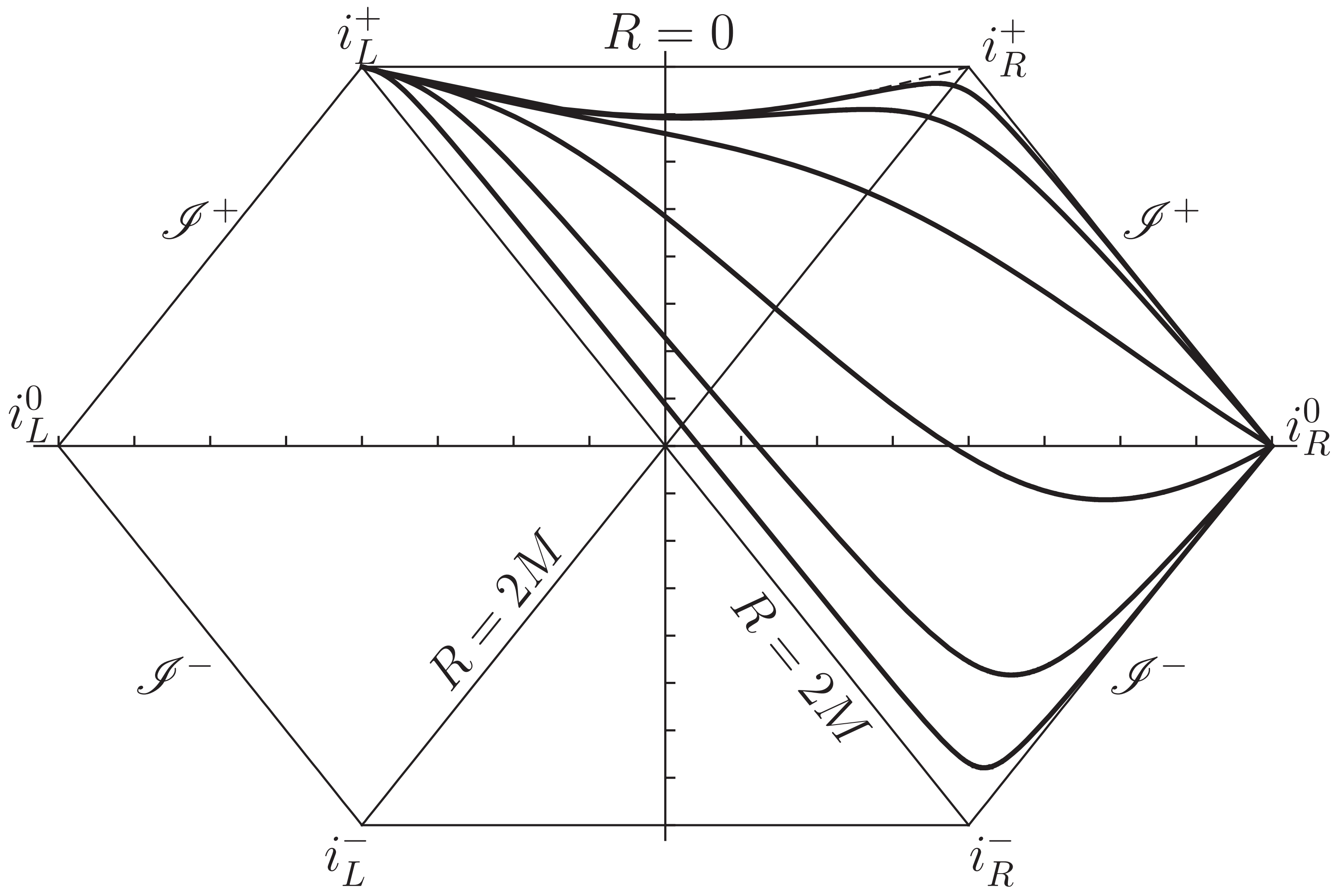
MOVING PUNCTURES

- As we evolve the initial data with the 1+log slicing condition, the topology of the slices changes for a short time before the topology settles down to a stationary "trumpet" geometry.
  - The conformal factor now diverges at $1/\sqrt{r}$.
  - Such a slice terminates on a finite limit surface that does not reach the singularity.



Trumpet geometry





HANDLING PHYSICAL SINGULARITIES: EXCISION

- Formulations that do not use moving punctures have to handle the physical singularity differently.

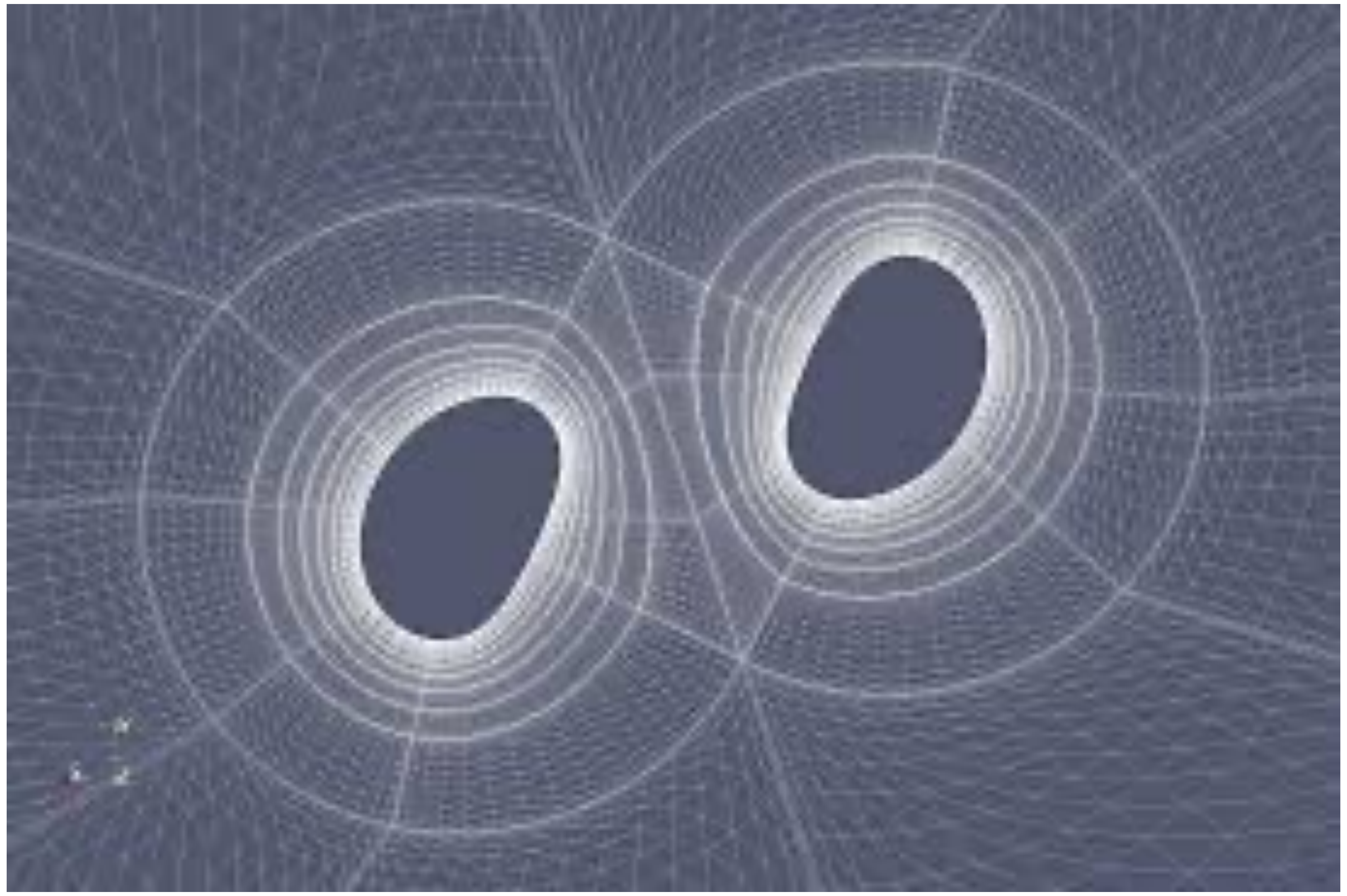
- The black hole horizon is a causal boundary. Anything inside the horizon is causally disconnected from the exterior region and hence cannot affect the physics there.
  - Singularity "hidden" inside the horizon.
  - Information cannot escape.

- Solution: Remove ("excise") the BH interior, and hence the physical (or coordinate) singularity, from the computational domain.

- Two ingredients needed:
  1. A boundary that lies inside the BH horizon such that the BH interior is removed.
  2. A non-zero shift vector β^i that keeps the horizon at a roughly constant coordinate location during the evolution.

BLACK HOLE EXCISION

- Problem 1: The BH event horizon is a globally defined quantity. To determine it, we require require the entire future development of the spacetime.
- Define a proxy instead:
  - Apparent horizon (AH) with $r_{\mathrm{AH}} < r_{\mathrm{EH}}$ define on every time slice.
  - Definition: The AH is the outmost 2-surface embedded in a spatial slice Σ_t, whose outgoing null geodesics have vanishing expansion everywhere.

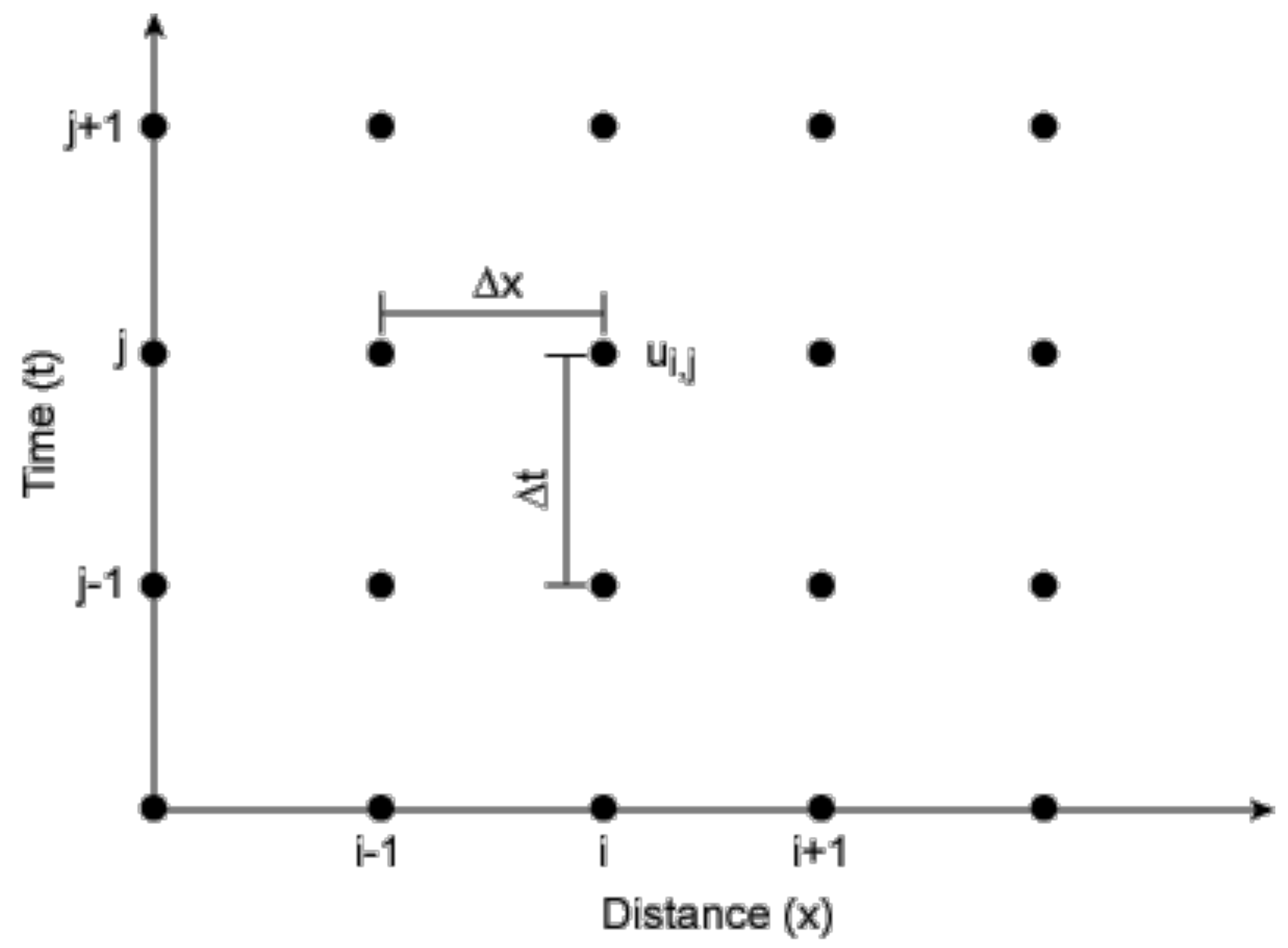


- Problem 2: Excision boundary must track the horizon which may develop a complex geometry.
- Problem 3: Gauge needs to be controlled precisely to ensure that no characteristic fields leak out of the horizon.

NUMERICAL METHODS: FINITE DIFFERENCING

- To solve PDEs on a computer, we need to discretise them.
- Given a continuous function $f(t, x)$, we approximate it by its value at a discrete set of points:

$$f_{ij} = f(t_j, x_i) + \text{truncation error}$$



  - Substitute a continuous spacetime with a set of discrete points (grid or mesh)
  - Replace differential operators by finite differences (algebraic equation)
  - Note: Numerical grids are not infinite - appropriate (outer) boundary conditions have to be chosen

Example 2nd order scheme (centred):

$$\partial_x f\big|_{x_i} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$
$$\partial_x^2 f\big|_{x_i} = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$$

FINITE DIFFERENCING

- Example: Advective equation (hyperbolic)

$$\partial_t u + v\partial_x u = 0$$

  - Solution: $u(t,x) = u(x - vt)$

- FD representation:

$$\partial_x u\Big|_{x_i}^{t_j} = \frac{u^j_{i+1} - u^j_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$ (centred spatial difference)

$$\partial_t u\Big|_{x_i}^{t_j} = \frac{u^{j+1}_i - u^j_i}{\Delta t} + \mathcal{O}(\Delta t)$$ (forward time difference)

- Explicit forward scheme:

$$u^{j+1}_i = u^j_i - \frac{v}{2}\frac{\Delta t}{\Delta x}(u^j_{i+1} - u^j_{i-1})$$

- Stability criterion: Courant-Friedrichs-Lewy (CFL) condition

$$|v|\frac{\Delta t}{\Delta x} \leq 1$$

Meaning: It ensures that a numerical signal does not travel further than 1 grid point during 1 timestep.

NUMERICAL METHODS: SPECTRAL METHODS

- Represent the solution to a differential equation, e.g. $u(x)$ as a truncated series in a complete set of basis function $\phi_k(x)$:

$$u(x) \simeq u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x)$$

spectral coefficients

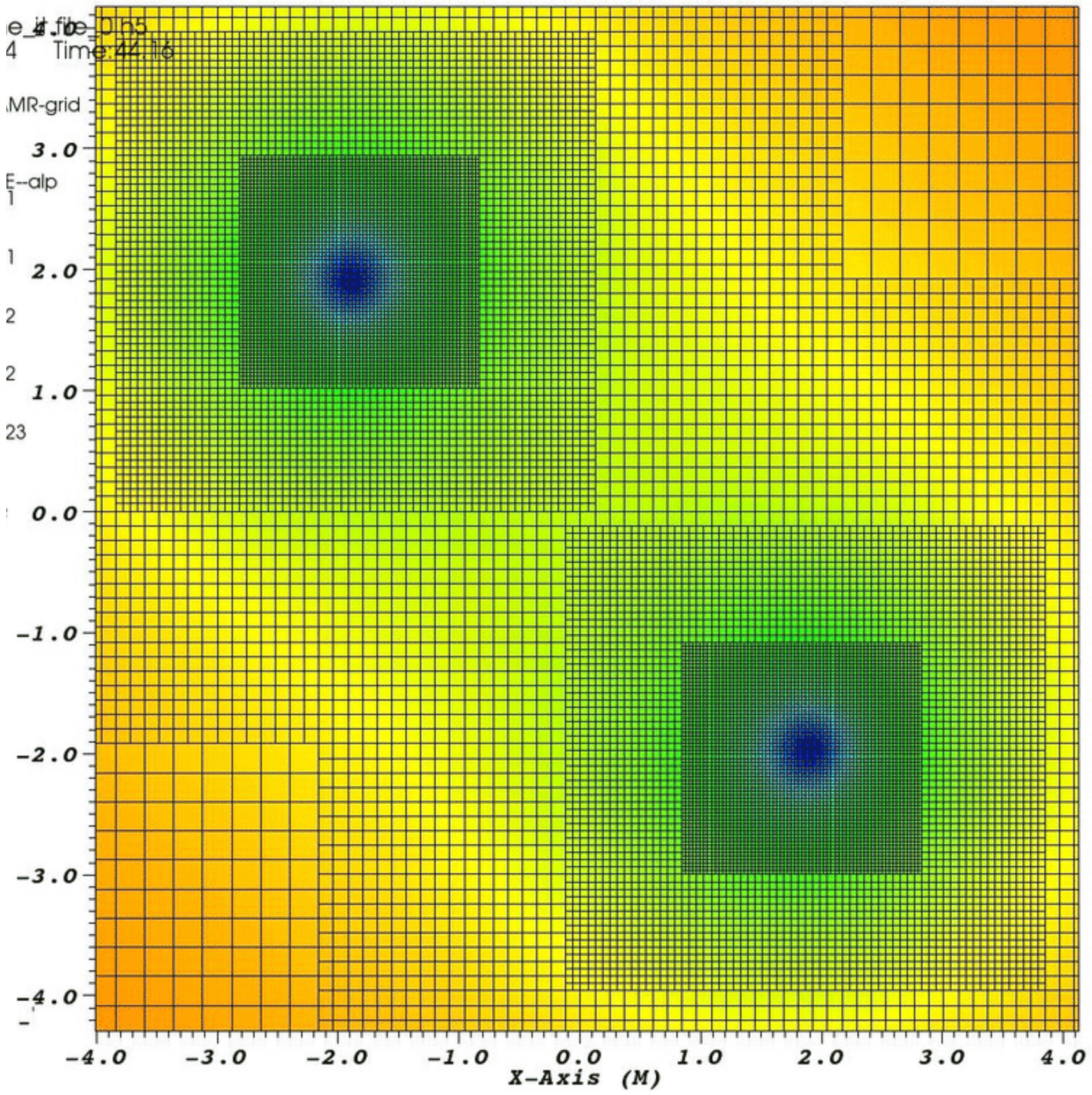
- Derivatives can be expressed analytically:

$$\partial_x u(x) \simeq \partial_x u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \partial_x \phi_k(x)$$

- Advantage: Numerical error often drops exponentially with N. Often fewer computational resources needed for the same accuracy as FD.
- Disadvantage: More difficult to implement. Discontinuities are a problem.

MESH REFINEMENT

- BBH simulations span a "dynamic range" that we want to resolve, i.e. the dynamics of the black holes in a relatively small area of the grid and gravitational-waves in the wave zone far from the black holes.
- BUT: A uniform numerical grid across the dynamic range is very expensive!
- Perform the simulation using a multi-grid structure with a different spatial resolution on each grid as required.
- Note: Special care needs to be taken at grid boundaries.



Adaptive mesh: boxes move with the BHs

GRAVITATIONAL WAVE EXTRACTION

- Different methods to extract GWs, e.g. Newman-Penrose formalism
  - 10 independent components of the Weyl tensor can be expressed as 5 scalars $\psi_0, \ldots, \psi_4$, which are formed by contracting the Weyl tensor with a null tetrad.
  - For certain tetrads (quasi-Kinnersley), we can interpret ψ_0 and ψ_4 as ingoing and outgoing null rays:

$$\psi_4 = -^{(4)}C_{abcd}k^a\bar{m}^bk^c\bar{m}^d$$

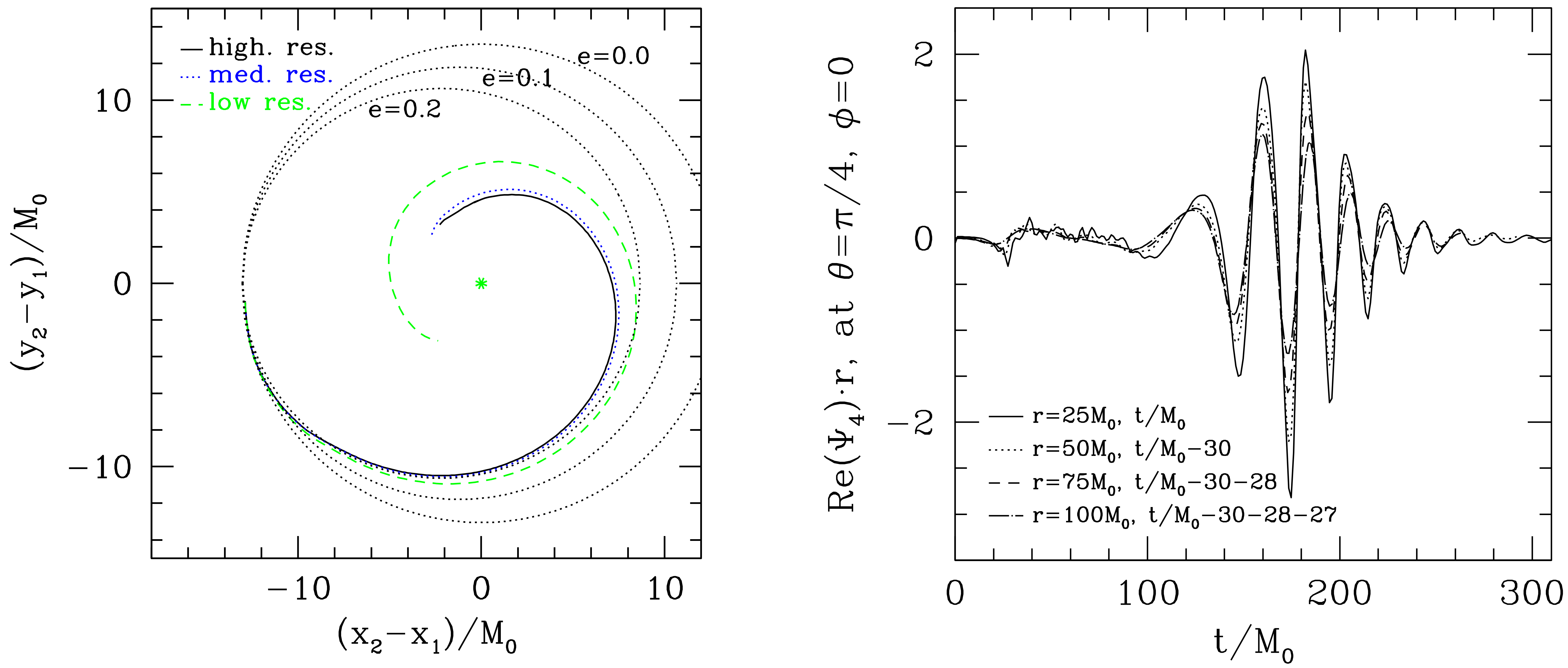
  - Where l and k are real radially outgoing and ingoing null vectors; m is a complex vector such that $-l^ak_a = 1 = m^a\bar{m}_a$.
  - In the TT gauge we find

$$\psi_4 = \ddot{h}_+ - i\ddot{h}_\times$$

  - Requires the 4D Riemann tensor, which is constructed from spatial 3D quantities on each slice.

THE BREAKTHROUGH IN 2005

- In 2005, Frans Pretorius produced the first successful numerical relativity simulation of two equal mass black holes
    - Generalised harmonic formulation, excision, finite differencing



Figures from F. Pretorius, PRL 95, 121101 (2005)

TWO MAIN APPROACHES TO BBH EVOLUTIONS

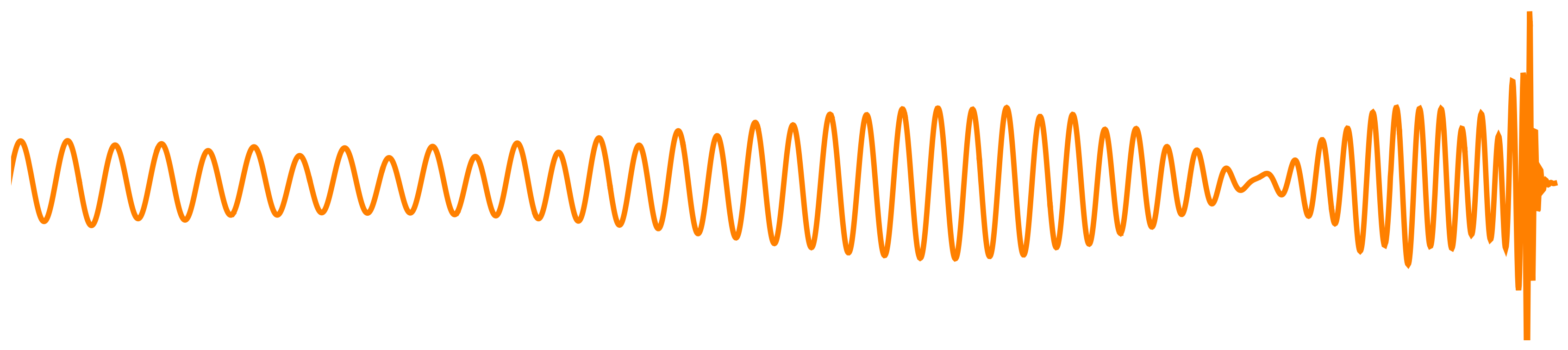
- Puncture initial data
- BSSN or C4z with moving punctures
- 1+log, Gamma-driver shift condition
- Sommerfeld outer boundary condition
- Finite differencing (FD) with adaptive mesh refinement (AMR)
- BAM, MayaKranc, LazEv, Einstein Toolkit, Lean, Goddard, Perimeter, GRChombo, …

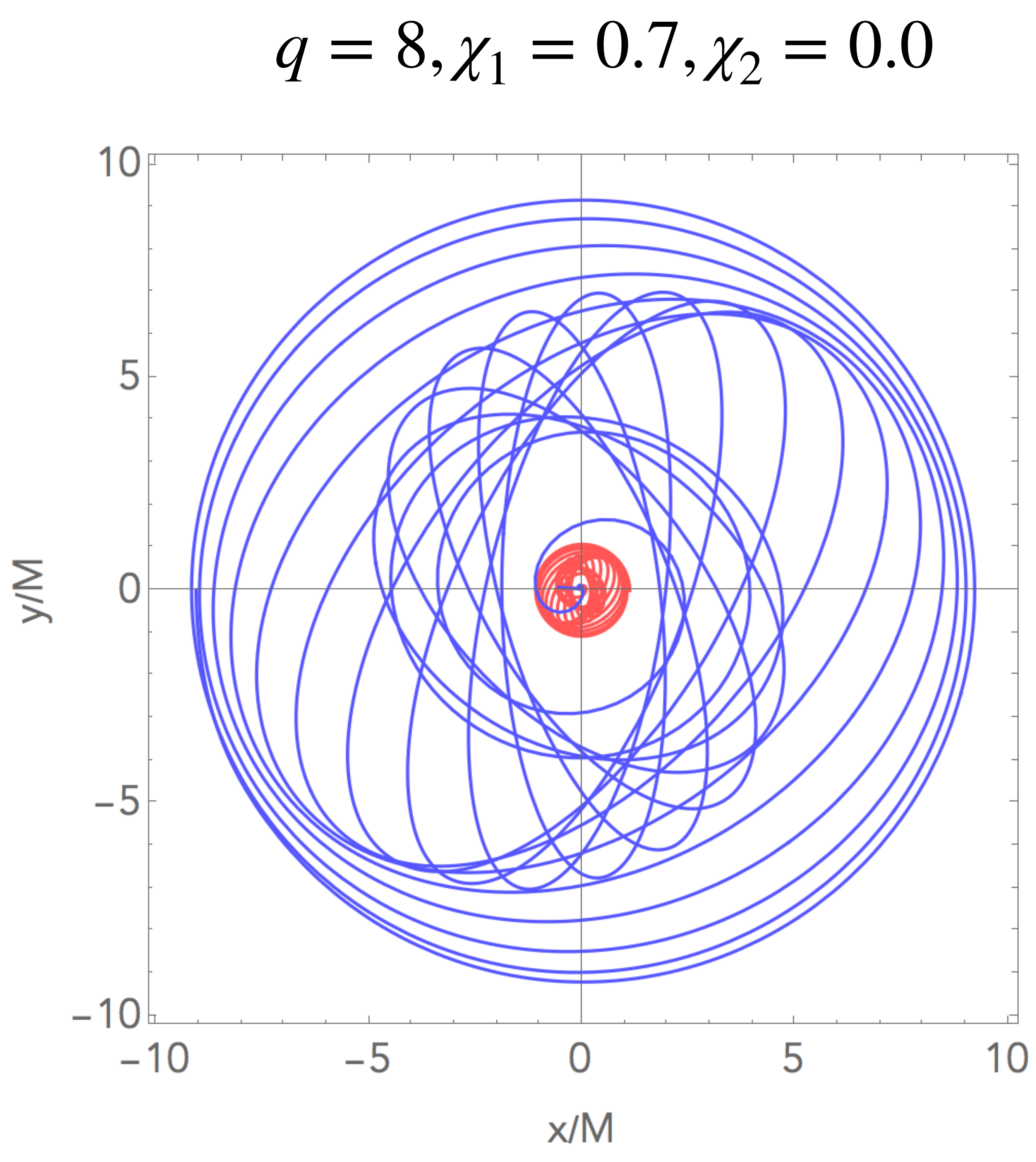
- Quasi-equilibrium excision initial data
- Generalised harmonic (GH) with constraint damping
- Damped harmonic gauge
- Constraint preserving, minimally reflective outer boundary condition
- Multi-domain spectral methods
- SXS Collaboration (SpEC)

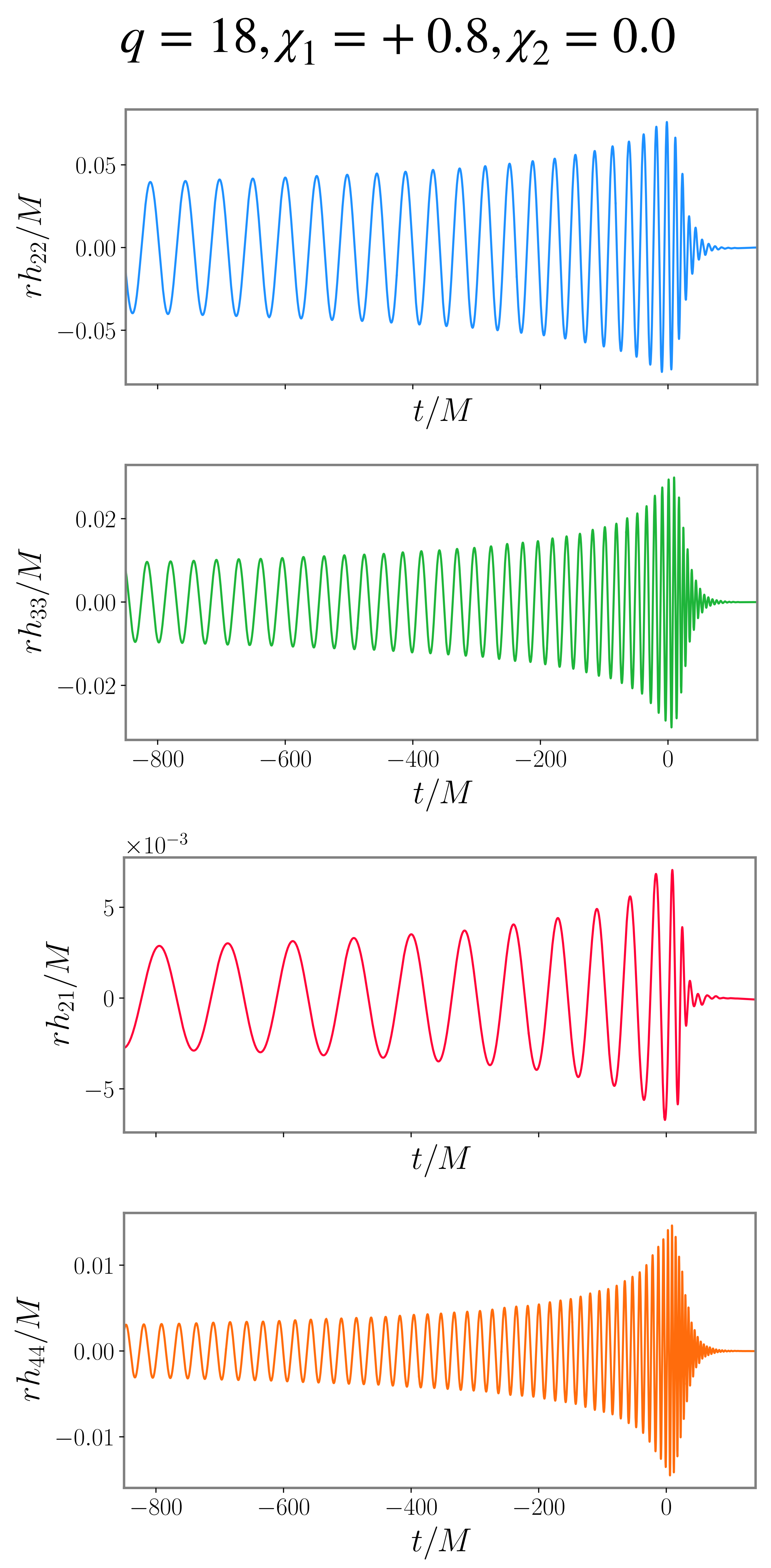
Pretorius:
FD, GH, AMR

CURRENT STATE-OF-THE-ART

- O(1000s) of NR simulations available today:
  - Predominantly quasi-circular ($e \lesssim 0.01$) bound systems
  - Moderate mass ratios ($m_1/m_2 \leq 18$)
    - A handful with larger mass ratios
  - Spin magnitudes up to $\chi_i \sim 0.85$
    - A few high-spin simulations
  - (Accurate) higher harmonics
  - Spin-precession



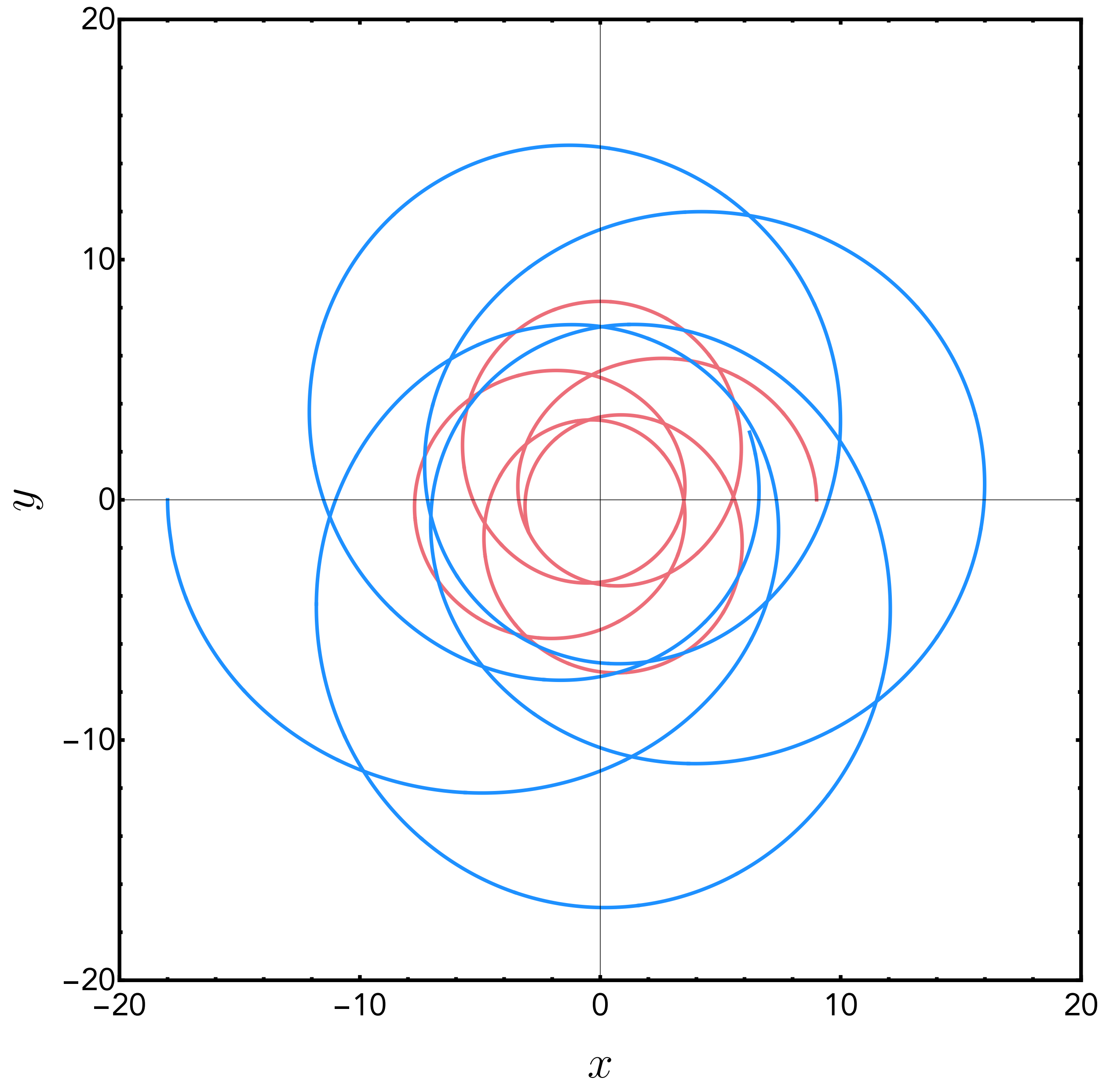


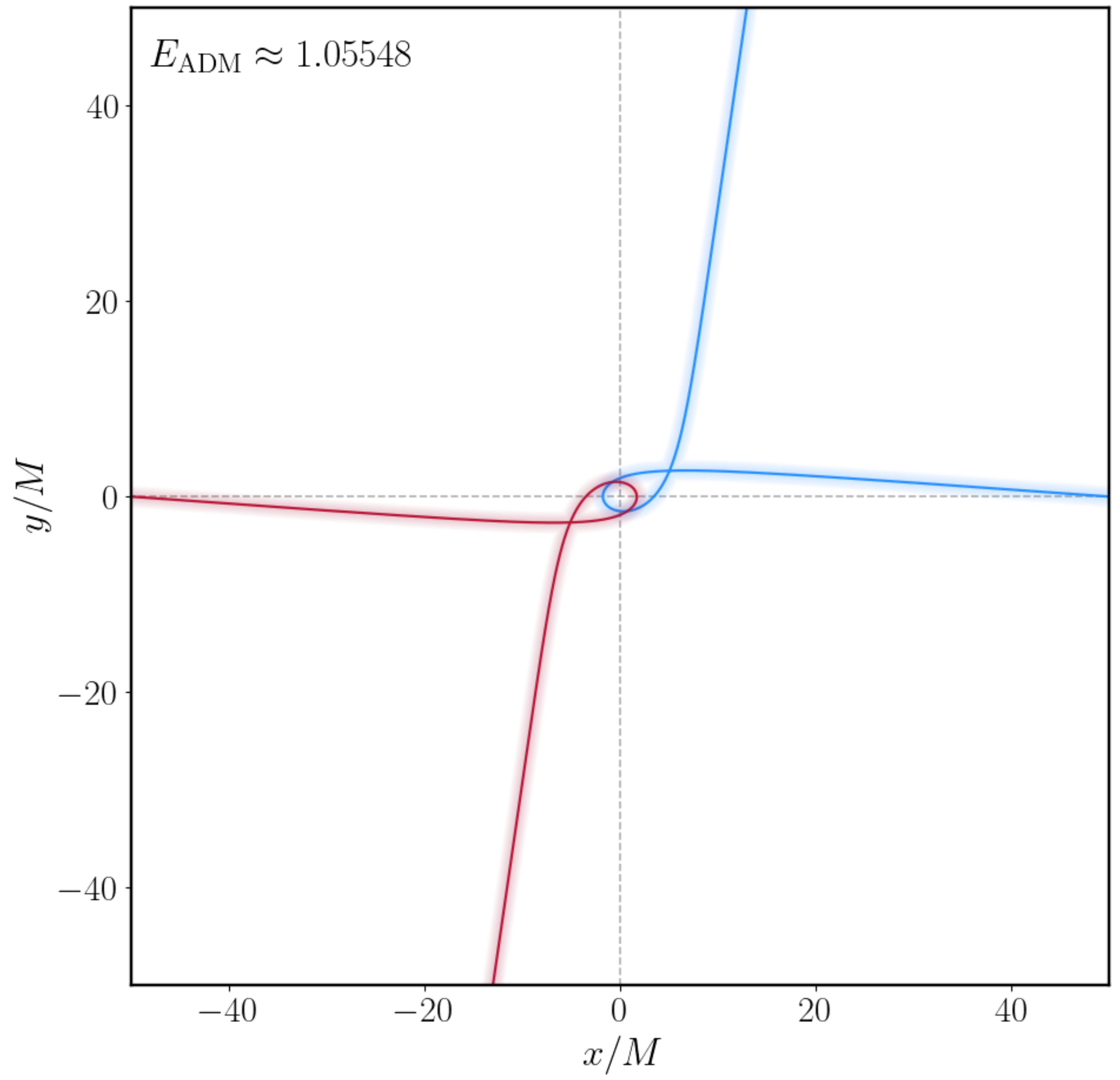


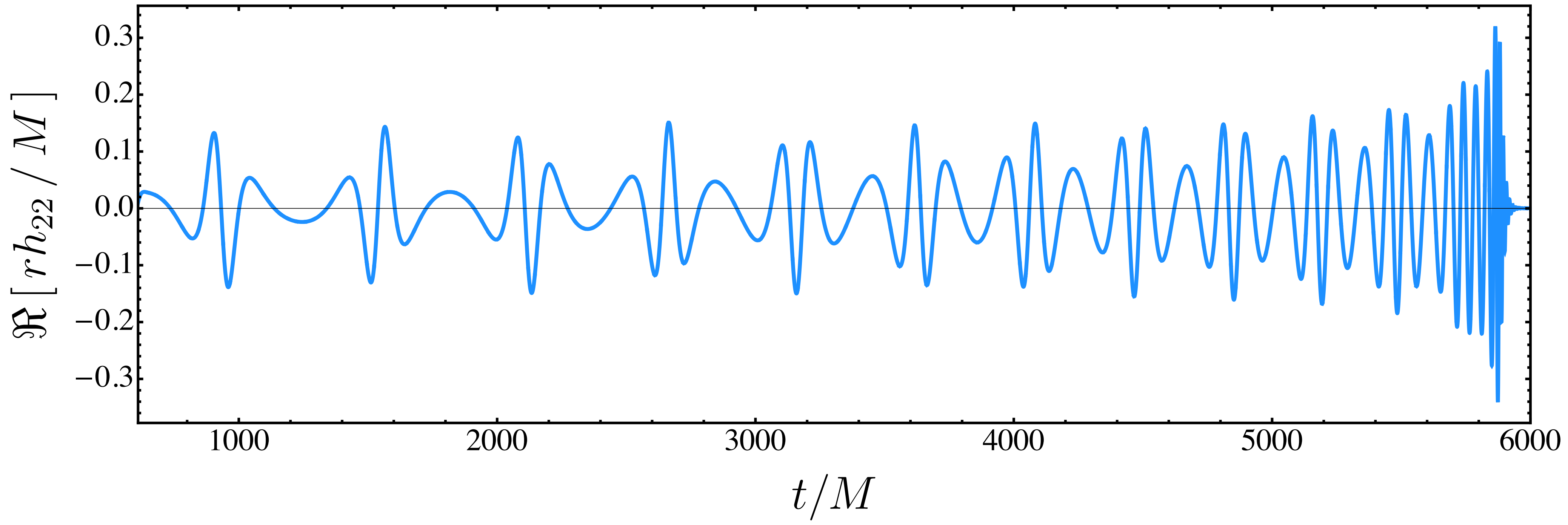
[Simulations: G. Pratten, P. Schmidt, S. Swain]

CURRENT STATE-OF-THE-ART

- Recent push to explore:
    - Binaries on eccentric bound orbits
    - Hyperbolic encounters
    - Strong-field black hole scattering
    - Higher mass ratios







[Simulations: G. Pratten, P. Schmidt, S. Swain]