# A *Brief* Introduction to the Effective One Body Formalism

Geraint Pratten

2026 Nordita Program: Amplitudes, Strong-Field Gravity and Resummation

# Anatomy of an Inspiral…

See Patricia's talk…

# Anatomy of an Inspiral…

Inspiral

Flux Balance:

$$\frac{dE_{\text{orbital}}}{dt} = \mathcal{L}_{\text{GW}} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

See Patricia's talk…

einstein toolkit

# Anatomy of an Inspiral…

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

Inspiral

Analytical approximations begin to break down

Flux Balance:

$$\frac{dE_{\rm orbital}}{dt} = \mathcal{L}_{\rm GW} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

See Patricia's talk…

einstein toolkit

# Anatomy of an Inspiral…

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY


Flux Balance:

$$\frac{dE_{\text{orbital}}}{dt} = \mathcal{L}_{\text{GW}} \approx \frac{32}{5} \frac{c^5}{G} \frac{(m_1 m_2)^2}{M^4} \left(\frac{v}{c}\right)^5$$

See Patricia's talk…

einstein toolkit

# Anatomy of an Inspiral…


Flux Balance:

$$\frac{dE_{\text{orbital}}}{dt} = \mathcal{L}_{\text{GW}} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

See Patricia's talk…

# Anatomy of an Inspiral…

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

**post-Newtonian**

$$h^{\alpha\beta}(t,r) = -\frac{4G}{c^4 r}\int T^{\alpha\beta}(t - r/c + \mathbf{n}\cdot\mathbf{r}'/c, \mathbf{r}')\, d^3r'$$

**scattering amplitudes**

$$\mathcal{M}^{\Delta+\nabla} = \frac{2\pi^2 G^2 \epsilon_1\cdot\epsilon_4 \epsilon_2\cdot\epsilon_3}{\sqrt{-q^2}} \sum_{n,i} \alpha^{(n,i)} \mathcal{O}^{(n,i)}$$

**numerical relativity**

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = \cdots$$

**BH perturbation theory**

$$h = \sum C_{[p]\ell mn}\, e^{-i\tilde{\omega}_{[p]\ell mn} t}\; {}_{-2}S_{[p]\ell mn}(\iota,\varphi)$$

**EFT**

$$S_{\rm eff} = -\frac{1}{16\pi G}\int d^4x\sqrt{\bar{g}}\, R[\bar{g}_{\mu\nu}] + \sum_{i=1}^{\infty} C_i(r_s)\int d\sigma\, \mathcal{O}_i(\sigma)$$

**gravitational self-force**

$$g_{\alpha\beta} + \sum_{m=-\infty}^{\infty}\left[q h^{1,m}_{\alpha\beta}(\Omega) + q^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\phi} + \mathcal{O}(q^3)$$

**post-Minkowski**

$$h^{\alpha\beta} = \sum_{n=1} G^{(n)} h^{\alpha\beta}_{(n)}$$

**effective one body**

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

+ other approaches

**Inspiral**

Analytical approximations begin to break down

**Merger**

**Ringdown**

Flux Balance:

$$\frac{dE_{\rm orbital}}{dt} = \mathcal{L}_{\rm GW} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

See Patricia's talk…

einstein toolkit

# Anatomy of an Inspiral…

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

post-Newtonian

$$h^{\alpha\beta}(t,r) = -\frac{4G}{c^4 r}\int T^{\alpha\beta}(t - r/c + \mathbf{n}\cdot\mathbf{r}'/c, \mathbf{r}')\, d^3r'$$

scattering amplitudes

$$\mathcal{M}^{\Delta+\nabla} = \frac{2\pi^2 G^2 \epsilon_1\cdot\epsilon_4 \epsilon_2\cdot\epsilon_3}{\sqrt{-q^2}} \sum_{n,i} \alpha^{(n,i)} \mathcal{O}^{(n,i)}$$

numerical relativity

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = \cdots$$

BH perturbation theory

$$h = \sum C_{[p]\ell mn}\, e^{-i\tilde{\omega}_{[p]\ell mn} t}\, {}_{-2}S_{[p]\ell mn}(\iota, \varphi)$$

EFT

$$S_{\rm eff} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\, R[\bar{g}_{\mu\nu}] + \sum_{i=1}^{\infty} C_i(r_s)\int d\sigma \mathcal{O}_i(\sigma)$$

gravitational self-force

$$g_{\alpha\beta} + \sum_{m=-\infty}^{\infty}\left[q h^{1,m}_{\alpha\beta}(\Omega) + q^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\phi} + \mathcal{O}(q^3)$$

post-Minkowski

$$h^{\alpha\beta} = \sum_{n=1} G^{(n)} h^{\alpha\beta}_{(n)}$$

effective one body

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

+ other approaches

**Inspiral**

Analytical approximations begin to break down

**Merger**

**Ringdown**

Flux Balance:

$$\frac{dE_{\rm orbital}}{dt} = \mathcal{L}_{\rm GW} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

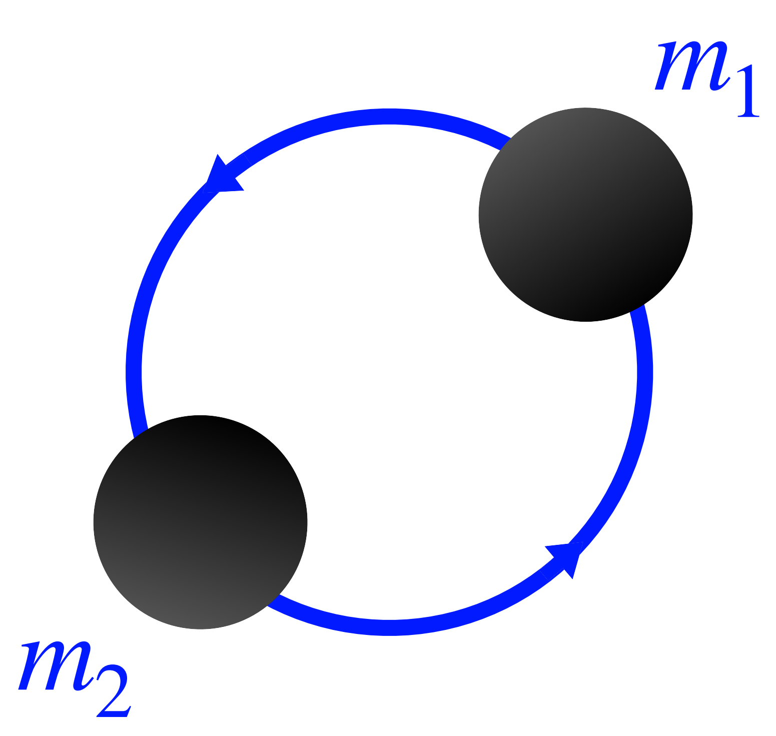
we will focus on the basics of EOB

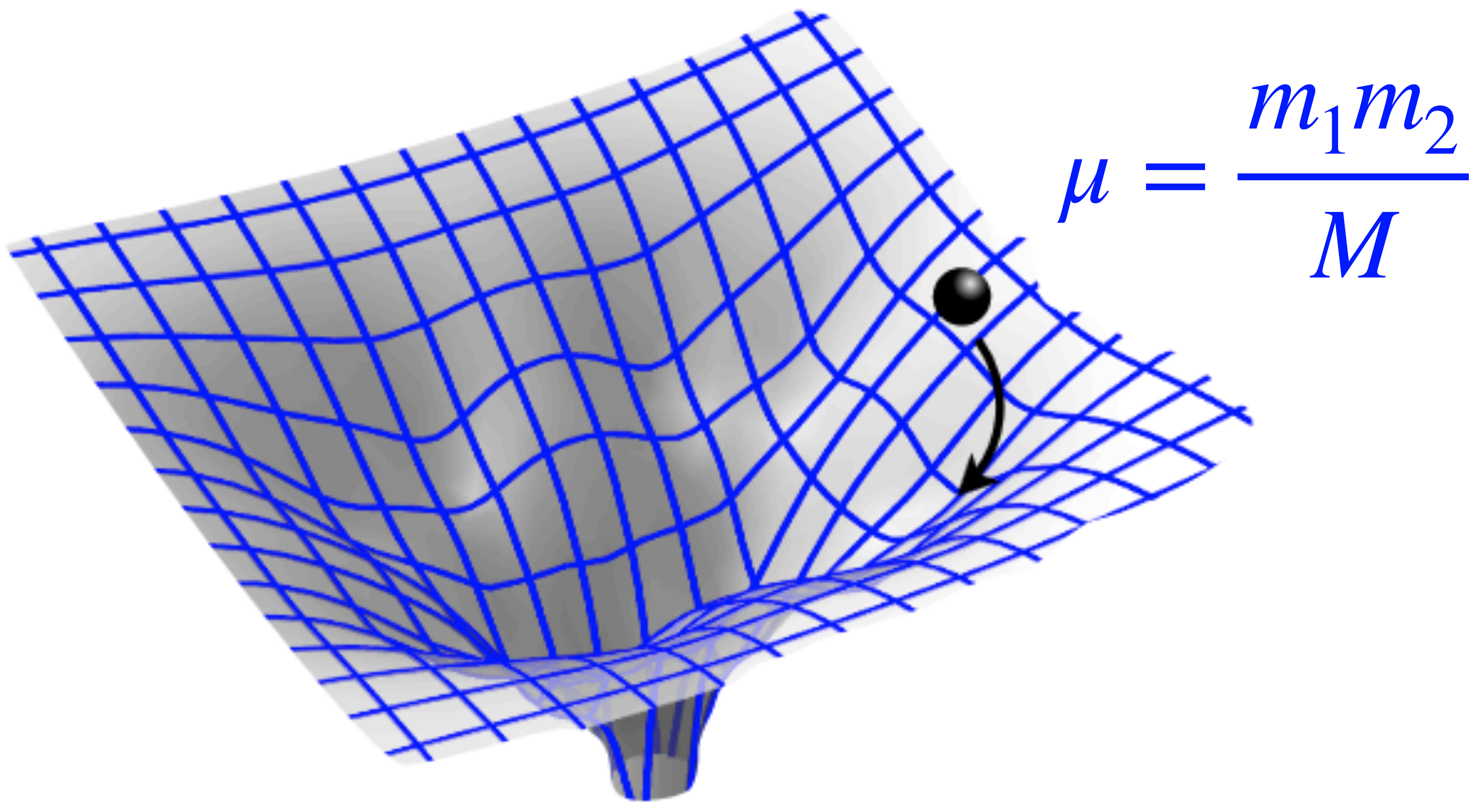
einstein toolkit

# The Effective One Body Framework

# Effective One Body

- Novel approach introduced by Buonanno and Damour in 1999
- Inspired by approach to interacting quantum two body problem [Brézin+ 1970]
- Basic idea is to map two-body problem onto an effective one-body problem via a canonical transformation




- Calculate motion of a test-particle in a deformed/effective external metric ↔ equations of motion

Buonanno and Damour 1999: arXiv:gr-qc/9811091
Buonanno and Damour 2000: arXiv:gr-qc/0001013

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Aim of these slides?
  - Provide a brief schematic introduction to the EOB framework
  - Key motivations and basic building blocks?
  - Highlight *very partial* recent developments and try to provide appropriate signposting
  - Time permitting make a connection to the recent scattering results

- What I cannot do…
  - Survey all the recent developments and true state-of-the-art
  - A hive of activity on many different aspects of EOB and source modelling in recent years
  - Provide detailed technical calculations in the space of ~ 1.5 hours [“left as an exercise of the reader”]

# Effective One Body

- Three main analytical components to an EOB model

# Effective One Body



- Three main analytical components to an EOB model

1. Hamiltonian to describe the *conservative* binary dynamics
2. Radiation reaction (RR) force to account for *loss of energy and angular momentum* via emission of GWs
3. Gravitational waveform for inspiral, merger, and ringdown

# Effective One Body



- Three main analytical components to an EOB model

1. Hamiltonian to describe the conservative binary dynamics
2. Radiation reaction (RR) force to account for loss of energy and angular momentum via emission of GWs
3. Gravitational waveform for inspiral, merger, and ringdown

Why?

- PN breaks down as an expansion in the strong-field regime
- Poor convergence near merger
- EOB resums PN information and maps to an effective problem with better strong field behaviour
- Model for merger and ringdown based on black hole perturbation theory

# Effective One Body



- Consider a test-particle orbiting a non-spinning BH of mass $M$

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

# Effective One Body



- Consider a test-particle orbiting a non-spinning BH of mass $M$

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- We can write down the Hamiltonian for a test-particle of mass $\mu$ orbiting the BH

$$H_{\mathrm{Schw}}(\boldsymbol{r}, \boldsymbol{p}) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[\mu^2 + \left(1 - \frac{2M}{r}\right) p_r^2 + \frac{p_\varphi^2}{r^2}\right]}$$

# Effective One Body

- Consider a test-particle orbiting a non-spinning BH of mass $M$

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- We can write down the Hamiltonian for a test-particle of mass $\mu$ orbiting the BH

$$H_{\text{Schw}}(\boldsymbol{r}, \boldsymbol{p}) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[\mu^2 + \left(1 - \frac{2M}{r}\right) p_r^2 + \frac{p_\varphi^2}{r^2}\right]}$$

- And the effective radial potential

$$\frac{V_{\text{eff}}^2(r)}{\mu^2} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{\mu^2 r^2}\right)$$

# Effective One Body

- Consider a test-particle orbiting a non-spinning BH of mass $M$

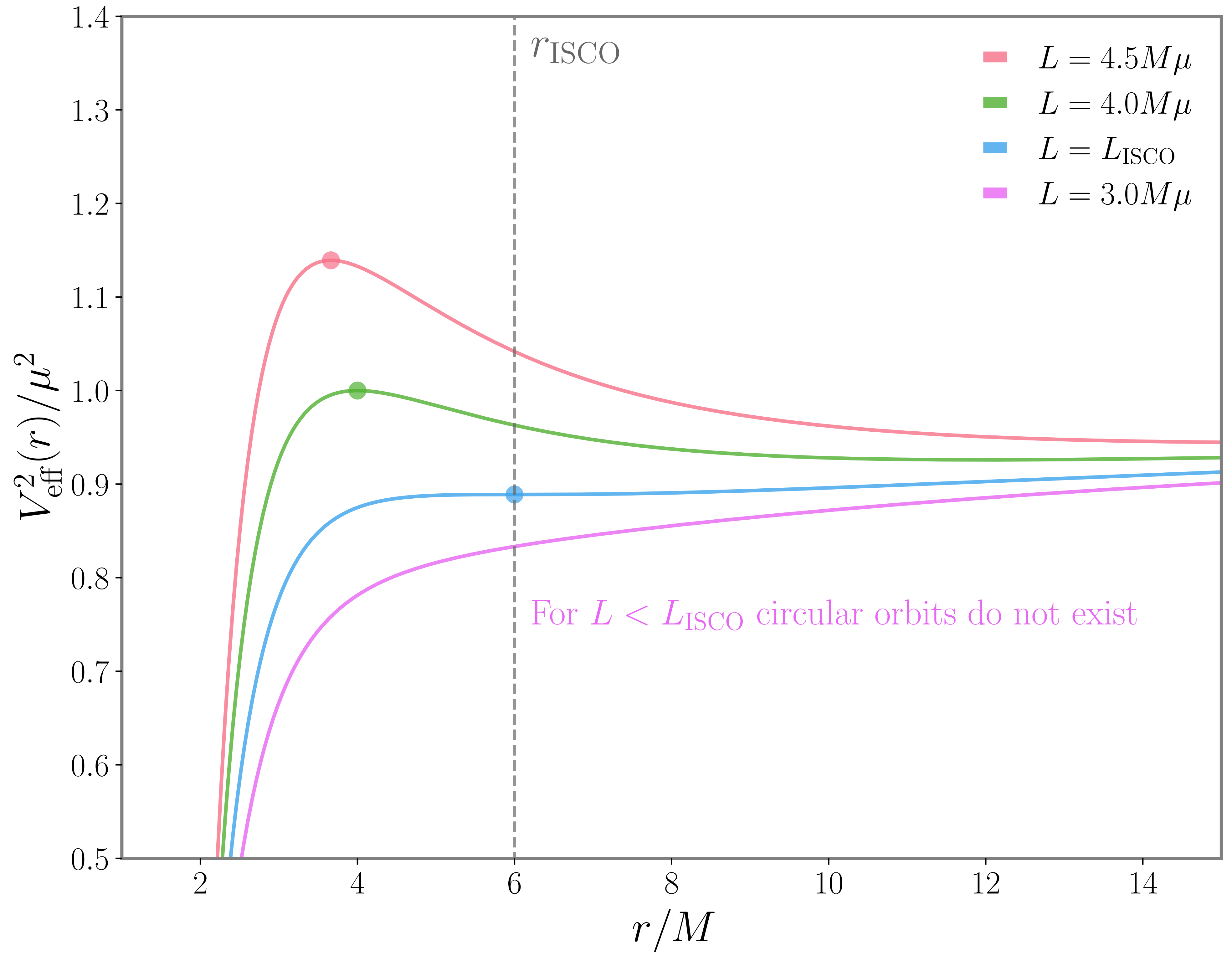
$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- We can write down the Hamiltonian for a test-particle of mass $\mu$ orbiting the BH

$$H_{\text{Schw}}(\boldsymbol{r}, \boldsymbol{p}) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[\mu^2 + \left(1 - \frac{2M}{r}\right) p_r^2 + \frac{p_\varphi^2}{r^2}\right]}$$

- And the effective radial potential

$$\frac{V_{\text{eff}}^2(r)}{\mu^2} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{\mu^2 r^2}\right)$$



# Effective One Body



- Let's work through an example of deriving the **conservative** EOB dynamics at 2PN

See Buonanno+99

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Let's work through an example of deriving the **conservative** EOB dynamics at 2PN

$$A^{\text{cons}} = A_0 + c^{-2}A_2 + c^{-4}A_4 + c^{-6}A_6$$

$$A^{\text{RR}} = c^{-5}A_5 + c^{-7}A_7$$

$$\frac{d^2 z_a^i}{dt^2} = A_a^{i\,\text{cons}} + A_a^{i\,\text{RR}}$$

time-asymmetric contributions from radiation reaction (beyond 2PN)

See Buonanno+99

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Let's work through an example of deriving the conservative EOB dynamics at 2PN

$$A^{\rm cons} = A_0 + c^{-2} A_2 + c^{-4} A_4 + c^{-6} A_6$$

$$A^{\rm RR} = c^{-5} A_5 + c^{-7} A_7$$

$$\frac{d^2 z_a^i}{dt^2} = \boxed{A_a^{i\,\rm cons}} + \boxed{A_a^{i\,\rm RR}}$$

time-asymmetric contributions from radiation reaction (beyond 2PN)

Equations of motion equivalently follow from a generalised Lagrangian in harmonic coordinates

$\Box x^\mu = 0$

[Damour & Deruelle]

$$L(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{a}_1, \boldsymbol{a}_2)$$

See Buonanno+99

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- Let's work through an example of deriving the conservative EOB dynamics at 2PN

$$A^{\text{cons}} = A_0 + c^{-2}A_2 + c^{-4}A_4 + c^{-6}A_6$$

$$A^{\text{RR}} = c^{-5}A_5 + c^{-7}A_7$$

$$\frac{d^2 z_a^i}{dt^2} = \boxed{A_a^{i\,\text{cons}}} + \boxed{A_a^{i\,\text{RR}}}$$

time-asymmetric contributions from radiation reaction (beyond 2PN)

Equations of motion equivalently follow from a generalised Lagrangian in harmonic coordinates

$\Box x^\mu = 0$

[Damour & Deruelle]

$$L(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \underline{\boldsymbol{a}_1, \boldsymbol{a}_2})$$

See Buonanno+99

UNIVERSITY OF BIRMINGHAM | THE ROYAL SOCIETY

- Let's work through an example of deriving the conservative EOB dynamics at 2PN

$$A^{\text{cons}} = A_0 + c^{-2}A_2 + c^{-4}A_4 + c^{-6}A_6$$

$$A^{\text{RR}} = c^{-5}A_5 + c^{-7}A_7$$

$$\frac{d^2 z_a^i}{dt^2} = \boxed{A_a^{i\,\text{cons}}} + \boxed{A_a^{i\,\text{RR}}}$$

time-asymmetric contributions from radiation reaction (beyond 2PN)

Equations of motion equivalently follow from a generalised Lagrangian in harmonic coordinates

$$\Box x^\mu = 0$$

[Damour & Deruelle]

$$L(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \underline{\boldsymbol{a}_1, \boldsymbol{a}_2})$$

Remove acceleration dependence through rewriting Lagrangian in ADM coordinates…

See Buonanno+99

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Let's work through an example of deriving the conservative EOB dynamics at 2PN

$$r_a^{\mathrm{ADM}}(t) = z_a^{\mathrm{HG}}(t) - \delta^* z_a(z, v)$$

$$\delta^* \mathbf{z}_1 = \frac{Gm_2}{c^4}\left\{\mathbf{n}\left[\frac{7}{8}v_2^2 - \frac{1}{8}(\mathbf{n}\cdot\mathbf{v}_2)^2\right] - \frac{7}{4}(\mathbf{n}\cdot\mathbf{v}_2)\,\mathbf{v}_2\right\} + \frac{1}{m_1}\frac{\partial F(\mathbf{z},\mathbf{v})}{\partial \mathbf{v}_1},$$

$$F = \sum \frac{Gm_1 m_2}{c^4}(\mathbf{n}\cdot\mathbf{v}_1)\left[-\frac{1}{4}v_2^2 + \frac{7}{4}\frac{Gm_1}{R} + \frac{1}{4}\frac{Gm_2}{R}\right]$$

See Buonanno+99

# Effective One Body



- Let's work through an example of deriving the conservative EOB dynamics at 2PN

$$r_a^{\mathrm{ADM}}(t) = z_a^{\mathrm{HG}}(t) - \delta^* z_a(z, v)$$

$$\delta^* \mathbf{z}_1 = \frac{Gm_2}{c^4}\left\{\mathbf{n}\left[\frac{7}{8}v_2^2 - \frac{1}{8}(\mathbf{n}\cdot\mathbf{v}_2)^2\right] - \frac{7}{4}(\mathbf{n}\cdot\mathbf{v}_2)\,\mathbf{v}_2\right\} + \frac{1}{m_1}\frac{\partial F(\mathbf{z},\mathbf{v})}{\partial \mathbf{v}_1},$$

$$F = \sum \frac{Gm_1m_2}{c^4}(\mathbf{n}\cdot\mathbf{v}_1)\left[-\frac{1}{4}v_2^2 + \frac{7}{4}\frac{Gm_1}{R} + \frac{1}{4}\frac{Gm_2}{R}\right]$$

But what is happening here… and why?

See Buonanno+99

# Effective One Body

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Transformation between gauges?

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad \varepsilon^{\mu}(\mathbf{x}, t) = \varepsilon^{\mu}[\mathbf{x}; \mathbf{y}_A(t), \mathbf{v}_A(t)]$$

G. Schaefer, Phys. Lett. A, 100, (1984)

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Transformation between gauges?

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad \varepsilon^{\mu}(\mathbf{x}, t) = \varepsilon^{\mu}[\mathbf{x}; \mathbf{y}_A(t), \mathbf{v}_A(t)]$$

Contact transformation induced by change of coordinates

$$\delta y_A^i(t) = \varepsilon^i(y_A, t) - \frac{v_A^i}{c}\,\varepsilon^0(y_A, t) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

G. Schaefer, Phys. Lett. A, 100, (1984)

# Effective One Body



- Transformation between gauges?

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad \varepsilon^{\mu}(\mathbf{x}, t) = \varepsilon^{\mu}[\mathbf{x}; \mathbf{y}_A(t), \mathbf{v}_A(t)]$$

Contact transformation induced by change of coordinates

$$\delta y_A^i(t) = \varepsilon^i(y_A, t) - \frac{v_A^i}{c}\,\varepsilon^0(y_A, t) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$L'[y_A, v_A, a_A, \cancel{b}_A] = L[y_A, v_A, a_A] + \cancel{\frac{dQ}{dt}} + \sum_A \frac{\delta L}{\delta y_A^i}\,\delta y_A^i + \mathcal{O}\left(\frac{1}{c^8}\right).$$

Adding total derivative to Lagrangian results in dynamically equivalent Lagrangian

G. Schaefer, Phys. Lett. A, 100, (1984)

- Transformation between gauges?

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad \varepsilon^{\mu}(\mathbf{x},t) = \varepsilon^{\mu}[\mathbf{x};\mathbf{y}_A(t),\mathbf{v}_A(t)]$$

Contact transformation induced by change of coordinates

$$\delta y_A^i(t) = \varepsilon^i(y_A,t) - \frac{v_A^i}{c}\,\varepsilon^0(y_A,t) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$L'[y_A,v_A,a_A,\cancel{b}_A] = L[y_A,v_A,a_A] + \cancel{\frac{dQ}{dt}} + \sum_A \frac{\delta L}{\delta y_A^i}\,\delta y_A^i + \mathcal{O}\left(\frac{1}{c^8}\right).$$

Adding total derivative to Lagrangian results in dynamically equivalent Lagrangian

Final Lagrangian is free from accelerations!

$$L'''[y_A,v_A] = L + \sum_A \frac{\delta L}{\delta y_A^i}\,\delta y_A^i + \frac{dF}{dt} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

$$L''' = L'' + \frac{dF}{dt}$$

G. Schaefer, Phys. Lett. A, 100, (1984), de Andrade+01

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Transformation between gauges?

Harmonic gauge generalised Lagrangian

$$L(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{a}_1, \boldsymbol{a}_2)$$

Perform contact transformation

$$r_a^{\mathrm{ADM}}(t) = z_a^{\mathrm{HG}}(t) - \delta^* z_a(z, v)$$

$$L^{\mathrm{ADM}}(q_1, q_2, \dot{q}_1, \dot{q}_2)$$

ADM Lagrangian is *ordinary*

$$\boldsymbol{p}_1 = -\boldsymbol{p}_2$$

(+ Legendre transformation + C.o.M frame)

$$H^{\mathrm{ADM}}(\boldsymbol{q}, \boldsymbol{p}) = \sum_A \boldsymbol{p}_A \cdot \boldsymbol{v}_A - L^{\mathrm{ADM}}(\boldsymbol{q}, \boldsymbol{v})$$

ADM Hamiltonian

See Buonanno+99

# Effective One Body



- Introduce reduced variables in ADM coordinates and the centre-of-mass frame

$$\boldsymbol{q} = \frac{\boldsymbol{Q}}{GM}, \quad \boldsymbol{p} = \frac{\boldsymbol{P}}{\mu}, \quad t = \frac{T}{GM}, \quad \hat{H} = \frac{H^{\mathrm{NR}}}{\mu} = \frac{H^{R} - Mc^2}{\mu}$$

Newtonian approximation ~ test particle of mass $\mu$ around external mass $GM$

$$\hat{H}(\boldsymbol{q},\boldsymbol{p}) = \hat{H}_0(\boldsymbol{q},\boldsymbol{p}) + \frac{1}{c^2}\hat{H}_2(\boldsymbol{q},\boldsymbol{p}) + \frac{1}{c^4}\hat{H}_4(\boldsymbol{q},\boldsymbol{p})$$

Invariance under time translations and spatial rotations leads to conserved quantities

$$\hat{H}^{\mathrm{NR}} = \hat{E}^{\mathrm{NR}} = \frac{E^{\mathrm{NR}}}{\mu}, \quad \boldsymbol{q} \times \boldsymbol{p} = \boldsymbol{j} = \frac{\boldsymbol{J}}{\mu GM}$$

See Buonanno+99

# Effective One Body

- With a PN Hamiltonian in place we can study classical dynamics of the system
- Introduce rescaled action

$$\hat{S} = \frac{S}{G\mu m}$$

- Which satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0$$

- Time-translation symmetry and rotational invariance isolate constants of motion

$$\partial_t S = -E \qquad \qquad \partial_\varphi S = \ell$$

See Buonanno+99

# Effective One Body

- This now becomes a standard problem in classical mechanics with a time-independent Hamiltonian

$$S(p,q,t) = W(q,p) - Et \qquad\qquad H\left(q, \partial_q W\right) = E$$

Hamilton's characteristic function

- Look for solutions using separation of variables (cyclic coordinates can always be separated)
- If motion is periodic in phase-space we can introduce an action variable

~ area enclosed by orbit in phase space

$$J = \frac{1}{2\pi}\oint q\, dq = \oint \frac{\partial W}{\partial q} dq = \text{const}$$

- Conjugate variable $\theta = \partial_J W$ evolves linearly at a characteristic frequency $\theta \propto \Omega\, t = \partial_J H\, t$

See Buonanno+99

# Effective One Body

$$\boldsymbol{q} = (r\cos\varphi, r\sin\varphi, 0)$$

$$\boldsymbol{p} = p_r^2 + \frac{j^2}{r^2}$$

- Back to EOB…
- Under planar motion we can decompose the rescaled action as

$$\hat{S} = \frac{S}{\mu G M} = -\hat{E}^{\mathrm{NR}}\hat{t} + j\varphi + \hat{S}_r(r, \hat{E}^{\mathrm{NR}}, j)$$

Solve perturbatively

$$\hat{S}_r(r, \hat{E}^{\mathrm{NR}}, j) = \int dr\ p_r(r, \hat{E}^{\mathrm{NR}}, j)$$

Introduce associated radial action variable

$$i_r(\hat{E}^{\mathrm{NR}}, j) = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} dr\ p_r\ (r, \hat{E}^{\mathrm{NR}}, j)$$

See Buonanno+99

# Effective One Body

- We can invert the (unscaled) radial action variable to find the energy perturbatively order-by-order

$$I_R(E^{\mathrm{NR}}, \mathcal{J}) = \alpha i_r(E^{\mathrm{NR}}/\mu, \mathcal{J}/\alpha) \qquad \alpha = \mu G M$$

$$E^{\mathrm{nr}}_{\mathrm{real}}(\mathcal{N}, \mathcal{J}) = -\frac{1}{2}\frac{\mu\alpha^2}{\mathcal{N}^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6}{\mathcal{N}\mathcal{J}} - \frac{15-\nu}{4\mathcal{N}^2}\right)\right.$$
$$\left. + \frac{\alpha^4}{c^4}\left(\frac{5(7-2\nu)}{2\mathcal{N}\mathcal{J}^3} + \frac{27}{\mathcal{N}^2\mathcal{J}^2} - \frac{3(35-4\nu)}{2\mathcal{N}^3\mathcal{J}} + \frac{145-15\nu+\nu^2}{8\mathcal{N}^4}\right)\right]$$

$$\mathcal{N} = I_R + \mathcal{J}$$

Delaunay variables where principle action ~ analogue to principal quantum number in hydrogen $\mathcal{N} = n\hbar$



See Buonanno+99

# Effective One Body



- We now have expressions for the real two-body problem at 2PN

See Buonanno+99

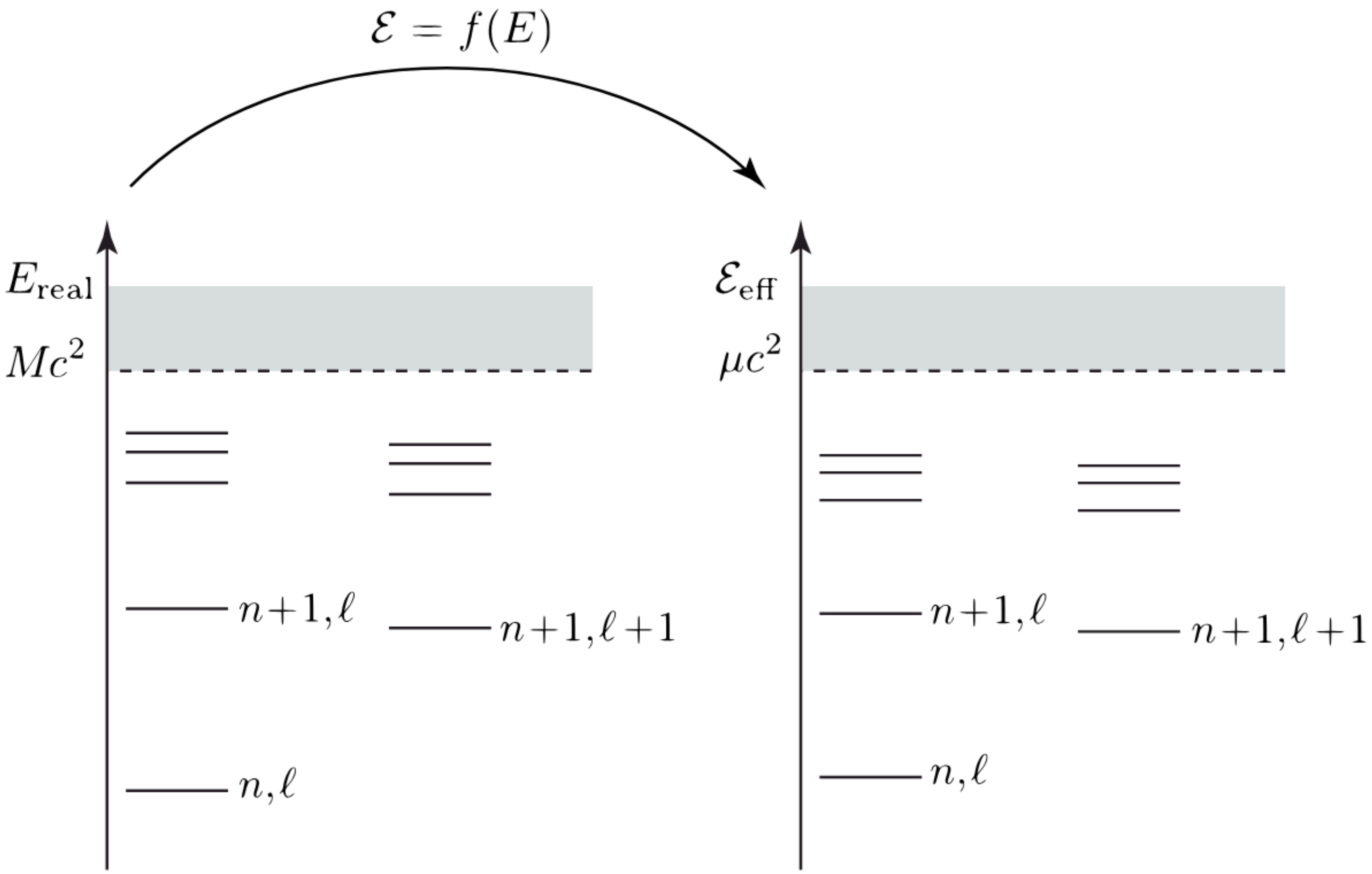
# Effective One Body

- We now have expressions for the real two-body problem at 2PN
- And we have the energy expressed in terms of Delaunay action variables

See Buonanno+99

# Effective One Body

- We now have expressions for the real two-body problem at 2PN
- And we have the energy expressed in terms of Delaunay action variables
- For EOB we need mapping (dictionary) between the real two-body dynamics and the EOB dynamics…?




$$E_{\rm real}(N,J) = Mc^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{C_1^{\rm real}}{NJ} + \frac{C_2^{\rm real}}{N^2}\right) + \cdots\right]$$

map from real to effective description

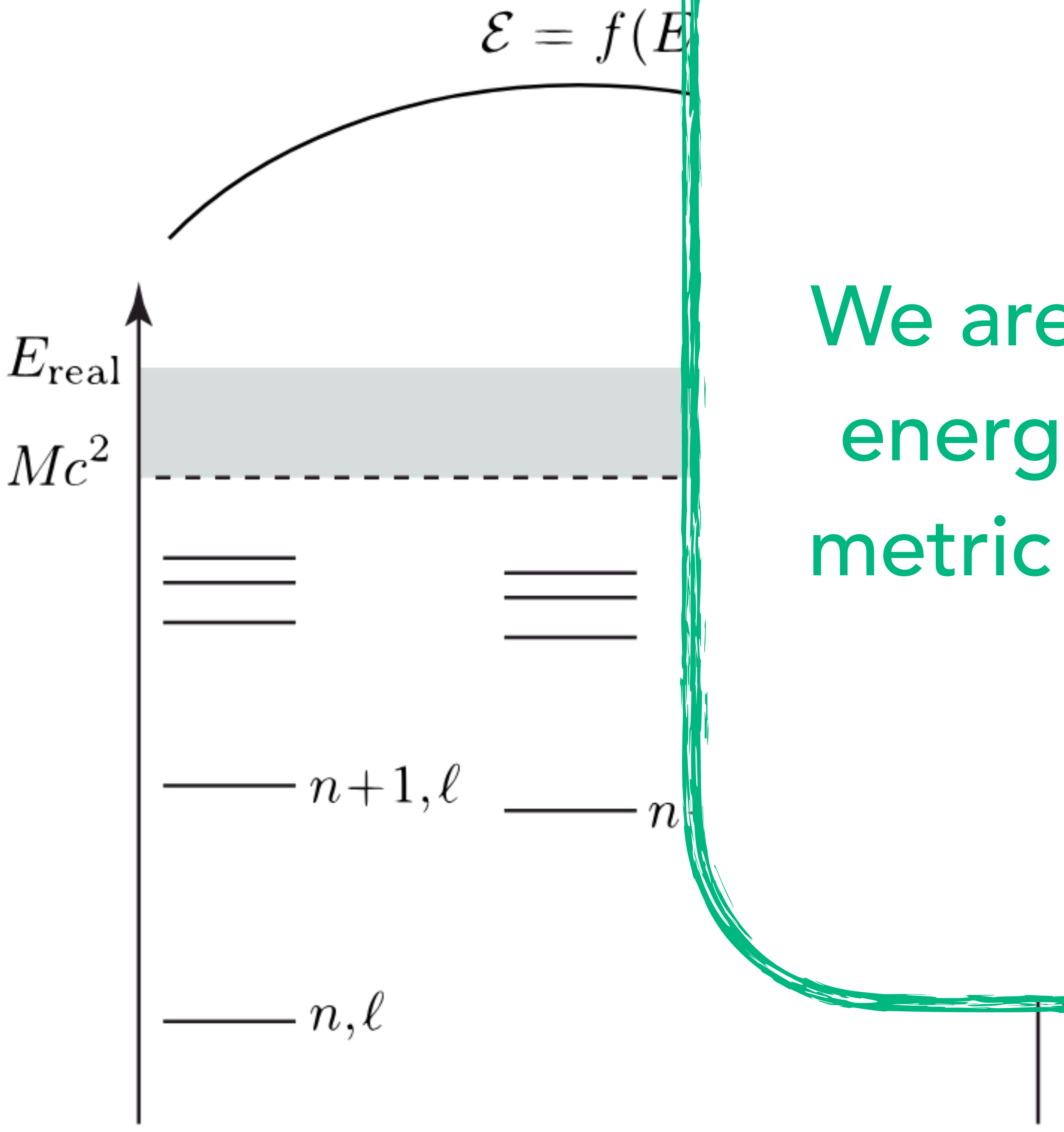
$$E_{\rm eff}(N,J) = \mu c^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{C_1^{\rm eff}}{NJ} + \frac{C_2^{\rm eff}}{N^2}\right) + \cdots\right]$$

See Buonanno+99

# Effective One Body



- We now have expressions for the real two-body problem at 2PN
- And we have the energy expressed in terms of Delaunay action variables
- For EOB we nee … OB dynamics…?




$\frac{C_1^{\rm real}}{NJ} + \frac{C_2^{\rm real}}{N^2}\Big) + \cdots\Big]$

…ctive description

$E_{\rm eff}(N, J) = \mu c^2 - \frac{\ }{2}\frac{\ }{N^2}\left[1 + \frac{\ }{c^2}\left(\frac{C_1^{\rm eff}}{NJ} + \frac{C_2^{\rm eff}}{N^2}\right) + \cdots\right]$

## Goal

We are going to look for an effective metric such that the energies of the bound states of a particle moving in this metric are in 1-1 correspondence with the energies of the real two-body bound state…

See Buonanno+99

# Effective One Body



- In EOB *map* the two-body dynamics to geodesic motion of a single particle in effective metric

$$S_{\text{tot}}\left[z_1^{\mu}, z_2^{\mu}, g_{\mu\nu}\right] = -\int m_1 ds_1 - \int m_2 ds_2 + S_{\text{field}}\left[g_{\mu\nu}(x)\right]$$

See Buonanno+99

# Effective One Body

- In EOB *map* the two-body dynamics to geodesic motion of a single particle in effective metric

$$S_{\rm tot}\left[z_1^\mu, z_2^\mu, g_{\mu\nu}\right] = -\int m_1 ds_1 - \int m_2 ds_2 + S_{\rm field}\left[g_{\mu\nu}(x)\right]$$

gauge-fixed Einstein-Hilbert action

$$ds_i = \sqrt{-g_{\mu\nu}(z_i^\lambda)dz_i^\mu dz_i^\nu}$$

Line element along trajectory of particle

See Buonanno+99

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- In EOB *map* the two-body dynamics to geodesic motion of a single particle in effective metric

$$S_{\rm tot}\left[z_1^\mu, z_2^\mu, g_{\mu\nu}\right] = -\int m_1 ds_1 - \int m_2 ds_2 + S_{\rm field}\left[g_{\mu\nu}(x)\right]$$

gauge-fixed Einstein-Hilbert action

$$ds_i = \sqrt{-g_{\mu\nu}(z_i^\lambda) dz_i^\mu dz_i^\nu}$$

Line element along trajectory of particle

- Associate real two-body dynamics to effective one-body dynamics in external spacetime

$$S_{\rm eff}\left[z_0^\mu\right] = -\int m_0\, ds_0 \qquad ds_0 = \sqrt{-g^{\rm eff}_{\mu\nu}(z_0^\lambda) dz_0^\mu dz_0^\nu}$$

Line element along trajectory of *effective* particle

deformed mass-shell condition
E.g. Damour 2001 arXiv:gr-qc/0103018

$$g^{\mu\nu}_{\rm eff} p_\mu p_\nu + \mu^2 + Q = 0 \quad \rightarrow \quad H_{\rm eff} = E_{\rm eff} = -p_0$$

See Buonanno+99

# Effective One Body

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- From effective Hamilton-Jacobi equations we play the same separation of variables game

$$S_{\rm eff} = -\mathcal{E}_{\rm eff} T_{\rm eff} + \mathcal{J}_{\rm eff}\varphi_{\rm eff} + S_{R_{\rm eff}}(R_{\rm eff}, \mathcal{E}_{\rm eff}, \mathcal{J}_{\rm eff})$$

$$\begin{aligned}\mathcal{E}^{\rm nr}_{\rm eff}(\mathcal{N}_{\rm eff}, \mathcal{J}_{\rm eff}) = -\frac{1}{2}\frac{\mu\alpha^2}{\mathcal{N}^2_{\rm eff}}\Bigg[1 + \frac{\alpha^2}{c^2}\left(\frac{2C_4}{\mathcal{N}_{\rm eff}\mathcal{J}_{\rm eff}} - \frac{C_2}{\mathcal{N}^2_{\rm eff}}\right) \\ + \frac{\alpha^4}{c^4}\left(\frac{2C_6}{\mathcal{N}_{\rm eff}\mathcal{J}^3_{\rm eff}} + \frac{3C_4^2}{\mathcal{N}^2_{\rm eff}\mathcal{J}^2_{\rm eff}} - \frac{4C_2C_4 + C_5}{\mathcal{N}^3_{\rm eff}\mathcal{J}_{\rm eff}} + \frac{5C_2^2 + 2C_3}{4\mathcal{N}^4_{\rm eff}}\right)\Bigg]\end{aligned}$$

See Buonanno+99

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- From effective Hamilton-Jacobi equations we play the same separation of variables game

$$S_{\rm eff} = -\mathcal{E}_{\rm eff} T_{\rm eff} + \mathcal{J}_{\rm eff}\varphi_{\rm eff} + S_{R_{\rm eff}}(R_{\rm eff}, \mathcal{E}_{\rm eff}, \mathcal{J}_{\rm eff})$$

$$\mathcal{E}^{\rm nr}_{\rm eff}(\mathcal{N}_{\rm eff}, \mathcal{J}_{\rm eff}) = -\frac{1}{2}\frac{\mu\alpha^2}{\mathcal{N}^2_{\rm eff}}\Bigg[1 + \frac{\alpha^2}{c^2}\left(\frac{2C_4}{\mathcal{N}_{\rm eff}\mathcal{J}_{\rm eff}} - \frac{C_2}{\mathcal{N}^2_{\rm eff}}\right) + \frac{\alpha^4}{c^4}\left(\frac{2C_6}{\mathcal{N}_{\rm eff}\mathcal{J}^3_{\rm eff}} + \frac{3C_4^2}{\mathcal{N}^2_{\rm eff}\mathcal{J}^2_{\rm eff}} - \frac{4C_2C_4 + C_5}{\mathcal{N}^3_{\rm eff}\mathcal{J}_{\rm eff}} + \frac{5C_2^2 + 2C_3}{4\mathcal{N}^4_{\rm eff}}\right)\Bigg]$$

$$\mathcal{N} = \mathcal{N}_{\rm eff}$$
$$\mathcal{J} = \mathcal{J}_{\rm eff}$$

We demand that the real and effective problem share a dictionary between energy levels

See Buonanno+99

# Effective One Body



- From effective Hamilton-Jacobi equations we play the same separation of variables game

$$S_{\rm eff} = -\mathcal{E}_{\rm eff} T_{\rm eff} + \mathcal{J}_{\rm eff}\varphi_{\rm eff} + S_{R_{\rm eff}}(R_{\rm eff}, \mathcal{E}_{\rm eff}, \mathcal{J}_{\rm eff})$$

$$\mathcal{E}^{\rm nr}_{\rm eff}(\mathcal{N}_{\rm eff}, \mathcal{J}_{\rm eff}) = -\frac{1}{2}\frac{\mu\alpha^2}{\mathcal{N}^2_{\rm eff}}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{2C_4}{\mathcal{N}_{\rm eff}\mathcal{J}_{\rm eff}} - \frac{C_2}{\mathcal{N}^2_{\rm eff}}\right)\right.$$
$$\left. + \frac{\alpha^4}{c^4}\left(\frac{2C_6}{\mathcal{N}_{\rm eff}\mathcal{J}^3_{\rm eff}} + \frac{3C_4^2}{\mathcal{N}^2_{\rm eff}\mathcal{J}^2_{\rm eff}} - \frac{4C_2C_4 + C_5}{\mathcal{N}^3_{\rm eff}\mathcal{J}_{\rm eff}} + \frac{5C_2^2 + 2C_3}{4\mathcal{N}^4_{\rm eff}}\right)\right]$$

$$\mathcal{N} = \mathcal{N}_{\rm eff}$$
$$\mathcal{J} = \mathcal{J}_{\rm eff}$$

$$\boxed{\frac{E^{\rm nr}_{\rm eff}}{\mu c^2} = \frac{E^{\rm nr}_{\rm real}}{\mu c^2}\left[1 + \sum_{n=1}^{+\infty}\alpha_n\left(\frac{E^{\rm nr}_{\rm real}}{\mu c^2}\right)^n\right]}$$

See Buonanno+99

# Effective One Body



$$\boxed{\frac{E_{\rm eff}^{\rm nr}}{\mu c^2} = \frac{E_{\rm real}^{\rm nr}}{\mu c^2}\left[1 + \sum_{n=1}^{+\infty} \alpha_n \left(\frac{E_{\rm real}^{\rm nr}}{\mu c^2}\right)^n\right]}$$

- How to solve for the free parameters? Impose physical constraints!
- Mass of the effective test particle should coincide with the reduced mass
- The linearised (i.e. one-graviton) effective metric should reduce to the Schwarzschild (or Kerr) metric in the test-mass limit $\nu \to 0$

See Buonanno+99

# Effective One Body



- Starting from the real nPN Hamiltonian

$$H_{\text{real}}^{\text{PN}} = H_{\text{Newt}} + H_{1\text{PN}} + H_{2\text{PN}} + \cdots$$

- Write down ansatz for relation between EOB Hamiltonian and effective Hamiltonian

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\nu} - 1\right)}$$

Not magical! Arises from inverting energy mapping. See Buonanno and Damour 1999

- $H_{\text{EOB}}$ related to $H_{\text{PN}}$ in a different gauge via canonical transformation

$$H_{\text{EOB}} = H_{\text{PN}} + \{\mathcal{G}, H_{\text{PN}}\} + \frac{1}{2!}\{\mathcal{G}, \{\mathcal{G}, H_{\text{PN}}\}\} + \cdots$$

Each bracket introduces factor of $c^{-2}$

- Find unknown coefficients in $H_{\text{eff}}$ by matching LHS to RHS

# Effective One Body



- Starting from the real nPN Hamiltonian

$$H^{\text{PN}}_{\text{real}} = H_{\text{Newt}} + H_{1\text{PN}} + H_{2\text{PN}} + \cdots$$

- Write down ansatz for relation between EOB Hamiltonian and effective Hamiltonian

$$\frac{\mathcal{E}^{\text{nr}}_{\text{eff}}}{\mu c^2} = \frac{\mathcal{E}^{\text{nr}}_{\text{real}}}{\mu c^2}\left(1 + \frac{\nu}{2}\frac{\mathcal{E}^{\text{nr}}_{\text{real}}}{\mu c^2}\right)$$

magical! Arises from
g energy mapping. See
nno and Damour 1999

- $H_{\text{EOB}}$ related to $H_{\text{PN}}$ in a

$$H_{\text{EOB}} = H_{\text{PN}} + \{\mathcal{G}, H_{\text{PN}}\} + \frac{1}{2!}\{\mathcal{G}, \{\mathcal{G}, H_{\text{PN}}\}\} + \cdots$$

Each bracket introduces factor of $c^{-2}$

- Find unknown coefficients in $H_{\text{eff}}$ by matching LHS to RHS

# Effective One Body



- Starting from the real nPN Hamiltonian

$$H^{\rm PN}_{\rm real} = H_{\rm Newt} + H_{\rm 1PN} + H_{\rm 2PN} + \cdots$$

- Write down ansatz for relation between EOB Hamiltonian and effective Hamiltonian

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\nu} - 1\right)}$$

Not magical! Arises from inverting energy mapping. See Buonanno and Damour 1999

- $H_{\rm EOB}$ related to $H_{\rm PN}$ in a different gauge via canonical transformation

$$H_{\rm EOB} = H_{\rm PN} + \{\mathcal{G}, H_{\rm PN}\} + \frac{1}{2!}\{\mathcal{G}, \{\mathcal{G}, H_{\rm PN}\}\} + \cdots$$

Each bracket introduces factor of $c^{-2}$

- Find unknown coefficients in $H_{\rm eff}$ by matching LHS to RHS

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H^{\nu}_{\text{eff}} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

# Effective One Body



- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\rm eff}^{\nu} = \mu \sqrt{A_{\nu}(r) \left[ \mu^2 + A_{\nu}(r) \bar{D}_{\nu}(r) p_r^2 + \frac{p_{\varphi}^2}{r^2} + Q_{\nu}(r, p_r) \right]}$$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\text{eff}}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\text{eff}}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

$$ds_{\text{eff}}^2 = -A_\nu(r)dt^2 + \frac{\bar{D}_\nu(r)}{A_\nu(r)}dr^2 + r^2 d\Omega^2$$

effective deformed metric

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\text{eff}}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

$$ds_{\text{eff}}^2 = -A_\nu(r)dt^2 + \frac{\bar{D}_\nu(r)}{A_\nu(r)}dr^2 + r^2 d\Omega^2$$

effective deformed metric

- The dynamics is encoded in the potentials $A_\nu$ and $D_\nu$

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\text{eff}}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r, p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

$$ds_{\text{eff}}^2 = -A_\nu(r)dt^2 + \frac{\bar{D}_\nu(r)}{A_\nu(r)}dr^2 + r^2 d\Omega^2$$

effective deformed metric

- The dynamics is encoded in the potentials $A_\nu$ and $D_\nu$

$$u = 1/r$$

$$A_{\text{non-spin}}^{\text{Taylor}}(u) = 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41\pi^2}{32}\right)u^4 + \left[\nu\left(\cdots\right) + \nu^2\left(\cdots\right) + \frac{64}{5}\nu\ln u\right]u^5 + \left[\nu a_6 + \cdots\right]u^6$$

E.g. Pompili+23 for SEOBNRv5

# Effective One Body

- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\rm eff}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

$$ds^2_{\rm eff} = -A_\nu(r)dt^2 + \frac{\bar{D}_\nu(r)}{A_\nu(r)}dr^2 + r^2 d\Omega^2$$

effective deformed metric

- The dynamics is encoded in the potentials $A_\nu$ and $D_\nu$

$$u = 1/r$$

$$A^{\rm Taylor}_{\rm non-spin}(u) = 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41\pi^2}{32}\right)u^4 + \left[\nu\left(\cdots\right) + \nu^2\left(\cdots\right) + \frac{64}{5}\nu\ln u\right]u^5 + \left[\nu a_6 + \cdots\right]u^6$$

$a_6$ is as-of-yet unknown PN coefficient that gets calibrated to NR

E.g. Pompili+23 for SEOBNRv5

# Effective One Body



- Effective Hamiltonian describes geodesic motion of test-particle in deformed external spacetime

$$H_{\rm eff}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r,p_r)\right]}$$

Differ from Schwarzschild due to PN corrections that depend on $\nu$

In non-spinning $\mu \to 0$ limit reduces to Hamiltonian of test-particle in Schwarzschild background

$$ds_{\rm eff}^2 = -A_\nu(r)dt^2 + \frac{\bar{D}_\nu(r)}{A_\nu(r)}dr^2 + r^2 d\Omega^2$$

effective deformed metric

- The dynamics is encoded in the potentials $A_\nu$ and $D_\nu$

$$u = 1/r$$

$$A^{\rm Taylor}_{\rm non-spin}(u) = 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41\pi^2}{32}\right)u^4 + \left[\nu\,(\cdots) + \nu^2\,(\cdots) + \frac{64}{5}\nu\ln u\right]u^5 + \left[\nu a_6 + \cdots\right]u^6$$

$$\bar{D}^{\rm Taylor}_{\rm non-spin}(u) = \cdots$$

$a_6$ is as-of-yet unknown PN coefficient that gets calibrated to NR

E.g. Pompili+23 for SEOBNRv5

# Effective One Body



- Brief aside on the structure of the EOB Hamiltonian…

$$H_{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\mathrm{eff}}}{\nu} - 1\right)} \qquad H_{\mathrm{eff}}^{\nu} = \mu\sqrt{A_\nu(r)\left[\mu^2 + A_\nu(r)\bar{D}_\nu(r)p_r^2 + \frac{p_\varphi^2}{r^2} + Q_\nu(r, p_r)\right]}$$

- EOB Hamiltonian has a double square-root structure → example of resummation
- Square roots contain infinite terms in their Taylor expansion…
- Odd that we started from a PN series and end up with an expression with seemingly more information?
- Yes… but also a well-known method in asymptotic series → must verify accuracy of resummation
- Another classic example is Padé approximant which we will introduce briefly…

# Effective One Body

- Quick detour… about 3PN
- Number of equations to solve establishing map between real and effective dynamics determined by number of combinations of scalars (and their powers) in Hamiltonian: $\boldsymbol{p}^2, \ (\boldsymbol{n}\cdot\boldsymbol{p}), \ 1/q$
- At 3PN we have 11 combinations… so 11 new equations to satisfy but only 10 free parameters?
- Proposed solution? Add higher-order terms to the action (i.e. stop assuming geodesic orbits)

$$S_{\mathrm{EOB}} = -\mu c \int ds_{\mathrm{eff}} \left[1 + Q^{(4)}_{\mu\nu\rho\sigma} u^\mu u^\nu u^\rho u^\sigma\right]$$

Leads to a deformed mass-shell condition

$$0 = \mu^2 c^2 + g^{\mu\nu}_{\mathrm{eff}}(x^\lambda_{\mathrm{eff}}) p^{\mathrm{eff}}_\mu p^{\mathrm{eff}}_\nu + Q^{\mu\nu\rho\sigma}_4 (x^\lambda_{\mathrm{eff}}) p^{\mathrm{eff}}_\mu p^{\mathrm{eff}}_\nu p^{\mathrm{eff}}_\rho p^{\mathrm{eff}}_\sigma + \mathcal{O}(p^6),$$

Constrained by test-mass limit, symmetry, and mapping to real PN dynamics

Damour/Jaranowski/Schäfer 2000

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc

# Effective One Body



- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc

We will come back to this later…

# Effective One Body



- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc
- Resummation of analytical information critical in reducing difference with NR information

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc
- Resummation of analytical information critical in reducing difference with NR information
- Padé resummation is an effective strategy

$$F(u) = P^1_n\left[F_{\rm PN}\right] = \frac{1 + n_1 u}{1 + d_1 u + \cdots + d_n u^n}$$

# Effective One Body

- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc
- Resummation of analytical information critical in reducing difference with NR information
- Padé resummation is an effective strategy

$$F(u) = P_n^1\left[F_{\rm PN}\right] = \frac{1 + n_1 u}{1 + d_1 u + \cdots + d_n u^n}$$

Damour & Nagar 09

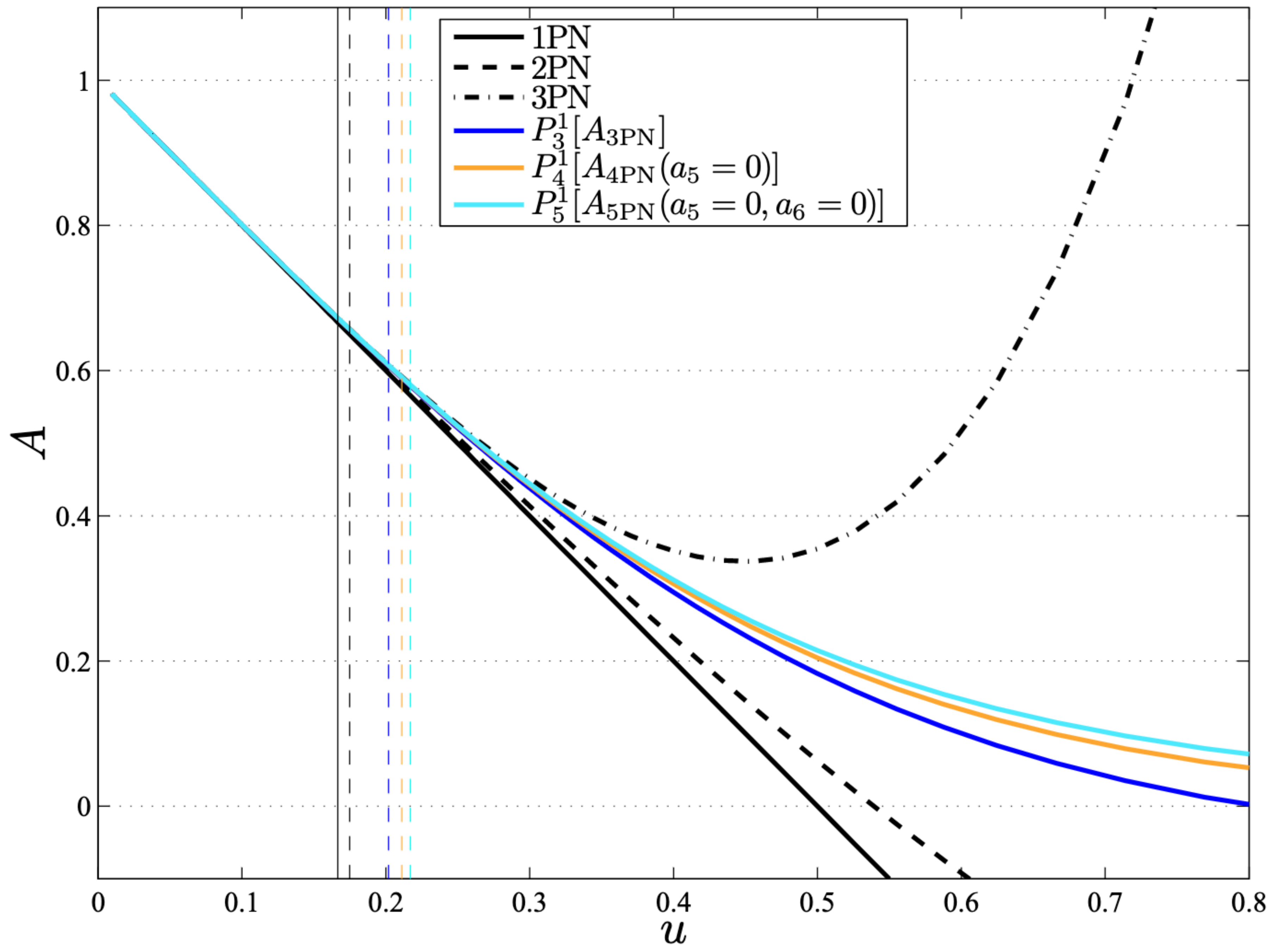



**Fig. 2** Various approximations and Padé resummation of the EOB radial potential $A(u)$, where $u = GM/(c^2R)$, for the equal-mass case $\nu = 1/4$. The vertical dashed lines indicate the corresponding (adiabatic) LSO location [2] defined by the condition $d^2\mathscr{E}^0_{\rm eff}/dR^2 = d\mathscr{E}^0_{\rm eff}/dR = 0$, where $\mathscr{E}^0_{\rm eff}$ is the effective energy along the sequence of circular orbits (*i.e.*, when $P_R^{\rm eff} = 0$).

# Effective One Body

UNIVERSITY OF BIRMINGHAM | THE ROYAL SOCIETY

- Significant freedom in structure of Hamiltonian, gauge, PN information included, resummation etc
- Resummation of analytical information critical in reducing difference with NR information
- Padé resummation is an effective strategy

$$F(u) = P_n^1\left[F_{\rm PN}\right] = \frac{1+n_1 u}{1+d_1 u+\cdots+d_n u^n}$$

- Example: resummation used in SEOBNRv5 (Pompili+23)

$$A_{\rm non-spin}(u) = P_5^1\left[A^{\rm Taylor}_{\rm non-spin}(u)\right]$$

$$\bar{D}_{\rm non-spin}(u) = P_3^2\left[\bar{D}^{\rm Taylor}_{\rm non-spin}(u)\right]$$

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Can generalise the Hamiltonian to include spin

# Effective One Body

- Can generalise the Hamiltonian to include spin

$$H_{\text{eff}} = \sqrt{A(r)\left[\mu^2 c^2 + A(r)\bar{D}(r)\frac{p_r^2}{c^2} + \frac{L^2}{c^2 r^2} + Q(r, p_r)\right]}$$

$$+ \frac{1}{c^3 r^3}\boldsymbol{L} \cdot \left[g_S(r, p_r)\boldsymbol{S} + g_{S^*}(r, p_r)\boldsymbol{S}^*\right].$$

# Effective One Body

- Can generalise the Hamiltonian to include spin

$$H_{\text{eff}} = \sqrt{A(r)\left[\mu^2 c^2 + A(r)\bar{D}(r)\frac{p_r^2}{c^2} + \frac{L^2}{c^2 r^2} + Q(r, p_r)\right]}$$
$$+ \frac{1}{c^3 r^3}\boldsymbol{L}\cdot\left[g_S(r, p_r)\boldsymbol{S} + g_{S^*}(r, p_r)\boldsymbol{S}^*\right].$$

inclusion of spin effects is not unique
(e.g. gauge choices)

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Can generalise the Hamiltonian to include spin

$$H_{\text{eff}} = \sqrt{A(r)\left[\mu^2c^2 + A(r)\bar{D}(r)\frac{p_r^2}{c^2} + \frac{L^2}{c^2r^2} + Q(r,p_r)\right]}$$
$$+ \frac{1}{c^3r^3}\boldsymbol{L}\cdot\left[g_S(r,p_r)\boldsymbol{S} + g_{S^*}(r,p_r)\boldsymbol{S}^*\right].$$

inclusion of spin effects is not unique (e.g. gauge choices)

- Include PN SO information through gyro-gravitomagnetic terms

# Effective One Body

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Can generalise the Hamiltonian to include spin

$$H_{\text{eff}} = \sqrt{A(r)\left[\mu^2c^2 + A(r)\bar{D}(r)\frac{p_r^2}{c^2} + \frac{L^2}{c^2r^2} + Q(r,p_r)\right]}$$
$$+ \frac{1}{c^3r^3}\boldsymbol{L}\cdot\left[g_S(r,p_r)\boldsymbol{S} + g_{S^*}(r,p_r)\boldsymbol{S}^*\right].$$

inclusion of spin effects is not unique
(e.g. gauge choices)

- Include PN SO information through gyro-gravitomagnetic terms

$$g_S = 2 - \frac{1}{c^2}\left[\frac{27\nu}{16}\frac{p_r^2}{\mu^2} + \frac{5\nu M}{16r}\right] + \cdots + \frac{1}{c^8}\left(g_S^{5.5\text{PN,loc}} + g_S^{5.5\text{PN,nonloc}}\right),$$
$$g_{S^*} = \frac{3}{2} - \frac{1}{c^2}\left[\left(\frac{3\nu}{2} + \frac{5}{4}\right)\frac{p_r^2}{\mu^2} + \left(\frac{3}{4} + \frac{\nu}{2}\right)\frac{M}{r}\right] + \cdots + \frac{1}{c^8}\left(g_{S^*}^{5.5\text{PN,loc}} + g_{S^*}^{5.5\text{PN,nonloc}}\right)$$

Damour 00, Barausse 11, Nagar 11, Khalil 23 + many others

# Effective One Body



- Can also generalise the Hamiltonian to arbitrary spins

E.g. Balmelli+15, Khalil+20, Khalil+23

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Can also generalise the Hamiltonian to arbitrary spins

$$ds^2 = g^{\mu\nu}_{\rm Kerr}\partial_\mu\partial_\nu$$

$$ds^2 = -\frac{\Lambda}{\Delta\Sigma}\partial_t^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\phi^2 - \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\phi$$

E.g. Balmelli+15, Khalil+20, Khalil+23

# Effective One Body

- Can also generalise the Hamiltonian to arbitrary spins

$$ds^2 = g^{\mu\nu}_{\mathrm{Kerr}} \partial_\mu \partial_\nu$$

$$ds^2 = -\frac{\Lambda}{\Delta\Sigma}\partial_t^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\phi^2 - \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\phi$$

$$\downarrow$$

$$H^{\mathrm{Kerr}} = \frac{2Mr}{\Lambda}\boldsymbol{L}\cdot\boldsymbol{a} + \left[A^{\mathrm{Kerr}}\left(\mu^2 + B^{\mathrm{Kerr}}_{\mathrm{np}}(\boldsymbol{n}\cdot\boldsymbol{p})^2 + B^{\mathrm{Kerr}}_{p}\boldsymbol{p}^2 + B^{\mathrm{Kerr}}_{\mathrm{npa}}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a})^2\right)\right]^{1/2}$$

E.g. Balmelli+15, Khalil+20, Khalil+23

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Can also generalise the Hamiltonian to arbitrary spins

$$ds^2 = g^{\mu\nu}_{\mathrm{Kerr}} \partial_\mu \partial_\nu$$

$$ds^2 = -\frac{\Lambda}{\Delta\Sigma}\partial_t^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\phi^2 - \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\phi$$

$$\downarrow$$

$$H^{\mathrm{Kerr}} = \frac{2Mr}{\Lambda}\boldsymbol{L}\cdot\boldsymbol{a} + \left[A^{\mathrm{Kerr}}\left(\mu^2 + B^{\mathrm{Kerr}}_{\mathrm{np}}(\boldsymbol{n}\cdot\boldsymbol{p})^2 + B^{\mathrm{Kerr}}_{p}\boldsymbol{p}^2 + B^{\mathrm{Kerr}}_{\mathrm{npa}}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a})^2\right)\right]^{1/2}$$

$$\downarrow$$

$$H^{\mathrm{prec}}_{\mathrm{eff}} = \frac{Mr}{\Lambda}\left[\boldsymbol{L}\cdot\left(g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta\boldsymbol{a}_-\right) + \mathrm{SO}_{\mathrm{calib}} + G^{\mathrm{prec}}_{a^3}\right]$$
$$+ \left[A^{\mathrm{prec}}\left(\mu^2 + B^{\mathrm{prec}}_{p}\boldsymbol{p}^2 + B^{\mathrm{prec}}_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2\right.\right.$$
$$\left.\left. + B^{\mathrm{Kerr}}_{npa}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a}_+)^2 + Q^{\mathrm{prec}}\right)\right]^{1/2}$$

Map to effective metric of deformed Kerr background with $\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2$

E.g. Balmelli+15, Khalil+20, Khalil+23

- Can also generalise the Hamiltonian to arbitrary spins

$$ds^2 = g^{\mu\nu}_{\rm Kerr}\partial_\mu\partial_\nu$$

$$ds^2 = -\frac{\Lambda}{\Delta\Sigma}\partial_t^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\phi^2 - \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\phi$$

$$\downarrow$$

$$H^{\rm Kerr} = \frac{2Mr}{\Lambda}\boldsymbol{L}\cdot\boldsymbol{a} + \left[A^{\rm Kerr}\left(\mu^2 + B^{\rm Kerr}_{\rm np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 + B^{\rm Kerr}_p\boldsymbol{p}^2 + B^{\rm Kerr}_{\rm npa}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a})^2\right)\right]^{1/2}$$

$$\downarrow$$

$$\begin{aligned} H^{\rm prec}_{\rm eff} = \frac{Mr}{\Lambda}\Big[\boldsymbol{L}\cdot\left(g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta\boldsymbol{a}_-\right) + {\rm SO}_{\rm calib} + G^{\rm prec}_{a^3}\Big] \\ + \Big[A^{\rm prec}\Big(\mu^2 + B^{\rm prec}_p\boldsymbol{p}^2 + B^{\rm prec}_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 \\ + B^{\rm Kerr}_{npa}\left(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a}_+\right)^2 + Q^{\rm prec}\Big)\Big]^{1/2} \end{aligned}$$

Now things are getting messy…

E.g. Balmelli+15, Khalil+20, Khalil+23

# Effective One Body



- Great, so we have a Hamiltonian…

# Effective One Body



- Great, so we have a Hamiltonian… now what?

- Great, so we have a Hamiltonian… now what?
- The Hamilton-Jacobi equations provide the equations of motion…

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{p}}\,, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}\,, \qquad \frac{d\boldsymbol{S}_{1,2}}{dt} = \frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2},$$

# Effective One Body

- Great, so we have a Hamiltonian… now what?
- The Hamilton-Jacobi equations provide the equations of motion…

radiation reaction force

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{p}}, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}, \qquad \frac{d\boldsymbol{S}_{1,2}}{dt} = \frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2},$$

# Effective One Body

- Great, so we have a Hamiltonian… now what?
- The Hamilton-Jacobi equations provide the equations of motion…

radiation reaction force

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H_{\rm EOB}}{\partial \boldsymbol{p}}\,, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H_{\rm EOB}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}\,, \qquad \frac{d\boldsymbol{S}_{1,2}}{dt} = \frac{\partial H_{\rm EOB}}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2},$$

impose that loss of the mechanical
system matches GW fluxes at infinity

$$\dot{E}_{\rm system} + \dot{E}_{\rm Schott} + \Phi_E = 0$$
$$\dot{J}_{\rm system} + \dot{J}_{\rm Schott} + \Phi_J = 0$$

Schott terms ~ near-zone non-radiative terms due to system's
interaction with its own radiation field

- Great, so we have a Hamiltonian… now what?
- The Hamilton-Jacobi equations provide the equations of motion…

radiation reaction force

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H_{\rm EOB}}{\partial \boldsymbol{p}}\,, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H_{\rm EOB}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}\,, \qquad \frac{d\boldsymbol{S}_{1,2}}{dt} = \frac{\partial H_{\rm EOB}}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2},$$

- The RR force ↔ flux of angular momentum expressed as sum of factorised and resummed multipoles

$$\mathcal{F}_\varphi = -\frac{M\Omega}{8\pi} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=1}^{\ell} m^2 |d_L\, h^F_{\ell m}|^2$$

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F}\, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)}\, S_{\mathrm{eff}}^{(\epsilon)}\, T_{\ell m}\, e^{i\delta_{\ell m}}\, (\rho_{\ell m})^{\ell}\, h_{\ell m}^{\mathrm{NQC}}$$

# Effective One Body

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

Newtonian contribution, $\epsilon$ denotes parity of mode

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$ non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

Effective source term: $$\hat{S}_{\mathrm{eff}} = \begin{cases} \dfrac{E_{\mathrm{eff}}(v_\Omega)}{\mu}, & \ell + m \text{ even}, \\ v_\Omega \dfrac{p_\phi(v_\Omega)}{M\mu}, & \ell + m \text{ odd}, \end{cases}$$

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$ non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

Resums infinite number of leading logarithms: $T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{k})}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2i\hat{k}\ln(2m\Omega r_0)}$

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

# Effective One Body

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

Residual phase correction due to sub-leading logarithms

# Effective One Body



- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

Residual amplitude correction such that expansion agrees with PN-expanded modes

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

Phenomenological NQC corrections  to shape waveform during late-plunge up to merger

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

- Complete EOB waveform by attaching a merger-ringdown

$$h_{\ell m}(t) = h_{\ell m}^{\mathrm{IM}}(t) \, \Theta\left(t_{\mathrm{match}}^{\ell m} - t\right) + h_{\ell m}^{\mathrm{R}}(t) \, \Theta\left(t - t_{\mathrm{match}}^{\ell m}\right) ,$$

- The inspiral-plunge EOB modes can be written as

factorised modes

$$h_{\ell m}^{\mathrm{IM}} = h_{\ell m}^{F} \, N_{\ell m}$$

non quasi-circular (NQC) correction

- Factorisation of each multipole, e.g. Damour+07, Damour+08, Pan+10

$$h_{\ell m}^{F}(t) = h_{\ell m}^{(N,\epsilon)} \, S_{\mathrm{eff}}^{(\epsilon)} \, T_{\ell m} \, e^{i\delta_{\ell m}} \, (\rho_{\ell m})^{\ell} \, h_{\ell m}^{\mathrm{NQC}}$$

- Complete EOB waveform by attaching a merger-ringdown

$$h_{\ell m}^{\mathrm{R}}(t) \sim \sum_{n=0}^{\infty} A_{\ell m n} e^{-i\sigma_{\ell m n} t}$$

$$h_{\ell m}(t) = h_{\ell m}^{\mathrm{IM}}(t)\,\Theta\left(t_{\mathrm{match}}^{\ell m} - t\right) + h_{\ell m}^{\mathrm{R}}(t)\,\Theta\left(t - t_{\mathrm{match}}^{\ell m}\right) \,,$$

Ringdown well described via BH perturbation theory as superposition of QNM

# Effective One Body:

- Summary: EOB is constructed from a number of key ingredients
  - Hamiltonian encoding the conservative dynamics
  - Radiation reaction force to account for loss of energy and angular momentum via emission of GWs
  - Waveform modes that describe the inspiral, merger, and bringdown

$$H_{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\mathrm{eff}}}{\nu} - 1\right)}$$

$$\mathcal{F} = \frac{\Omega}{16\pi}\frac{p}{L}\sum_{\ell m} m^2 |h_{\ell m}|^2$$

Solve Hamilton-Jacobi Equations

$$\dot{\boldsymbol{r}} = \frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{p}}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H_{\mathrm{EOB}}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}$$

$$h^{\mathrm{IM}}_{\ell m} = h^{F}_{\ell m} N_{\ell m}$$

$$h^{F}_{\ell m}(t) = h^{(N,\epsilon)}_{\ell m}\, S^{(\epsilon)}_{\mathrm{eff}}\, T_{\ell m}\, e^{i\delta_{\ell m}}\, (\rho_{\ell m})^{\ell}\, h^{\mathrm{NQC}}_{\ell m}$$

$$h_{\ell m}(t) = h^{\mathrm{IM}}_{\ell m}(t)\,\Theta\left(t^{\ell m}_{\mathrm{match}} - t\right) + h^{\mathrm{R}}_{\ell m}(t)\,\Theta\left(t - t^{\ell m}_{\mathrm{match}}\right)$$



Adapted from slides by: Buonanno, Pompili

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS
  - Pompili+23 https://arxiv.org/abs/2303.18039
  - Khalil+23 https://arxiv.org/abs/2303.18143
  - Ramos-Buades+23 https://arxiv.org/abs/2303.18046
  - Maarten+23 https://arxiv.org/abs/2303.18026

  - Damour+14 https://arxiv.org/abs/1406.6913
  - Nagar+18 https://arxiv.org/abs/1806.01772
  - Nagar+21 https://arxiv.org/abs/2108.02043
  - Nagar+23 https://arxiv.org/abs/2304.09662

Some starter references (but highly incomplete and out-of-date with latest developments)

# Effective One Body

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\mathrm{noS}}$ and $\bar{D}_{\mathrm{noS}}$, 5.5PN in $Q_{\mathrm{noS}}$ | 4PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ | 4PN with partial 5PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ |
| $A_{\mathrm{noS}}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\mathrm{noS}}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\mathrm{noS}} \equiv 1/\bar{D}_{\mathrm{noS}}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | **SEOBNRv5** | **SEOBNRv4** [99, 100, 107, 111] | **TEOBResumS** [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\mathrm{noS}}$ and $\bar{D}_{\mathrm{noS}}$, 5.5PN in $Q_{\mathrm{noS}}$ | 4PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ | 4PN with partial 5PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ |
| $A_{\mathrm{noS}}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\mathrm{noS}}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\mathrm{noS}} \equiv 1/\bar{D}_{\mathrm{noS}}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

# Effective One Body

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23



Different analytical information included in the various potentials

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log re-summation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor ex-panded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for cir-cular orbits) |

Khalil+23

Same resummation of A-potential

# Effective One Body

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of SEOBNRv4 and TEOBResumS.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\mathrm{noS}}$ and $\bar{D}_{\mathrm{noS}}$, 5.5PN in $Q_{\mathrm{noS}}$ | 4PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ | 4PN with partial 5PN in $A_{\mathrm{noS}}$, 3PN in $\bar{D}_{\mathrm{noS}}$ and $Q_{\mathrm{noS}}$ |
| $A_{\mathrm{noS}}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\mathrm{noS}}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\mathrm{noS}} \equiv 1/\bar{D}_{\mathrm{noS}}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

Different resummation of D-potential

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of SEOBNRv4 and TEOBResumS.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

Different limits of Hamiltonian in test-particle limit

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23



Different gauge choices and resummation for spin-orbit interactions

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of SEOBNRv4 and TEOBResumS.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

Differing analytical spin information

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the **SEOBNRv5** Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of **SEOBNRv4** and **TEOBResumS**.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log re-summation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

SEOBNRv5 has incorporates precession into the Hamiltonian

# Effective One Body

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Two main flavours of EOB are SEOBNR and TEOBResumS

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian derived here, which builds on the results of Refs. [103, 104], compared to that of SEOBNRv4 and TEOBResumS.

| | SEOBNRv5 | SEOBNRv4 [99, 100, 107, 111] | TEOBResumS [102, 112, 113] |
|---|---|---|---|
| nonspinning part | 4PN with partial 5PN in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$, 5.5PN in $Q_{\rm noS}$ | 4PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ | 4PN with partial 5PN in $A_{\rm noS}$, 3PN in $\bar{D}_{\rm noS}$ and $Q_{\rm noS}$ |
| $A_{\rm noS}$ resummation | (1,5) Padé | horizon factorization and log resummation | (1,5) Padé |
| $\bar{D}_{\rm noS}$ resummation | (2,3) Padé | log | Taylor expanded ($D_{\rm noS} \equiv 1/\bar{D}_{\rm noS}$ is inverse-Taylor resummed) |
| Hamiltonian in the $\nu \to 0$ limit | reduces to Kerr Hamiltonian for a *test mass* in a generic orbit | reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit | the $A$ potential reduces to Kerr, but not the full Hamiltonian |
| spin-orbit part | 3.5PN, in $(r, L^2)$ gauge, Taylor expanded | 3.5PN, added in the spin map | 3.5PN, in $(r, p_r^2)$ gauge, inverse-Taylor resummed |
| higher-order spin information | NNLO SS (4PN), LO $S^3$ (3.5PN), LO $S^4$ (4PN) | LO SS (2PN) | NNLO SS (4PN) for circular orbits |
| precessing-spin Hamiltonian | yes | yes | no |
| spin-multipole constants included | yes | no | yes (in the SS contributions for circular orbits) |

Khalil+23

Again, differing analytical information in multipole moments

# Effective One Body: Precession Information?

- Partial-precession effects in Hamiltonian

$$H_{\text{eff}}^{\text{pprec}} = \frac{M p_\phi \boldsymbol{l} \cdot (g_{a_+} \boldsymbol{a}_+ + g_{a_-} \delta \boldsymbol{a}_-) + \text{SO}_{\text{calib}} + \left\langle G_{a^3}^{\text{pprec}} \right\rangle}{r^3 + a_+^2 (r + 2M)} + \left[ A^{\text{pprec}} \left( \mu^2 + B_p^{\text{pprec}} \frac{p_\phi^2}{r^2} + \left(1 + B_{np}^{\text{pprec}}\right) (\boldsymbol{n} \cdot \boldsymbol{p})^2 \right. \right.$$

# Effective One Body: Precession Information?

- Partial-precession effects in Hamiltonian

$$H_{\rm eff}^{\rm pprec} = \frac{M p_\phi \boldsymbol{l}\cdot(g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta \boldsymbol{a}_-) + \mathrm{SO}_{\rm calib} + \langle G_{a^3}^{\rm pprec}\rangle}{r^3 + a_+^2(r+2M)} + \left[A^{\rm pprec}\left(\mu^2 + B_p^{\rm pprec}\frac{p_\phi^2}{r^2} + \left(1+B_{np}^{\rm pprec}\right)(\boldsymbol{n}\cdot\boldsymbol{p})^2\right.\right.$$

(Khalil+2023)

full-precessing-spin (prec) Hamiltonian
$H_{\rm EOB}^{\rm prec}(\mathbf{r}, \mathbf{p}, \mathbf{a}_+, \mathbf{a}_-)$

orbit-averaging in-plane spins for circular orbits

simplifying PN resummation, requiring that $H_{\rm EOB}^{\rm pprec}$ reduces to $H_{\rm EOB}^{\rm prec}$ when PN expanded through $\mathcal{O}(a_\pm^3)$

(Khalil+2023)

partial-precessing-spin (pprec) Hamiltonian
$H_{\rm EOB}^{\rm pprec}(r, p_r, \mathbf{l}^2, \mathbf{a}_\pm\cdot\mathbf{l}_{\rm N}, \mathbf{a}_\pm\cdot\mathbf{l}, \mathbf{a}_\pm^2, \mathbf{a}_+\cdot\mathbf{a}_-)$

$\mathbf{r}, \mathbf{p} \rightarrow$ canonical variables in center of mass

$\mathbf{a}_{1,2} \rightarrow$ body's spins $\quad \mathbf{a}_\pm = \mathbf{a}_1 \pm \mathbf{a}_2$

$\nu \rightarrow$ symmetric mass ratio

$\mathbf{l} \rightarrow$ orbital angular momentum

$\mathbf{l}_{\rm N} \rightarrow$ Newtonian angular momentum

PN-expanded EOB equations for spins, angular momentum, velocity parameter $\mathrm{v} = (M\omega)^{1/3}$

$$\dot{\mathbf{a}}_\pm = \mathbf{\Omega}_{\mathbf{a}_\pm} \times \mathbf{a}_\pm$$

$$\mathbf{l} = \mathbf{l}(\mathbf{l}_{\rm N}, \mathrm{v}, \mathbf{a}_\pm)$$

$$\dot{\mathbf{l}}_{\rm N} = \dot{\mathbf{l}}_{\rm N}(\mathbf{l}_{\rm N}, \mathrm{v}, \mathbf{a}_\pm)$$

$$\dot{\mathrm{v}} = \left[\frac{\dot{E}(\mathrm{v})}{dE_{\rm EOB}(\mathrm{v})/d\mathrm{v}}\right]_{\text{PN expanded}}$$

EOB orbital equations in co-precessing (P) frame

$$\dot{r} = \frac{\partial H_{\rm EOB}^{\rm pprec}}{\partial p_r} \qquad \omega \equiv \dot{\phi} = \frac{\partial H_{\rm EOB}^{\rm pprec}}{\partial p_\phi}$$

$$\dot{p}_r = -\frac{\partial H_{\rm EOB}^{\rm pprec}}{\partial r} + \mathcal{F}_r \qquad \dot{p}_\phi = \mathcal{F}_\phi$$

Co-precessing frame EOB waveform multipoles
$$h_{\ell m}^{\rm P}(t) = h_{\ell m}^{\rm P}(r, \mathrm{v}, \mathbf{a}_\pm\cdot\mathbf{l}_{\rm N})$$

Follow map outlined in Schmidt+11

Inertial-frame EOB waveform multipoles
$$h_{\ell m}^{\rm I}(t) = \sum_{m',m''} \mathbf{R}_{mm'}^{\rm JI}\,\mathbf{R}_{m'm''}^{\rm PJ}\,h_{\ell m''}^{\rm P}(t)$$

Adapted from slides by Antoni Ramos-Buades

# Effective One Body: Examples of GSF?

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Calibration to 2nd order gravitational-self-force (GSF) flux

# Effective One Body: Examples of GSF?

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Calibration to 2nd order gravitational-self-force (GSF) flux

- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

# Effective One Body: Examples of GSF?

- Calibration to 2nd order gravitational-self-force (GSF) flux
- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[ \epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega) \right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

Schwarzschild metric

# Effective One Body: Examples of GSF?

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Calibration to 2nd order gravitational-self-force (GSF) flux
- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

metric amplitudes

# Effective One Body: Examples of GSF?

- Calibration to 2nd order gravitational-self-force (GSF) flux

- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

- Flux from $(\ell, m)$ modes of $\epsilon h^{1,m}_{\alpha\beta} + \epsilon^2 h^{2,m}_{\alpha,\beta}$ at $\mathscr{I}^+$

$$\mathcal{F}^{\mathrm{GSF}\epsilon}_{\ell m}(\epsilon, y) = \epsilon^2 \mathcal{F}^{\mathrm{GSF1}\epsilon}_{\ell m}(y) + \epsilon^3 \mathcal{F}^{\mathrm{GSF2}\epsilon}_{\ell m}(y) + \mathcal{O}(\epsilon^4)$$

# Effective One Body: Examples of GSF?

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Calibration to 2nd order gravitational-self-force (GSF) flux
- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

- Flux from $(\ell, m)$ modes of $\epsilon h^{1,m}_{\alpha\beta} + \epsilon^2 h^{2,m}_{\alpha,\beta}$ at $\mathscr{I}^+$

$$\mathcal{F}^{\mathrm{GSF}\epsilon}_{\ell m}(\epsilon, y) = \epsilon^2 \mathcal{F}^{\mathrm{GSF}1\epsilon}_{\ell m}(y) + \epsilon^3 \mathcal{F}^{\mathrm{GSF}2\epsilon}_{\ell m}(y) + \mathcal{O}(\epsilon^4)$$

- Want to compare to EOB flux

$$\mathcal{F}^{\mathrm{EOB}}_{\ell m} = d_L^2 \frac{(mM\Omega)^2}{8\pi} \left|h^{\mathrm{N}}_{\ell m}\right|^2 \left|\hat{S}_{\ell m}\right|^2 \left|T_{\ell m}\right|^2 \left|\rho_{\ell m}\right|^{2\ell}$$

# Effective One Body: Examples of GSF?



- Calibration to 2nd order gravitational-self-force (GSF) flux

- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

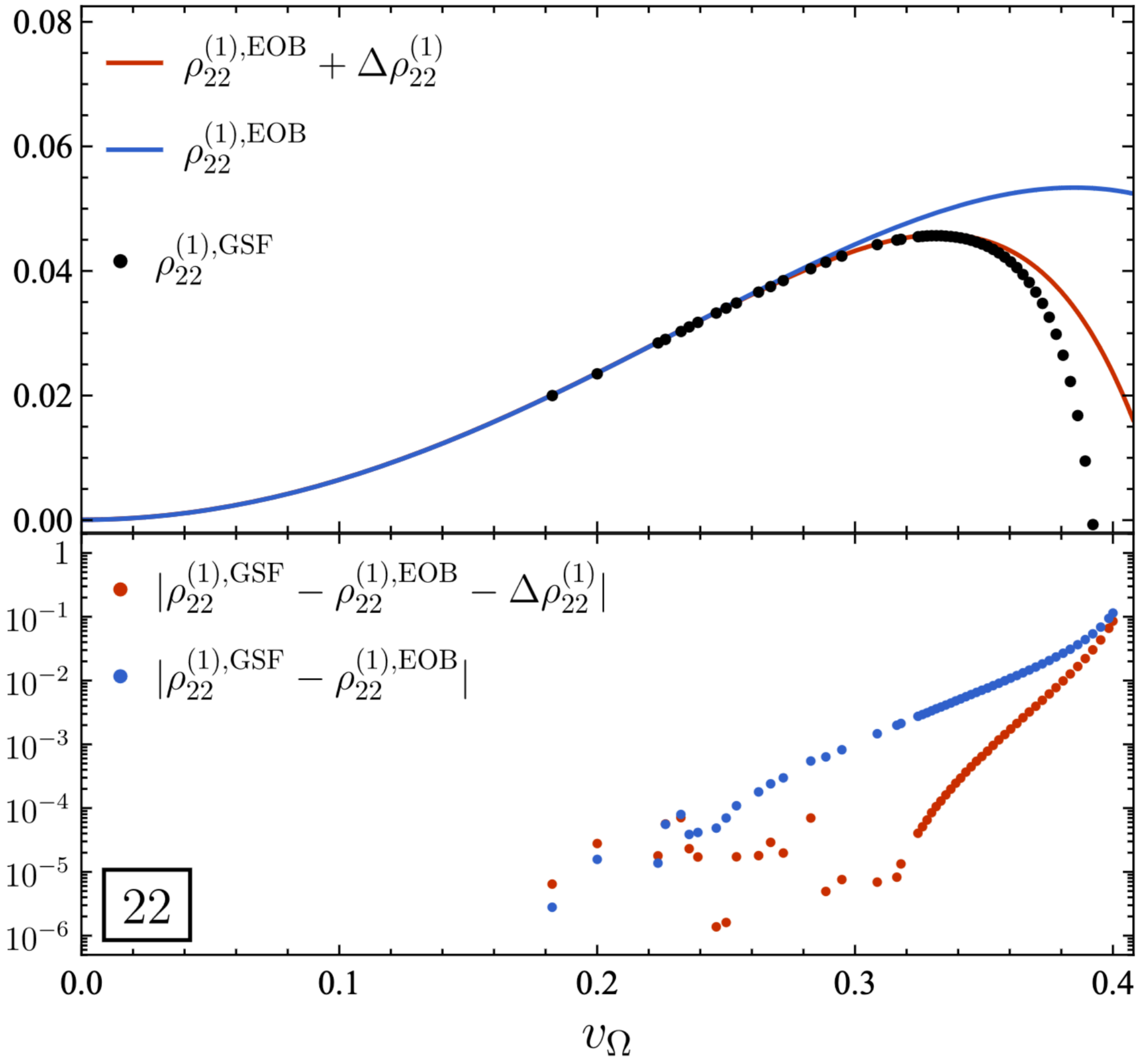
- Flux from $(\ell, m)$ modes of $\epsilon h^{1,m}_{\alpha\beta} + \epsilon^2 h^{2,m}_{\alpha,\beta}$ at $\mathscr{I}^+$

$$\mathcal{F}^{\mathrm{GSF}\epsilon}_{\ell m}(\epsilon, y) = \epsilon^2 \mathcal{F}^{\mathrm{GSF}1\epsilon}_{\ell m}(y) + \epsilon^3 \mathcal{F}^{\mathrm{GSF}2\epsilon}_{\ell m}(y) + \mathcal{O}(\epsilon^4)$$

- Want to compare to EOB flux

$$\mathcal{F}^{\mathrm{EOB}}_{\ell m} = d_L^2 \frac{(mM\Omega)^2}{8\pi} \left|h^{\mathrm{N}}_{\ell m}\right|^2 \left|\hat{S}_{\ell m}\right|^2 \left|T_{\ell m}\right|^2 \left|\rho_{\ell m}\right|^{2\ell}$$

Maarten+23



# Effective One Body: Examples of GSF?



Maarten+23

- Calibration to 2nd order gravitational-self-force (GSF) flux
- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

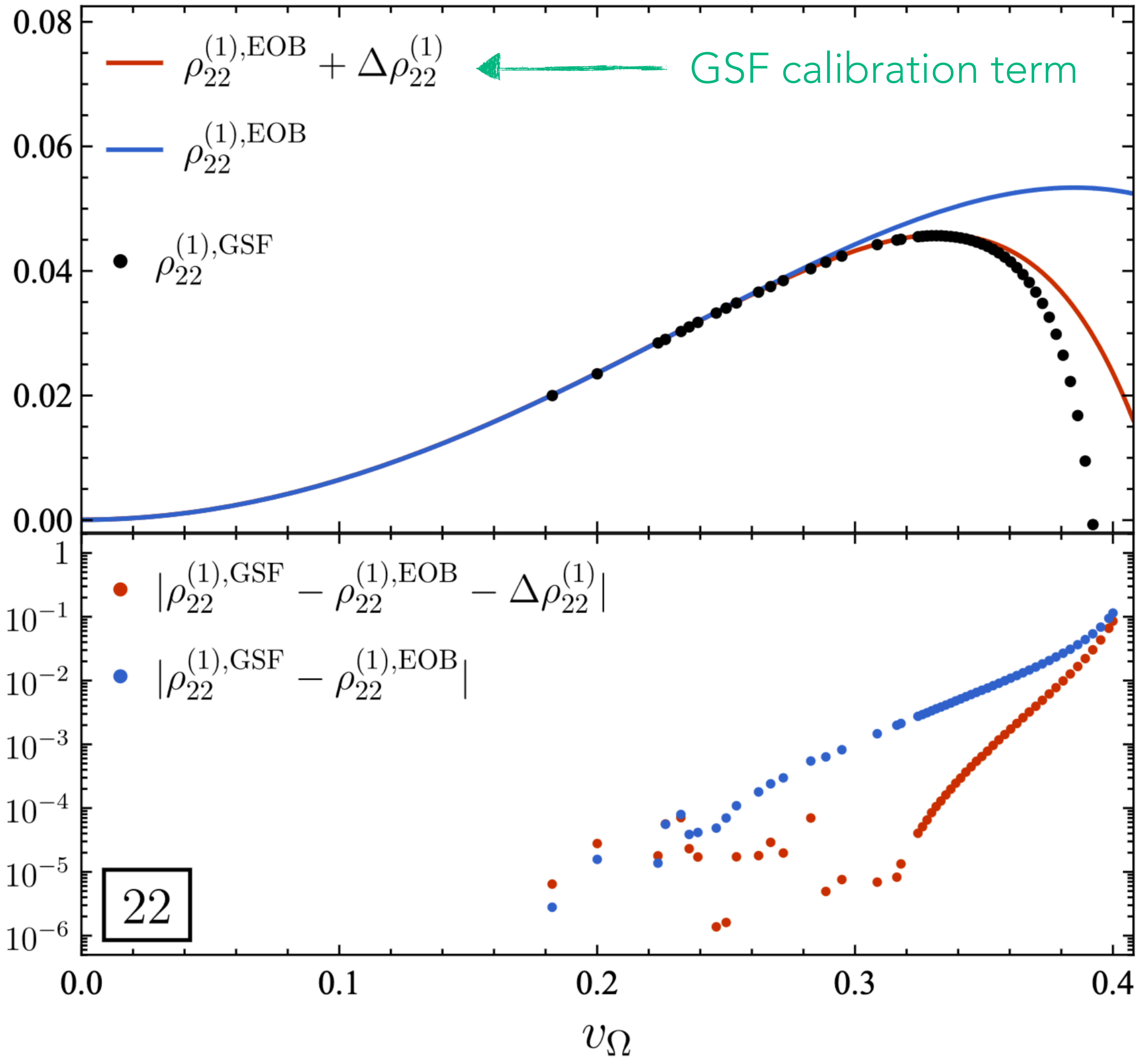
$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

- Flux from $(\ell, m)$ modes of $\epsilon h^{1,m}_{\alpha\beta} + \epsilon^2 h^{2,m}_{\alpha,\beta}$ at $\mathscr{I}^+$

$$\mathcal{F}^{\mathrm{GSF}\epsilon}_{\ell m}(\epsilon, y) = \epsilon^2 \mathcal{F}^{\mathrm{GSF}1\epsilon}_{\ell m}(y) + \epsilon^3 \mathcal{F}^{\mathrm{GSF}2\epsilon}_{\ell m}(y) + \mathcal{O}(\epsilon^4)$$

- Want to compare to EOB flux

$$\mathcal{F}^{\mathrm{EOB}}_{\ell m} = d_L^2 \frac{(mM\Omega)^2}{8\pi} \left|h^{\mathrm{N}}_{\ell m}\right|^2 \left|\hat{S}_{\ell m}\right|^2 \left|T_{\ell m}\right|^2 \left|\rho_{\ell m}\right|^{2\ell}$$



# Effective One Body: Examples of GSF?

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Calibration to 2nd order gravitational-self-force (GSF) flux

Maarten+23

- GSF models metric of binary as expansion in $\epsilon = m_2/m_1$

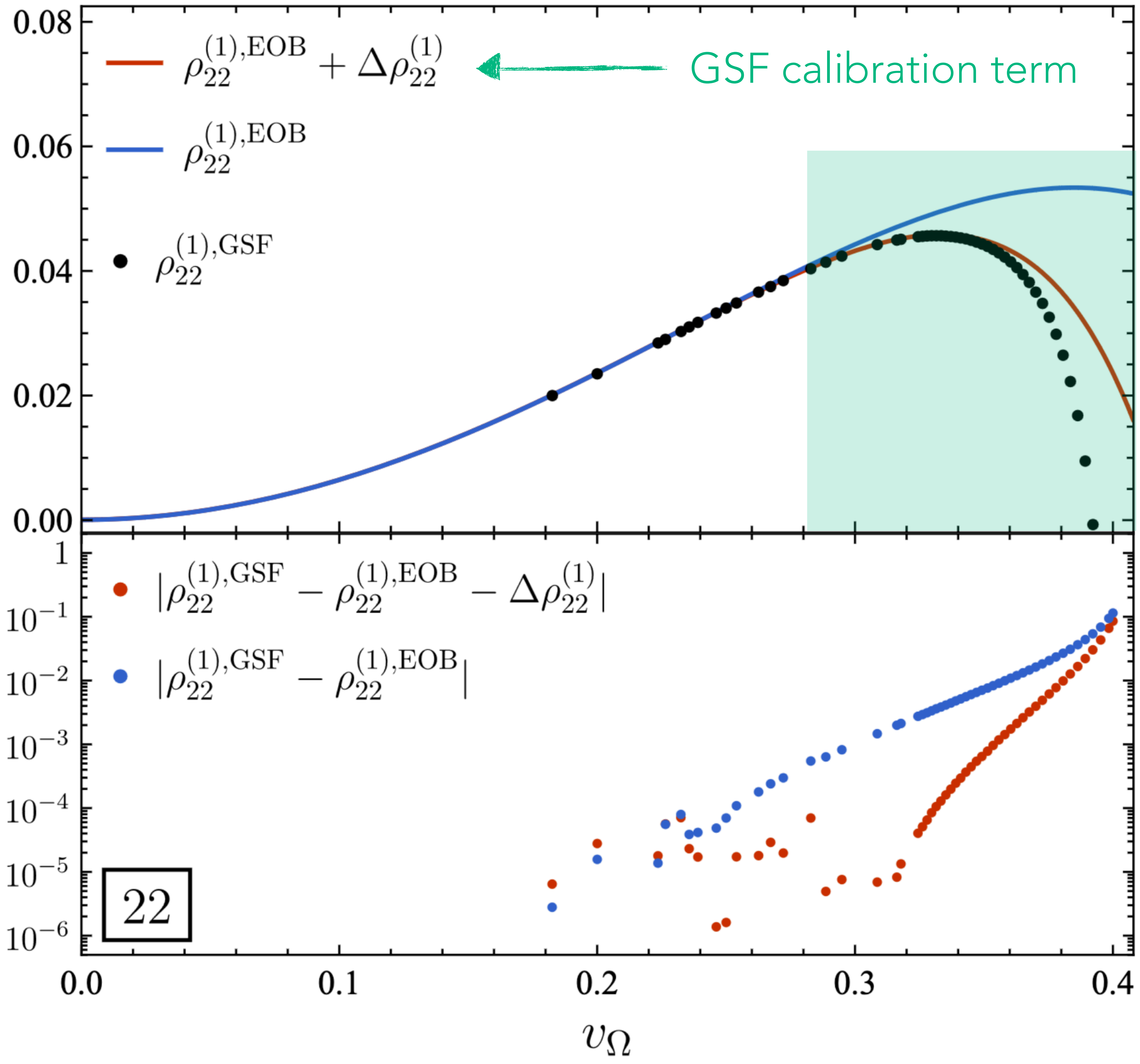
$$g_{\alpha\beta} + \sum_{m=\infty}^{\infty} \left[\epsilon h^{1,m}_{\alpha\beta}(\Omega) + \epsilon^2 h^{2,m}_{\alpha\beta}(\Omega)\right] e^{-im\varphi} + \mathcal{O}(\epsilon^3)$$

- Flux from $(\ell, m)$ modes of $\epsilon h^{1,m}_{\alpha\beta} + \epsilon^2 h^{2,m}_{\alpha,\beta}$ at $\mathscr{I}^+$

$$\mathcal{F}^{\mathrm{GSF}\epsilon}_{\ell m}(\epsilon, y) = \epsilon^2 \mathcal{F}^{\mathrm{GSF}1\epsilon}_{\ell m}(y) + \epsilon^3 \mathcal{F}^{\mathrm{GSF}2\epsilon}_{\ell m}(y) + \mathcal{O}(\epsilon^4)$$

- Want to compare to EOB flux

$$\mathcal{F}^{\mathrm{EOB}}_{\ell m} = d_L^2 \frac{(mM\Omega)^2}{8\pi} \left|h^{\mathrm{N}}_{\ell m}\right|^2 \left|\hat{S}_{\ell m}\right|^2 \left|T_{\ell m}\right|^2 \left|\rho_{\ell m}\right|^{2\ell}$$



# EOB and Scattering?

# Effective One Body: Scattering

- Historical motivation for EOB based on a few key ingredients:
  - PN description of real two-body dynamics
  - Dictionary between PN-expanded knowledge of two-body bound states and bound states of test particle moving in external metric

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Historical motivation for EOB based on a few key ingredients:
  - PN description of real two-body dynamics
  - Dictionary between PN-expanded knowledge of two-body bound states and bound states of test particle moving in external metric
- Analytical approximations traditionally carried out within post-Newtonian framework
  - Assumes weak field $[GM/(rc^2)] \ll 1$ and small velocities $[v/c] \ll 1$
- Recent interest in revisiting the post-Minkowski framework [Damour+16, Damour+17]
  - Only assumes weak fields $[GM/(rc^2)] \ll 1$ with causal constraints on velocity

# Effective One Body: Scattering

## High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour
Phys. Rev. D **97**, 044038 – Published 26 February 2018

Article | References | Citing Articles (208) | PDF | HTML | Export Citation

### ABSTRACT

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.



https://arxiv.org/abs/1710.10599

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

## High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour

Phys. Rev. D **97**, 044038 – Published 26 February 2018

Article | References | Citing Articles (208) | PDF | HTML | Export Citation

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in

(possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

# Effective One Body: Scattering

## High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour
Phys. Rev. D **97**, 044038 – Published 26 February 2018

Article | References | Citing Articles (208) | PDF | HTML | Export Citation

### ABSTRACT

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a

of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.



https://arxiv.org/abs/1710.10599

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- Dictionary for scattering states?
- For bound-orbits we had $I_R$, $I_\varphi$ but this no longer holds in the scattering case…
- For bound orbits the radial motion oscillates between periapsis and apoapsis but test particle returns to its starting point
- For scattering states the radial motion is *unbounded* so the orbit is *open* → phase space trajectory is an open curve not a closed loop
- The angular action variable remains periodic and well-defined

$$I_\varphi \equiv \frac{1}{2\pi} \oint P_\varphi d\varphi = P_\varphi \equiv J$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Replacement for radial half of the bound-state dictionary?
- Use Hamilton-Jacobi theory [Damour 2016]

$$S_0(R, \varphi; \mathcal{E}, J) = J\varphi + \int dR\, P_R(R; \mathcal{E}, J)$$

Variable conjugate to J

$$\frac{\partial S_0}{\partial J} = \varphi + \int dR \frac{\partial P_R}{\partial J} \longrightarrow \varphi = -\int dR \frac{\partial P_R(R; \mathcal{E}, J)}{\partial J} + \text{const.}$$

Which is just the angle ~ functional of the radial trajectory

See also Kälin+20

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Replacement for radial half of the bound-state dictionary?
- Use Hamilton-Jacobi theory [Damour 2016]

$$\Phi_{\text{bound}} = -\oint dR \frac{\partial P_R(R; \mathcal{E}, J)}{\partial J} = -2\pi \frac{\partial I_R(\mathcal{E}, J)}{\partial J}$$

$$\longmapsto \quad \mathcal{E}_{\text{eff}}(\chi, J) = f\left[\mathcal{E}_{\text{real}}(\chi, J)\right]$$

$$\chi = -\int_{-\infty}^{+\infty} dR \frac{\partial P_R(R; \mathcal{E}, J)}{\partial J} - \pi$$

- Let's build an EOB Hamiltonian in PM gravity

$$\hat{H}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{B(r)} + Q(r, p_r, p_\phi)\right)}$$

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Let's build an EOB Hamiltonian in PM gravity

$$\hat{H}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{B(r)} + Q(r, p_r, p_\phi)\right)}$$

Choice of Hamiltonian defines our deformed mass-shell condition

$$1 + g^{\mu\nu}_{\text{eff}} p_\mu p_\nu + Q(r, \gamma, \nu) = 0$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Let's build an EOB Hamiltonian in PM gravity

$$\hat{H}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{B(r)} + Q(r, p_r, p_\phi)\right)}$$

$$1 + g_{\text{eff}}^{\mu\nu} p_\mu p_\nu + Q(r, \gamma, \nu) = 0$$

Solve in terms of $p_r$

$$p_r^2 = \underbrace{(\gamma^2 - 1)}_{p_\infty^2} - \frac{p_\varphi^2}{r^2} + w(r, p_\varphi, \gamma, \nu)$$

- Let's build an EOB Hamiltonian in PM gravity

$$\hat{H}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{B(r)} + Q(r, p_r, p_\phi)\right)}$$

$$1 + g_{\text{eff}}^{\mu\nu} p_\mu p_\nu + Q(r, \gamma, \nu) = 0$$

$$p_r^2 = (\gamma^2 - 1) - \frac{p_\varphi^2}{r^2} + \underline{w(r, p_\varphi, \gamma, \nu)}$$

$$w(\bar{u}, p_\infty) = \gamma^2 \left(\frac{B(\bar{u})}{\bar{A}(\bar{u})} - 1\right) - (\bar{B}(\bar{u}) - 1) - \bar{B}(\bar{u})\widehat{Q}^E(\bar{u}, \gamma)$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Let's build an EOB Hamiltonian in PM gravity

$$\hat{H}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{B(r)} + Q(r, p_r, p_\phi)\right)}$$

$$1 + g_{\text{eff}}^{\mu\nu} p_\mu p_\nu + Q(r, \gamma, \nu) = 0$$

$$p_r^2 = (\gamma^2 - 1) - \frac{p_\varphi^2}{r^2} + \underline{w(r, p_\varphi, \gamma, \nu)}$$

$$w(\bar{r}, \gamma) = \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \frac{w_3(\gamma)}{\bar{r}^3} + \frac{w_4(\gamma)}{\bar{r}^4} + O\left[\frac{1}{\bar{r}^5}\right].$$

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\rm eff}, J) = -\int_{-\infty}^{+\infty} d\bar{R}\,\frac{\partial P_R(\bar{R};\, \mathcal{E}_{\rm eff}, J)}{\partial J}$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\text{eff}}, J) = -\int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R};\, \mathcal{E}_{\text{eff}}, J)}{\partial J}$$

$$\frac{\pi}{2} + \frac{1}{2}\chi(\gamma, j) = +\int_0^{u_{\max}(\gamma, j)} \frac{j\, d\bar{u}}{\sqrt{p_\infty^2 + w(\bar{u}, p_\infty) - j^2 \bar{u}^2}}$$

# Effective One Body: Scattering

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\text{eff}}, J) = -\int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R}; \mathcal{E}_{\text{eff}}, J)}{\partial J}$$

$$\frac{\pi}{2} + \frac{1}{2}\chi(\gamma, j) = +\int_0^{u_{\max}(\gamma,j)} \frac{j\, d\bar{u}}{\sqrt{p_\infty^2 + w(\bar{u}, p_\infty) - j^2\bar{u}^2}}$$

Schematically
- PM-expand $w$ and integrand
- Take finite part of integral
- Solve order by order

$$w(\bar{r}, \gamma) = \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \frac{w_3(\gamma)}{\bar{r}^3} + \frac{w_4(\gamma)}{\bar{r}^4} + O\left[\frac{1}{\bar{r}^5}\right].$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\rm eff}, J) = -\int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R}; \mathcal{E}_{\rm eff}, J)}{\partial J}$$

$$\chi_1(\gamma) = \frac{1}{2}\frac{w_1(\gamma)}{p_\infty}$$

$$\chi_2(\gamma) = \frac{\pi}{4} w_2(\gamma)$$

$$\chi_3(\gamma) = -\frac{1}{24}\left[\frac{w_1(\gamma)}{p_\infty}\right]^3 + \frac{1}{2}\frac{w_1(\gamma) w_2(\gamma)}{p_\infty} + p_\infty w_3(\gamma)$$

$$\chi_4(\gamma) = \frac{3\pi}{8}\left[\frac{1}{2} w_2^2(\gamma) + w_1(\gamma) w_3(\gamma) + p_\infty^2 w_4(\gamma)\right]$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\rm eff}, J) = -\int_{-\infty}^{+\infty} d\bar{R}\,\frac{\partial P_R(\bar{R};\, \mathcal{E}_{\rm eff}, J)}{\partial J}$$

Potentials are fixed by PM-expanded analytical scattering angles

$$\chi_{n\rm PM}(\gamma, j) \equiv \sum_{i=1}^{n} 2\frac{\chi_i(\gamma)}{j^i}$$

$$\chi_1(\gamma) = \frac{1}{2}\frac{w_1(\gamma)}{p_\infty}$$

$$\chi_2(\gamma) = \frac{\pi}{4}w_2(\gamma)$$

$$\chi_3(\gamma) = -\frac{1}{24}\left[\frac{w_1(\gamma)}{p_\infty}\right]^3 + \frac{1}{2}\frac{w_1(\gamma)w_2(\gamma)}{p_\infty} + p_\infty w_3(\gamma)$$

$$\chi_4(\gamma) = \frac{3\pi}{8}\left[\frac{1}{2}w_2^2(\gamma) + w_1(\gamma)w_3(\gamma) + p_\infty^2 w_4(\gamma)\right]$$

# Effective One Body: Scattering

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\rm eff}, J) = - \int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R}; \mathcal{E}_{\rm eff}, J)}{\partial J}$$

Potentials are fixed by PM-expanded analytical scattering angles

$$\chi_{n\rm PM}(\gamma, j) \equiv \sum_{i=1}^{n} 2 \frac{\chi_i(\gamma)}{j^i}$$

$$\chi_1(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}$$

$$\chi_2(\gamma) = \frac{3\pi}{8} \frac{(5\gamma^2 - 1)}{h(\gamma; \nu)}$$

$$\chi_3(\gamma) = \chi_3^{\rm cons}(\gamma) + \chi_3^{\rm rr}(\gamma)$$

$$\chi_4(\gamma) = \chi_4^{\rm cons}(\gamma) + \chi_4^{\rm rr,odd}(\gamma) + \chi_4^{\rm rr,even}(\gamma)$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- For un-bound orbits energy map defined by scattering angle… derive $\theta$ from EOB Radial potentials

$$\pi + \chi(\mathcal{E}_{\rm eff}, J) = -\int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R}; \mathcal{E}_{\rm eff}, J)}{\partial J}$$

Potentials are fixed by PM-expanded analytical scattering angles

$$\chi_{n\rm PM}(\gamma, j) \equiv \sum_{i=1}^{n} 2 \frac{\chi_i(\gamma)}{j^i}$$

$$w_1(\gamma) = 2(2\gamma^2 - 1)$$

$$w_2(\gamma) = \frac{3}{2} \frac{(5\gamma^2 - 1)}{h(\gamma; \nu)}$$

$$w_3(\gamma) = w_3^{\rm cons}(\gamma) + w_3^{\rm rr}(\gamma)$$

$$w_4(\gamma) = w_4^{\rm cons}(\gamma) + w_4^{\rm rr, odd}(\gamma) + w_4^{\rm rr, even}(\gamma)$$

# Effective One Body: Scattering

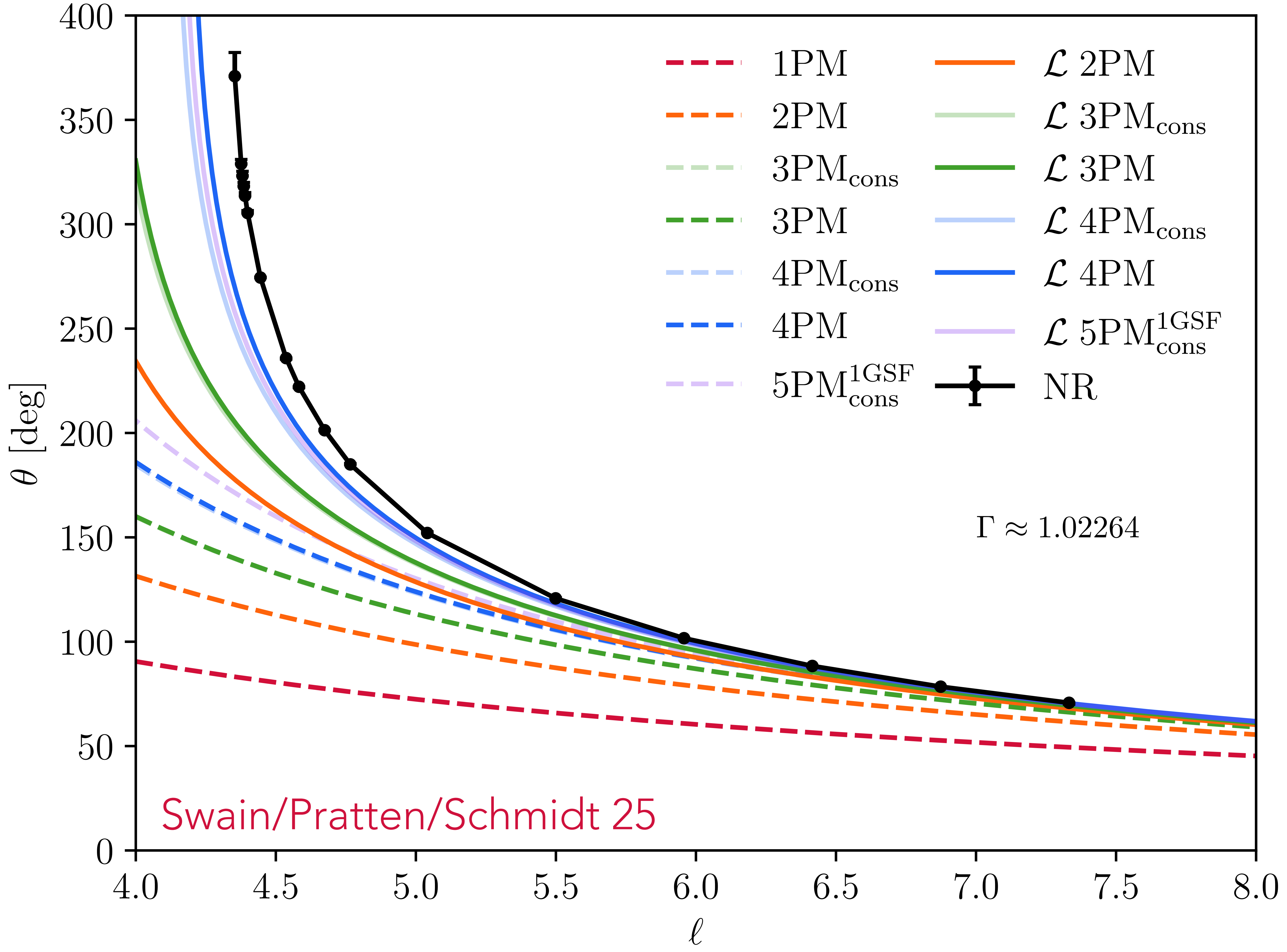


- Further resummation using singular structure [Damour+22, Long+24, Swain+25]

Just signposting recent EOB + PM work (not exhaustive at all)

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]



# Effective One Body: Scattering

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]

- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]

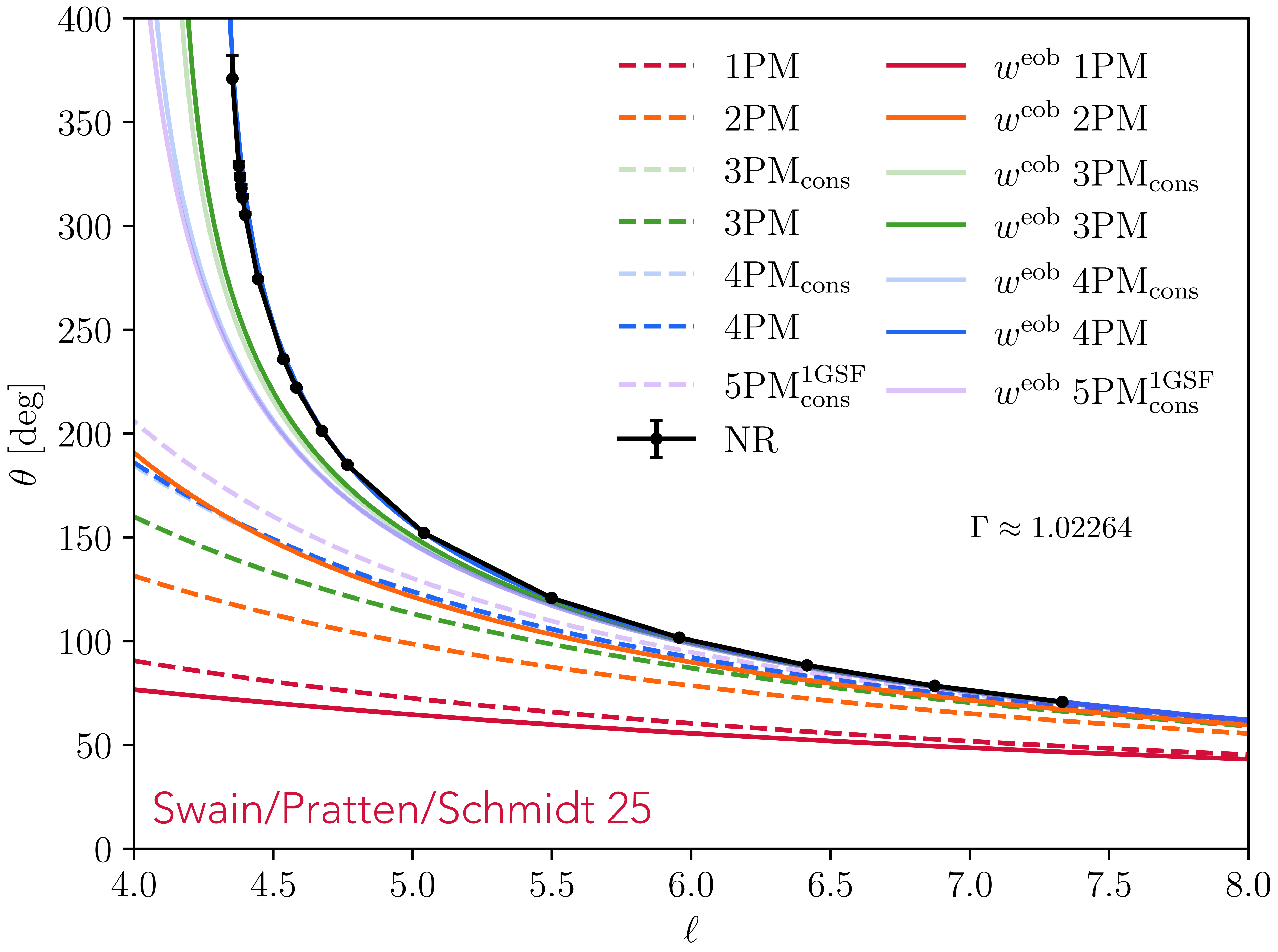
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]

$$\chi_{n\mathrm{PM}}^{\mathrm{weob}}(\gamma, j) \equiv 2j \int_0^{\bar{u}_{\max}(\gamma,j)} \underbrace{\frac{d\bar{u}}{\sqrt{p_\infty^2 + w_{n\mathrm{PM}}(\bar{u},\gamma) - j^2\bar{u}^2}}}_{\text{Solve non-perturbatively}} - \pi$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]



# Effective One Body: Scattering

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]

# Effective One Body: Scattering

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]

Deform radial potentials

$$w_{n\mathrm{PM}}(\bar{r},\gamma,\ell,S_i) = w^{\mathrm{orb}}(\bar{r},\gamma) + \frac{\ell\, w^{\mathrm{S}}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^2} + \frac{w^{\mathrm{S}^2}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^2} + \frac{\ell\, w^{\mathrm{S}^3}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^4} + \frac{w^{\mathrm{S}^4}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^4}$$

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]

$$H_{\rm eff} = \frac{ML(g_{a_+}a_+ + g_{a_-}\delta a_-)}{r^3 + a_+^2(r+2M)} + \sqrt{A\left(\mu^2 + \frac{L^2}{r^2} + (1+B^{\rm Kerr}_{\rm np})p_r^2 + B^{\rm Kerr}_{\rm npa}\frac{L^2 a_+^2}{r^2}\right)}$$

PM-expand gyro-gravitomagnetic terms

Side note: SEOB-PM builds in probe-limit

# Effective One Body: Scattering

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]

Centrifugal radius [Damour+14]

$$r_c^2 = r^2 + a^2 + \frac{2Ma^2}{r}$$

Explores range of approaches to incorporating spin

$$A_{\text{eff}}^{\text{eq}}(r, \gamma, \nu) = \left[1 - \frac{2}{r_c} + \Delta A\right] \frac{\left(1 + \frac{2}{r_c}\right)}{\left(1 + \frac{2}{r}\right)}$$

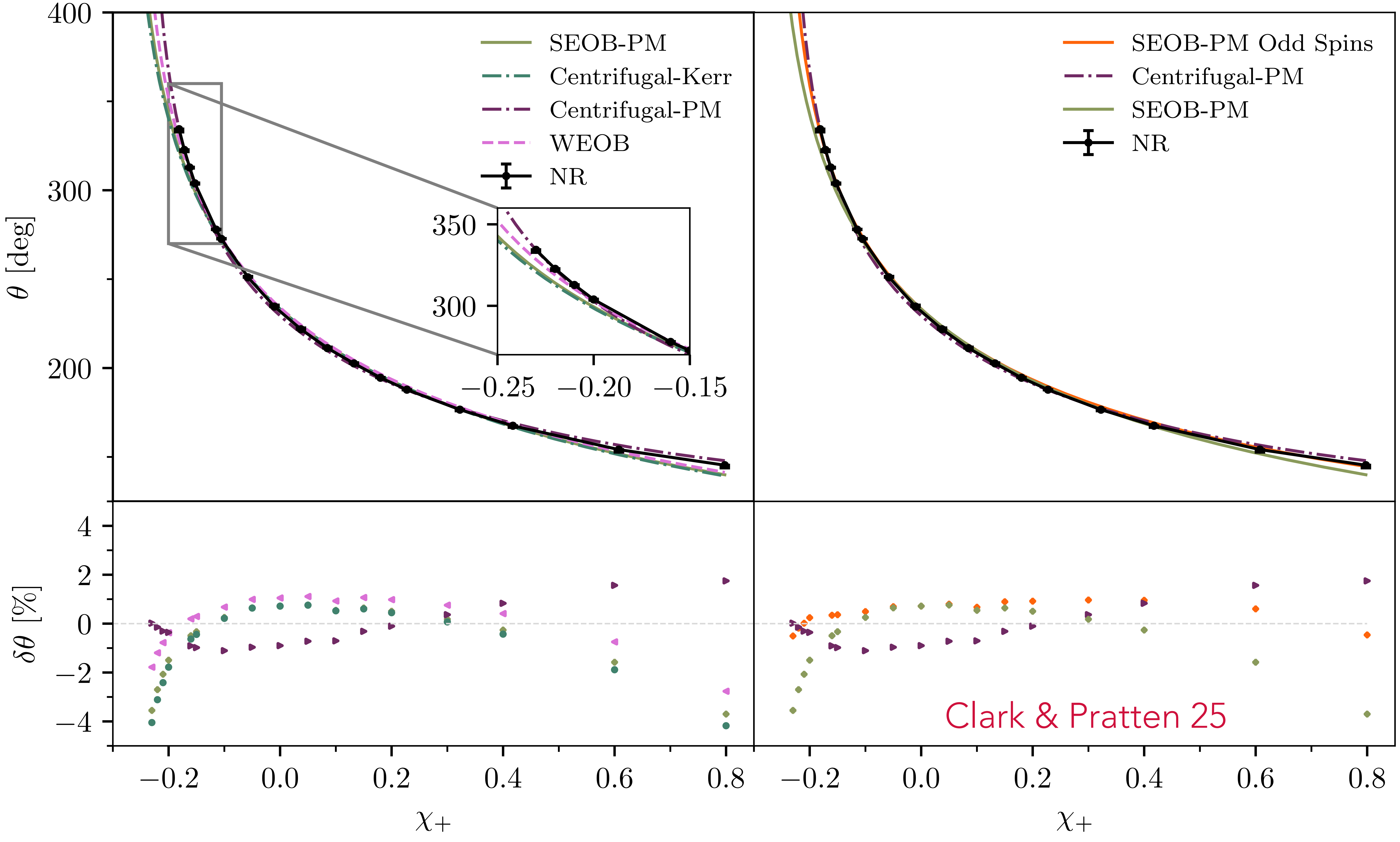
PM deformations into $r_c$ or $A$-potentials?

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear tran ... our+22]
- Alternative ga
- Inclusion of sp



# Effective One Body: Scattering



- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]
- Include tidal effects [Fontbuté+25, Schulze+26]

# Effective One Body: Scattering



- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]
- Include tidal effects [Fontbuté+25, Schulze+26]
- Incorporation of PM information into bound-orbit models [Buonanno+24, Damour+25]

# Effective One Body: Scattering

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Further resummation using singular structure [Damour+22, Long+24, Swain+25]
- Nonlinear transformations to define resummation of PM-expanded scattering angles [Damour+22]
- Alternative gauges [Buonanno+24, Clark/Pratten 25, Damour+25]
- Inclusion of spin information [Rettegno+23 (inc GP, PS), Buonnano+24, Clark/Pratten25]
- Include tidal effects [Fontbuté+25, Schulze+26]
- Incorporation of PM information into bound-orbit models [Buonanno+24, Damour+25]
- Coverage of NR parameter space [Rettegno+23 (inc GP, PS), Swain/Pratten/Schmidt+25, Long+25]

UNIVERSITY OF BIRMINGHAM
THE ROYAL SOCIETY

- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25

- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:

$$S_{\rm nonminimal} = \sum_A \sum_{\ell \geq 2} \frac{1}{2\ell!} \left[ \mu_A^{(\ell)} \int d\tau_A \left(G_L^A(\tau_A)\right)^2 + \frac{\ell}{\ell+1} \sigma_A^{(\ell)} \int d\tau_A \left(H_L^A(\tau_A)\right)^2 \right.$$
$$\left. + \mu_A^{\prime(\ell)} \int d\tau_A \left(\frac{dG_L^A(\tau_A)}{d\tau_A}\right)^2 + \frac{\ell}{\ell+1} \sigma_A^{\prime(\ell)} \int d\tau_A \left(\frac{dH_L^A(\tau_A)}{d\tau_A}\right)^2 + \ldots \right]$$

# Effective One Body: Bedtime Reading

UNIVERSITY OF BIRMINGHAM THE ROYAL SOCIETY

- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:

$$A_T(u) = \sum_{A=1}^{2} \sum_{\ell \geq 2} A_A^{(\ell+)\mathrm{LO}}(u)\, \hat{A}_A^{(\ell+)}(u) + A_A^{(\ell-)\mathrm{LO}}(u)\, \hat{A}_A^{(\ell-)}(u) \tag{12}$$

$$A_A^{(\ell+)\mathrm{LO}}(u) = -\kappa_A^{(\ell+)} u^{2\ell+2}$$

$$A_A^{(\ell-)\mathrm{LO}}(u) = -\kappa_A^{(\ell-)} u^{2\ell+3}$$

A hugely incomplete snapshot of recent EOB work

# Effective One Body: Bedtime Reading

UNIVERSITY OF BIRMINGHAM

THE ROYAL SOCIETY

- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:
  - Damour/Nagar 10, Bernuzzi+12, Steinhoff+16, Hinderer+16, Gamba+23, Haberland+25, Schulze+26, …

A hugely incomplete snapshot of recent EOB work

# Effective One Body: Bedtime Reading



- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:
  - Damour/Nagar 10, Bernuzzi+12, Steinhoff+16, Hinderer+16, Gamba+23, Haberland+25, Schulze+26, …
- Small mass ratio limit
  - van de Meent+25, Leather+25, Albertina+25, Albanesi+26, Gamba+26, Nagni+26, Nishimura+26, Faggioli+26

A hugely incomplete snapshot of recent EOB work

# Effective One Body: Bedtime Reading



- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:
  - Damour/Nagar 10, Bernuzzi+12, Steinhoff+16, Hinderer+16, Gamba+23, Haberland+25, Schulze+26, …
- Small mass ratio limit
  - van de Meent+25, Leather+25, Albertina+25, Albanesi+26, Gamba+26, Nagni+26, Nishimura+26, Faggioli+26
- Beyond GR
  - Julié+23, Jain+23, Chiaramello+25, Pompili+25, Julié+25, Jain+25

A hugely incomplete snapshot of recent EOB work

# Effective One Body: Bedtime Reading

- Extensions to eccentricity
  - TEOBResumS: Chiaramello+20, Nagar+22, Albanesi+25, Gonzales+25, Albanesi+26
  - SEOB: Ramos-Buades+21, Gamboa+ 24, Faggioli+25
- Tidal effects:
  - Damour/Nagar 10, Bernuzzi+12, Steinhoff+16, Hinderer+16, Gamba+23, Haberland+25, Schulze+26, …
- Small mass ratio limit
  - van de Meent+25, Leather+25, Albertina+25, Albanesi+26, Gamba+26, Nagni+26, Nishimura+26, Faggioli+26
- Beyond GR
  - Julié+23, Jain+23, Chiaramello+25, Pompili+25, Julié+25, Jain+25

A hugely incomplete snapshot of recent EOB work

Thank You!