- Lecture 1:
  - Introduction
    - * Two-body Problem in GR
    - * Comparison with different methods
    - * Why scattering?
  - Worldline EFT formalism
    - * Organizing Principles
    - * Conservative sector
    - * Dissipation sector
    - * Scalar Toy Model
- Lecture 2:
  - Scattering Calculation
    - * Waveform & Deflection at leading order (LO)
    - * Extracting effective potential
    - * Dimensional Regularization
    - * Sketch of next-to-leading order (NLO)

△ Additional Reading Materials :

(Far from complete)

- Goldberger , hep-ph/0701129
- Porto , 1601.04914

Research papers

- Goldberger , Ross , 0912.4254
- Goldberger , Ridgway , 1611.03493
- amplitude & EFT , 1908.01493
- Worldline QFT , 2010.02865
- Self-Force EFT , 2406.14770 , 2308.15304

⋮

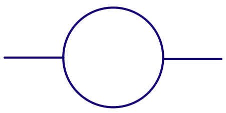
(and references therein)

△ Useful Integrals

1)

$$\int \frac{d^d\vec{k}}{(2\pi)^d} \, \frac{e^{i\vec{k}\cdot\vec{r}}}{(\vec{k}^2)^n} = \frac{\Gamma(d/2-n)}{4^n \cdot \pi^{d/2} \, \Gamma(n)} \, \frac{1}{r^{d-2a}}$$

2)



$$I(a,b) = \int \frac{d^d\vec{\ell}}{(2\pi)^d} \, \frac{1}{(\vec{\ell}^2)^a \left((\vec{q}-\vec{\ell})^2\right)^b}$$

$$= \frac{\pi^{d/2} \, \Gamma(a+b-d/2) \, \Gamma(d/2-a) \, \Gamma(d/2-b)}{(2\pi)^d \, \Gamma(a) \, \Gamma(b) \, \Gamma(d-a-b)} \, (\vec{q}^2)^{d/2-a-b}$$

Δ Scalar Toy Model :

To do calculation in this lecture, we will use a scalar toy model.

$$S_{bulk} = \int d^dx \left(\tfrac{1}{2} (\partial\phi)^2 \right)$$

$$S_{pp} = \sum_{\alpha=1,2} -m_\alpha \int d\lambda_\alpha \left(\frac{1}{2e} + \frac{1}{2} e V_\alpha^2 + \frac{1}{e} f(\phi) \right)$$

$$= \sum_{\alpha=1,2} -m_\alpha \int d\lambda_\alpha \left(\frac{1}{2} + \frac{1}{2} V_\alpha^2 + q_\alpha \frac{\phi}{\Lambda_\alpha} + \frac{\tilde{C}_\alpha}{2\Lambda_\alpha^2} \phi^2 + \dots \right)$$

To mimic GR, we pick $\Lambda = m_{pl}$, $q_\alpha = 1$

Δ EOM :

ϕ : $\quad S = \int \left(\tfrac{1}{2} (\partial\phi)^2 + \dots \right)$

$$\underbrace{(-\partial_t^2 + \nabla^2)}_{\equiv -\Box} \phi + \frac{\delta S_{int}}{\delta \phi} = 0$$

$$\Rightarrow \quad \Box\phi = \frac{\delta S_{int}}{\delta \phi} = \sum_\alpha -m_\alpha \int d\lambda \cdot \left\{ \frac{1}{\Lambda} \cdot \delta(x - x(\lambda)) + \frac{\tilde{C}_\alpha}{\Lambda^2} \phi \cdot \delta(x - x(\lambda)) + \dots \right\}$$

x^μ : $\quad \dot{V}_{\alpha\mu} = \frac{1}{\Lambda} \partial_\mu \phi + \frac{\tilde{C}_\alpha}{\Lambda^2} \phi \, \partial_\mu \phi + \dots$

e : $\quad V^2 = \frac{1}{e^2} (1 + 2 f(\phi)) \xrightarrow[e=1]{\text{choose}} 1 + \frac{2}{\Lambda} \phi + \dots$

△ Scattering in Worldline EFT :

Setup : Consider the two particles

$$x^\mu_\alpha = \underbrace{b^\mu_\alpha}_{\text{impact parameter}} + \underbrace{v^\mu_\alpha}_{\text{initial velocity}} \cdot \lambda + \underbrace{\delta x^\mu_\alpha}_{\text{correction due to } \phi} \qquad (\alpha = 1, 2)$$

We will solve δx^μ perturbatively in q_α and in momentum space.

$$f(x) = \int \frac{d^dk}{(2\pi)^d}\, e^{-ikx} f(k) \ ; \qquad f(k) = \int d^dx\, e^{ikx} f(x)$$

$$\equiv \int_k e^{-ikx} f(k) \qquad\qquad \delta\!\!\!^{-}(x) \equiv (2\pi)^d \delta(x)$$

Solve ϕ :
$$\Box \phi = \sum_\alpha -m_\alpha \int d\lambda_\alpha \frac{1}{\Lambda} \delta(x - x(\lambda))$$

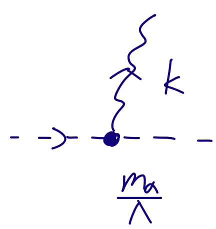
$$\phi = \int_k e^{-ikx} \phi(k)$$

$$(-k^2)\,\phi(k) = \sum_\alpha \int d\lambda_\alpha \frac{(-m_\alpha)}{\Lambda} \cdot e^{ik\cdot x_\alpha(\lambda)}$$

$$= \sum_\alpha \int d\lambda_\alpha \frac{(-m_\alpha)}{\Lambda} \cdot e^{ik(b_\alpha + v_\alpha \lambda_\alpha + \cancel{\delta x_\alpha})}$$

$$= \sum_\alpha \frac{(-m_\alpha)}{\Lambda} e^{ikb_\alpha}\, \delta\!\!\!^{-}(k\cdot v_\alpha)$$

$$\Rightarrow \quad \phi(k) = \frac{1}{k^2} \sum_\alpha \left\{ \frac{m_\alpha}{\Lambda} e^{ikb_\alpha}\, \delta\!\!\!^{-}(k\cdot v_\alpha) \right\}$$



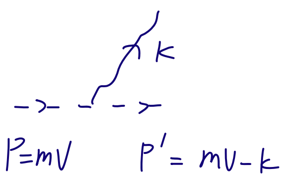
Remarks :

1) $\frac{1}{k^2}$ $\Rightarrow$ Need to choose a boundary condition

Classical physics : retarded BC (v.s. Feynman in QFT)

$$\frac{1}{k^2} \longrightarrow \frac{1}{(\omega + i0)^2 - \vec{k}^2}$$

2) $\delta(k \cdot V_\alpha)$ $\Rightarrow$ This is the classical version of on-shell condition coming from a straight worldline.



$$\delta(P'^2 - m^2)$$

$$= \delta(\underbrace{P^2 - m^2}_{=0} + 2P\cdot k + k^2)$$

$$\underset{k \ll P}{=} \delta(m\, 2V\cdot k) \sim \frac{1}{2m}\delta(V\cdot k)$$

Classical limit

3) Since V is time-like, $(V^0, V^i) \sim (1, v)$

k must be space-like $(k^0, k^i) \sim (v, 1)$ $\Leftarrow$ "potential mode"

(Rest frame : $V = (1, 0)$, $\delta(k\cdot V) = \delta(\omega)$)

$\Rightarrow$ k^2 never on-shell, $i0$ (BC) doesn't matter.

$$\phi(x) = \int_k e^{-ik x}\phi(k) = \left(-\frac{m}{\Lambda}\right)\cdot \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\vec{k}^2}\cdot e^{i\vec{k}\cdot(\vec{x}-\vec{b})}$$

$$= \left(-\frac{m}{\Lambda}\right)\cdot \frac{1}{4\pi|\vec{x}-\vec{b}|}$$

$\Rightarrow$ This is the standard Coulomb potential !

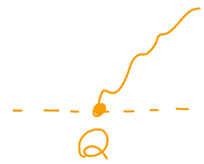
$\Rightarrow$ No physical wave ! (k not on-shell)

Δ This is in contrast with the ϕ generated by a time-dependent source $S_{pp} \supset -\int d\lambda\, Q(\lambda)\, \phi(x(\lambda))$

$$\Box \phi = J(x) = -\int d\lambda \cdot Q(\lambda)\, \delta(x - x(\lambda))$$

$$\phi(k) = \frac{1}{k^2} \int d\lambda\, Q(\lambda)\, e^{ik \cdot x(\lambda)}$$

$$\xrightarrow[\text{rest frame}]{} \frac{1}{k^2} \int d\lambda\, Q(\lambda)\, e^{i\omega\lambda} = \frac{1}{k^2} \cdot Q(\omega)$$



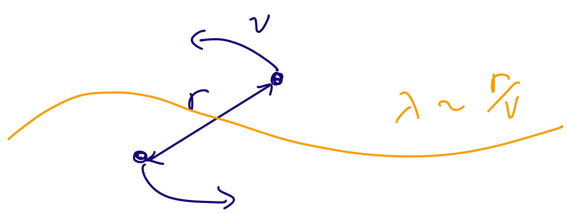
$$\phi(x) = \int_k \frac{1}{(\omega + i0)^2 - \vec{k}^2} e^{-ik \cdot x} Q(\omega)$$

$$\propto \frac{1}{4\pi r} \int \frac{d\omega}{2\pi} e^{-i\omega u} Q(\omega)$$

using Retarded Green's function

$$G_R(\Delta t, \Delta\vec{x}) = \Theta(\Delta t) \frac{1}{4\pi \Delta r} \delta(\Delta t - \Delta r)$$

Δ We call the momentum scaling



$k^\mu = (k^0, k^i) \sim (v/r, 1/r) \Rightarrow$ potential mode

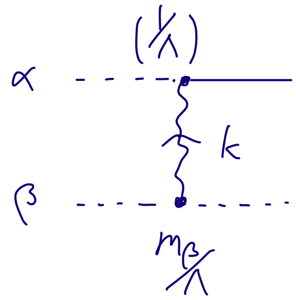
gives instataneous effective potential

$$\frac{-1}{k^2} = \frac{1}{\vec{k}^2 - \omega^2} = \frac{1}{\vec{k}^2} + \frac{\omega^2}{\vec{k}^4} + \cdots$$

$\sim (v/r, v/r) \Rightarrow$ radiation mode

gives the gravitational wave

△ For the worldline



$$\dot{V}_{\alpha\mu} = \frac{1}{\Lambda}\,\partial_\mu \phi$$

$$= \frac{1}{\Lambda}\int_k (-ik^\mu)\, e^{-ik\,x(\lambda)}\,\phi_k$$

$$= \frac{1}{\Lambda}\int_k (-ik^\mu)\, e^{-ikb_\alpha}\, e^{-i(k\cdot V_\alpha)\lambda}$$

$$\times\ \underline{\frac{1}{k^2}\left\{\sum_\beta \frac{m_\beta}{\Lambda}\, e^{ikb_\beta}\, f(k\cdot V_\beta)\right\}}$$

$$= \sum_{\beta\neq\alpha}\frac{m_\beta}{\Lambda^2}\cdot\int_k f(k\cdot V_\beta)\,\frac{-ik^\mu}{k^2}\cdot e^{-ik(b_\alpha-b_\beta)}\, e^{-i(k\cdot V_\alpha)\lambda}$$

This is a function of λ.

We will need, $\delta V_\alpha(\lambda)$, $\delta X_\alpha(\lambda)$

$$\delta V_\alpha^\mu(\lambda) = \int_{-\infty}^{\lambda} d\lambda'\ \dot{V}_\alpha(\lambda')$$

$$= \left(\frac{m_\beta}{\Lambda^2}\right)\cdot\int_k f(k\cdot V_\beta)\,\frac{-ik^\mu}{k^2}\cdot e^{-ik(b_\alpha-b_\beta)}$$

$$\int_{-\infty}^{\lambda} d\lambda'\cdot e^{-i(k\cdot V_\alpha)\lambda'}$$

∘ Since $\underline{\delta V_\alpha^\mu(\lambda\to-\infty)=0}$, we promote $k\cdot V_\alpha \to (k\cdot V_\alpha + i0)$

(retarded BC)

$$\int_{-\infty}^{\lambda} d\lambda'\cdot e^{-i(k\cdot V_\alpha+i0)\lambda'} = \frac{i}{k\cdot V_\alpha + i0}\, e^{-i(k\cdot V_\alpha+i0)\lambda'}\Big|_{-\infty}^{\lambda}$$

$$= \frac{i}{k\cdot V_\alpha+i0}\, e^{-i(k\cdot V_\alpha+i0)\lambda}$$

$$\Rightarrow \delta V_\alpha^\mu = \left(\frac{m_\beta}{\Lambda^2}\right) \cdot \int_k f(k\cdot V_\beta) \frac{-ik^\mu}{k^2} \cdot e^{-ik(b_\alpha - b_\beta)} \frac{i}{k\cdot V_\alpha + i0} e^{-i(k\cdot V_\alpha + i0)\lambda}$$

Like wise

$$\delta x_\alpha^\mu(\lambda) = \int_{-\infty}^{\lambda} d\lambda' \, \delta V_\alpha^\mu(\lambda')$$

$$= \left(\frac{m_\beta}{\Lambda^2}\right) \cdot \int_k f(k\cdot V_\beta) \frac{-ik^\mu}{k^2} \cdot e^{-ik(b_\alpha - b_\beta)} \left(\frac{i}{k\cdot V_\alpha + i0}\right)^2 e^{-i(k\cdot V_\alpha + i0)\lambda}$$

We are interested in the asymptotic resut $\lambda \to \infty$

$$\int_{-\infty}^{\lambda} d\lambda' \cdot e^{-i(k\cdot V_\alpha + i0)\lambda'} = \frac{i}{k\cdot V_\alpha + i0} e^{-i(k\cdot V_\alpha + i0)\lambda} \xrightarrow{\lambda\to\infty} f(k\cdot V_\alpha)$$

$$\delta V_\alpha^\mu(\lambda\to\infty) = \left(\frac{m_\beta}{\Lambda^2}\right) \cdot \int_k f(k\cdot V_\alpha)\, f(k\cdot V_\beta) \frac{-ik^\mu}{k^2} \cdot e^{-ik(b_\alpha - b_\beta)}$$

The $\frac{i}{k\cdot V_\alpha + i0}$ appears as we integrate over λ,

It is the propagator in the classical limit

$$\frac{1}{(mV_\alpha + k)^2 - m^2} = \frac{1}{2m(k\cdot V_\alpha) + k^2}$$

$$\xrightarrow[k \ll mV]{} \frac{1}{2m}\left(\frac{1}{k\cdot V_\alpha} - \frac{k^2}{(k\cdot V_\alpha)^2} + \cdots\right)$$

From the einbein choice, we have $V^2 = 1 + \frac{2q_\alpha}{\Lambda}\phi(X(\lambda))$

$$\Rightarrow \quad (V_\alpha + \delta V_\alpha)^2 = 1 + 2V_\alpha \cdot \delta V_\alpha + \mathcal{O}(\delta V_\alpha^2)$$

$$= 1 + \frac{2m_\beta}{\Lambda^2} \cdot \int_k \delta(k\cdot V_\beta) \cdot e^{-ik\cdot(b_\alpha + k\cdot V_\alpha \lambda)} \frac{\cancel{k\cdot V_\alpha}}{k^2(\cancel{k\cdot V_\alpha + i0})} e^{ik\cdot b_\beta}$$

$$= 1 + \frac{2}{\Lambda}\int_k e^{-ik\cdot x_\alpha(\lambda)} \cdot \phi(k)$$

$$= 1 + \frac{2}{\Lambda}\phi(x_\alpha(\lambda)) \qquad \text{Consistent !}$$

△ Worldline Feynman Rules

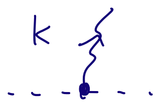
(see WQFT 2010.02865)

① $$\phi(k) = \frac{1}{k^2}\,\frac{m}{\Lambda}\int d\lambda\, \underbrace{e^{ik\cdot x(\lambda)}}_{= e^{ik\cdot b}\, e^{i(k\cdot v)\lambda}\,(1 + ik\cdot\delta x(\lambda) + \cdots)}$$

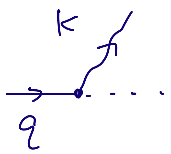
Propagator :


: $\frac{1}{k^2}$

Interaction :



: $\left(\frac{m}{\Lambda}\right) e^{ik\cdot b}\ \delta(k\cdot v)$



: $\left(\frac{m}{\Lambda}\right) e^{ik\cdot b}\ \delta((k-q)\cdot v)\cdot(ik\cdot\delta x(q))$

② $$\dot{V}_\mu = \frac{1}{\Lambda}\partial_\mu \phi$$

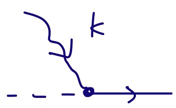
$$= \int_k \frac{m}{\Lambda}(-ik_\mu)\, e^{-ik\cdot b}\, \frac{e^{-i(k\cdot V)\lambda}}{m}\, e^{-ik\cdot\delta x}\, \phi_k$$

Propergator for δx :

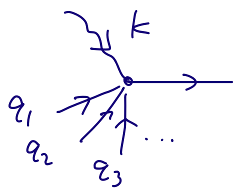
$$\xrightarrow{\ \delta x^\mu\ } \quad : \quad \frac{1}{m}\left(\frac{i}{k\cdot V + i0}\right)^2 .$$

(k is the total momentum injected to the world line .)

Interaction :



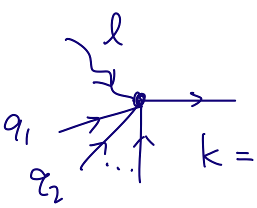
$$: \int_k \frac{m}{\Lambda}(-ik^\mu)\, e^{-ik\cdot b}$$



$$: \int_k \frac{m}{\Lambda}(-ik^\mu)\, e^{-ikb}\, \frac{1}{n!}(-ik\cdot\delta x)^n$$

If the worldline propagates to $\lambda \to \infty$,

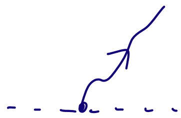
need to add $\frac{1}{m}\delta(k\cdot V)$, where k is the total momentum injected to the worldline .



$$k = l + q_1 + q_2 \cdots$$

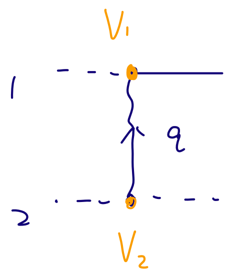
△ Examples :

①



$$\phi(k) = \frac{1}{k^2} \times \frac{m}{\Lambda}\, e^{ik\cdot b}\, \delta(k\cdot v)$$

②

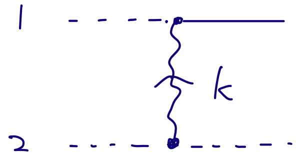


$$\delta v_1^\mu = \int_q \underbrace{\left(\frac{m_1}{\Lambda}\,(-iq^\mu)\cdot e^{-iqb_1}\right)}_{V_1} \times \underbrace{\frac{1}{m_1}\,\delta(q\cdot v_1)}_{\text{propagator for } \delta v_1}$$

$$\times \underbrace{\frac{1}{q^2}}_{} \times \underbrace{\frac{m_2}{\Lambda}\, e^{iqb_2}\, \delta(q\cdot v_2)}_{V_2}$$

$$= \int_q \delta(q\cdot v_1)\, \delta(q\cdot v_2) \cdot \frac{-iq^\mu}{q^2} \times \left(\frac{m_2}{\Lambda^2}\right) \cdot e^{-iq(b_1-b_2)}$$

To see the deflection more concretely, we choose a frame with the initial conditions



$$V_1 = (\gamma,\ \gamma v,\ 0,\ 0)\ ,\quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$V_2 = (1,\ 0,\ 0,\ 0)$$

$$b_1 = (0,\ 0,\ b,\ 0)$$

$$b_2 = (0,\ 0,\ 0,\ 0)$$

$$\delta V_1^\mu = \left(\frac{m_2}{\Lambda^2}\right) \int_k \delta(\omega)\, \delta(\gamma(\omega - v k_x)) \cdot \frac{-i k^\mu}{k^2}\, e^{i k_y \cdot b}$$

$\Rightarrow$ k only lives in $k_\perp = (k_y, k_z)$

$$\delta V_{1\perp} = \left(\frac{m_2}{\Lambda^2}\right) \cdot \frac{1}{\gamma v} \int_{\vec{k}_\perp} \frac{i \vec{k}_\perp}{\vec{k}_\perp^{\,2}}\, e^{i \vec{k}_\perp \cdot \vec{b}}$$

$$= \left(\frac{m_2}{\Lambda^2}\right) \cdot \frac{1}{\gamma v}\, \partial_{\vec{b}} \left(\int_{\vec{k}_\perp} \frac{e^{i \vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}} \right)$$

$$= \frac{1}{m_1}\, \partial_{\vec{b}} \left(\underbrace{\frac{m_1 m_2}{\gamma v\ \Lambda^2} \int_{\vec{k}_\perp} \frac{e^{i \vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}}}_{\equiv \text{ eikonal phase } \chi(b)\ \sim \text{ radial action}} \right)$$

△ The eikonal phase $\chi(b)$ is symmetric in α, β

$\chi(b)$ naturally arise from amplitudes.

Suppose α is a massless particle,

$\chi(b)$ can be interpreted as the phase shift

in the eikonal limit in BHPT (large ℓ ~ large b limit)

△ The integral $I(b) \equiv \int_{\vec{k}_\perp} \frac{e^{i\vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}}$ is infrared divergent as $\vec{k}_\perp^{\,2} \to 0$

We can regulate this by $d = 4-2\varepsilon$

$$I(b) = \int \frac{d^{2-2\varepsilon}\vec{k}_\perp}{(2\pi)^{2-2\varepsilon}} \frac{e^{i\vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}}$$

$$= -\frac{1}{4\pi} b^{2\varepsilon} \left(\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0) \right)$$

$$= -\frac{1}{4\pi} \left(\frac{1}{\varepsilon} + \ln b^2 \right)$$

$$\delta V_{\alpha\perp} = \partial_{\vec{b}} \chi$$

$$= \frac{1}{m_\alpha} \left(\frac{m_\alpha m_\beta}{\gamma v \Lambda^2} \right) \underbrace{\left(-\frac{1}{2\pi} \frac{\vec{b}}{b^2} \right)}_{\text{IR finite}}$$

leading order deflection but naturally all-order in v.

△ For GR scaling ($q_i = 1$, $\Lambda = m_{pl}$)

$\Delta V \propto \left(\frac{Gm}{b} \right) \Rightarrow$ deflection in weak-field limit.

Δ Dimensional Regularization :

It is convenient to extend $d = 4 - 2\varepsilon$ to regulate divergences without introducing other scales to the integrals.

1) IR divergence :

$$I(b) = \int_{\vec{k}_\perp} \frac{e^{i\vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}} \longrightarrow \int \frac{d^{d-2}\vec{k}_\perp}{(2\pi)^{d-2}} \frac{e^{i\vec{k}_\perp \cdot \vec{b}}}{\vec{k}_\perp^{\,2}}$$

$k \to 0$

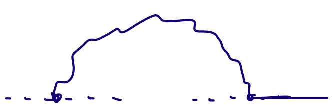
$=$

$$\int d^2k/k^2 \sim \ln k \to \infty$$

2) UV divergence : Very common as we shrink finite objects into points

e.g.

Self-deflection ?



$$\delta \dot{v}^\mu \propto \int_k f(k \cdot v)\, e^{-i(k \cdot v)\lambda} \frac{k^\mu}{k^2}$$

$$\propto \int_k f(k \cdot v) \frac{k^\mu}{k^2}$$

$$I = \int \frac{d^dk}{(2\pi)^d} f(k \cdot v) \frac{k^\mu}{k^2} \propto k^{d-2} = k^{2-2\varepsilon}$$

But we don't have any scale available !

$\Rightarrow$ $I = 0$ is the only mathematical consistent choice.

$\divideontimes$ Similarly, $\frac{1}{r^{n+\varepsilon}}$ in the potential $\longrightarrow 0$ as $r \to 0$

△ Detailed Explanation :

Consider

$$I = \int d^d l \frac{1}{(l^2)^a (l \cdot v)^b}$$

a, b are integers

$d = n - 2\varepsilon$

$$[I] = d - 2a - b$$

$$= (n - 2a - b) - 2\varepsilon$$

If we rescale l as $l' = \lambda \cdot l$

$$I = \int d^d l \frac{1}{(l^2)^a (l \cdot v)^b} = \int d^d l' \frac{1}{(l'^2)^a (l' \cdot v)^b}$$

$$= \lambda^{n-2a-b-2\varepsilon} \cdot I$$

$\Rightarrow$ If $I \neq 0 \Rightarrow \lambda^{n-2a-b-2\varepsilon} = 1$ (False)

By contraction , $\underline{I = 0}$

• Lemma : $\left. \frac{1}{r^{a+b\cdot\varepsilon}} \right|_{r \to 0} = 0$ ($a, b \in$ integers)

Since

$$\frac{1}{r^{d-2n}} = \underbrace{\frac{1}{C_{d,n}} \int d^d k \frac{1}{k^{2n}} e^{ikr}}_{\text{becomes scaleless when } r=0} \xrightarrow[r=0]{} 0$$

△ Matching to effective potential

- While doing scattering is fun, bound systems are more relevant to observation. We are interested in the effective potential V_{eff}

- Using scattering angle, we can reverse engineer a V_{eff}

  E.g. in the CM frame

$$H = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + V(\vec{p}^2, r)$$

$$= \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + \frac{a_1(\vec{p}^2)}{r} + \frac{a_2(\vec{p}^2)}{r^2} + \cdots$$

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial r} \end{cases} \Rightarrow \text{ derive scattering angle}$$

$\Rightarrow$ fix $(a_1, a_2, \cdots)$ from the known data.

* We implicitly pick a gauge a_n is only a function of $\vec{p}^2$ to make the inversion possible. (isotropic gauge)

* We assume V_{eff} is instantaneous (until tail appears)

- This matching procedure gives PM V_{eff} from scattering data.

$\triangle$ We can compare this "on-shell" matching procedure with more traditional "off-shell" method by "integrating" out ϕ

$$e^{iS_{eff}} = \int \mathcal{D}\phi \; e^{i(S_{bulk} + S_{pp})}$$

Since this is Gaussian, effectively we evaluate the action on the support of EOM

$$\phi(k) = \frac{1}{k^2}\left(\frac{m_\alpha}{\Lambda}\right)\int d\lambda \cdot e^{ik\cdot x(\lambda)}$$

No $x(\lambda) = b + v\lambda + \delta x$ assumed
arbitrary trajectory !

$$S_{eff} \propto \left(\frac{m_\alpha m_\beta}{\Lambda^2}\right)\int d\lambda_\alpha \, d\lambda_\beta \cdot \int_k \frac{1}{k^2} e^{ik(x_\alpha - x_\beta)}$$

$$= \left(\frac{m_\alpha m_\beta}{\Lambda^2}\right)\int d\lambda_\alpha \, d\lambda_\beta$$

$$\times \int_k \underbrace{\frac{-1}{\vec{k}^2 - \omega^2}} e^{i\omega\cdot(t_\alpha - t_\beta)} e^{-i\vec{k}\cdot(\vec{x}_\alpha - \vec{x}_\beta)}$$

potential $\Rightarrow \dfrac{-1}{\vec{k}^2}\left(1 + \dfrac{\omega^2}{\vec{k}^2} + \cdots\right)$

$$\Rightarrow \frac{-1}{\vec{k}^2}\left(1 + \frac{1}{\vec{k}^2}(-\partial_{t_\alpha}^2) + \cdots\right)$$

$$= \frac{-1}{\vec{k}^2}\left(1 + \frac{1}{\vec{k}^2}(-\partial_{t_\beta}^2) + \cdots\right)$$

off-shell
gauge ambiguity

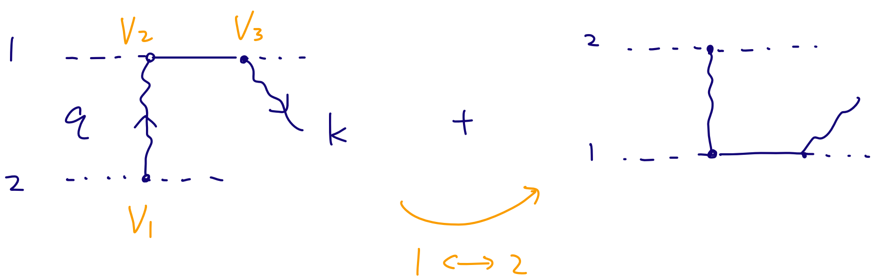
△ Leading Order Wave form :

It is straightforward to go to higher orders.

$$\phi(k) = \frac{1}{k^2} \sum_\alpha \int d\lambda_\alpha \, \frac{m_\alpha}{\Lambda} \cdot e^{ik\cdot(b_\alpha + V_\alpha \lambda_\alpha + \delta x_\alpha)}$$

$$= \frac{1}{k^2} \sum_\alpha \int d\lambda_\alpha \, \frac{m_\alpha}{\Lambda} \cdot e^{ik(b_\alpha + V_\alpha \lambda_\alpha)} \underbrace{(1 + ik\cdot\delta x_\alpha + \cdots)}_{\text{perturbative expansion}}$$

Diagrams :



$$\text{Diagram 1} = \frac{1}{k^2} \int_q \underbrace{\left(\frac{m_2}{\Lambda}\right) e^{iq\cdot b_2} f(q\cdot V_2)}_{V_1} \times \frac{1}{q^2}$$

$$\underbrace{\frac{m_1}{\Lambda} (-iq_\mu) \, e^{-iq b_1}}_{V_2} \times \frac{1}{m_1} \left(\frac{i}{q\cdot V_1 + i0}\right)^2$$

$$\underbrace{\frac{m_1}{\Lambda} e^{ik\cdot b_1} \quad (ik^\mu) \times f((k-q)\cdot V_1)}_{V_3}$$

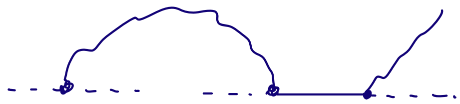
$$= \frac{1}{k^2} \left(\frac{m_1 m_2}{\Lambda^3}\right) e^{ik\cdot b_1}$$

$$\times \int_q f(q\cdot V_2) \, f((k-q)\cdot V_1) \, e^{-iq(b_1 - b_2)} \frac{(-k\cdot q)}{q^2 (q\cdot V_1 + i0)^2}$$

$\phi(k)$ = diagram 1 + diagram 2

Remarks :

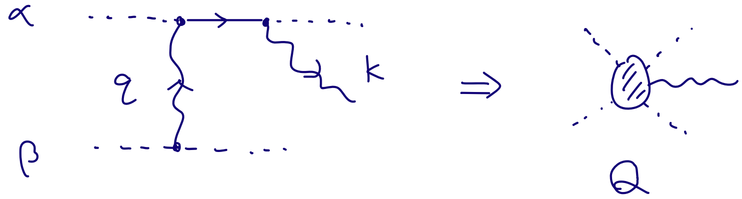
1) When $l=2$



The integral in q becomes scaleless.

2) q & $q-k$ are constrained to be in the potential mode,

But k can be in the radiation mode $\Rightarrow$ leading-order waveform



OR

potential mode $\Longrightarrow$ NLO deflection angle

3) $\phi(k)$ is homogeneous in m_1, m_2

$\Rightarrow$ No radiation in the probe limit !

△ Deflection at NLO : (Sketch)

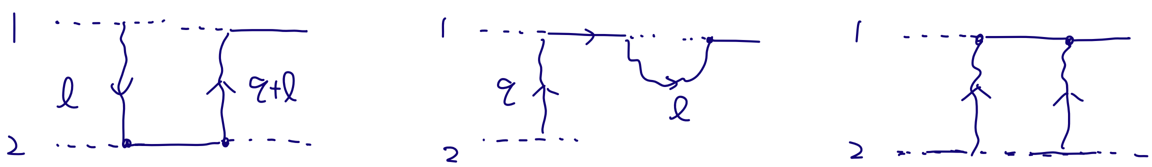
$$\dot{v}_{\alpha\,\mu} = \frac{1}{\Lambda}\int_k (-ik^\mu)\, e^{-ik\,x(\lambda)}\, \phi_k$$

$$= \frac{1}{\Lambda}\int_k (-ik^\mu)\, e^{-ik(b_\alpha + v_\alpha \lambda)}\, \phi_k \left(1 - ik\delta x + \frac{1}{2!}(-ik\delta x)^2 + \dots\right)$$

① $\mathcal{O}(1/\Lambda^3) \times 1$

② $\mathcal{O}(1/\Lambda) \times \mathcal{O}(1/\Lambda^2)$

Diagrams :



1−1 ① 1−2 ②

(1−1)

$$\int_{\ell,q} \frac{-i(q+\ell)^\mu}{\ell^2\cdot(q+\ell)^2}\left(\frac{i}{\ell\cdot v_2 + i0}\right)^2 \cdot \delta(q\cdot v_1)\,\delta(q\cdot v_2)\,\delta(\ell\cdot v_1)$$

$$\times \frac{m_1}{\Lambda}\,\frac{m_2}{\Lambda}\cdot\frac{1}{\Lambda^2}\cdot\left(i(q+\ell)\cdot(-i\ell)\right) \times e^{i\ell(b_1-b_2)}\, e^{-i(q+\ell)(b_1-b_2)}$$

$$= \frac{m_1 m_2}{\Lambda^4} \times \int_q \delta(q\cdot v_1)\,\delta(q\cdot v_2)\, e^{-iq(b_1-b_2)}$$

$$\times \int_\ell \frac{-i(q+\ell)^\mu}{\ell^2 (q+\ell)^2}\left(\frac{i}{\ell\cdot v_2 + i0}\right)^2 (q+\ell)\cdot\ell$$

(1−2)

$$\int_{\ell,q} \frac{-i\ell^\mu}{q^2\cdot\ell^2}\cdot\left(\frac{i}{q\cdot v_1 + i0}\right)^2 \cdot \delta(q\cdot v_2)\,\delta((\ell-q)\cdot v_1)\,\delta(\ell\cdot v_1)$$

$$\times (q\cdot\ell) \times e^{-iq(b_1-b_2)} \times \frac{m_1 m_2}{\Lambda^4}$$

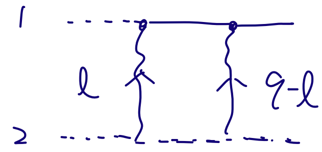
ℓ-integral becomes scaleless !

$= 0$

(2)

$$\int_{\ell,q} \frac{(-iq^\mu)}{\ell^2(q-\ell)^2}\left(\frac{i}{\ell\cdot v_1+i0}\right)^2 \delta(\ell\cdot v_2)\,\delta((q-\ell)\cdot v_2)\,\delta(q\cdot v_1)$$

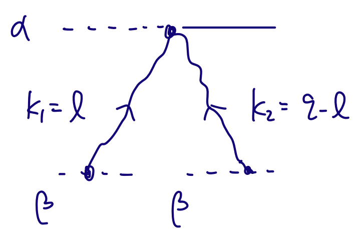
$$\times\left(\frac{m_2^2}{\Lambda^4}\right)\times e^{-iq(b_1-b_2)}$$



If we include $S_{pp} \supset -m\int d\lambda \left(\frac{C_\alpha}{2\Lambda^2}\phi^2\right)$, we also have

$$\dot{v}_\mu = \frac{C_\alpha}{2\Lambda^2}\partial_\mu(\phi^2)$$

$$= \frac{C_\alpha}{2\Lambda^2}\partial_\mu\left(\int_{k_1,k_2}\left(\frac{m_\beta}{\Lambda}\right)^2 \delta(k_1\cdot v_\beta)\frac{1}{k_1^2}e^{-ik_1x(\lambda)}e^{ik_1b_\beta}\,\delta(k_2\cdot v_\beta)\frac{1}{k_2^2}e^{-ik_2x(\lambda)}e^{ik_2b_\beta}\right)$$



※ Change variables to $k_1=\ell$, $k_2=q-\ell$

$$\delta v_\mu = \frac{C_\alpha}{2\Lambda^2}\left(\frac{m_\beta}{\Lambda}\right)^2 \int_q (-iq^\mu)\cdot e^{-iq(b_\alpha-b_\beta)}\cdot\delta(q\cdot v_\beta)\,\delta(q\cdot v_\alpha)$$

↳ similar to LO deflection

$$\times\int_\ell \delta(\ell\cdot v_\beta)\frac{1}{\ell^2(q-\ell)^2}$$

New!

Δ When $v_\beta=(1,\vec{0})$, $q\cdot v_\beta=\ell\cdot v_\beta=0 \Rightarrow q^0=\ell^0=0$

$\Rightarrow$ q and ℓ are both potential

$$\int_\ell \delta(\ell\cdot v_\beta)\frac{1}{\ell^2(q-\ell)^2} = \int\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{\vec{\ell}^2(\vec{q}-\vec{\ell})^2} = \frac{1}{8|\vec{q}|}$$

Remarks :

△ Compare to LO, the $\frac{1}{\vec{q}^2}$ is replaced with $\frac{1}{|\vec{q}|}$.

In position space $\frac{1}{\vec{q}^2} \longrightarrow 1/r$ potential

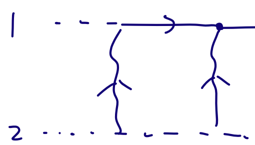
$\frac{1}{|\vec{q}|} \longrightarrow 1/r^2$ potential

△ The "loop" is being cut, opened as a tree.

$\Rightarrow$ phase-space integrals.

△ In the GR case, the $1/r^2$ term can be determined from the Schwarzschild background, bypassing the need to calculate such diagrams

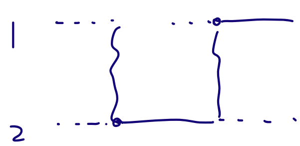
△



No recoil of 2

fixed by Schwartzschild

(OSF)



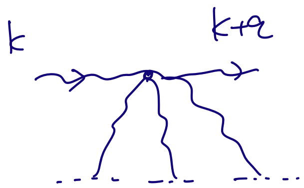
recoil of 2 $\neq 0$

(ISF)

(See Nabha's lectures !)

Δ Renormalization :

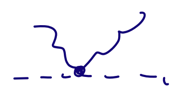
Consider a diagram with ϕ^5 self interaction



$$\int_{l_1, l_2} \frac{1}{l_1^2\, l_2^2\, (q-l_1-l_2)^2} = \frac{1}{8\pi^2}\left(\frac{1}{\varepsilon} - 2\ln \vec{q}^{\,2} + \dots \right)$$

This diagram diverges as $l_1 \sim l_2 \to \infty$ $\Rightarrow$ Typical in point-particle limit.

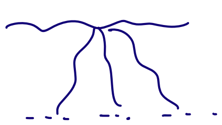
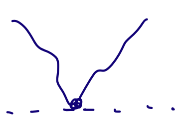
$\Rightarrow$ UV divergence shows up as $\frac{1}{8\pi^2 \varepsilon}$

The divergence can be removed by the coupling in 

However, since $[\phi] = 1-\varepsilon$

We should write

$$S_{pp} = -m \int d\lambda \left(\frac{1}{2} (\tilde{C} + \delta \tilde{C}) \cdot \underbrace{\mu^{2\varepsilon}}_{\text{New scale to keep } [\tilde{C}] = 0} \frac{\phi^2}{\Lambda^2} \right)$$

( + ) $\propto$ $\underbrace{\left(\tilde{C} + \overset{\text{some constant from calculation}}{a} \ln\left(\vec{q}^{\,2}/\mu^2\right) + \dots \right)}_{\text{finite but depend on } \mu}$

We can cancel the μ dependence by demanding

$$\frac{d\tilde{C}}{d\ln\mu} = 2a \Rightarrow$$ The coupling runs !

Classical RG effect !