# High-Precision Dynamics and Waveform in Two-Body Scattering

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**"Amplitudes, Strong-Field Gravity and Resummation"**

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# Acronyms

**PN** **Post-Newtonian** approximation
(**expansion in 1/c; ie $v^2/c^2$ and $GM/(c^2r)$**)

**PM** **Post-Minkowskian** approximation (**expansion in G; ie in $GM/(c^2b)$**)
and its recent **Worldline EFT avatars**

**MPM** **Multipolar post-Minkowskian** approximation

**SF** **Gravitational Self-Force**: expansion in m1/m2,

**EOB** **Effective One-Body** Approach

**NR** **Numerical Relativity**

**EFT** various flavours:
**NRGR Effective Field Theory** (EFT) à la Goldberger-Rothstein
**Classical/Eikonal limit of Quantum scattering amplitude**
**Worldline** EFT

**TF** **Tutti Frutti** method

# Based on:

High Precision Black Hole Scattering: Tutti Frutti vs Worldline Effective Field Theory
(Bini,TD) (arXiv:2504.20204)

High-post-Newtonian-order dynamical effects induced by tail-of-tail interactions
in a two body system (Bini,TD, Geralico )

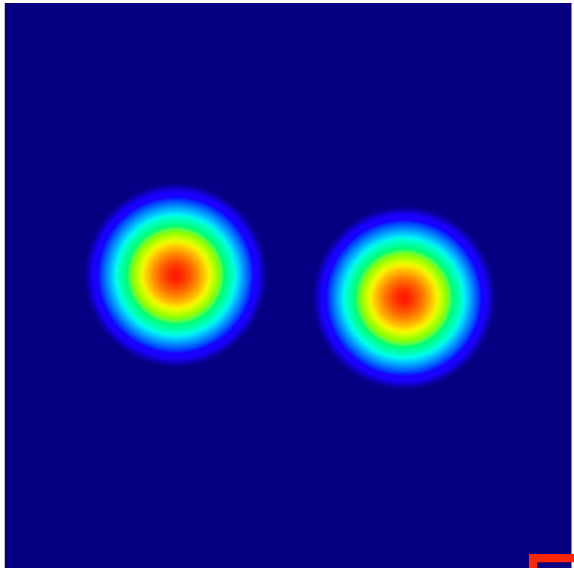
Gravitational scattering of solitonic boson stars: Analytics vs Numerics
(TD, Jain, Sperhake)

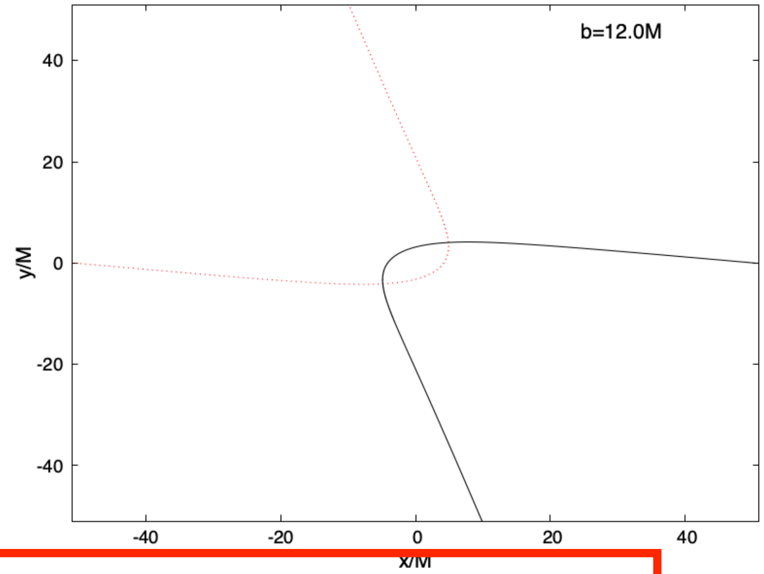
High-order effective-one-body tidal interactions and gravitational scattering
(Malte Schulze, Sebastiano Bernuzzi, Piero Rettegno, Joan Fontbut´e, Andrea Placidi,TD)

Quadrupolar bremsstrahlung waveform at the third-and-a-half post-Newtonian accuracy
(WIP) (Bini,TD, Geralico )

# Gravitational scattering of solitonic boson stars

(TD-Jain-Sperhake, 2512.00945)






complex scalar field $\varphi(t,r) = A(r)e^{\mathrm{i}(\epsilon\omega t+\delta\Phi)}$

$$S = \int \frac{\sqrt{-g}}{2}\left\{\frac{R}{8\pi G} - [g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + V(|\varphi|)]\right\}\mathrm{d}^4x, \qquad V(|\varphi|) = \mu^2|\varphi|^2\left(1-2\frac{|\varphi|^2}{\sigma_0^2}\right)^2$$

**NR results for various systems: BS-BS, BS-antiBS, BS-BSpi/2,BS-BSpi**

$68 \text{ deg} \leq \chi \leq 259 \text{ deg}$

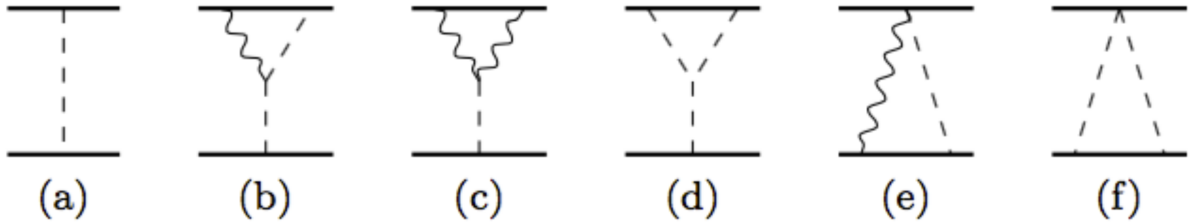
| $\frac{b}{GM}$ | $\frac{J_{\rm in}^{\rm NR}}{GM^2}$ | $\chi_{\rm NR}^{\rm BS-BS}$ | $\chi_{\rm NR}^{\rm BS-\overline{BS}}$ | $\chi_{\rm NR}^{\rm BS-BS^{\frac{\pi}{2}}}$ | $\chi_{\rm NR}^{\rm BS-BS^{\pi}}$ |
|---|---|---|---|---|---|
| 9.9 | 1.15315 | 258.79(95) | 216.41(1.41) | 214.18(1.40) | 198.41(1.26) |
| 10.5 | 1.22360 | 170.51(89) | 165.42(81) | 165.28(81) | 161.23(75) |
| 11.0 | 1.28186 | 143.91(62) | 142.32(60) | 142.28(60) | 140.83(58) |
| 12.0 | 1.39839 | 114.43(64) | 114.20(63) | 114.33(53) | 113.96(65) |
| 13.0 | 1.51492 | 96.89(84) | 96.89(84) | 96.75(60) | 96.81(82) |
| 14.1 | 1.64982 | 83.39(1.14) | 83.38(1.14) | 83.32(1.10) | 83.32(1.11) |
| 15.0 | 1.74798 | 75.51(1.23) | 75.73(1.37) | 75.62(1.30) | 75.62(1.30) |
| 16.0 | 1.86450 | 68.16(1.27) | 68.44(1.44) | 67.97(1.14) | 68.50(1.47) |

# BS scattering: Analytics vs Numerics

$$\chi(\gamma,j)=\chi^{\mathrm{BH}}(\gamma,j)+\chi^{\mathrm{tidal}}(\gamma,j)+\chi^{\mathrm{scalar}}(\gamma,j)$$

$$S_{\mathrm{EFT}}=\int\frac{\sqrt{-g}}{2}\left\{\frac{R}{8\pi G}-\left[g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi+\mu^2\bar{\varphi}\varphi\right]\right\}\mathrm{d}^4x-\sum_A\int\left\{m_A-2\pi\left[\varphi(z_A)\bar{s}_A(\tau_A)+\bar{\varphi}(z_A)s_A(\tau_A)\right]\right\}\mathrm{d}\tau_A\,.$$

$$s_A(\tau_A)=c_Ae^{i\omega_A\tau_A}$$



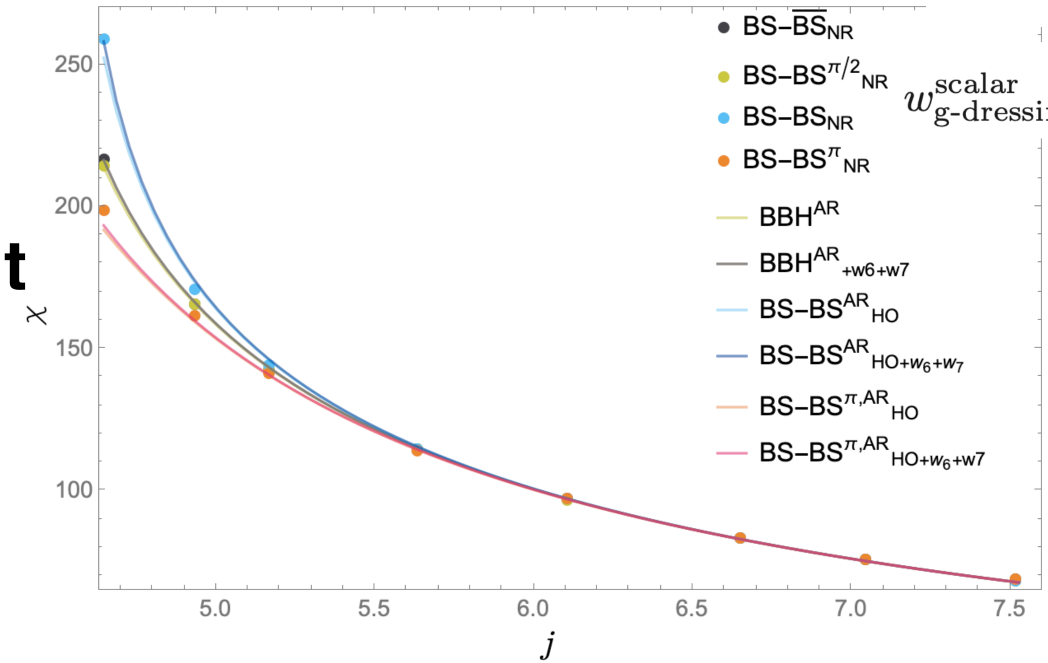

$$w(\gamma,\bar{r})=w^{\mathrm{BH}}(\gamma,\bar{r})+w^{\mathrm{tidal}}(\gamma,\bar{r})+w^{\mathrm{scalar}}(\gamma,\bar{r})$$

two difficulties:
massive propagators+
exp(i omega t) worldline
sources

$$w^{\mathrm{scalar}}_{\mathrm{LO}}=\frac{8\pi c_1c_2}{Gm_1m_2}\frac{e^{-\bar{m}\bar{r}}}{\bar{r}}\cos(\Phi_{21})$$

$$w^{\mathrm{scalar}}_{\text{g-dressing}}=e^{GM((2\omega^2-\mu^2)/\tilde{\mu})\log(2e^{\gamma_{\mathrm{E}}}GM\tilde{\mu}\bar{r})}$$

**good agreement using w-EOB**



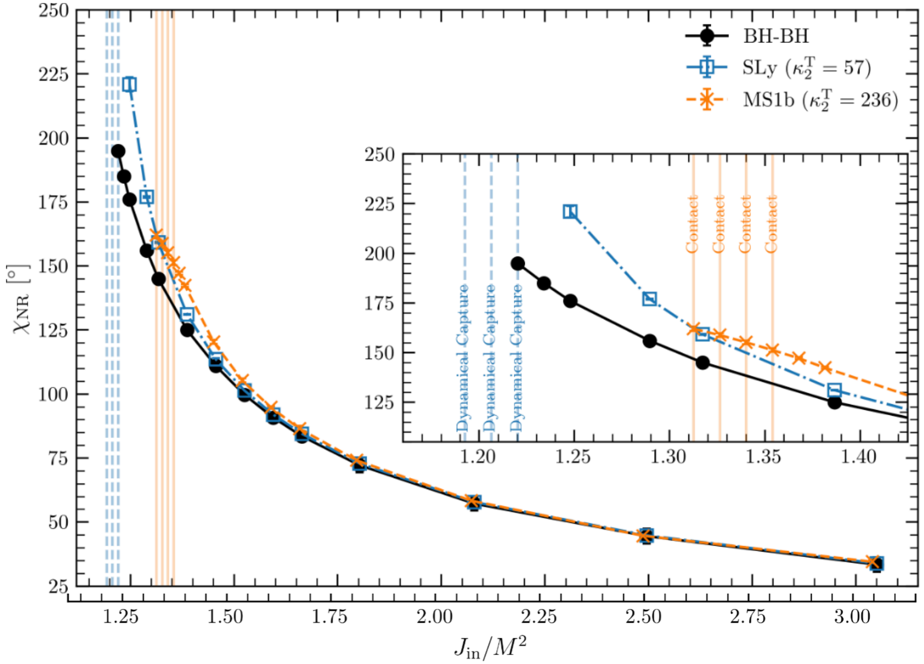
# High-order effective-one-body tidal interactions and gravitational scattering

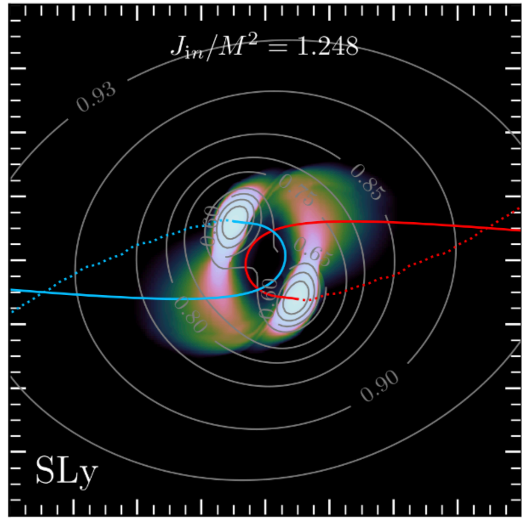
(Malte Schulze, Sebastiano Bernuzzi, Piero Rettegno, Joan Fontbuté, Andrea Placidi,TD,26)

**NR data**

**NS scattering**

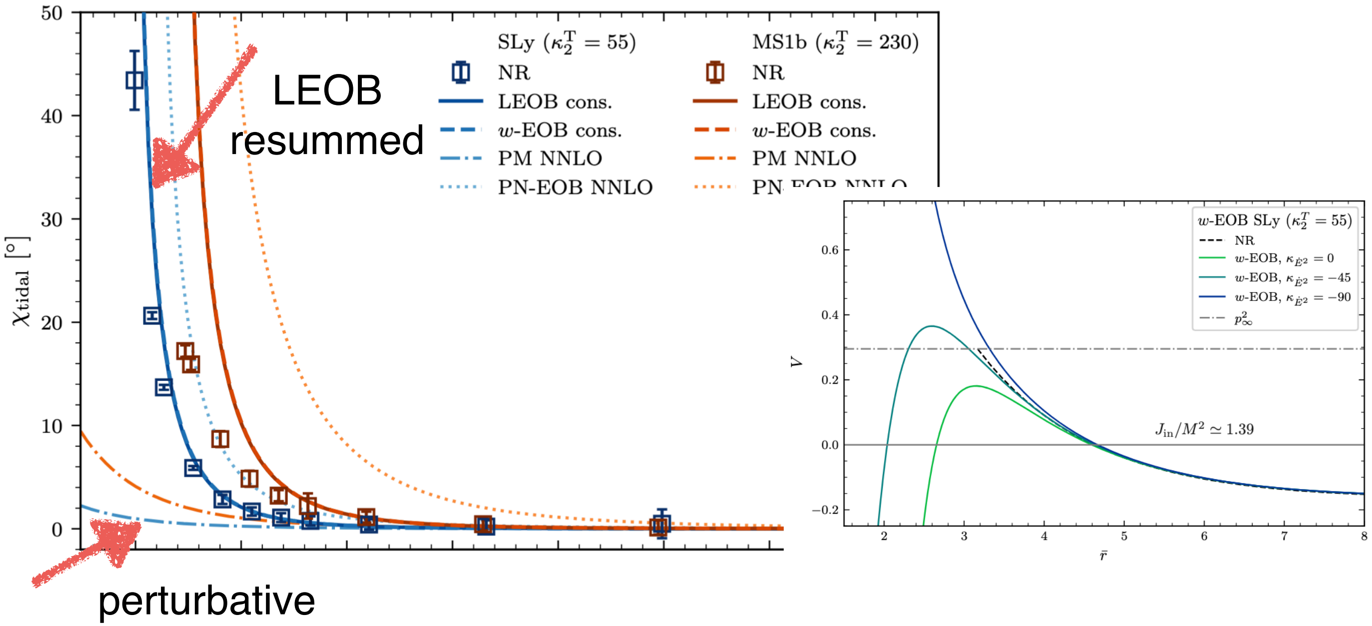
Fontbuté,Bernuzzi+…., 2025




**NR/AR comparisons good agreement for SLy when using resummed EOB potential and NNLO PM-tidal result**

(Jakobsen, G. Mogull, J. Plefka, and B. Sauer)



**High Precision Black Hole Scattering: Tutti Frutti vs Worldline Effective Field Theory**

(Bini,TD)

**High-post-Newtonian-order dynamical effects induced by tail-of-tail interactionsin a two body system**

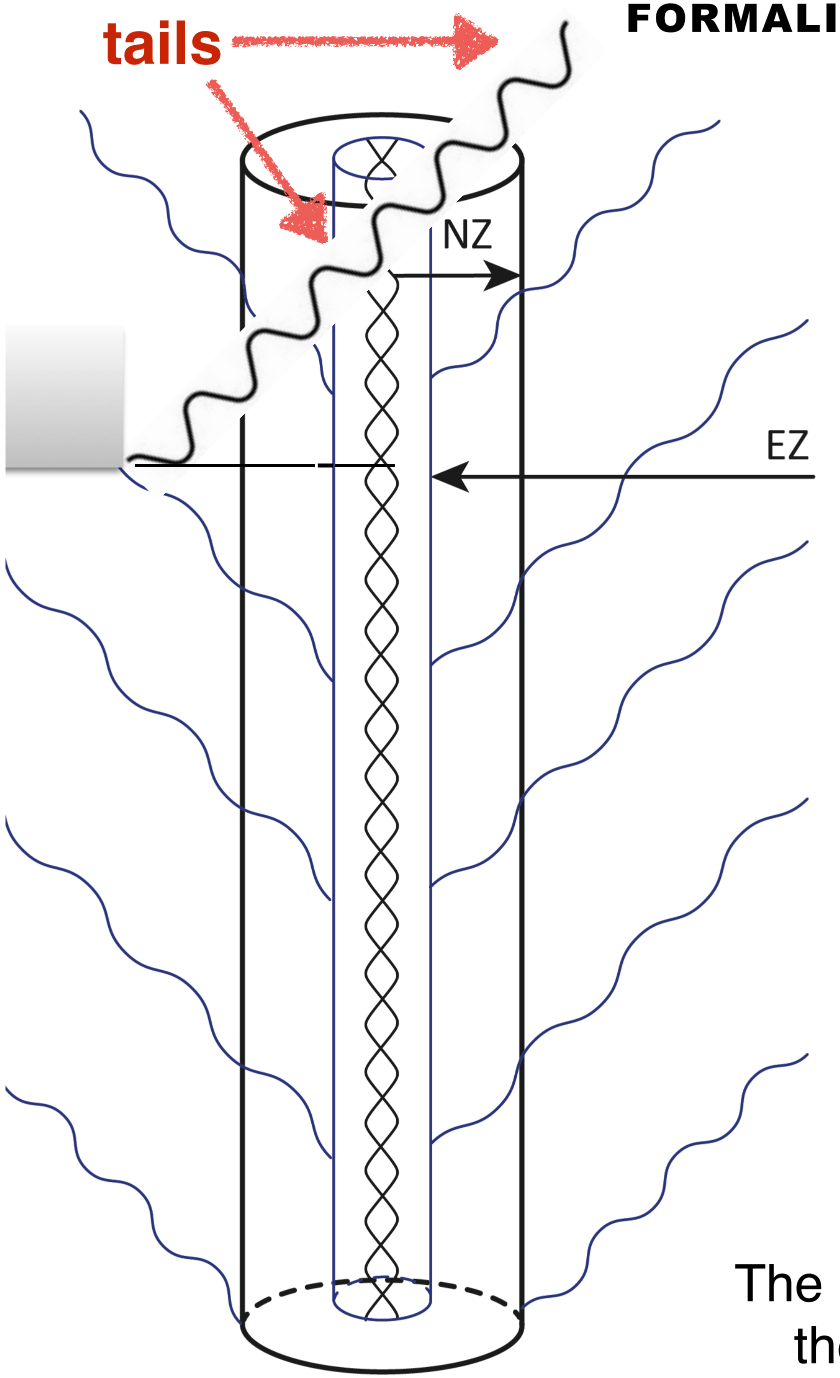
(Bini,TD, Geralico )

# Tutti-Frutti method

(Bini,TD, Geralico ,19-20-21)

**combines PN, MPM, EOB,**
**Delaunay, Self-Force,**
**and mass-polynomiality**
**of scattering angle**

# GRAVITATIONAL WAVE GENERATION: MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER) (1986,88,89,90,...)




**Decomposition of space-time in various overlapping regions:**

1. **near-zone: r << lambda : PN matched to MPM**
2. **exterior zone: r >> r_source: MPM**
3. **far wave-zone: Bondi-type expansion**

**then matching between the zones**

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$g = \eta + Gh_1 + G^2 h_2 + G^3 h_3 + \dots,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \dots i_\ell} \left( \frac{M_{i_1 i_2 \dots i_\ell}(t - r/c)}{r} \right) + \partial\partial\dots\partial \left( \frac{\epsilon_{j_1 j_2 k} S_{k j_3 \dots j_\ell}(t - r/c)}{r} \right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret} \left( \left( \frac{r}{r_0} \right)^B \partial\partial h_1 h_1 \right) + \dots,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \dots$$

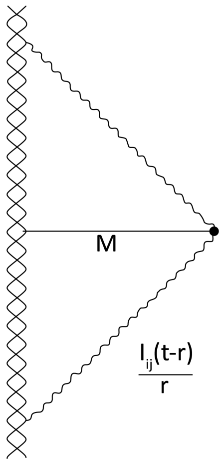
**STF tensors encoding multipole moments**

mass-type and spin-type multipole moments

The PN-matched MPM formalism has allowed to compute the GW emission to very high accuracy (Blanchet et al)

# Tail-transported Nonlocal Contributions to Binary Dynamics

(Blanchet-TD 88, TD 10, DJS 14, BDG 19)




$$\mathcal{R}'_i = -\frac{2}{5c^5}\bar{\rho}x^j \frac{d^5}{dt^5}(\bar{I}_{ij} + \delta \bar{I}_{ij})$$

**4PN level rad reac**

$$\delta \bar{I}_{ab}(t) = \frac{4}{c^3}\bar{I} \int_0^{+\infty} dv \ln \left| \frac{v}{2P} \right| {}^{(2)}\bar{I}_{ab}(t-v)$$

external

equivalent to (FoffaSturani 13)

$$L[t] = \frac{4G^2\mathcal{M}}{5c^8}\mathrm{Pf}_{2\mathrm{P}} \int_{-\infty}^{t_0} dt_0 \frac{I_{ij}^{(3)}(t) I^{\mathrm{ext}\,(3)}_{ij}(t_0)}{t - t_0}$$

Corresponding 4PN **conservative time-symmetric** action (DJS 14)

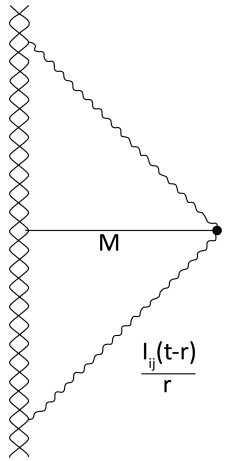
$$S = \frac{G^2\mathcal{M}}{5c^8}\mathrm{Pf}_{2s/c} \int\int dt\, dt' \frac{I_{ij}^{(3)}(t) I_{ij}^{(3)}(t')}{|t - t'|}$$

# Tail-transported Nonlocal Contributions to Binary Dynamics (2)

Generalization to **5PN level**
(TD 10, DJS 15, BDG 19)
using Damour-Soffel-Xu approach,
and the fact that **retarded QQQ terms are local in time** (BD 88)

$$S_{\rm int} = \sum_\ell \frac{1}{\ell!}\int {\rm d}t\Big[G_L^{\rm ext}(t)M_L^{\rm BD}[y(t)] + \frac{\ell}{c^2(\ell+1)}H_L^{\rm ext}(t)S_L^{\rm BD}[y(t)]\Big],$$

system in external nonlocal « tidal » field



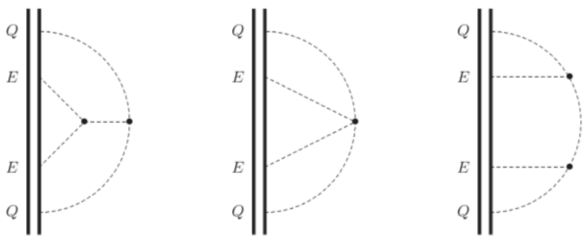

time-symmetric projection

$$S_{\rm nonloc}^{4+5\rm PN}[x_1(s_1),x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt {\rm PF}_{2r_{12}^h(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}_{1\rm PN}^{\rm split}(t,t').$$

$$\mathcal{F}_{1\rm PN}^{\rm split}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I_{ab}^{(3)}(t)I_{ab}^{(3)}(t') + \frac{1}{189c^2}I_{abc}^{(4)}(t)I_{abc}^{(4)}(t') + \frac{16}{45c^2}J_{ab}^{(3)}(t)J_{ab}^{(3)}(t')\right). \quad (6)$$

The mass and spin multipole moments $I_{ab}$, $I_{abc}$, and $J_{ab}$ entering the latter expression are the Blanchet-Damour (1PN-accurate) source multipole moments defined by explicit integrals over the stress-energy tensor of the source [18].

Extension to **5.5PN level**
(DJS 15, BDG 20)
tail-of-tail nonlocal involving




(a) (b) (c)

time-symmetric projection

$$B = -\frac{107}{105},$$

$$S^{\rm tail^2} = -\frac{B}{10}\frac{G}{c^5}\left(\frac{G\mathcal{M}}{c^3}\right)^2\int {\rm d}t\int_{-\infty}^{+\infty}\frac{{\rm d}\tau}{\tau}M_{ij}^{(3)}(t) \times [M_{ij}^{(4)}(t+\tau) - M_{ij}^{(4)}(t-\tau)].$$

# Mass polynomiality structure in scattering (TD'20)

$$\frac{dx_a^\mu}{ds_a} = g^{\mu\nu}(x_a)u_{a\nu},$$

$$\frac{du_{a\mu}}{ds_a} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x_a)u_{a\alpha}u_{a\beta},$$

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu},$$

$$T^{\mu\nu}(x) = \sum_{a=1,2} m_a \int ds_a u_a^\mu u_a^\nu \frac{\delta^4(x - x_a(s_a))}{\sqrt{g}}$$

$$\Delta p_{a\mu} = -\frac{m_a}{2}\int_{-\infty}^{+\infty} ds_a \partial_\mu g^{\alpha\beta}(x_a)u_{a\alpha}u_{a\beta}$$

$$\Delta p_{1\mu} = -2Gm_1m_2 \frac{2(u_{10}\cdot u_{20})^2 - 1}{\sqrt{(u_{10}\cdot u_{20})^2 - 1}}\frac{b_\mu}{b^2} + \frac{Gm_1m_2}{b}\Delta_\mu.$$

**polynomial in Gm1/b and Gm2/b**

**When considering any mass-polynomial conservative scattering impulse, i.e. Delta p1+Delta p2=0**

$$\frac{1}{2}\chi(E_{\rm real}, J) = \frac{\chi_1(\gamma,\nu)}{j} + \frac{\chi_2(\gamma,\nu)}{j^2} + \frac{\chi_3(\gamma,\nu)}{j^3} + \frac{\chi_4(\gamma,\nu)}{j^4} + \cdots,$$

**at G^n**

$$h^{n-1}(\gamma,\nu)\chi_n(\gamma,\nu) = P^\gamma_{d(n)}(\nu),$$

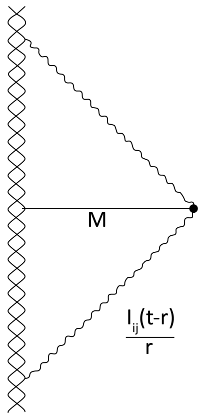
**polynomial in nu of degree [(n-1)/2]**

**0SF scattering gives access to full G^2 dynamics !**
**1SF scattering gives access to full G^3 and G^4 conservative dynamics !**
**2SF scattering gives access to full G^5 and G^6 conservative dynamics**

# Tutti-Frutti method (Bini-TD-Geralico,'19,20,21)

with PN local Hamiltonian

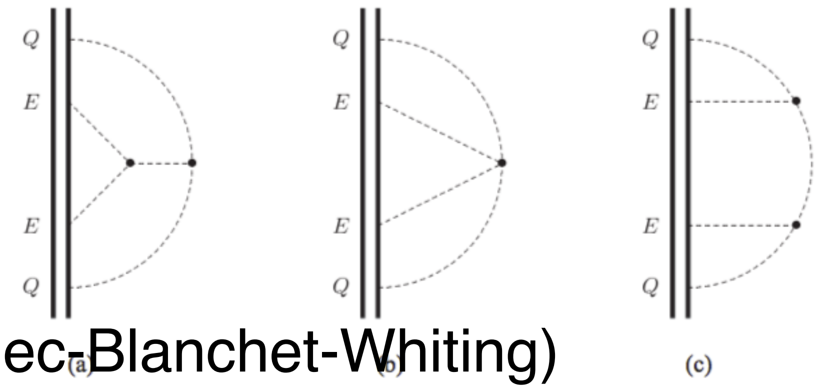
$$S_{\rm tot}^{\leq n{\rm PN}}[x_1(s_1),x_2(s_2)] = S_{\rm loc}^{\leq n{\rm PN}}[x_1(s_1),x_2(s_2)] + S_{\rm nonloc}^{\leq n{\rm PN}}[x_1(s_1),x_2(s_2)].$$




$$S_{\rm nonloc}^{4+5{\rm PN}}[x_1(s_1),x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt\,{\rm PF}_{2r_{12}^h(t)/c}\times\int\frac{dt'}{|t-t'|}\mathcal{F}^{\rm split}_{\rm 1PN}(t,t').$$

$$\mathcal{F}^{\rm split}_{\rm 1PN}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I_{ab}^{(3)}(t)I_{ab}^{(3)}(t') + \frac{1}{189c^2}I_{abc}^{(4)}(t)I_{abc}^{(4)}(t') + \frac{16}{45c^2}J_{ab}^{(3)}(t)J_{ab}^{(3)}(t')\right).$$

starting at 5.5PN, G^5



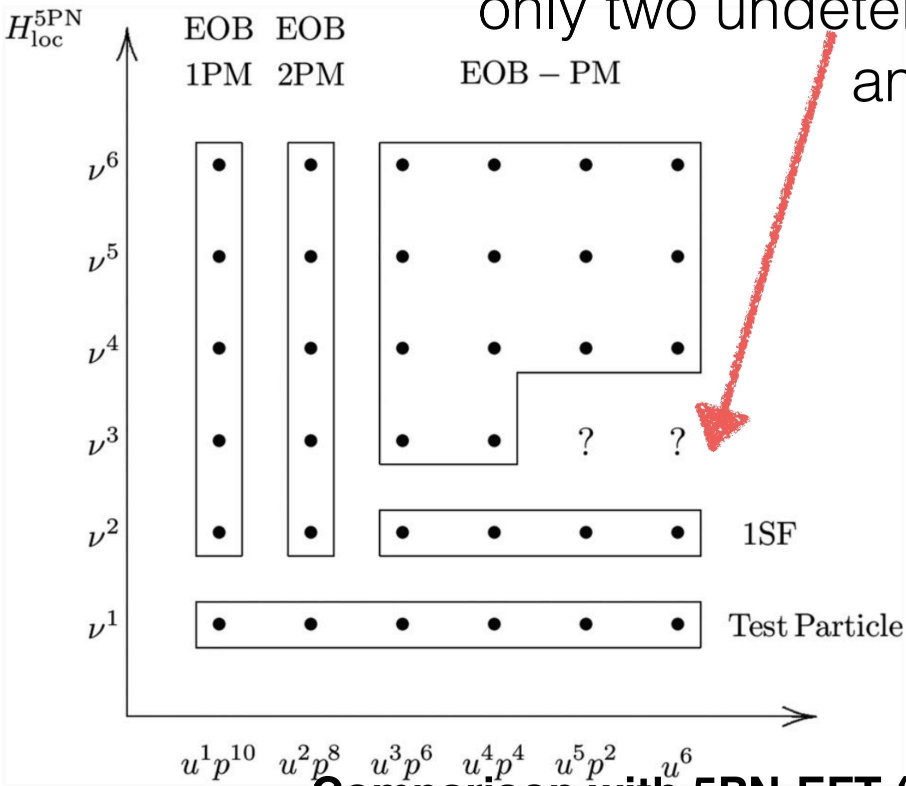

Then combining:
1SF computations of
Detweiler redshift+ first law (LeTiec-Blanchet-Whiting)
polynomiality in nu
Delaunay averaging,
EOB reformulation
one determines H_loc modulo a few 2SF parameters

# 5PN Tutti-Frutti Hamiltonian in EOB-DJS gauge

$$H^{\rm 5PN}_{\rm loc} = \sum_{m+n=6} h_{2mn}(\nu)(p^2)^m u^n. \qquad H = Mc^2\sqrt{1+2\nu(\hat{H}_{\rm eff}-1)}.$$

$$\hat{H}_{\rm eff}{}^2 = A(u;\nu)[1 + A(u;\nu)\bar{D}(u;\nu)p_r^2 + p_\phi^2 u^2 + Q(u,p;\nu)].$$

only two undetermined numerical coefficients among 36 at 5PN




$$a_6^{\rm loc} = \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3,$$

G^6

$$\bar{d}_5^{\rm loc} = \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3$$

G^5

$$q_{44}^{\rm loc} = \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{2075}{3} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3,$$

G^4

**Comparison with 5PN-EFT (Bluemlein+..): agreement in most terms in q44 apart from rational nu^2 term, pi^2 parts of d5 and a6 determined from potential graviton part**

# Subtleties linked to radiation reaction

Rad-reac force starts at G^2/c^5: 2.5PN and 2PM (harmonic TD-Deruelle'81)

$$\frac{\mathbf{F}_1^{\text{rad}}}{m_1} = -\frac{4}{5}\frac{G^2}{c^5}m_1 m_2 \frac{v_{12}^2}{r_{12}^3}\left[\mathbf{v}_{12} - 3(\mathbf{v}_{12}\cdot\mathbf{n}_{12})\mathbf{n}_{12}\right] + O\left(\frac{G^3}{c^5}\right) + O\left(\frac{G^2}{c^7}\right), \qquad (1$$

Causes J loss at G^2 (TD-Deruelle '81, TD'20, Bini-TD 22)

Both Erad and Jrad contribute to rad-reacted impulse via the linear-response formula (Bini-TD 12)

$$\chi^{\text{rr}} = -\frac{1}{2}\frac{\partial\chi^{\text{cons}}(E,J)}{\partial J}J^{\text{rad}} - \frac{1}{2}\frac{\partial\chi^{\text{cons}}(E,J)}{\partial E}E^{\text{rad}}$$

Loss of linear momentum (recoil) starts at G^3/c^7 and be treated linearly at G^4 and G^5 (BDG '21…)

Quadratic Rad-Reac effects start contributing **conservative-like** terms at G^4/c^10: 5PN (Bini-TD-Geralico 21)

$$\delta^{\mathcal{F}^2_{rr}}\chi \sim \frac{G^4 m_1^2 m_2^2}{c^8 b^4}\frac{v_{12}^2}{c^2}\,. \qquad \lim_{t\to+\infty}\epsilon_{rr}^2\mathbf{v}_1^{(2)}(t) = -\frac{3\pi}{8}\epsilon_{rr}^2\frac{v_0^3}{b^4}\hat{\mathbf{b}}\,,$$

# Tutti Frutti vs Worldline Effective Field Theory (BDG 23, Bini-TD'25)

$$\Delta p_a^\mu = \Delta p_a^{\mathrm{cons}\,\mu} + \Delta p_a^{\mathrm{rrlin}J_{\mathrm{rad}}\,\mu} + \Delta p_a^{\mathrm{rrlin}P_{\mathrm{rad}}\,\mu} + \Delta p_1^{\mathrm{rr\,remain\,sup}\,\mu}$$

$$\Delta p_a^{\mathrm{rr\,lin}\,P_{\mathrm{rad}}}{}_\mu \equiv -\frac{1}{2}\frac{\partial\chi^{\mathrm{cons}}}{\partial E} E_{\mathrm{rad}} \frac{d}{d\chi_{\mathrm{cons}}} \Delta p^{\mathrm{cons}}_{a\mu} + \frac{\Delta P_{\mathrm{c.m.}}}{P_{\mathrm{c.m.}}} p^{\mathrm{out}}_{a\mu} - \frac{m_a^2}{E_a}\frac{\Delta P_{\mathrm{c.m.}}}{P_{\mathrm{c.m.}}} U_\mu - \frac{E_a}{E} P^{\mathrm{rad}}_\mu - \frac{(p^{\mathrm{out}}_{a\nu} P^\nu_{\mathrm{rad}})}{E} U_\mu + \Delta p^{\mathrm{rr\,remain}\,P_{\mathrm{rad}}}_{a\mu}.$$

$$\Delta p_a^{\mathrm{rr\,lin}J_{\mathrm{rad}}} \equiv -\frac{1}{2}\frac{\partial\chi^{\mathrm{cons}}}{\partial J} J_{\mathrm{rad}} \frac{d}{d\chi_{\mathrm{cons}}} \Delta p^{\mathrm{cons}}_{a\mu}$$

**only uncertain part linked to rad-reac^2**

$$\Delta p_1^{\mathrm{rr\,remain\;\,sup}\;\mu} = \frac{G^5}{b^5} m_1^3 m_2^3 f_b^{G^5\,\mathrm{remain}} \hat{b}^\mu + O(G^6).$$

# Explicit TF predictions and checks at G^4 and G^5

$$\Delta p_{a\mu} = \Delta p_{a\mu}^{\rm cons} + \Delta p_{a\mu}^{\rm rr\,lin} + \Delta p_{a\mu}^{\rm rr\,nonlin}.$$

**at G^4**
**fully predictive**

$$\Delta p^{\rm rr}_{1\ \mu G^4} = \Delta p^{\rm rr\,lin-odd}_{1\ \mu G^4} + \frac{G^4}{b^4} m_1^3 m_2^2 p_x^{G^4}(\gamma)\hat{b}^{\mu}_{12},$$

$$\Delta p^{\rm rr}_{1\,\mu G^4} = \Delta p^{\rm rr\,lin-odd}_{1\,\mu G^4} + \frac{m_1}{m_2 - m_1} P^{\rm rad}_{x\,G^4}\hat{b}^{\mu}_{12}.$$

relation between the rad-reac^2 term and P^rad_x

direct link rad-rad^2 linear recoil

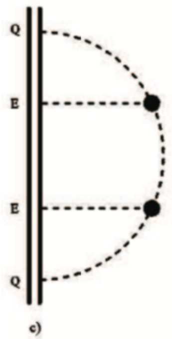
$$c^{\rm 4diss}_{\rm 1b,2rad} = \frac{m_1}{m_2 - m_1} P^{\rm rad}_{xG^4}$$

in **agreement** with full G^4 impulse of Dlapa+ 23, and Damgaard+ 23

in addition determines J3 from cb1rad, agrees with Manohar+…

$$c^{\rm 4diss}_{\rm 1b,1rad} = \frac{G^4}{b^4} m_1^2 m_2^2 \left\{ (m_1+m_2)\left[\mathcal{E}(\gamma)\frac{\gamma(6\gamma^2-5)}{(\gamma^2-1)^{3/2}} - \pi\frac{3}{4}\hat{J}_2(\gamma)\frac{(5\gamma^2-1)}{(\gamma^2-1)^{3/2}} - \hat{J}_3(\gamma)\frac{(2\gamma^2-1)}{(\gamma^2-1)^2}\right] - m_1\mathcal{E}(\gamma)\frac{2\gamma^2-1}{(\gamma+1)\sqrt{\gamma^2-1}} \right\}. \quad (3.2$$

**at G^5 1SF**
agrees with Driesse+



$$f_b^{G^5,1\rm SF} = -\frac{2\tilde{\chi}_5^{\rm 1cons}}{(\gamma^2-1)^2} - \frac{\chi_1 \hat{J}_4^0}{b^5(\gamma^2-1)^2} + K$$

TF determined to 6PN including a **tail-of-tail** term

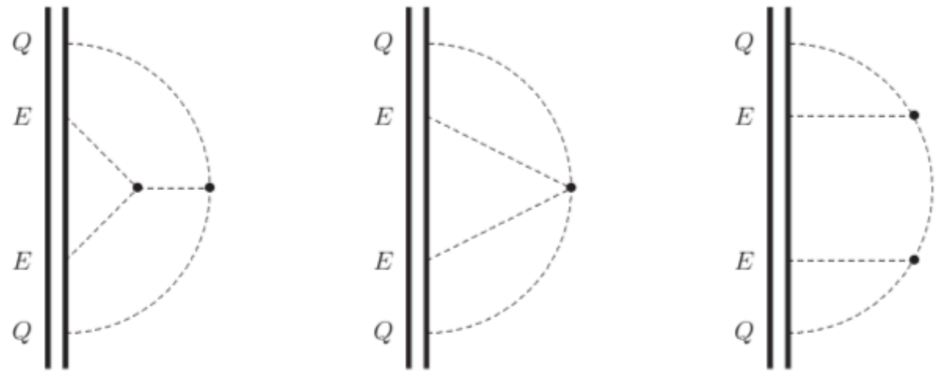
determined to 6PN by TF + Heissenberg'24

TF determined P_mu^rad to 5.5 PN

# High-post-Newtonian-order dynamical effects induced by tail-of-tail interactions in a two body system (BDG'25)

**tail-of-tail conservative action at all multipole orders**

(DJS'14,BD'25,using Blanchet'05,Goldberger-Ross'10,…)




(a) (b) (c)

$$S^{\rm tail-of-tail}_{\rm time-sym} = \frac{1}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 G\sum_{l\geq 2}\frac{1}{c^{2l+1}}a_l\beta_l^{\rm even}\int dt \times \int_{-\infty}^{\infty} dt' I_L^{(l+2)}(t) I_L^{(l+2)}(t') \ln\frac{c|t-t'|}{2r_0}$$
$$+ \frac{1}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 G\sum_{l\geq 2}\frac{1}{c^{2l+3}}b_l\beta_l^{\rm odd}\int dt \times dt' J_L^{(l+2)}(t) J_L^{(l+2)}(t') \ln\frac{c|t-t'|}{2r_0}\,,$$

**1SF confirmations and 2SF new results at the 6.5PN level**

$$A(u,\nu) = 1-2u+\sum_{n\geq 3} a_n(\nu,\ln u)u^n\,,$$
$$\bar{D}(u,\nu) = 1+\sum_{n\geq 2}\bar{d}_n(\nu,\ln u)u^n\,.$$

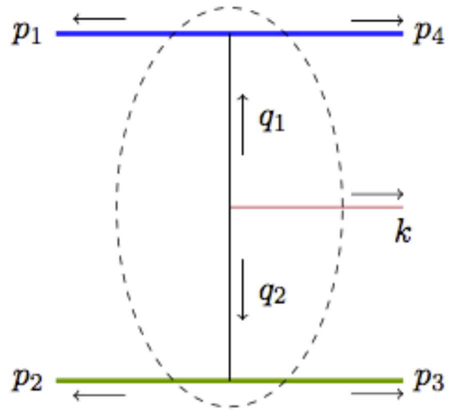
$$a_{6.5} = \frac{13696}{525}\nu\pi\,,$$
$$a_{7.5} = -\frac{10052}{225}\nu^2\pi - \frac{512501}{3675}\nu\pi\,,$$

**agreement at 6.5PN with 1SF Driesse et al** (using a recent result of Geralico'25)

**prediction of the « conservative » G^5 scattering at 2SF in terms of TF undetermined parameters**

# PM waveform computation $W(k^\mu) = \epsilon^\mu \epsilon^\nu h_{\mu\nu}(\omega, \theta, \phi)$

G^1=1PM (linearized,Einstein 1918) stationary $\propto \delta(\omega)$




## LO (tree level) waveform

G^2=2PM: classical time-domain W(t,n): Kovacs-Thorne 1977

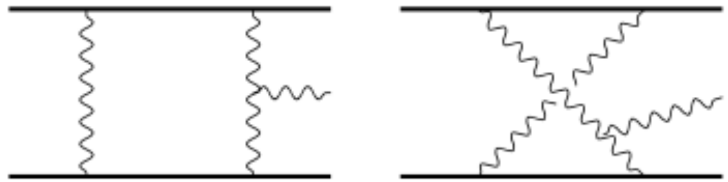
quantum-based: yields W(k,p1,p2,p3,p4)=W(k,p1,p2,q1)

Johansson-Ochirov'15, GoldbergerRidgway'17 Luna-Nicholson-OConnellWhite'18
Mougiakakos-Riva-Vernizzi'21,Bautista-Siemonsen'22, De Angelis-Gonzo-Novichkov'23

## Recent NLO (one-loop) waveform

G^3=3PM

Brandhuber+'23, Herderschee+'23, Georgoudis+'23, Bohnenblust+'24



**5-point HEFT one-loop amplitude**
**—-> O(G^3) waveform via KMOC**

**asymptotic metric**

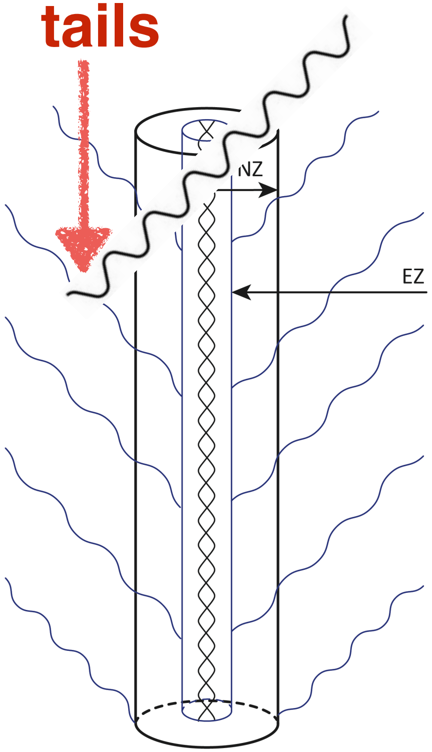
$$\begin{aligned}
&\mathscr{M}(\varepsilon, k, p_1, p_2, q_1, q_2) \\
&\quad \equiv i\langle p_3 p_4 | \hat{a}(k) \mathbb{T} | p_1 p_2 \rangle + \langle p_3 p_4 | \mathbb{T}^\dagger \hat{a}(k) \mathbb{T} | p_1 p_2 \rangle \\
&\quad = i\langle p_3 p_4 k | \mathbb{T} | p_1 p_2 \rangle + \langle p_3 p_4 | \mathbb{T}^\dagger \hat{a}(k) \mathbb{T} | p_1 p_2 \rangle \,,
\end{aligned}$$

**« cut term » important (Caron-Huot+'23)**

# Comparing one-loop amplitude to MPM waveform

(Bini-TD-Geralico'23)

**tails**




**algorithmic STF tensors encoding multipole moments (related to the source moments I_L,J_L)**

$$g = \eta + Gh_1 + G^2 h_2 + G^3 h_3 + \ldots,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \ldots i_\ell}\left(\frac{M_{i_1 i_2 \ldots i_\ell}(t-r/c)}{r}\right) + \partial\partial\ldots\partial\left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t-r/c)}{r}\right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret}\left(\left(\frac{r}{r_0}\right)^B \partial\partial h_1 h_1\right) + \ldots,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \ldots.$$

**radiative multipole moments (observable at infinity) U_L, V_L**

$$r h_{ij}^{\rm TT} = \frac{4G}{c^2} P(n)_{ijab} \sum_{l=2}^{\infty} \frac{1}{c^l}\frac{1}{l!}\left(U_{abL-2} n_{L-2} - \frac{2l}{c(l+1)} n_{cL-2}\epsilon_{cd(a} V_{b)dL-2}\right)$$

$$\mathcal{M}^{\rm MPM}(k,b,u_1,u_2,m_1,m_2) = -i\frac{\kappa}{2}\epsilon^\mu\epsilon^\nu h^{\rm MPM}_{\mu\nu}(\omega,\theta,\phi) = -i\frac{\kappa}{2}\int dt e^{i\omega t}\epsilon^\mu\epsilon^\nu h^{\rm MPM}_{\mu\nu}(t,\theta,\phi)$$

$$\mathcal{M}^{\rm HEFT}(k,b,u_1,u_2,m_1,m_2) =$$
$$e^{i\frac{b_1+b_2}{2}\cdot k}\int \frac{d^D q}{(2\pi)^{D-2}}\,\delta\left(2p_1\cdot\left(q+\frac{k}{2}\right)\right)\delta\left(2p_2\cdot\left(-q+\frac{k}{2}\right)\right)\, e^{iq\cdot(b_1-b_2)}\mathcal{M}^{(1)}_{5,\rm HEFT}\left(q+\frac{k}{2},-q+\frac{k}{2};h\right)$$

# Previous comparisons one-loop amplitude vs MPM waveform

$$W(t,\theta,\phi) \sim \frac{1}{c^4}\left(G\,(\text{stationary}) + G^2(1+\frac{1}{c^1}+\frac{1}{c^2}+\frac{1}{c^3}+\cdots) + G^3(1+\frac{1}{c^1}+\frac{1}{c^2}+\frac{1}{c^3}+\cdots) + O(G^4)\right)$$

**tree-level** **one-loop**

**Aim: accuracy up to radiation-reaction effects: O(1/c^5) beyond LO quadrupole**

$$U_{ij}(\omega) \sim \left(G(1+\frac{1}{c^2}+\frac{1}{c^4}) + G^2(1+\frac{1}{c^2}+\frac{1}{c^3}+\frac{1}{c^4}+\frac{1}{c^5}) + O(G^4)\right) + O(\frac{1}{c^6})$$

**Newtonian G^2** **LO tail** **rad-reac plus similar effects**

$$U_{ij}^{\rm tail}(t) = \frac{2G\mathcal{M}}{c^3}\int_0^\infty d\tau I_{ij}^{(4)}(t-\tau)\left(\ln\left(\frac{\tau}{2b_0}\right)+\frac{11}{12}\right)$$

**Main results of the initial EFT-MPM comparison (Bini-TD-Geralico, 2023): mismatch at the Newtonian level, except if one refers the one-loop amp. to classical averaged momenta, rather than incoming momenta; then the terms linked to time-even PN corrections to multipoles agree but there are many mismatches at the G^2/c^5 level**

**Updated comparisons (Georgoudis et al.'23,'24, Bini et al. '24) lead to perfect agreement after taking into account three subtle effects:**

**(1) the bilinear-in-amplitude KMOC term generates the needed rotation**
**(2) IR divergences generate an additional (D-4)/(D-4) contribution**
**(3) zero-frequency gravitons contribute additional terms at h~G and h~G^3**
**(4) interesting links beween zero-freq gravitons and BMS frame** (Veneziano-Vilkovisky)

**Quadrupolar bremsstrahlung waveform at the third-and-a-half post-Newtonian accuracy**
(WIP) (Bini,TD, Geralico )

Extending the MPM accuracy to 3.5PN and to the 2-loop level:
i.e. O(G^4) contributions to h(omega), or O(G^3) contribs to W(omega)=h/(4 G)

Main motivation: high-precision MPM bremsstrahlung waveform at **2-loop** to provide future comparison

**Surprises**: added MPM complexity of reaching the 3.5PN level, and an interesting finding about **BMS**!

$$\begin{aligned} h_c(t_r, \theta, \phi) &= \lim_{R\to\infty} (R(h_+ - i h_\times)) \\ &= \lim_{R\to\infty} \bar{m}^\mu \bar{m}^\nu R\, h_{\mu\nu}\,. \end{aligned}$$

$$W(t_r, \theta, \phi) \equiv \frac{c^4}{4G} h_c(t_r, \theta, \phi)$$

$$t_r = t - \frac{r}{c} - \frac{2G\mathcal{M}}{c^3} \ln \frac{r}{b_0}$$

$$\hat{W}(\omega, \theta, \phi) = \int_{-\infty}^{\infty} dt_r e^{i\omega t_r} W(t_r, \theta, \phi)$$

**Focus on the quadrupolar waveform l=2^+** $\hat{U}_2(\omega,\theta,\phi) = \frac{1}{2!}\bar{m}^i\bar{m}^j\hat{U}_{ij}(\omega)$

$$U_{ij} = M^{(2)}_{ij} + \eta^3 U^{\overline{1.5\mathrm{PN}}}_{ij} + \eta^5 U^{\overline{2.5\mathrm{PN}}}_{ij} + \eta^6 U^{\overline{3\mathrm{PN}}}_{ij} + \eta^7 U^{\overline{3.5\mathrm{PN}}}_{ij} + O\left(\eta^8\right)$$

radiative moment at infinity

source moment

extra contributions from gauge moments, e.g:

$$\begin{aligned} M_{ij} = {} & I_{ij} + 4G\eta^5 M^{W_0 I_2}_{ij} \\ & + 4G\eta^7 [M^{W_2 I_2}_{ij} + M^{Y_2 I_2}_{ij} + M^{X_0 I_2}_{ij} + M^{W_1 I_3}_{ij} \\ & + M^{Y_1 I_3}_{ij} + M^{W_0 W_2}_{ij} + M^{W_1 W_1}_{ij} + M^{W_0 Y_2}_{ij} + M^{W_1 Y_1}_{ij} \\ & + M^{Z_1 I_2}_{ij} + M^{W_1 J_2}_{ij} + M^{Y_1 J_2}_{ij}] \,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} M^{W_0 I_2}_{ij} &= W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij} \,, \\ M^{W_2 I_2}_{ij} &= \frac{4}{7} W^{(1)}_{a\langle i} I^{(3)}_{j\rangle a} + \frac{6}{7} W_{a\langle i} I^{(4)}_{j\rangle a} \end{aligned}$$

hereditary terms: tails, memory, instantan.

$$\begin{aligned} U^{\overline{1.5\mathrm{PN}}}_{ij}(t_r) &= 2G\mathcal{M}\eta^3 \int_0^{+\infty} d\tau\, M^{(4)}_{ij}(t_r-\tau)\left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12}\right] , \\ U^{\overline{2.5\mathrm{PN}}}_{ij}(t_r) &= G\eta^5 \left\{ U^{\overline{2.5\mathrm{PN}(\mathrm{mem})}}_{ij} + U^{\overline{2.5\mathrm{PN} I_2 I_2}}_{ij} + \mathrm{U}^{2.5\mathrm{PN} I_2 J_1}_{ij} \right\} , \\ U^{\overline{2.5\mathrm{PN}(\mathrm{mem})}}_{ij}(t_r) &= -\frac{2}{7}\int_0^{+\infty} d\tau \left[M^{(3)}_{a\langle i} M^{(3)}_{j\rangle a}\right](t_r-\tau) \,, \\ U^{\overline{2.5\mathrm{PN} I_2 I_2}}_{ij}(t_r) &= \frac{1}{7} M^{(5)}_{a\langle i} M_{j\rangle a} - \frac{5}{7} M^{(4)}_{a\langle i} M^{(1)}_{j\rangle a} - \frac{2}{7} M^{(3)}_{a\langle i} M^{(2)}_{j\rangle a} \,, \\ U^{\overline{2.5\mathrm{PN} I_2 J_1}}_{ij}(t_r) &= \frac{1}{3}\epsilon_{ab\langle i} M^{(4)}_{j\rangle a} S_b \,, \\ U^{\overline{3\mathrm{PN}}}_{ij}(t_r) &= 2G^2\mathcal{M}^2\eta^6 \int_0^{+\infty} d\tau\, M^{(5)}_{ij}(t_r-\tau)\left[\ln^2\left(\frac{c\tau}{2b_0}\right) + \frac{11}{6}\ln\left(\frac{c\tau}{2b_0}\right) - \frac{107}{105}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \end{aligned}$$

$$U_{ij}^{\overline{3.5\mathrm{PN}}} = G\eta^7 \Big[\mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}(\mathrm{mem})} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}I_2 I_4} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}I_3 I_3} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}J_1 J_3} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}J_2 J_2} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{PN}}J_3 I_2} + \mathrm{U}_{ij}^{\overline{3.5\mathrm{P}}}$$

$$\begin{aligned}
U_{ij}^{\overline{3.5\mathrm{PN}}(\mathrm{mem})} &= \int_0^{+\infty} d\tau \left[-\frac{5}{756} M_{ab}^{(4)} M_{ijab}^{(4)} - \frac{32}{63} S_{a\langle i}^{(3)} S_{j\rangle a}^{(3)}\right](t_r - \tau) \\
&+ \epsilon_{ac\langle i} \int_0^{+\infty} d\tau \left[\frac{5}{42} S_{j\rangle cb}^{(4)} M_{ab}^{(3)} - \frac{20}{189} M_{j\rangle cb}^{(4)} S_{ab}^{(3)}\right](t_r - \tau)\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}I_2 I_4} &= -\frac{1}{432} M_{ab} M_{ijab}^{(7)} + \frac{1}{432} M_{ab}^{(1)} M_{ijab}^{(6)} - \frac{5}{756} M_{ab}^{(2)} M_{ijab}^{(5)} + \frac{19}{648} M_{ab}^{(3)} M_{ijab}^{(4)} + \frac{1957}{3024} M_{ab}^{(4)} M_{ijab}^{(3)} \\
&+ \frac{1685}{1008} M_{ab}^{(5)} M_{ijab}^{(2)} + \frac{41}{28} M_{ab}^{(6)} M_{ijab}^{(1)} + \frac{91}{216} M_{ab}^{(7)} M_{ijab}\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}I_3 I_3} &= -\frac{5}{252} M_{ab\langle i} M_{j\rangle ab}^{(7)} + \frac{5}{189} M_{ab\langle i}^{(1)} M_{j\rangle ab}^{(6)} + \frac{5}{126} M_{ab\langle i}^{(2)} M_{j\rangle ab}^{(5)} + \frac{5}{2268} M_{ab\langle i}^{(3)} M_{j\rangle ab}^{(4)}\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}J_1 J_3} &= \frac{5}{42} S_a S_{ija}^{(5)}\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}J_2 J_2} &= \frac{80}{63} S_{a\langle i} S_{j\rangle a}^{(5)} + \frac{16}{63} S_{a\langle i}^{(1)} S_{j\rangle a}^{(4)} - \frac{64}{63} S_{a\langle i}^{(2)} S_{j\rangle a}^{(3)}\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}J_3 I_2} &= \epsilon_{ac\langle i} \bigg(\frac{1}{168} S_{j\rangle bc}^{(6)} M_{ab} + \frac{1}{24} S_{j\rangle bc}^{(5)} M_{ab}^{(1)} + \frac{1}{28} S_{j\rangle bc}^{(4)} M_{ab}^{(2)} - \frac{1}{6} S_{j\rangle bc}^{(3)} M_{ab}^{(3)} + \frac{3}{56} S_{j\rangle bc}^{(2)} M_{ab}^{(4)} \\
&+ \frac{187}{168} S_{j\rangle bc}^{(1)} M_{ab}^{(5)} + \frac{65}{84} S_{j\rangle bc} M_{ab}^{(6)}\bigg)\,, \\
U_{ij}^{\overline{3.5\mathrm{PN}}I_3 J_2} &= \epsilon_{ac\langle i} \bigg(\frac{1}{189} M_{j\rangle bc}^{(6)} S_{ab} - \frac{1}{189} M_{j\rangle bc}^{(5)} S_{ab}^{(1)} + \frac{10}{189} M_{j\rangle bc}^{(4)} S_{ab}^{(2)} + \frac{32}{189} M_{j\rangle bc}^{(3)} S_{ab}^{(3)} + \frac{65}{189} M_{j\rangle bc}^{(2)} S_{ab}^{(4)} \\
&- \frac{5}{189} M_{j\rangle bc}^{(1)} S_{ab}^{(5)} - \frac{10}{63} M_{j\rangle bc} S_{ab}^{(6)}\bigg)\,.
\end{aligned}$$

in addition to the MPM nonlinear contributions one must explicitly solve the 3.5PN-accurate hyperbolic motion, including rad-reac effects, and finally compute the Fourier transform from t_r->omega

e.g.

$$
\begin{aligned}
\delta^{\mathrm{rr},\mathrm{G}^3} x \; = \; & \left(-\frac{8}{5}\frac{1}{(1+T^2)^{3/2}}\mathrm{arcsinh}(T) - \frac{47}{15}\mathrm{arctan}(T) - \frac{47}{30}\pi \right.\\
& \left. - \frac{8}{5}(1+T^2)^{1/2} + \frac{8}{5(1+T^2)^{1/2}} - \frac{8}{5}T - \frac{59}{15(1+T^2)}T\right)\frac{G^3M^3\nu}{p_\infty b^2}\eta^5 \\
& + \left[\left(\left(\frac{464}{105}+\frac{4}{5}\nu\right)\frac{1}{(1+T^2)^{3/2}} + \left(-\frac{12}{5}\nu - \frac{98}{5}\right)\frac{1}{(1+T^2)^{5/2}} + \frac{16}{(1+T^2)^{7/2}}\right)\mathrm{arcsinh}(T)\right.\\
& + \left(-\frac{172}{35} - \frac{12}{5}\nu\right)(1+T^2)^{1/2} - \frac{172}{35}T - \frac{4997}{1680}\pi - \frac{4997}{840}\mathrm{arctan}(T)\\
& - \frac{12}{5}\nu T + \frac{40}{7(1+T^2)^{1/2}} + \left(-\frac{2357}{840} - 2\nu\right)\frac{T}{(1+T^2)} + \left(\frac{12}{5}\nu - \frac{4}{5}\right)\frac{1}{(1+T^2)^{3/2}}\\
& \left. + \left(-\frac{47}{5}\nu - \frac{97}{12}\right)\frac{T}{(1+T^2)^2} + \frac{16T}{(1+T^2)^3}\right]\frac{\nu G^3M^3 p_\infty}{b^2}\eta^7 \,,
\end{aligned}
$$

iterated Bessel functions:

$$
\begin{aligned}
Q_1^{\mathrm{as}}(u) \; & \equiv \; \int_{-\infty}^{+\infty} dT e^{iuT}\frac{\mathrm{arcsinh}(T)}{(1+T^2)} \,,\\
Q_{1/2}^{\mathrm{as2}}(u) \; & \equiv \; \int_{-\infty}^{+\infty} dT e^{iuT}\frac{\mathrm{arcsinh}^2(T)}{(1+T^2)^{1/2}} \,,\\
Q_{1/2}^{\mathrm{at}}(u) \; & \equiv \; \int_{-\infty}^{+\infty} dT e^{iuT}\frac{\mathrm{arctan}(T)}{(1+T^2)^{1/2}} \,.
\end{aligned}
$$

# sample two-loop results:

| | |
|---|---|
| $U_2^{\bar{m}_1^2 G^3 \eta^0}$ | $\frac{\nu}{b^2 p_\infty^5}\left[(2+u^2)K_0(u)+2uK_1(u)-\frac{1}{2}u^3 Q_{1/2}^{\mathrm{as2}\,\prime}(u)\right]$ |
| $U_2^{\bar{m}_1 \bar{m}_2 G^3 \eta^0}$ | $\frac{\nu}{b^2 p_\infty^5}\left[6iuK_0(u)+(4i+2iu^2)K_1(u)+iu^3 Q_{1/2}^{\mathrm{as2}}(u)-iu^2 Q_{1/2}^{\mathrm{as2}\,\prime}(u)\right]$ |
| $U_2^{\bar{m}_2^2 G^3 \eta^0}$ | $\frac{\nu}{b^2 p_\infty^5}\left[(-2-u^2)K_0(u)-3uK_1(u)-\frac{1}{2}u^2 Q_{1/2}^{\mathrm{as2}}(u)+\frac{1}{2}u^3 Q_{1/2}^{\mathrm{as2}\,\prime}(u)\right]$ |

| | |
|---|---|
| $U_2^{\bar{m}_1^2 G^3 \eta^7}$ | $\frac{\nu p_\infty^2}{b^2}[A_{K_0}^{\bar{m}_1^2} K_0(u) + A_{K_1}^{\bar{m}_1^2} K_1(u) + A_{e^{-u}}^{\bar{m}_1^2} e^{-u}]$ |
| $A_{K_0}^{\bar{m}_1^2}$ | $\left(\frac{32\nu^2}{7}-\frac{1472\nu}{105}\right)u^2+\left(\left(\left(-\frac{\pi^2}{7}-\frac{3853i\pi}{630}\right)\nu^2+\left(-\frac{59\pi^2}{42}+\frac{1117i\pi}{180}\right)\nu+\frac{47\pi^2}{84}-\frac{517i\pi}{504}\right)u^4+\mathcal{L}\left(-\frac{2}{7}i\pi\nu^2-\frac{59i\pi\nu}{21}+\frac{47i\pi}{42}\right)u^4$ |
| $A_{K_1}^{\bar{m}_1^2}$ | $\left(\left(-\frac{41\pi^2}{189}+\frac{1592i\pi}{567}\right)\nu^2+\left(\frac{247\pi^2}{378}-\frac{16523i\pi}{5670}\right)\nu-\frac{71\pi^2}{378}+\frac{781i\pi}{2268}\right)u^5$ |
| | $\left(\left(-\frac{96}{35}+\frac{386i\pi}{945}-\frac{8\pi^2}{63}\right)\nu^2+\left(\frac{152}{105}-\frac{541i\pi}{2520}-\frac{31\pi^2}{252}\right)\nu+\frac{2071\pi^2}{504}-\frac{22781i\pi}{3024}\right)u^3$ |
| | $+\mathcal{L}\left(\left(\left(-\frac{82}{189}i\pi\nu^2+\frac{247i\pi\nu}{189}-\frac{71i\pi}{189}\right)u^5+\left(-\frac{16}{63}i\pi\nu^2-\frac{31i\pi\nu}{126}+\frac{2071i\pi}{252}\right)u^3\right)$ |
| $A_{e^{-u}}^{\bar{m}_1^2}$ | $-\frac{409i\pi\nu^2}{1344}-\frac{3397i\pi\nu}{2688}+\left(-\frac{16073i\pi\nu^2}{12096}+\left(-\frac{39\pi^2}{14}+\frac{114349i\pi}{17280}\right)\nu+\frac{61\pi^2}{28}-\frac{671i\pi}{168}\right)u^3$ |
| | $+\left(\frac{14221i\pi\nu^2}{10080}+\left(-\frac{33\pi^2}{14}-\frac{84703i\pi}{60480}\right)\nu-\frac{97\pi^2}{28}+\frac{1067i\pi}{168}\right)u^2+\left(-\frac{477}{320}i\pi\nu^2+\left(-\frac{33\pi^2}{14}+\frac{13939i\pi}{2688}\right)\nu-\frac{97\pi^2}{28}+\frac{1067i\pi}{168}\right)u$ |
| | $+\frac{-\frac{409i\pi\nu^2}{1344}-\frac{3397i\pi\nu}{2688}}{u}+\mathcal{L}\left(\left(\frac{61i\pi}{14}-\frac{39i\pi\nu}{7}\right)u^3+\left(-\frac{33}{7}i\pi\nu-\frac{97i\pi}{14}\right)u^2+\left(-\frac{33}{7}i\pi\nu-\frac{97i\pi}{14}\right)u\right)$ |

# Comparison with 3.5PN-accurate one-loop waveform

(Heissenberg-Russo 25, De Angelis, 25)

Besides checks from log-dependence, SF (against Geralico'26), and soft limit-memory term (including nonlinear memory)

$$\delta U_2 = U_2^{\rm EFT} - U_2^{\rm MPM}$$

using

$$\ln(2 b_0 \mu_{\rm IR}) + \gamma_E = 0$$

we found the following **mismatches**

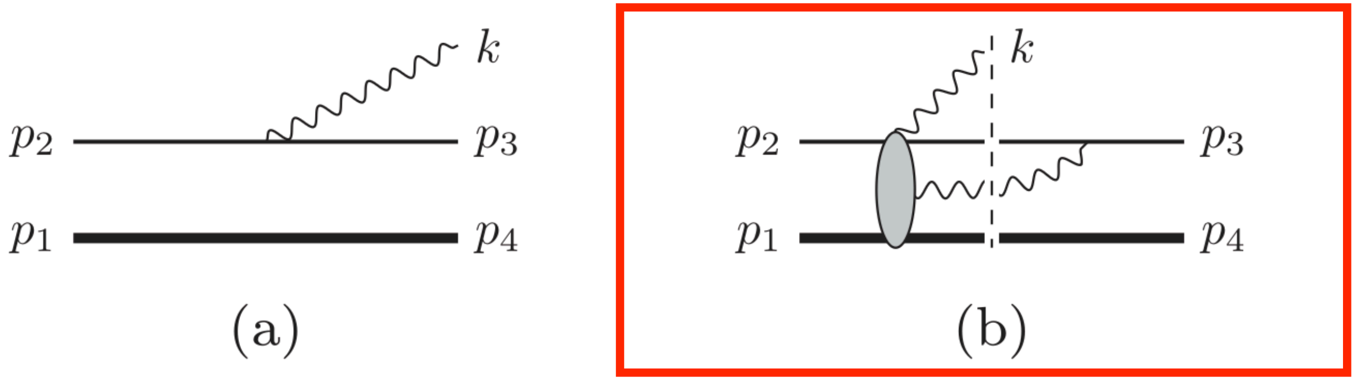
$$\begin{aligned}
\delta U_{22} &= i\nu^2 G^2 \sqrt{\frac{\pi}{5}} p_\infty^4 \frac{4}{105 b}(1-4\nu)u[2(3+2u+5u^2)K_0(u) \\
&+ (16+9u+10u^2)K_1(u)]\,, \\
\delta U_{20} &= i\nu^2 G^2 \sqrt{\frac{2\pi}{15}} p_\infty^4 \frac{4}{35 b}(1-4\nu)u[2K_0(u) \\
&+ 11uK_1(u)] \\
\delta U_{2\bar{2}} &= i\nu^2 G^2 \sqrt{\frac{\pi}{5}} p_\infty^4 \frac{4}{105 b}(1-4\nu)u[2(3-2u+5u^2)K_0(u) \\
&+ (-16+9u-10u^2)K_1(u)]
\end{aligned} \tag{9.10}$$

# Structure of the one-loop waveform (from Tale of Two Formalism paper)

(D-4)/(D-4) term !

$$W^{\mathrm{EFT}}(\omega,\mathbf{n}) = W^{G^1} + W^{G^2(0)} + W^{G^2\epsilon/\epsilon} + W^{G^2\mathrm{cut}} + W^{G^2\mathrm{disc,reg}'} + \delta W^{G^2\mathrm{rot}}$$

delicate term linked to disconnected diagrams involving zero-energy gravitons




$$\mathcal{M}_{\mathrm{disc}} = -i\kappa^2 \mathcal{M}^{\mathrm{tree}}[m_2^2 w_2^2 I(u_3,k) + m_1^2 w_1^2 I(u_4,k)],$$

$$I(u_3,k) = \frac{1}{m_2}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\hat{\delta}(2u_3\cdot\ell)\hat{\delta}(\ell^2)\Theta(\ell^0)}{2\ell\cdot k} = \frac{1}{m_2\omega}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\hat{\delta}(2u_3\cdot\ell)\hat{\delta}(\ell^2)\Theta(\ell^0)}{2\ell\cdot \hat{k}}$$

logarithmically divergent integral which needed an additional regularization besides dim-reg

$$\mathscr{M}_{\text{disc}} = \mathcal{S}^{1 \text{ loop disc}} - i\omega G E\left[\frac{1}{\epsilon} - \log\frac{\beta^2}{\pi}\right]\mathcal{M}^{\text{tree}},$$

$$\mathcal{S}^{1 \text{ loop disc}} = iG\left[m_1 w_1 \log\frac{w_1^2}{\omega^2} + m_2 w_2 \log\frac{w_2^2}{\omega^2}\right]\mathcal{M}^{\text{tree}}$$
$$= +i\omega\beta^{\text{VV}}(\theta,\phi)\mathcal{M}^{\text{tree}}$$

yields the **Veneziano-Vilkovisky supertranslation** aligning the EFT BMS frame with the MPM BMS frame

However, we found that

$$\delta U_2 = +i\omega\beta^{\text{VV}}_{l=1}(\theta,\phi)W_{\text{tree}}$$

ie one needs (at 3.5PN) to subtract the dipolar part of beta^VV from the disconnected cut contribution (or add beta1^VV to W^MPM)

$$W^{G^2\text{disc,new}} = i\omega(\beta^{\text{VV}}(\theta,\phi) - \beta_1^{\text{VV}}(\theta,\phi)n^Y)W^{\text{tree}}($$

This is fully compatible with the fact that beta^VV is, actually, only defined modulo l=0 and l=1 homogeneous contributions

# Conclusions

**Comparisons between various approaches to the 2-body pb are important and allow one to understand new, subtle aspects of gravitational interactions**

**For GW generation the « traditional » MPM+PN approach is the only one which has reached a (LIGO-relevant) high precision:**
**4PN for quasi-circular motions (Blanchet et al)**
**and 3.5PN+ 2-loop for scattering motions**

**The many subtle effects linked to radiation-reaction and the decomposition « conservative + rad reac » remain to be fully clarified**