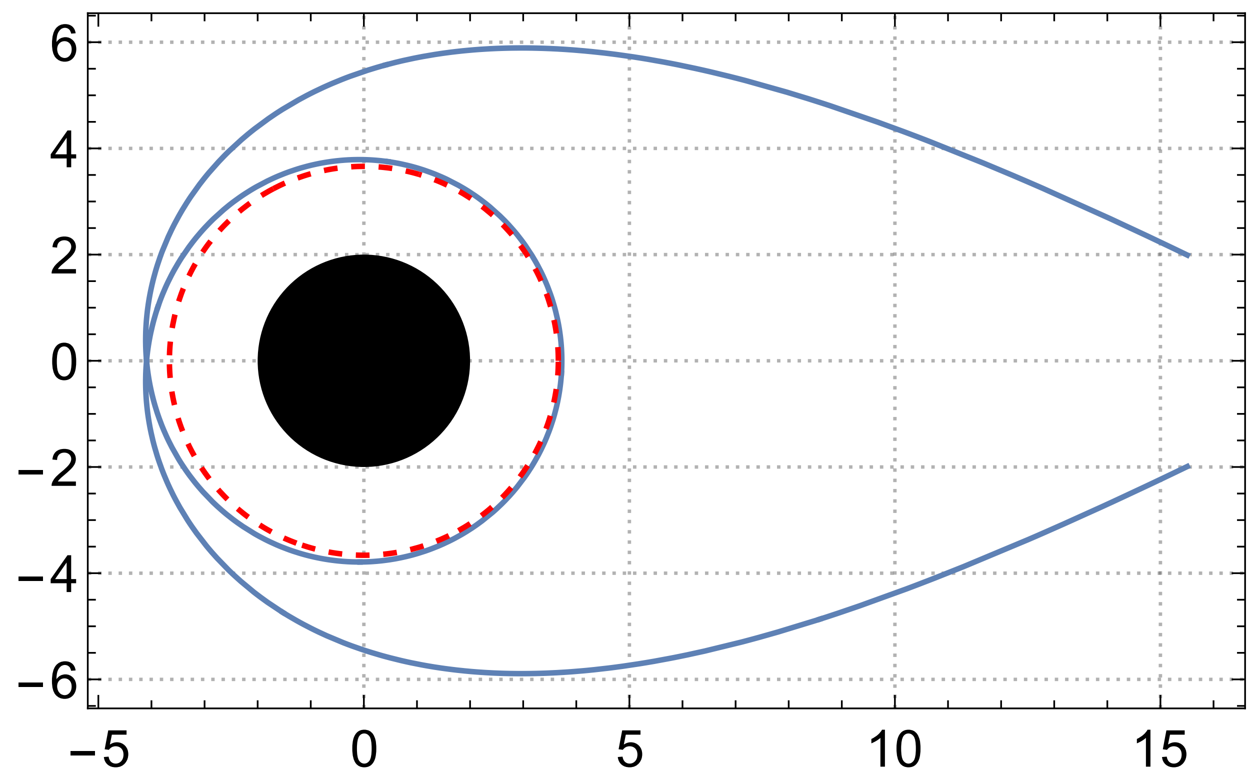
# Gravitational radiation from hyperbolic orbits: comparison between SF, PM, PN and NR





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Nordita, Stockholm
14th April 2026

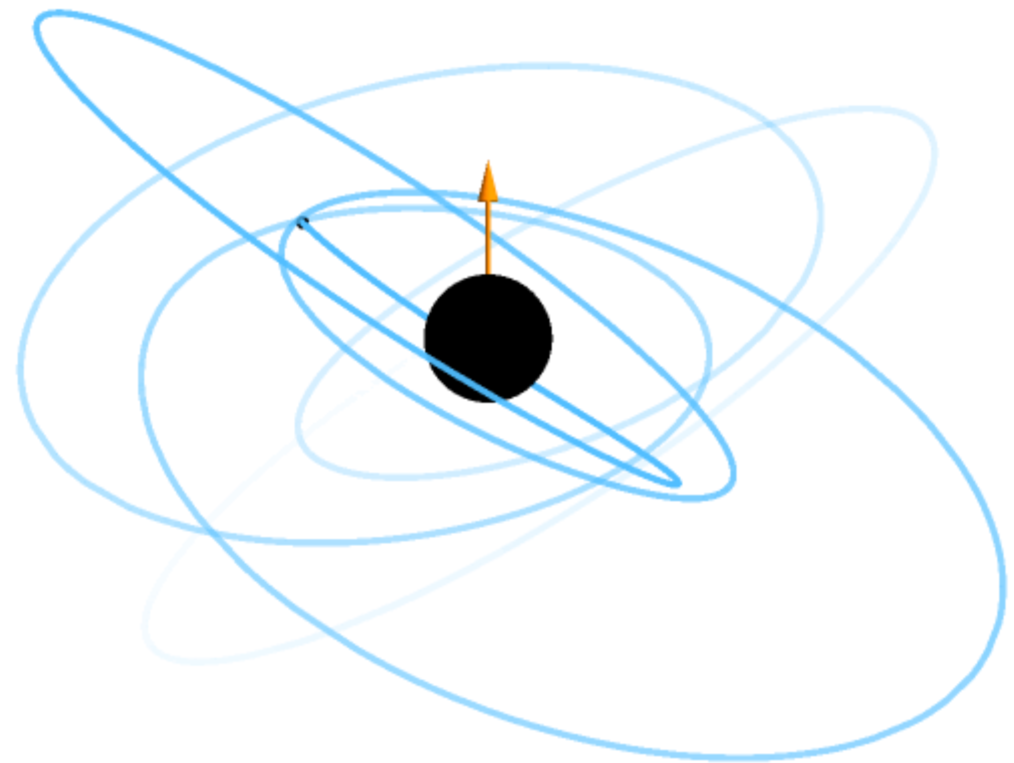
# Self-force approach

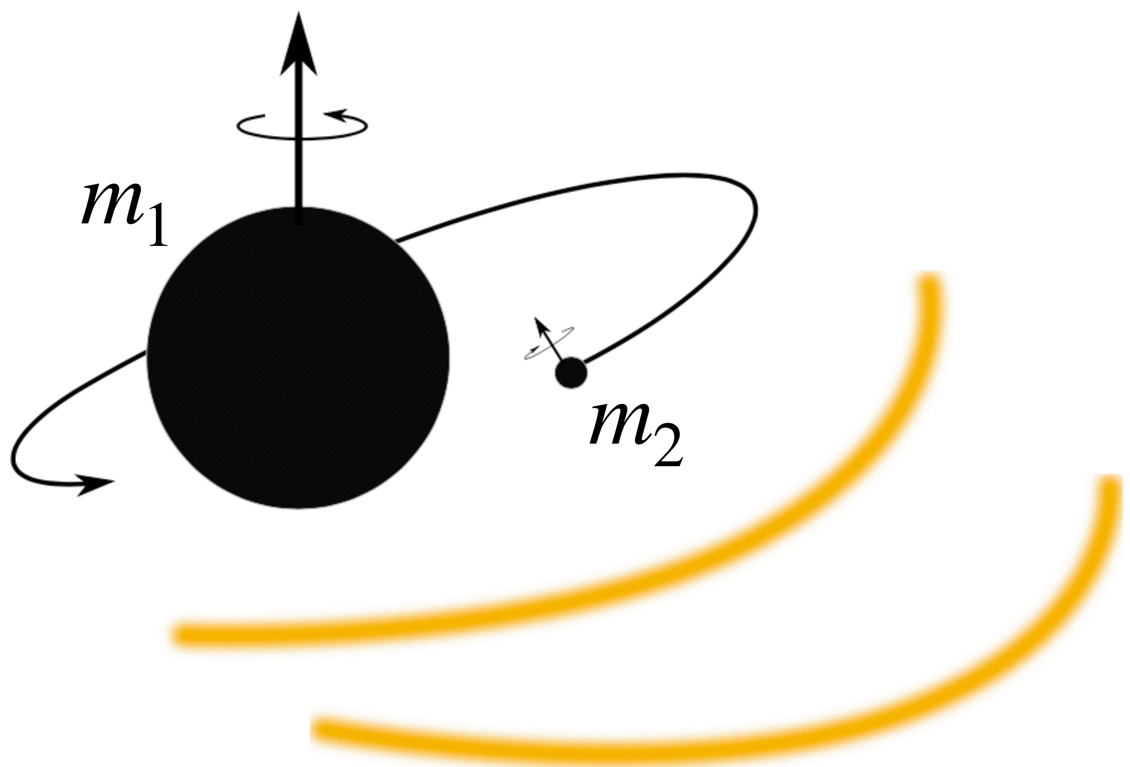
- $\epsilon = 1/q = m_2/m_1 \ll 1$
- Small body perturbs spacetime:

$$g_{\alpha\beta} = g_{\alpha\beta}^{\mathrm{Kerr}} + \epsilon h_{\alpha\beta}^{(1)} + \epsilon^2 h_{\alpha\beta}^{(2)} + \ldots$$

- Perturbation affects $m_2$'s motion:

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f_{(1)}^\mu + \epsilon^2 f_{(2)}^\mu + \ldots$$





# Scattered orbits: my motivation

- Lots of analytic progress modelling scattering orbits within the post-Minkowskian (PM) expansion
- There has been much focus on the (conservative) scattering angle
- On the self-force side work has been on scalar-field models
- I wanted to compare with the latest result in the dissipative for gravity
- Led to NW, arXiv:2512.02274, and Barack, Gonzo, Leather, Long, NW, arXiv:2602.10089


Driesse+, arXiv:2411.11846

# Motivation

$G = c = 1$

<u>Notation:</u>

- $\nu$, symmetric mass ratio, $b$, impact parameter, $v_\infty$, velocity at infinity

$$\Delta E^{\infty}_{PM} = \nu^2 \left[ \frac{F^{3PM}(v_\infty)}{b^3} + \frac{F^{4PM}(v_\infty)}{b^4} + \frac{F^{5PM}(v_\infty)}{b^5} + \mathcal{O}\left(\frac{1}{b^6}\right) \right] + \mathcal{O}(\nu^3)$$

E. Herrmann+
arXiv:2101.07255

Leaf count:
295

C. Dlapa+
arXiv:2210.05541

Leaf count:
1,351

M. Driesse+
arXiv:2411.11846

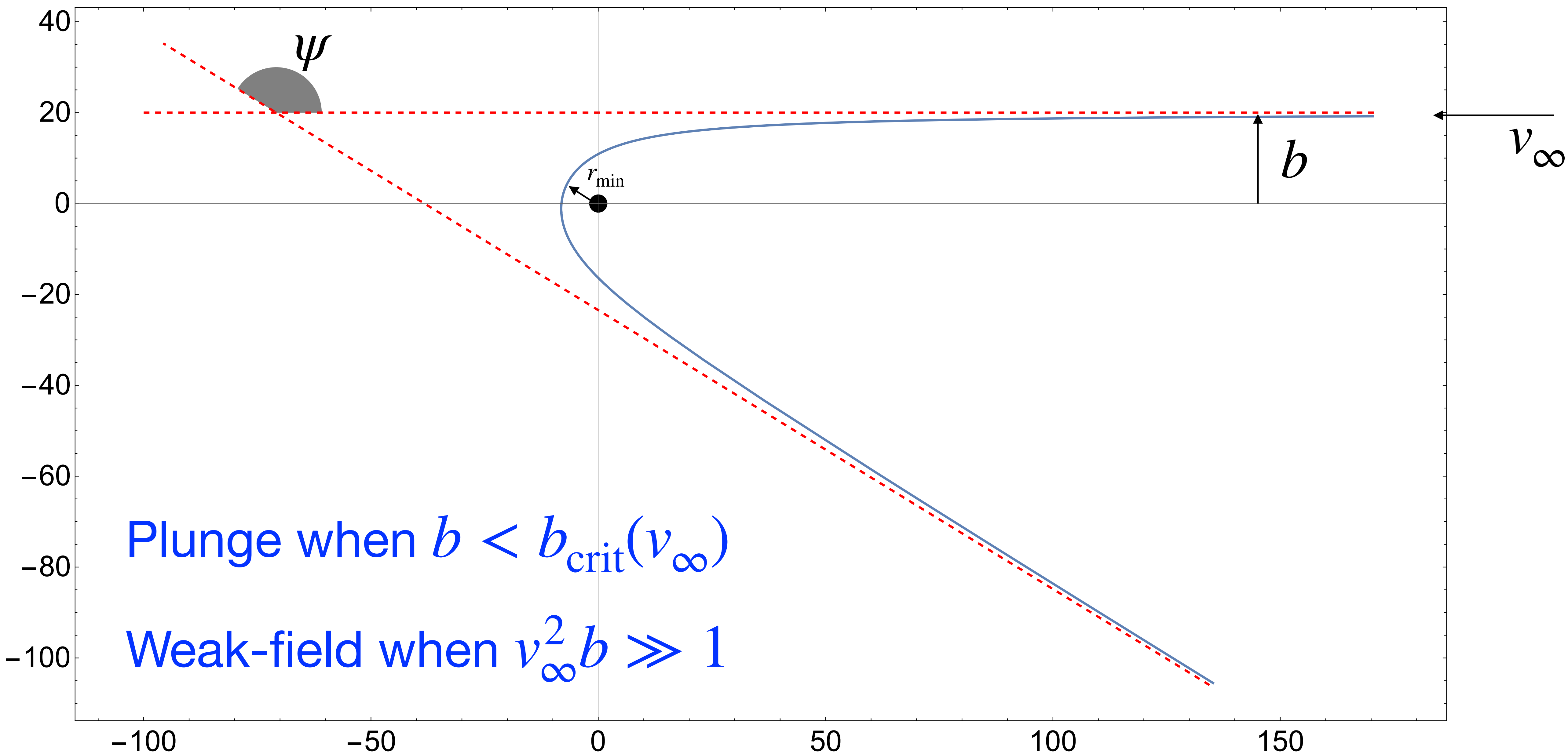
1000s of Feynman Diagrams

300,000 CPU hours

Leaf count:
290,331


<u>Original goals:</u>
- Cross-check with 5PM result
- Extract 6PM coefficient

# Hyperbolic orbits




$$r_p(\chi) = \frac{p}{1 + e\cos\chi}$$

$$(E, L) \leftrightarrow (p, e) \leftrightarrow (b, v_\infty)$$

Hyperbolic: $e > 1$

Parabolic: $e = 1$

# Perturbations of Schwarzschild

Hopper, arXiv:1706.05455
Hopper and Evans, arXiv:1006.4907

Solve the Regge-Wheeler-Zerilli (ZM and CPM) equations

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_\ell^{\mathrm{ZM/CPM}}(r)\right]\Psi_{\ell m}(t,r)=S_{\ell m}(t,r)=\tilde{G}_{\ell m}(t)\delta(r-r_p(t))+\tilde{F}_{\ell m}(t)\delta'(r-r_p(t))$$

Transform to the Fourier domain:

$$\Psi_{\ell m}(t,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{\ell m\omega}(r)e^{-i\omega t}\,d\omega$$

$$S_{\ell m}(t,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z_{\ell m\omega}(r)e^{-i\omega t}\,d\omega$$

$$\left[\frac{d^2}{dr_*^2}+\omega^2-V_\ell^{\mathrm{ZM/CPM}}(r)\right]X_{\ell m\omega}(r)=Z_{\ell m\omega}^{\mathrm{ZM/CPM}}(r)$$

$$X_{\ell m\omega}(r)=C^+_{\ell m\omega}(r)\hat{X}^+_{\ell m\omega}(r)+C^-_{\ell m\omega}(r)\hat{X}^-_{\ell m\omega}(r)$$

Weighting coefficients:

$$C^\pm_{\ell m\omega}(r)=\pm\frac{1}{W_{\ell m\omega}}\int_{r_{\min}}^{r}\frac{\hat{X}^\mp_{\ell m\omega}(r')Z_{\ell m\omega}(r')}{f(r')}\,dr'$$

# Perturbations of Schwarzschild

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$$\left[\frac{d^2}{dr_*^2}+\omega^2-V_\ell^{\mathrm{ZM/CPM}}(r)\right]X_{\ell m\omega}(r)=Z_{\ell m\omega}^{\mathrm{ZM/CPM}}(r) \qquad X_{\ell m\omega}(r)=C^{+}_{\ell m\omega}(r)\hat{X}^{+}_{\ell m\omega}(r)+C^{-}_{\ell m\omega}(r)\hat{X}^{-}_{\ell m\omega}(r)$$

Weighting coefficients:

$$C^{\pm}_{\ell m\omega}=\frac{1}{W_{\ell m\omega}}\int_{-\infty}^{\infty}\left[\frac{1}{f_p}\hat{X}^{\mp}_{\ell m\omega}(r_p)\tilde{G}_{\ell m}(t)\left(\frac{2M}{r_p^2f_p^2}\hat{X}^{\mp}_{\ell m\omega}(r_p)-\frac{1}{f_p}\frac{d\hat{X}^{\mp}_{\ell m\omega}(r_p)}{dr}\right)\tilde{F}_{\ell m}(t)\right]e^{i\omega t}\,dt$$

# Perturbations of Schwarzschild

Hopper, arXiv:1706.05455
Hopper and Evans, arXiv:1006.4907

Write the waveform as:

$$h_+ - ih_\times = \frac{1}{2r}\sum_{lm}\sqrt{\frac{(l+2)!}{(l-2)!}}\left(\Psi^{ZM}_{lm} - i\Psi^{CPM}_{lm}\right)\,{}_{-2}Y_{lm}$$

Waveform at retarded time $u = t - r_*$ :

$$\Psi_{lm}(u, r_* \to \infty) = \int_{-\infty}^{\infty} C^+_{lm\omega}\, d\omega$$

as $\hat{X}_{lm\omega}$ are asymptotically unit normalised

Problem: the integrand for the weighting coefficients falls off too slowly. Hopper (arXiv:1706.05455) constructed higher-order master functions:

$$\Psi^{(n+1)}_{\ell m}(t,r) = \dot{\Psi}^{(n)}_{\ell m}(t,r) + \frac{\dot{r}_p E^2}{f_p^2 V_p^2} F^{(n)}_{\ell m}\delta(r - r_p(t))$$

By construction these have satisfy the same RWZ equations but with sources which fall off more rapidly:

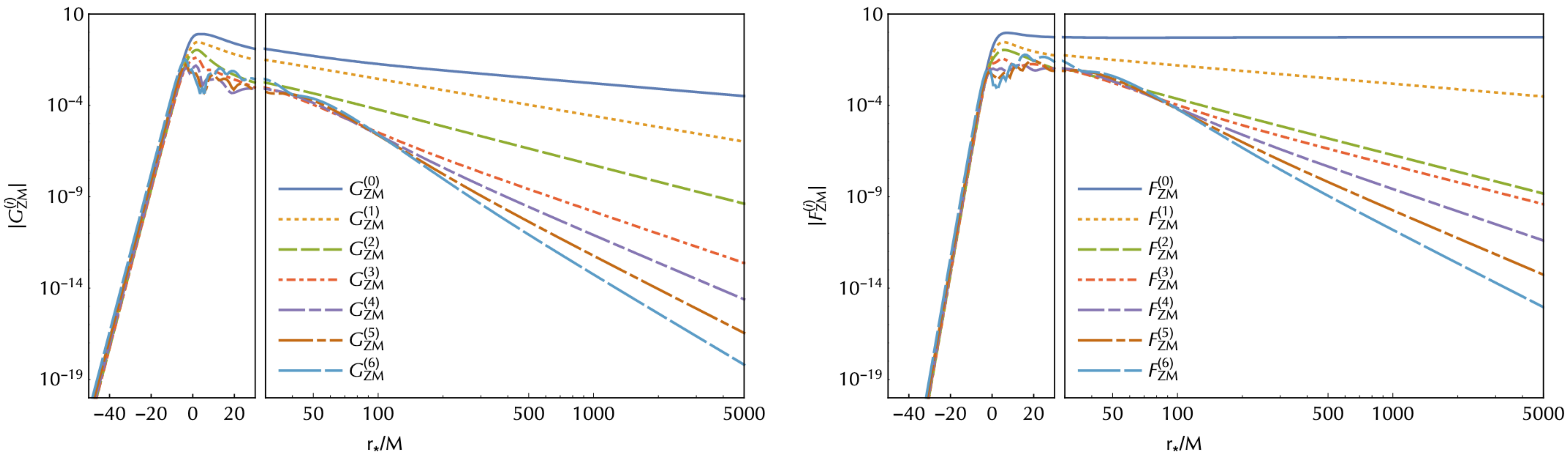



Figure credit: Hopper, arXiv:1706.05455

Can then relate the weighting coefficients of the new master function to the old via:

$$C^{\pm}_{lm\omega} \equiv C^{(0)\pm}_{lm\omega} = \frac{C^{(n)\pm}_{lm\omega}}{(-i\omega)^n}$$

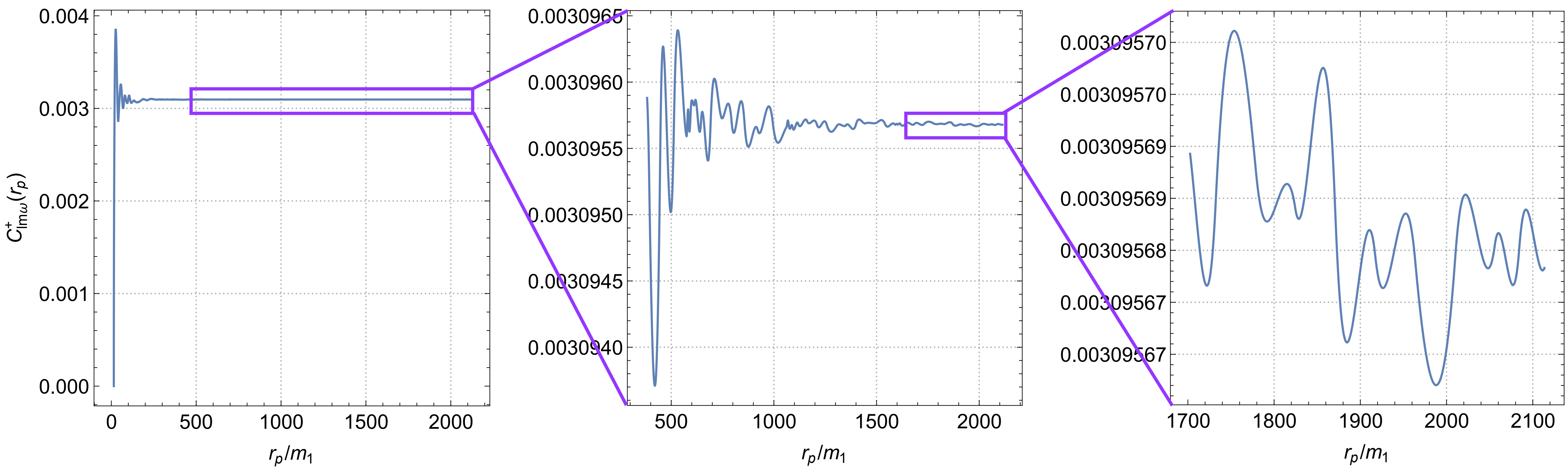
# Radiated energy and sources of error

Radiated energy given by:

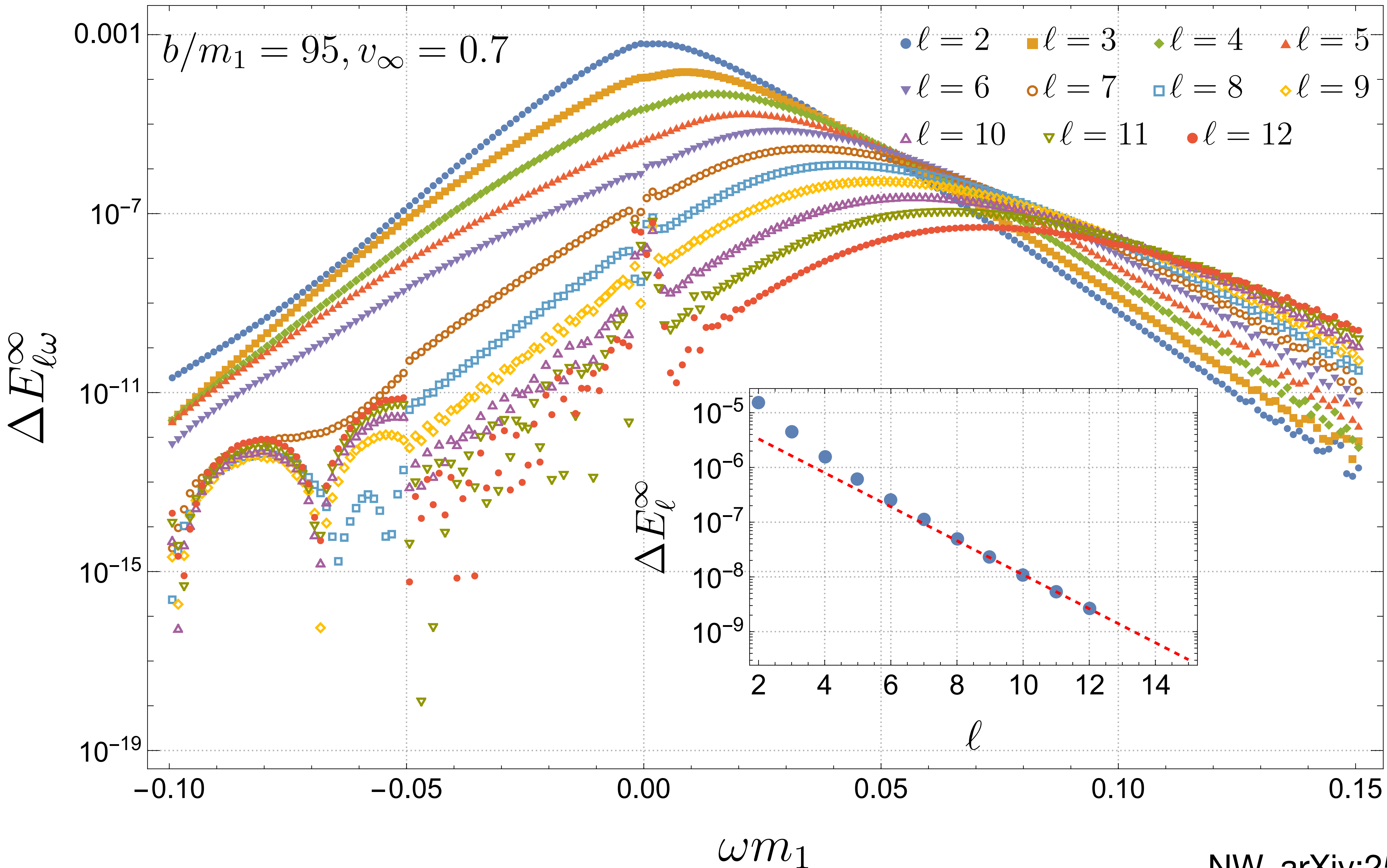
$$\Delta E^{\pm} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{64} \frac{(l+2)!}{(l-2)!} \int_{-\infty}^{\infty} \omega^2 \, | C^{\pm}_{\ell m \omega} |^2 \, d\omega$$

Four main sources of error:

- truncation of $\ell m$-mode sum
- sampling density of $\omega$-modes
- truncation of $\omega$ integral
- convergence of $C^{\pm}_{\ell m \omega}$ integrals



# Results: mode data




NW, arXiv:2512.02274

# PN results

1PN result derived by Blanchet and Schäfer (1989)

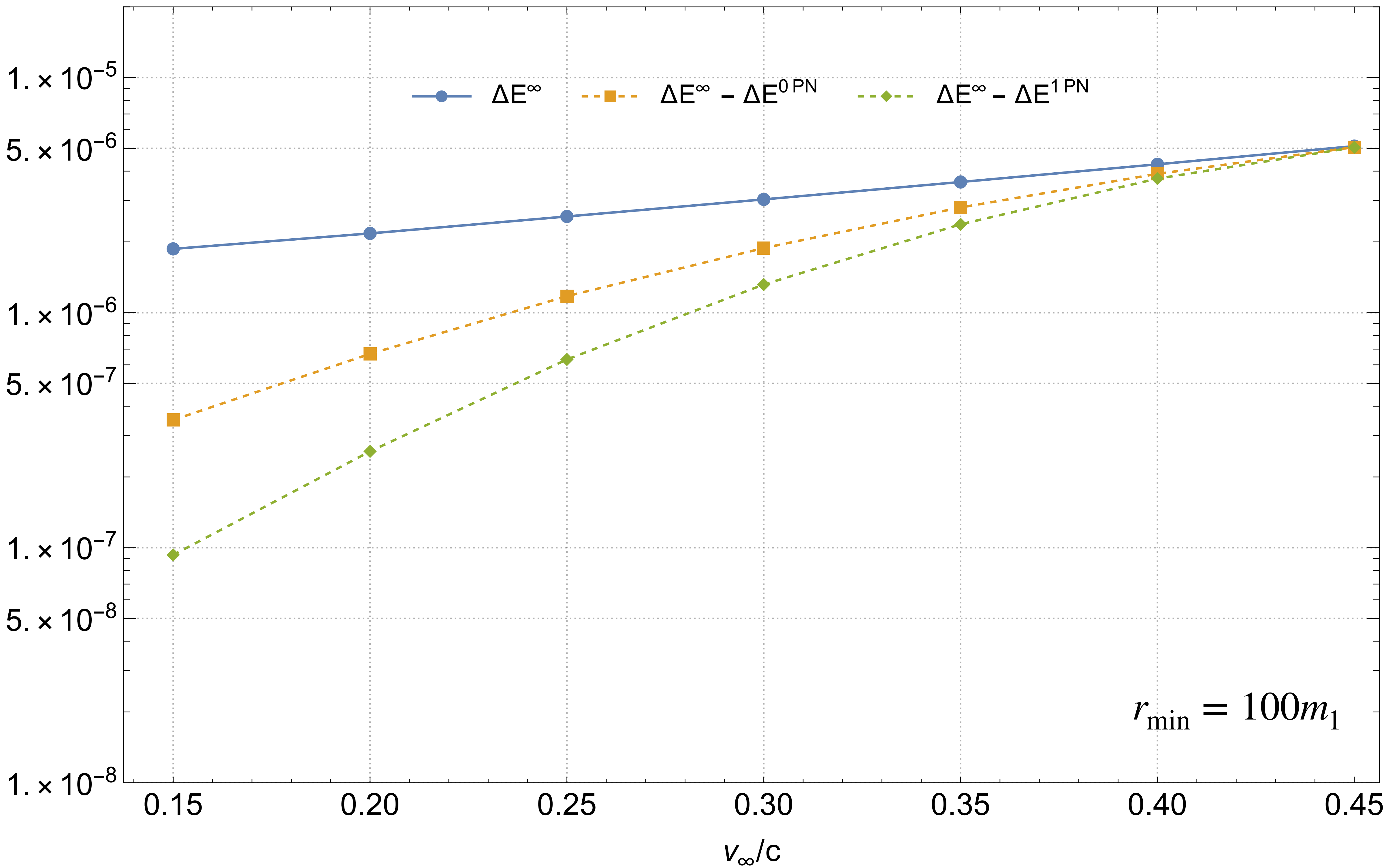
$$\Delta E^{1PN} = \frac{2\left(\frac{1}{3}\sqrt{e^2-1}\left(673e^2+602\right)+\left(37e^4+292e^2+96\right)\cos^{-1}\left(-\frac{1}{\sqrt{e^2}}\right)\right)}{15L^7} + \frac{2\left(\frac{1}{840}\sqrt{e^2-1}\left(1271421e^4+1447788e^2+1516596\right)+\frac{1}{56}\left(17933e^6+94542e^4+117288e^2+52624\right)\cos^{-1}\left(-\frac{1}{\sqrt{e^2}}\right)\right)}{15L^9}$$

We can expand these results as a PM series

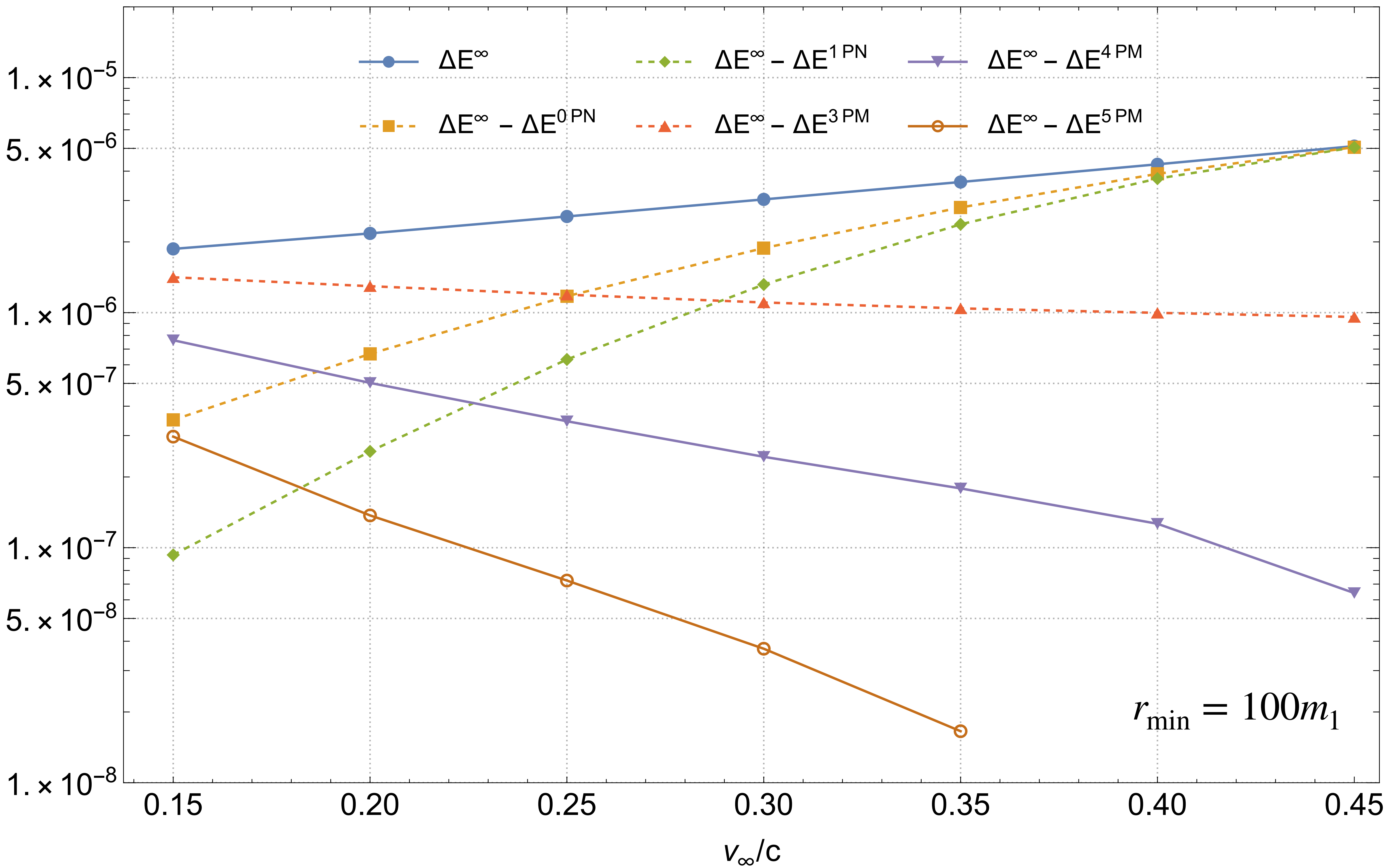
$$\Delta E^{1PN} = \frac{\frac{2393\pi v_\infty^3}{840}+\frac{37\pi v_\infty}{15}}{b^3} + \frac{\frac{28384 v_\infty}{1575}+\frac{1568}{45 v_\infty}}{b^4} + \frac{\frac{122\pi}{5v_\infty^3}+\frac{3583\pi}{280 v_\infty}}{b^5} + \frac{\frac{4672}{45 v_\infty^5}+\frac{60352}{315 v_\infty^3}}{b^6} + \frac{\frac{85\pi}{3v_\infty^7}+\frac{4397\pi}{24 v_\infty^5}}{b^7} + \mathcal{O}(b^{-8})$$

2PN is known from Bini, Damour, Geralico, arXiv:2007.11239
3PN expanded through 15PM is known from Cho, arXiv:2203.10872

# Comparison with PN



# Comparison with PN and PM

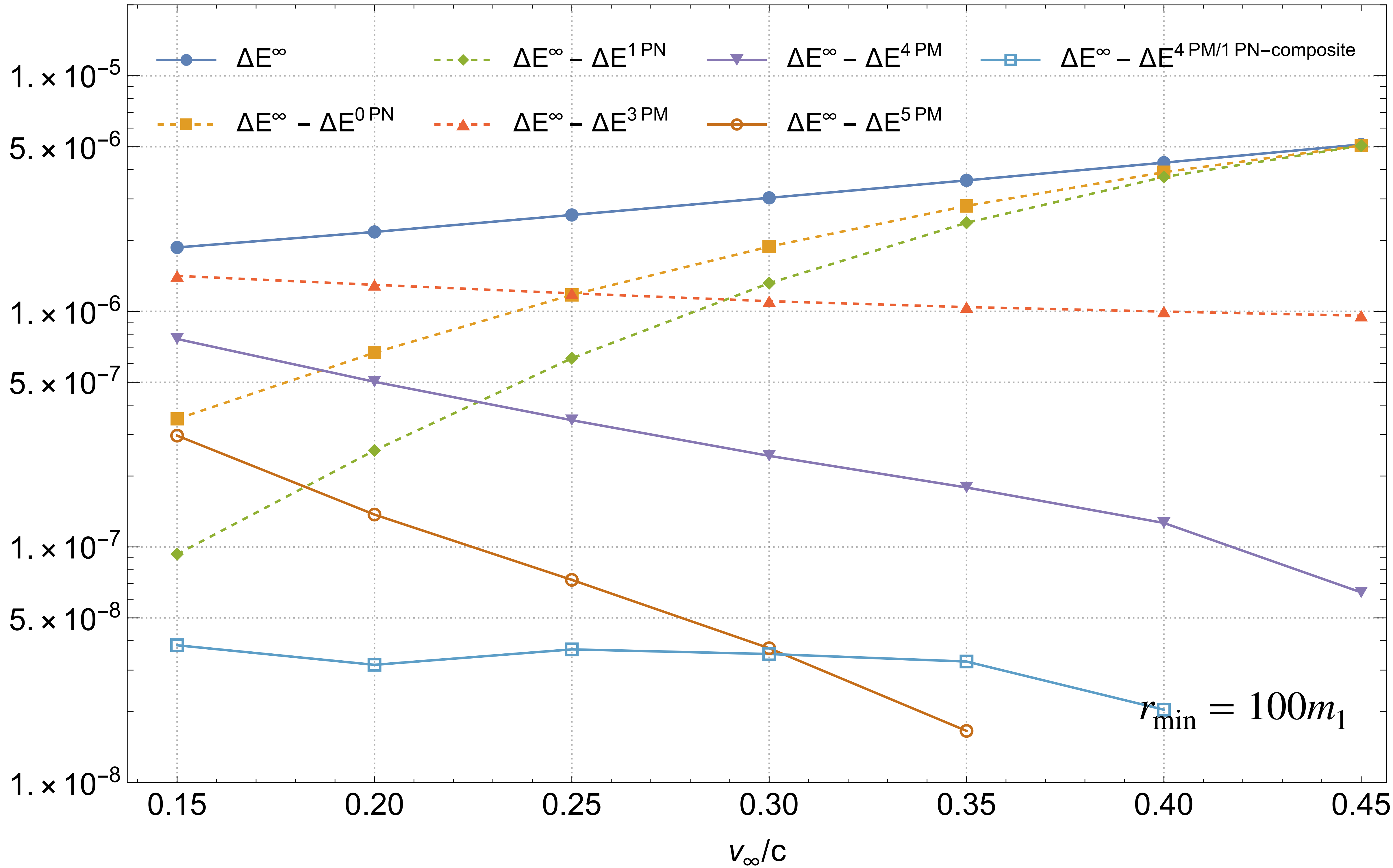


# A PN/PM composite

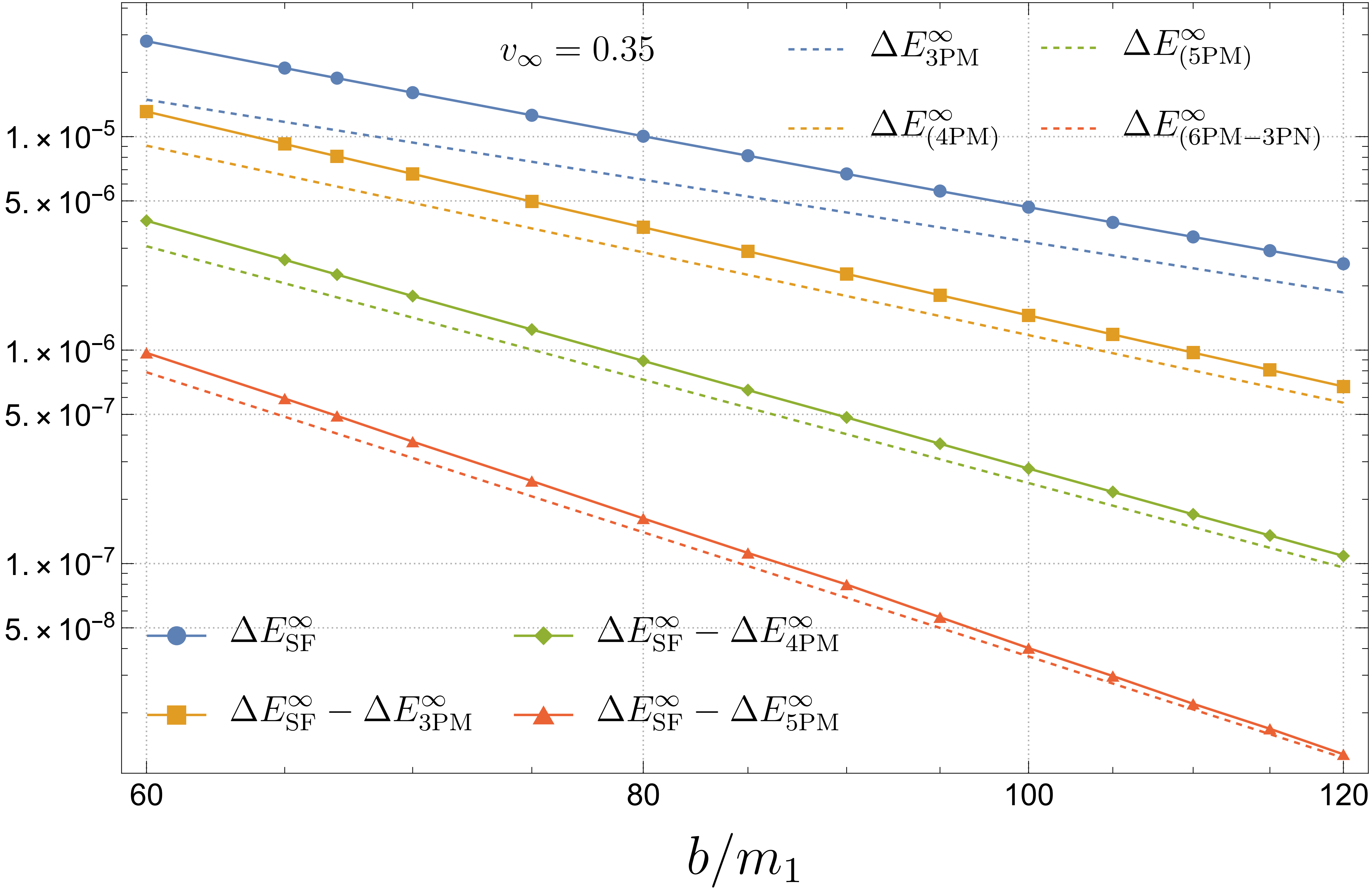
| | PM3 | PM4 | PM5 | PM6 | PM7 | ... |
|---|---|---|---|---|---|---|
| PN0 | $\mathcal{O}(G^3, v)$ | $\mathcal{O}(G^4, v^{-1})$ | $\mathcal{O}(G^5, v^{-3})$ | $\mathcal{O}(G^6, v^{-5})$ | $\mathcal{O}(G^7, v^{-7})$ | |
| PN1 | $\mathcal{O}(G^3, v^3)$ | $\mathcal{O}(G^4, v)$ | $\mathcal{O}(G^5, v^{-1})$ | $\mathcal{O}(G^6, v^{-3})$ | $\mathcal{O}(G^7, v^{-5})$ | |
| PN2 | $\mathcal{O}(G^3, v^5)$ | $\mathcal{O}(G^4, v^3)$ | $\mathcal{O}(G^5, v)$ | $\mathcal{O}(G^6, v^{-1})$ | $\mathcal{O}(G^7, v^{-3})$ | |
| PN3 | $\mathcal{O}(G^3, v^7)$ | $\mathcal{O}(G^4, v^5)$ | $\mathcal{O}(G^5, v^3)$ | $\mathcal{O}(G^6, v)$ | $\mathcal{O}(G^7, v^{-1})$ | |
| ... | | | | | | |

$$\Delta E^{4PM/1PN\text{-composite}} = \Delta E^{3PM} + \Delta E^{4PM} + \frac{\frac{85\pi}{3v^7} + \frac{4397\pi}{24v^5}}{b^7} + \frac{\frac{4672}{45v^5} + \frac{60352}{315v^3}}{b^6} + \frac{\frac{122\pi}{5v^3} + \frac{3583\pi}{280v}}{b^5}$$

# Comparison with PN, PM, and PN/PM composite



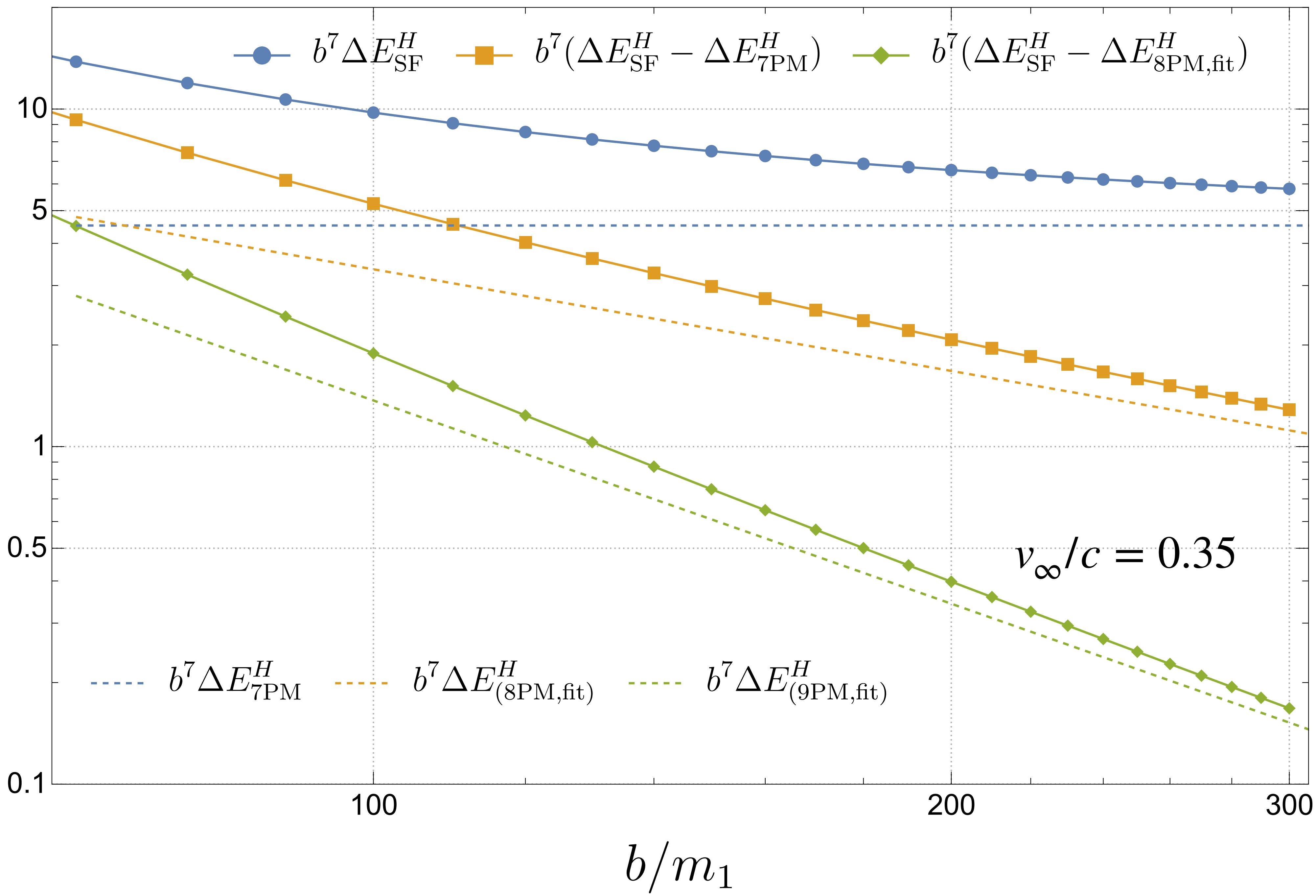
# Residual with 5PM



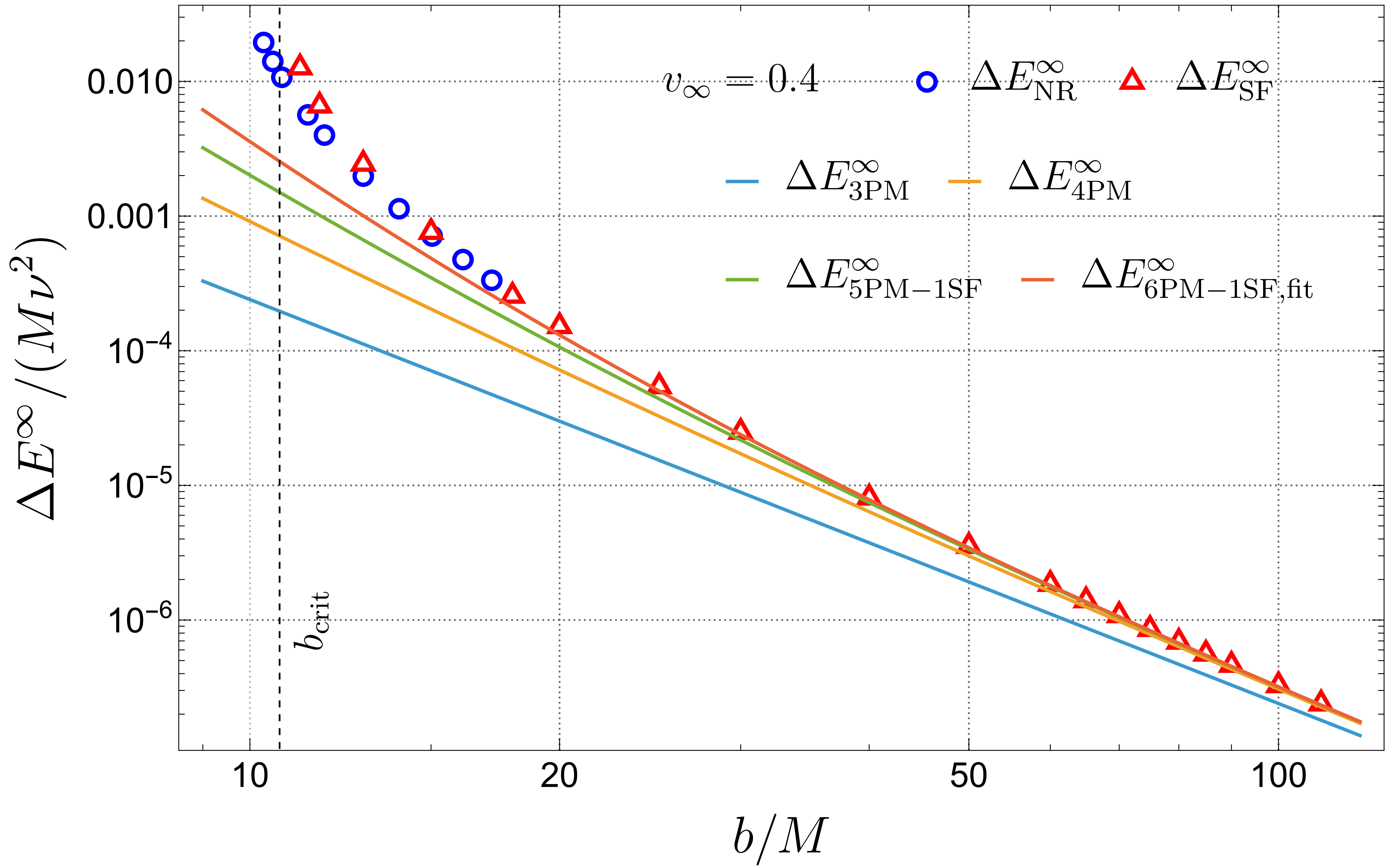
# Horizon flux: comparison with PM

$$\Delta E_H^{PM} = \frac{5\pi m_1^6 m_2^2}{16 b^7}(21E^4 - 14E^2 + 1)\sqrt{E^2 - 1} + \mathcal{O}(b^{-8})$$

Jones and Ruf, arXiv:2310.00069



# Comparison with NR




NR data for $\nu = 1/4$ $(q = 1)$ from Damour+, arXiv:1402.7307

# New results and comparisons

After computing the total radiated energy, I turned my attention to the total change in the angular momentum

This is much more subtle. PM community has studied these subtleties extensively. I am just discovering them.

Heissenberg and Russo: arXiv:2406.03937

$$\Delta J = \Delta J^{\mathrm{rad}} + \Delta J^{\mathrm{static}} + \Delta J^{mem}$$

Radiative: $\mathcal{O}(G^3)$ Static: $\mathcal{O}(G^2)$ Non-linear memory: $\mathcal{O}(G^5)$

Everything in this ‘new results’ part is preliminary…

# New results and comparisons: zero frequency limit

The ZM/CPM coefficients have a pole at $\omega = 0$

$$C^{+}_{\ell m} = \frac{\mathscr{A}_{lm}}{\omega} + \mathscr{B}_{lm} \log \omega + \mathscr{C} \omega (\log \omega)^2 + \dots$$

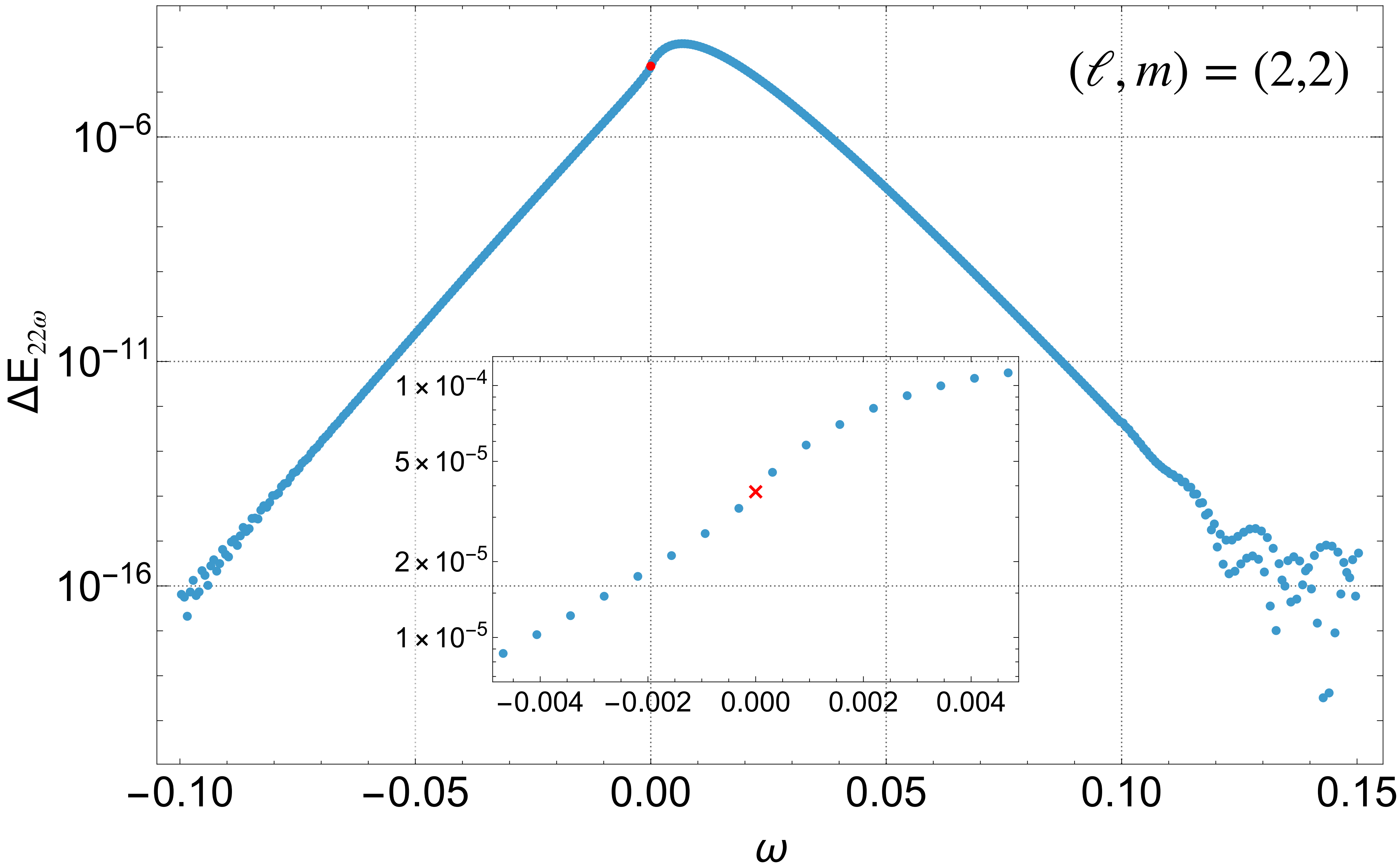
This does not cause a problem for $\Delta E_{lm\omega} \sim |\dot{h}_{lm\omega}|^2 = \omega^2 |C^{+}_{\ell m\omega}|^2$

This does cause a problem for $\Delta J_{lm} \sim \dot{h}_{lm\omega} \partial_\phi h_{lm\omega} = m\omega |C^{+}_{\ell m\omega}|^2$

For $\Delta J$ we also need to worry about the frame, recoil, etc

Side note: my code computes $C^{(n)}_{lm\omega}$. Recall $C^{+}_{lm\omega} \equiv C^{(0)}_{lm\omega} = \dfrac{C^{(n)}_{lm\omega}}{(-i\omega)^n}$

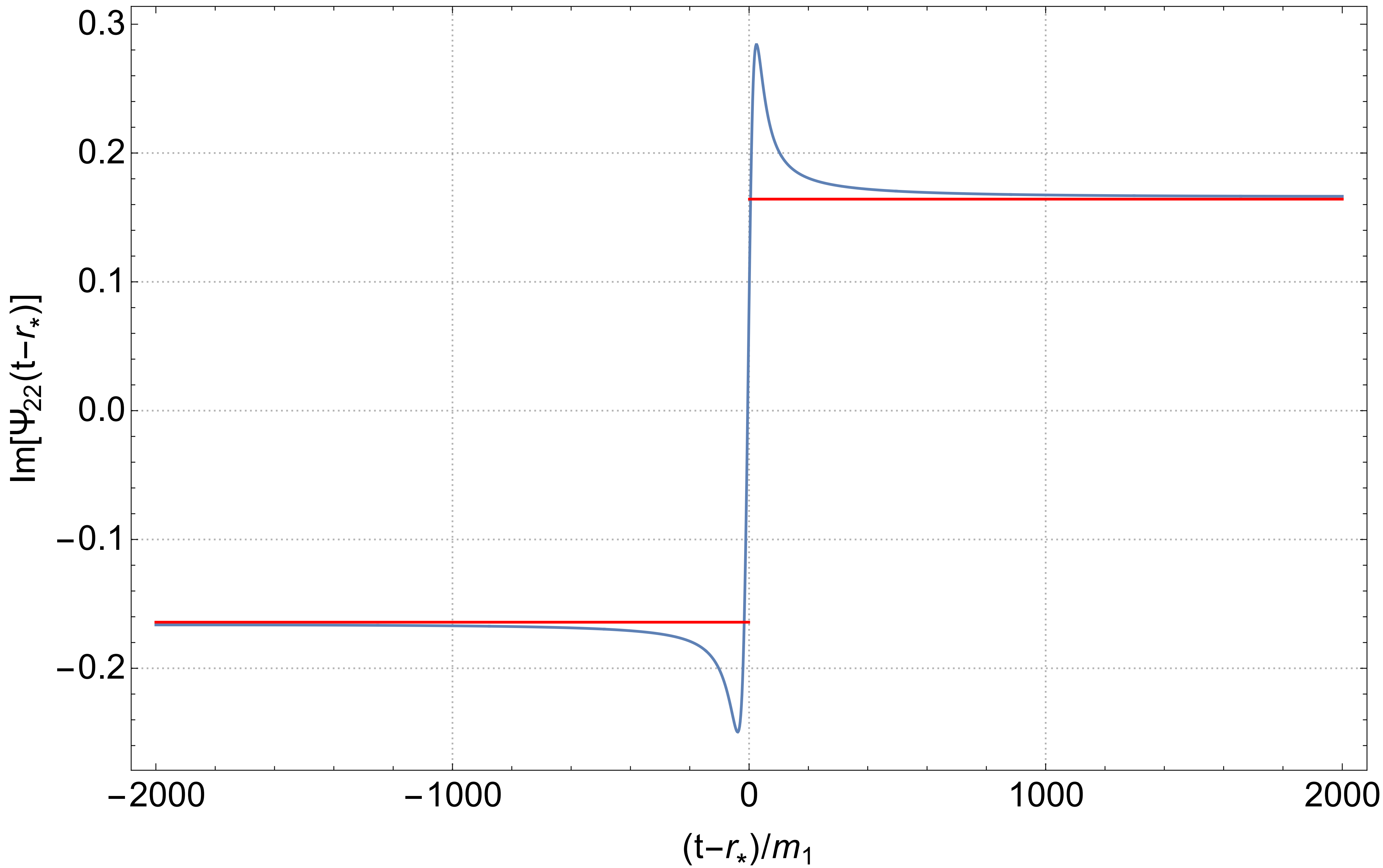
# New results and comparisons: zero frequency limit




Recently, Fucito, Morales, Russo (arXiv:2408.07329) computed $\mathscr{A}_{lm}$ analytically

# New results and comparisons: waveform

## Waveform with linear memory



# New results and comparisons: $\Delta J$

Total change in angular momentum:

$$\Delta J = \Delta J^{\rm rad} + \Delta J^{\rm static} + \Delta J^{mem}$$

Split $C^{+}_{\ell m\omega}$ into a regular and singular piece

$$C_{lm\omega} = \left( C_{lm\omega} - \frac{\mathscr{A}_{lm}}{\omega} F(\omega) \right) + \frac{\mathscr{A}_{lm}}{\omega} F(\omega) \equiv C^{\rm reg}_{lm\omega} + C^{\rm sing}_{lm\omega}$$

where $F(\omega)$ is a regulator, e.g., $F(\omega) = \exp(-\omega^2\sigma^2)$

$$\Delta J^{\pm,\rm rad}_{lm} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{64} \frac{(l+2)!}{(l-2)!} \int_{-\infty}^{\infty} m\omega \, | \, C^{\pm}_{\ell m\omega} |^2 \, d\omega$$

# New results and comparisons: $\Delta J$

$$\Delta J_{lm}^{\pm,\mathrm{rad}} = \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{1}{64}\frac{(l+2)!}{(l-2)!}\int_{-\infty}^{\infty} m\omega\,|\,C_{\ell m\omega}^{\pm}\,|^2\,d\omega$$

As $\omega \to 0$

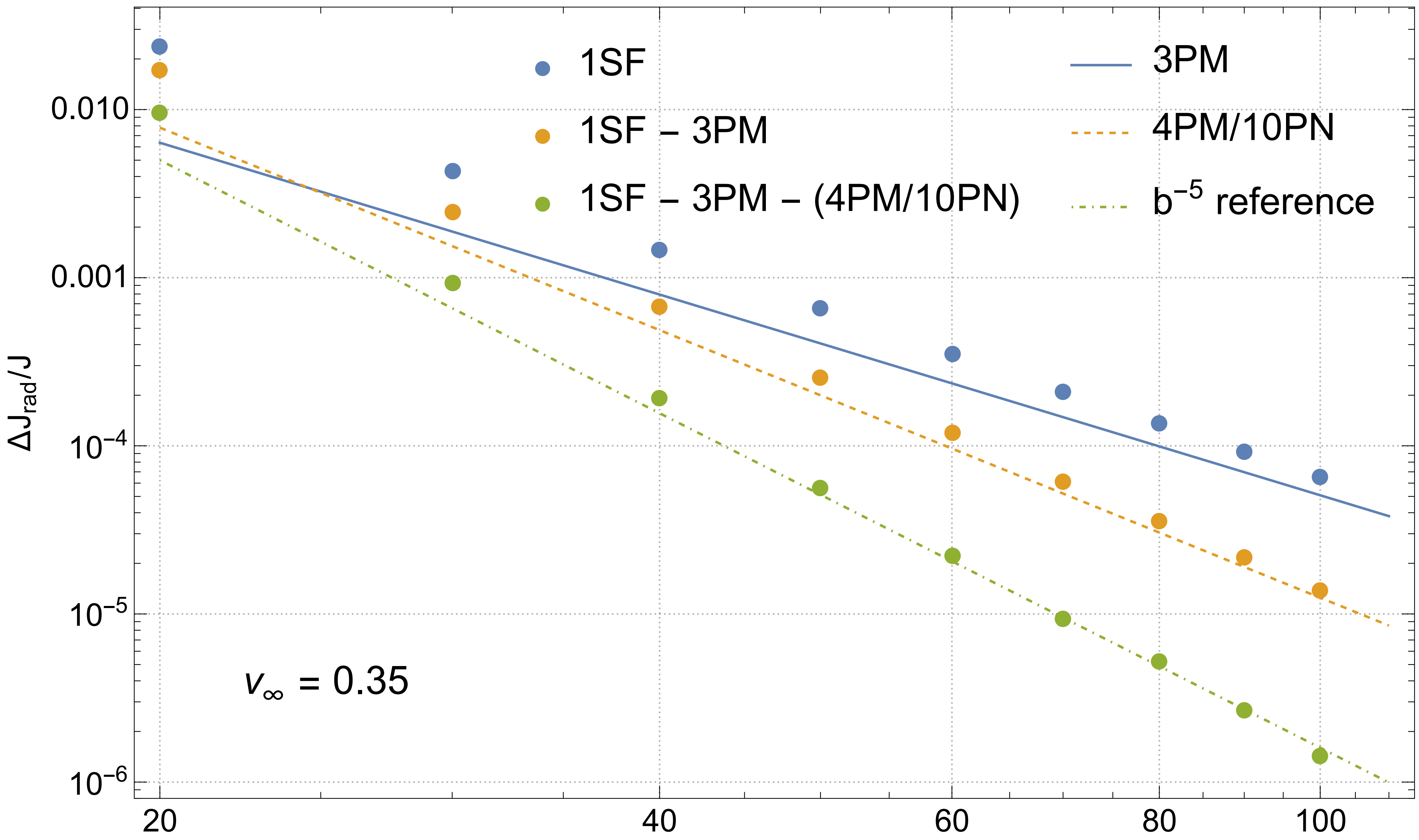
$$m\omega\,|\,C_{lm\omega}^{\mathrm{reg}}\,|^2 \sim m\omega\,|\log\omega\,|^2$$ ✓

$$m\omega\cdot 2\Re[C_{lm\omega}^{\mathrm{reg}}\bar{C}_{lm\omega}^{\mathrm{sing}}] \sim \Re[\log\omega]$$ ✓

$$m\omega\,|\,C_{lm\omega}^{\mathrm{sing}}\,|^2 \sim m\frac{|\,\mathscr{A}_{lm}\,|^2}{\omega}$$ The +m and -m modes cancel ✓

Note: I do not currently account for the recoil or CoM motion

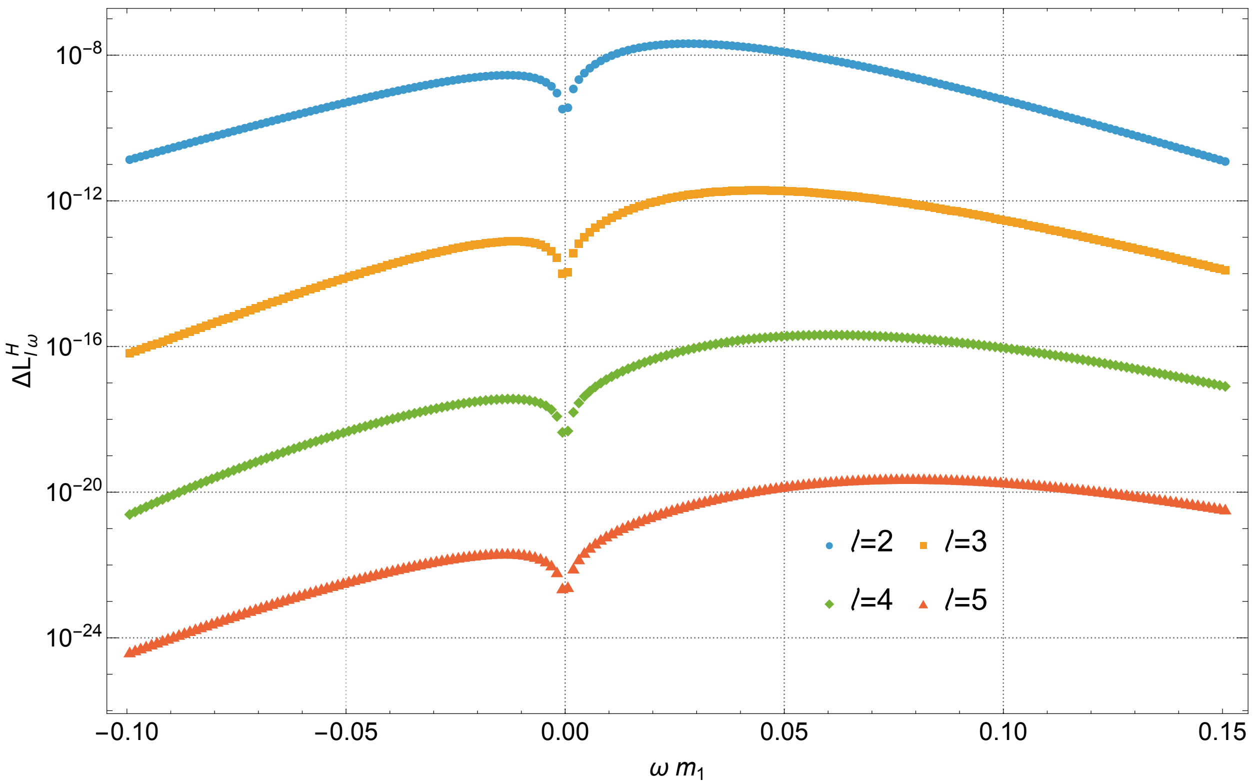
# New results and comparisons: $\Delta J$

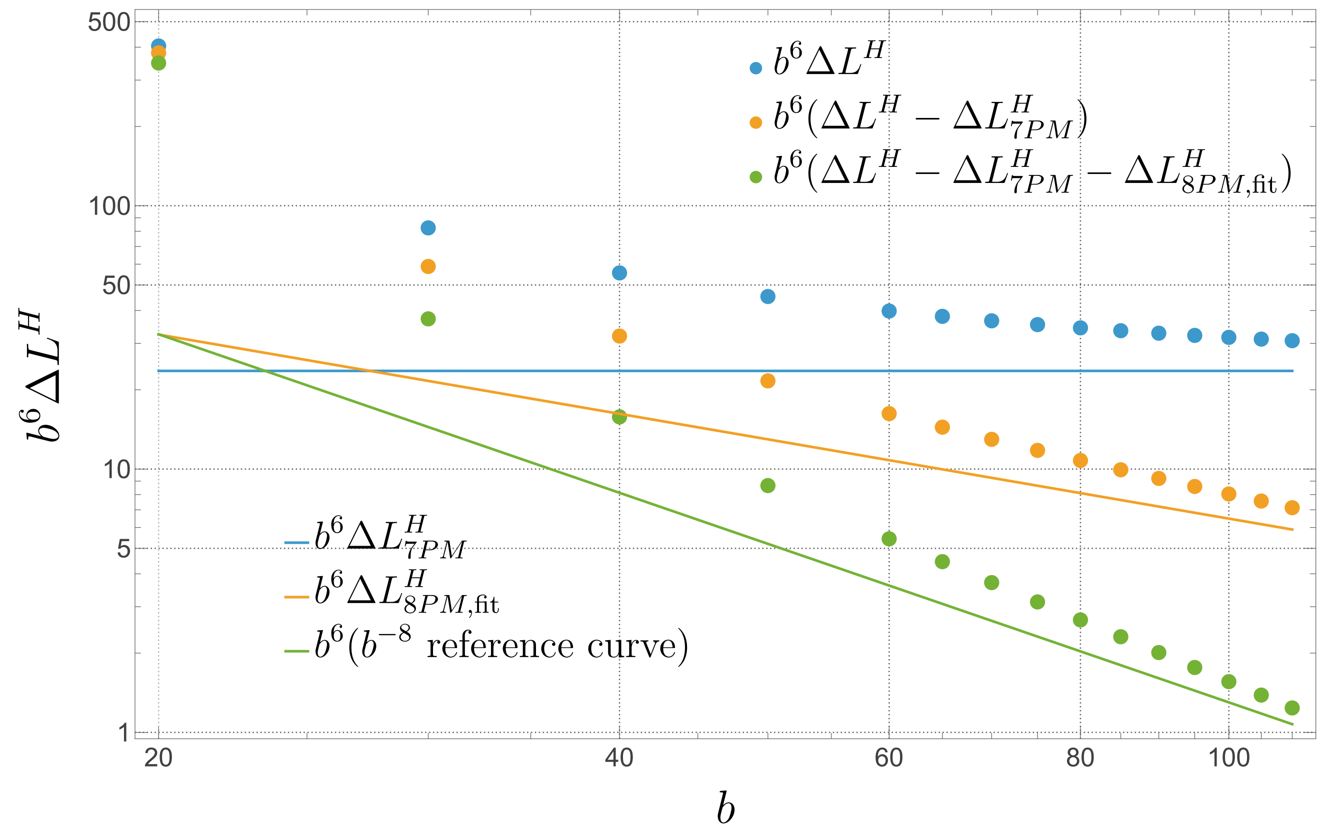



I find agreement with 3PM (Manohar, Ridgway, Shen, arXiv:2203.04283) for the radiative part. Evidence of agreement with radiative part of 4PM-10PN from Heissenberg+Russo (arXiv:2511.13835)

# New results and comparisons

## Angular momentum absorbed by the horizon




$$J_{\mathrm{abs}} = \frac{\pi G^7 \mu^2 M_{\mathrm{BH}}^6 \sigma \left(7\sigma^2 - 3\right)}{2b^6}$$

Spectrum is free of poles. Find agreement with leading-order PM from Cipriani+ (arXiv:2602.05766) and I can fit for higher-order terms.

# New results and comparisons: non-linear memory

To compute the total $\Delta J$ I also need the contribution from the non-linear memory

$$h_{lm} = -\frac{1}{\sqrt{2}}\left(\mathcal{U}_{lm} - i\mathcal{V}_{lm}\right) \qquad \dot{\mathcal{U}}_{lm}^{(\mathrm{mem})} = \sqrt{\frac{2(l-2)!}{(l+2)!}} \sum_{l'=2}^{\infty}\sum_{l''=2}^{\infty}\sum_{m'=-l'}^{l'}\sum_{m''=-l''}^{l''} (-1)^{m+m''}\, \dot{h}_{l'm'}\, \dot{h}^{*}_{l''m''}\, G^{2,-2,0}_{l',l'',m',-m'',-m}\,,$$

$$G^{s_1,s_2,s_3}_{l_1,l_2,l_3,m_1,m_2,m_3} \equiv \int \mathrm{d}\Omega_2 \;\; {}_{-s_1}Y_{l_1 m_1}(\Theta,\Phi) \;\; {}_{-s_2}Y_{l_2 m_2}(\Theta,\Phi) \;\; {}_{-s_3}Y_{l_3 m_3}(\Theta,\Phi)\;.$$

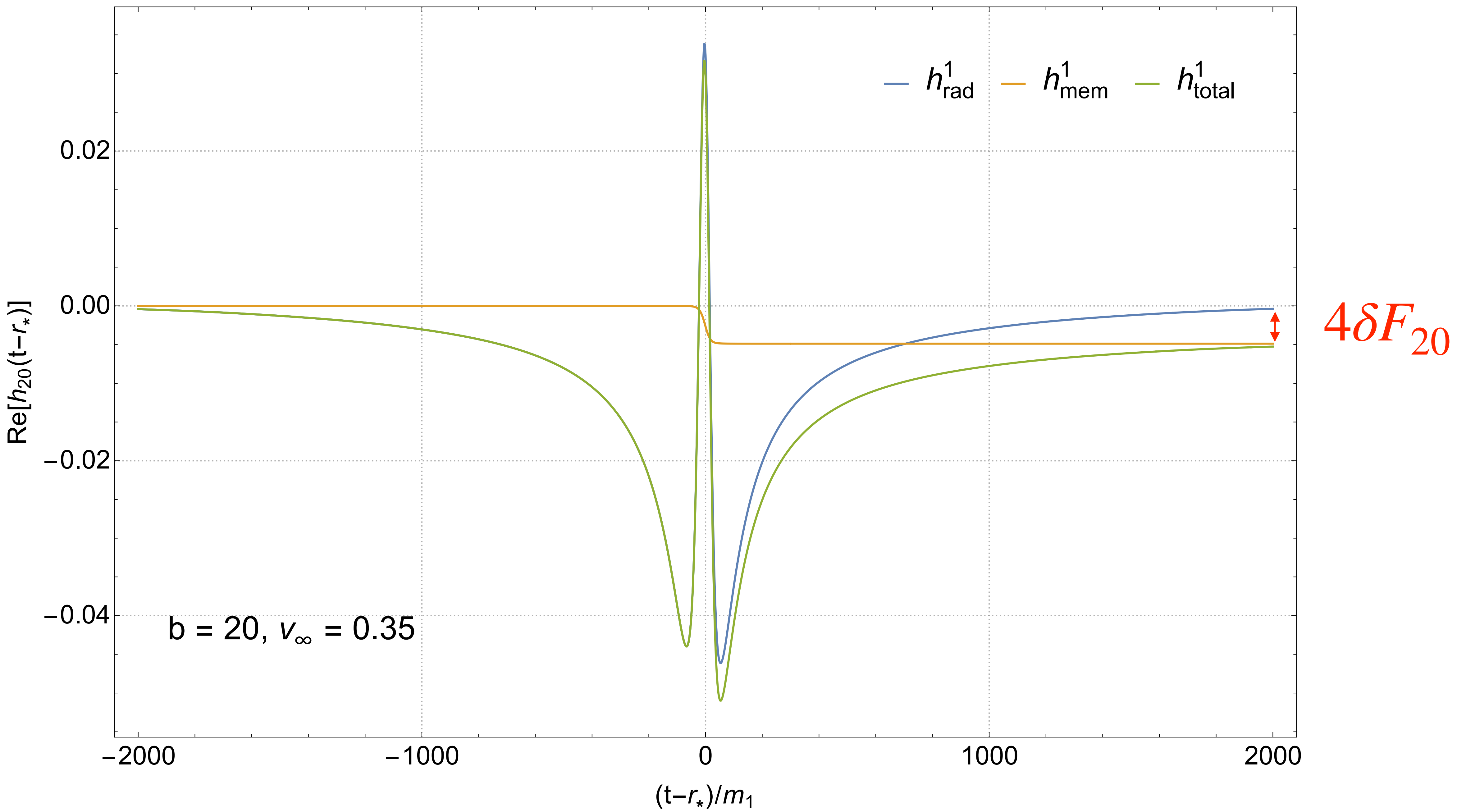
$$G^{s_1,s_2,s_3}_{l_1,l_2,l_3,m_1,m_2,m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Cunningham+ (including NW), arXiv:2410.23950

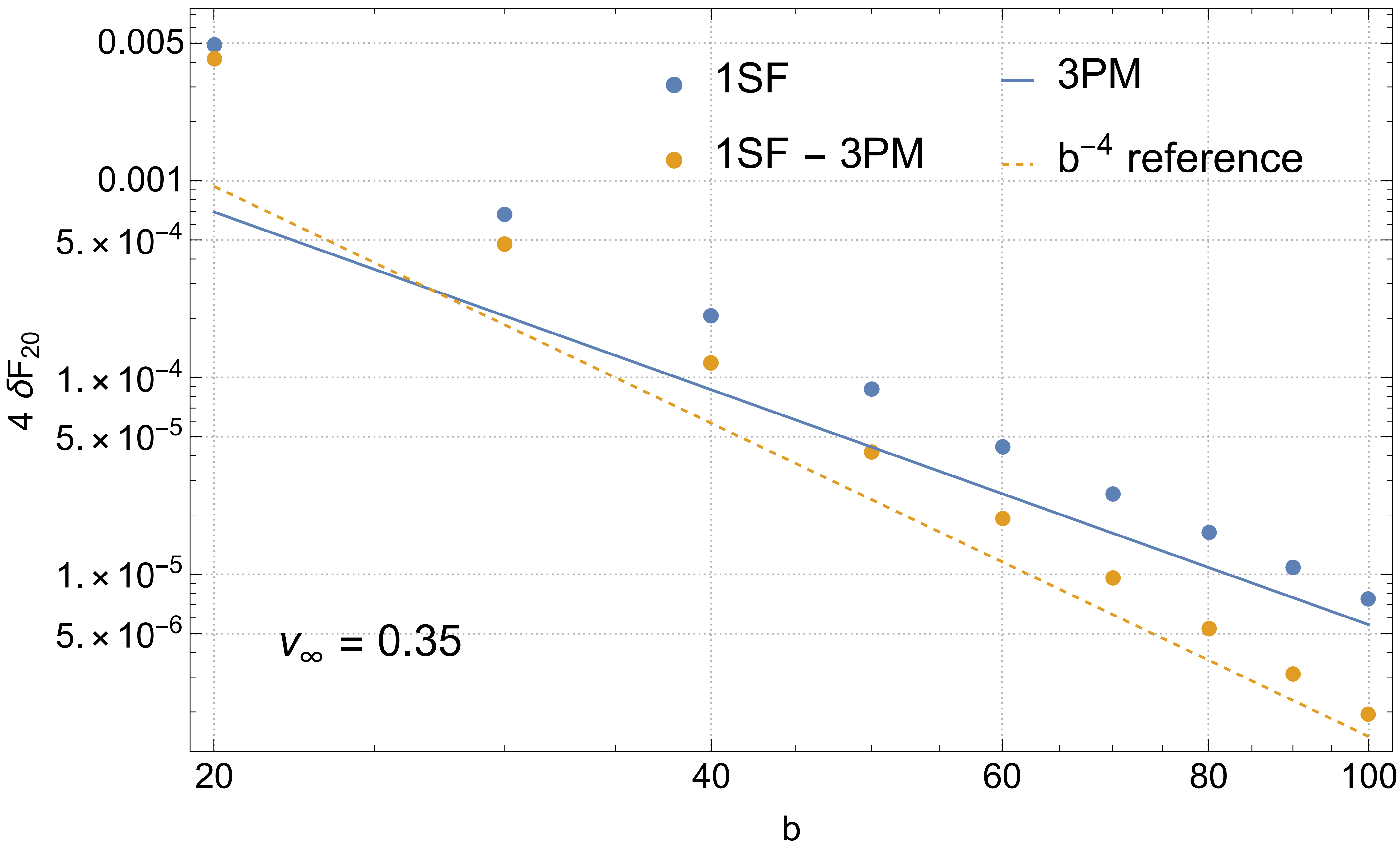
Recently, Georgoudis+ (arXiv:2506.20733) computed the $\omega = 0$ contribution to the memory

$$h_{\mu\nu} \sim \frac{4G}{r}\omega_{\mu\nu} \qquad \tilde{\omega}_{\mu\nu} \sim \frac{i}{\omega}\left(f_{\mu\nu} + \delta F_{\mu\nu}\right) + \mathcal{O}(\log\omega), \quad \text{as} \quad \omega \to 0$$

# New results and comparisons: non-linear memory

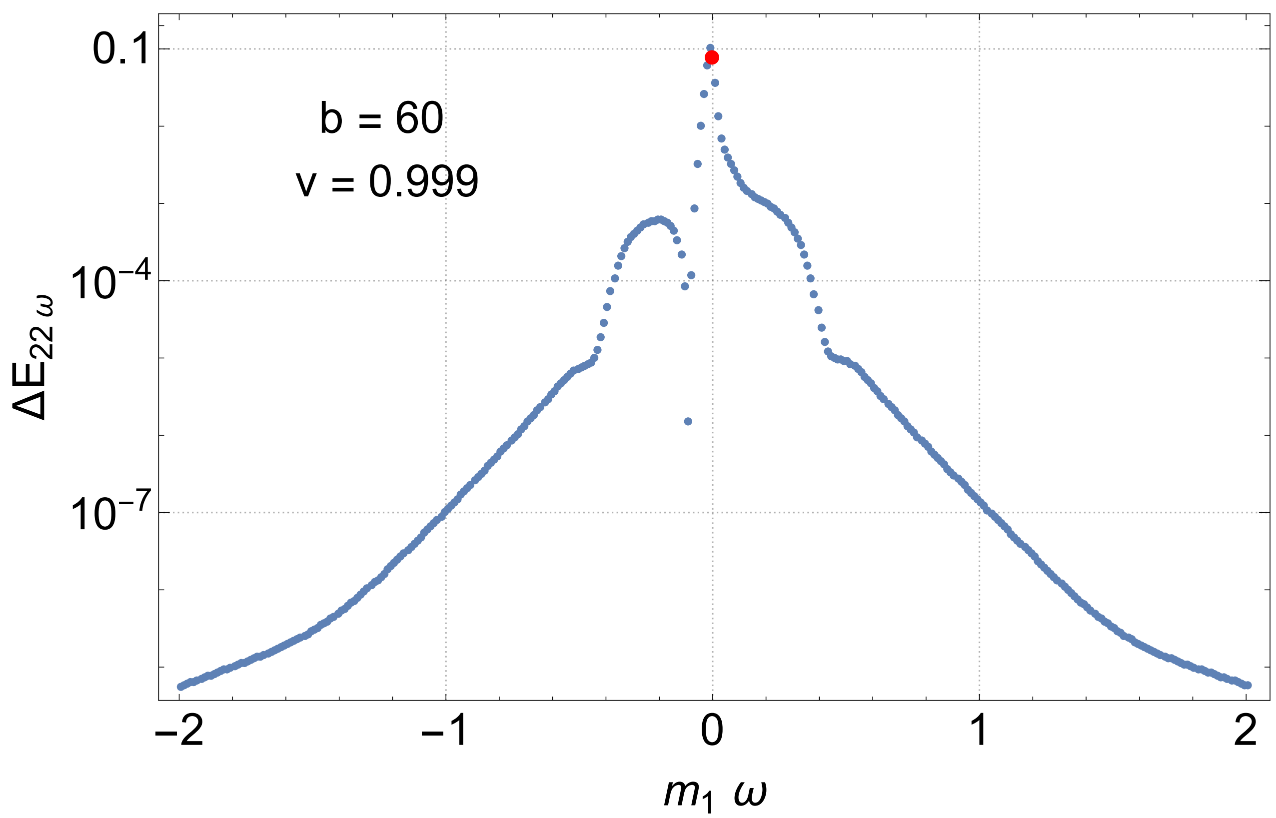


# New results and comparisons: non-linear memory

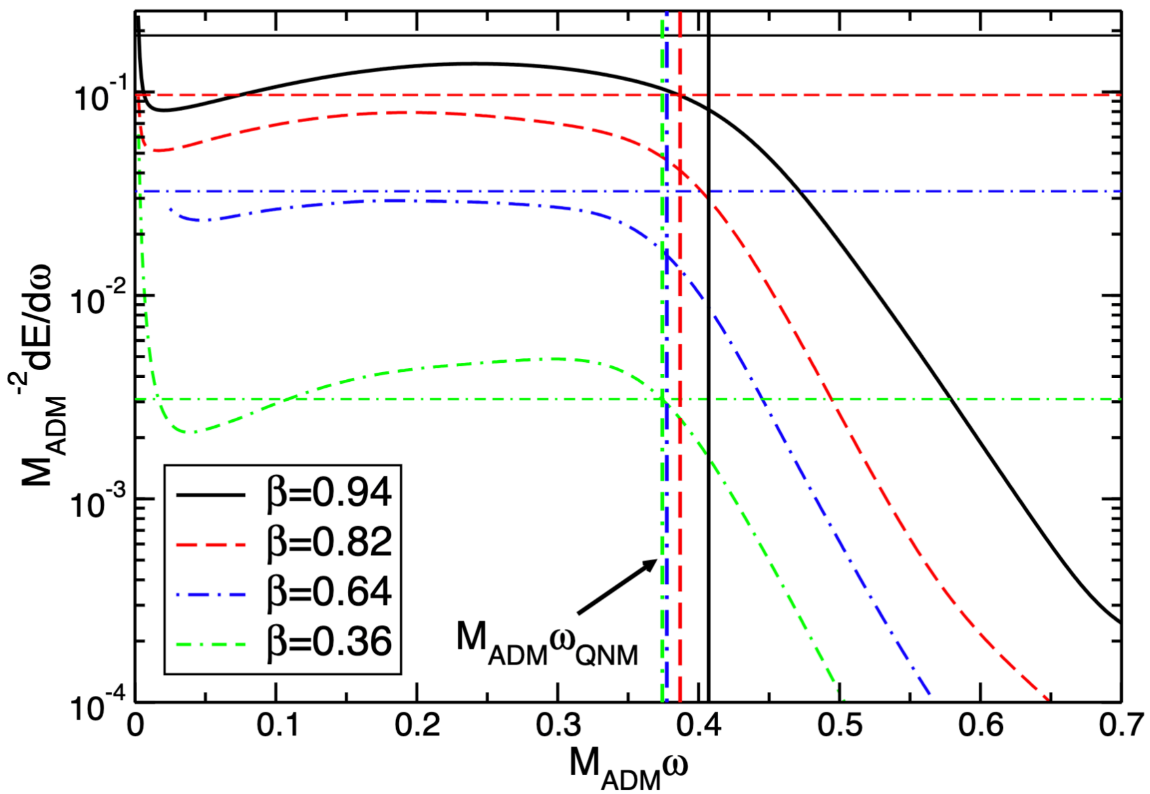


# Ultra-relativistic limit

## GSF scattering




## NR head-on




Berti+, arXiv:1003.0812

Numerically costly. Summing over modes prohibitively expensive. Would be very useful to have lm-mode results from PM/NR to compare against.

# Conclusions and future work

This talk:

- Computation of radiated energy for hyperbolic orbits around a Schwarzschild black hole
- Comparison with PN, PM, and NR
- Creation of a PN-PM composite
- Progress in angular momentum and memory calculations
- Initial exploration of ultra-relativistic limit

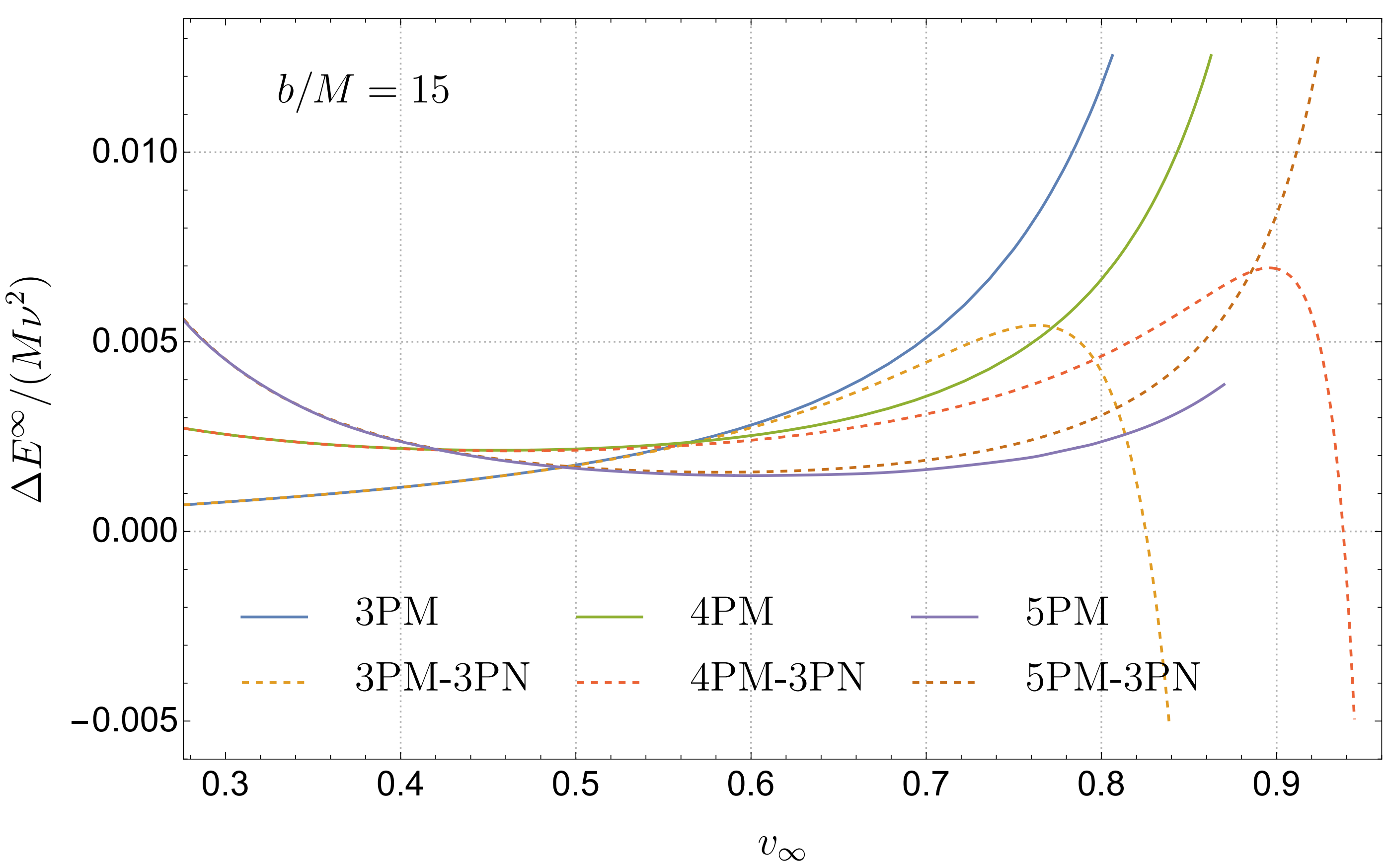
Future work:

- Complete angular momentum calculation
- Calculate the dissipative correction to the scattering angle
- Extension to Kerr
- Conservative effects
- More PM/PN composites

Extra slides

# Future: High-order PM-PN expansions and resummation

- Latest PM results are impressive but going to higher order is challenging
- Potential for rapid progress with PM-PN expansions. See, e.g., Geralico (arXiv:2603.11774) for $\Delta E$ to 6PM-6PN, works by Usseglio, Kavanagh, Bini+

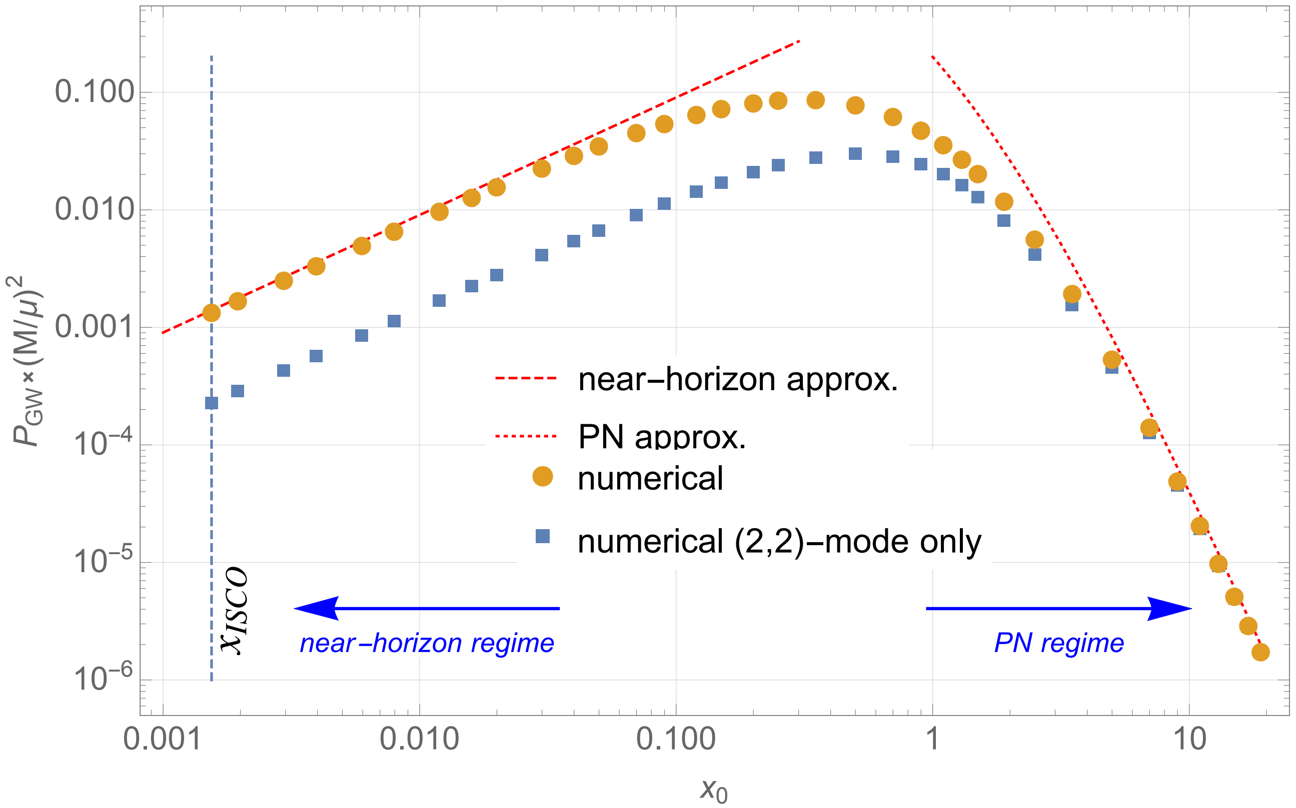



More speculative: Is there scope for resumming PM-PN results using information from the ultra relativistic limit?

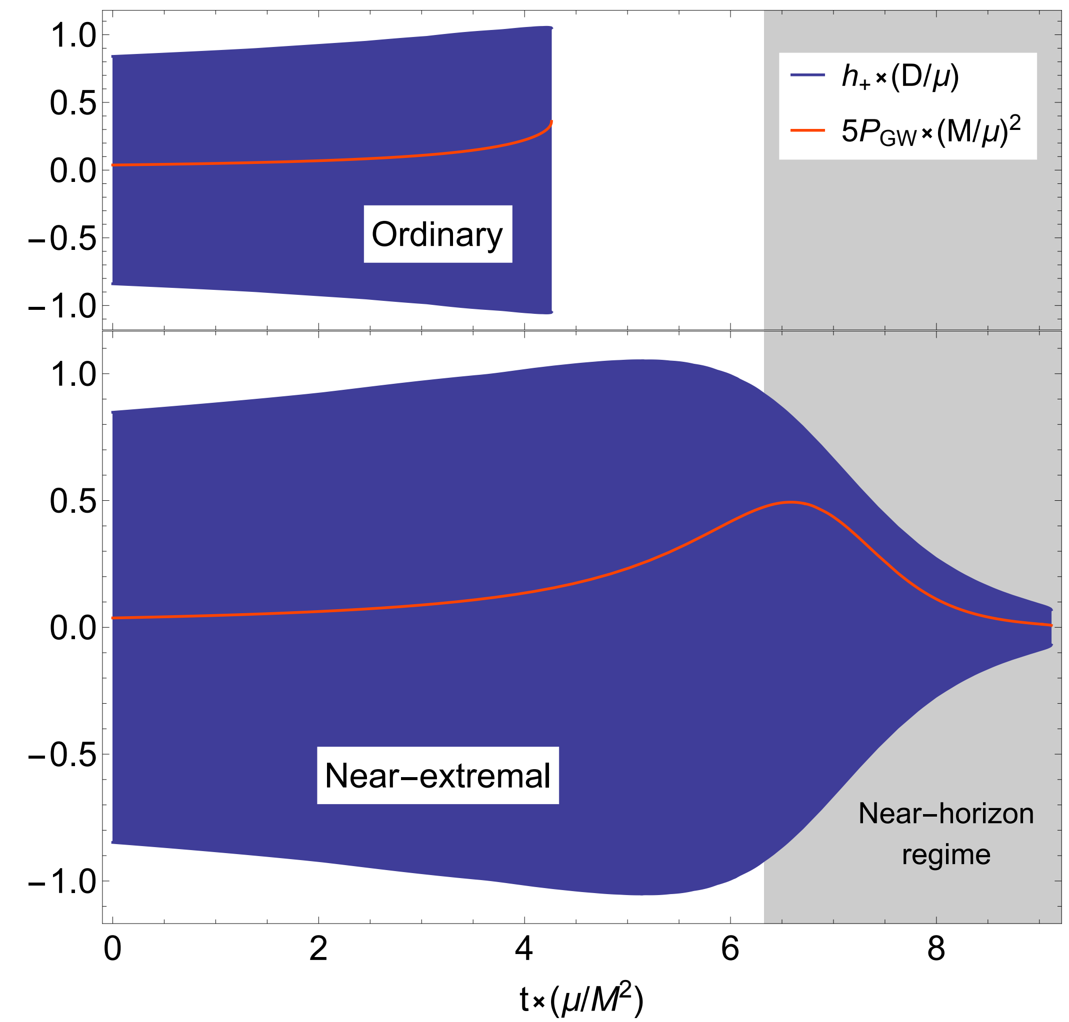
# Future: Kerr

Now at least three groups with Teukolsky codes for scattering orbits

Particularly interesting: scattering in near-extremal Kerr



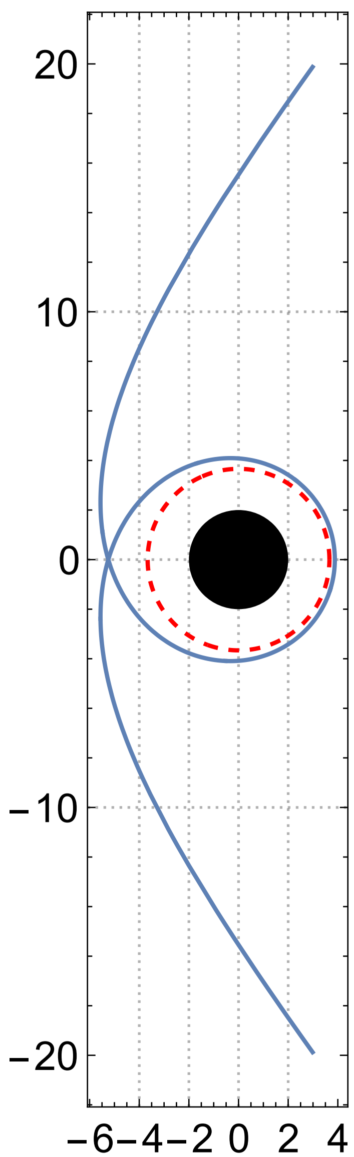

Gralla, Porfyriadis, NW, arXiv:1506.08496



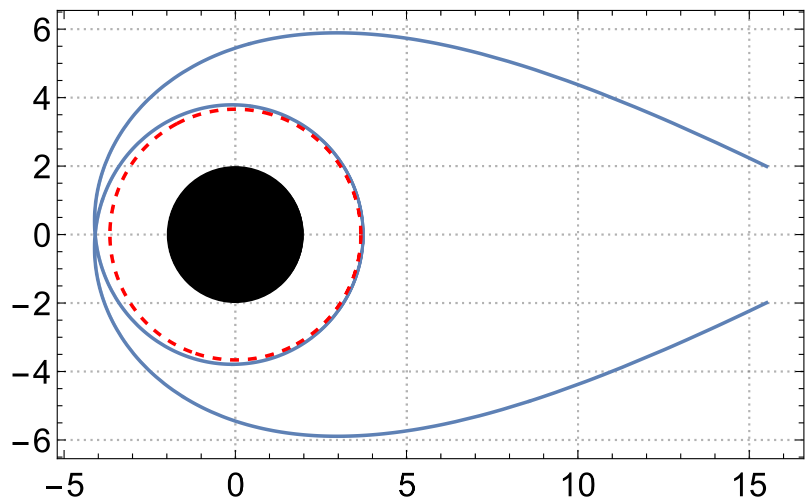

Gralla, Hughes, NW, arXiv:1603.01221

# Source of the divergence at the critical orbit

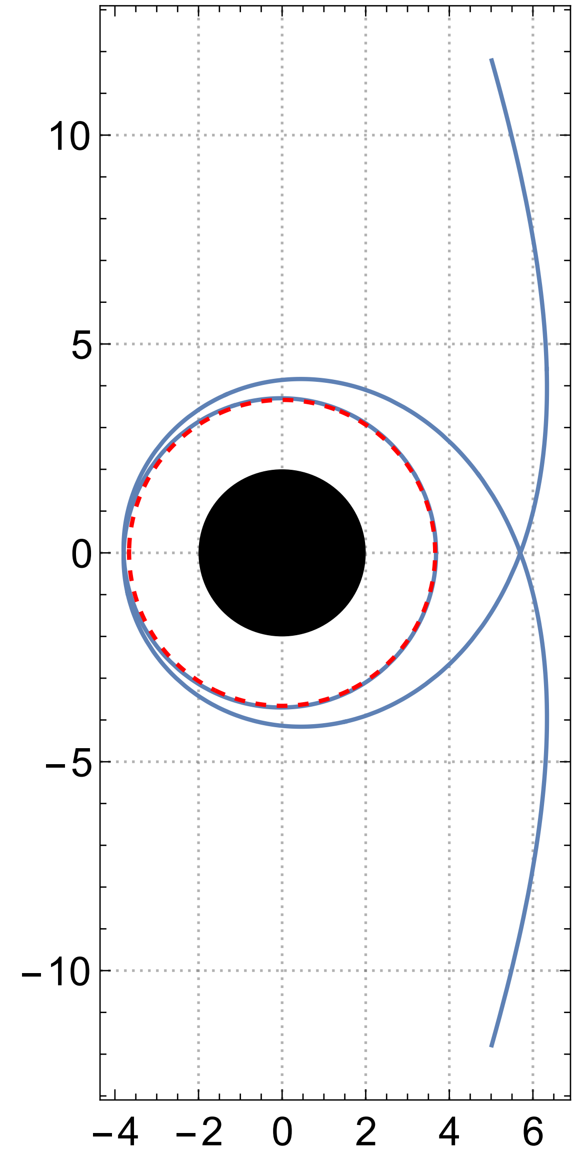
$$j = \frac{L}{m_1} \qquad \delta j = j - j_{\text{crit}}$$



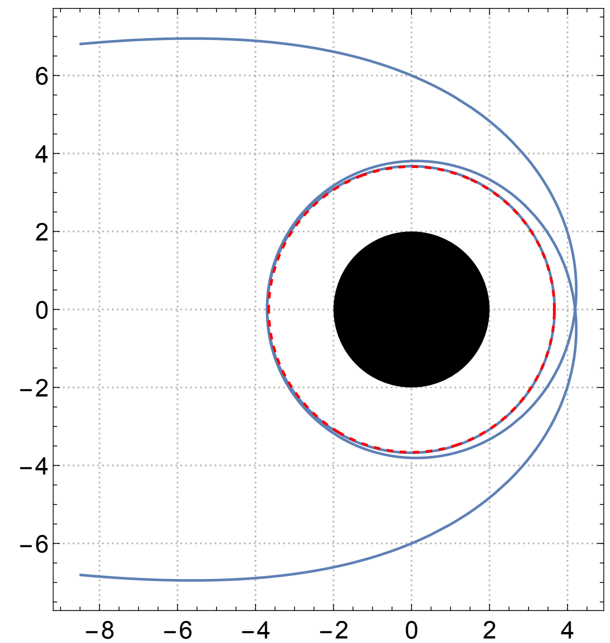

$\delta j = 10^{-1}$




$\delta j = 10^{-2}$




$\delta j = 10^{-3}$




$\delta j = 10^{-4}$

Whirl radius of critical (homoclinic) orbit $:= R$

# Resummation: model

$$E^+ \equiv \Delta E^\infty, \quad E^- \equiv \Delta E^H$$

As the critical orbit is approached we have:

$$E^\pm_{\rm sing}(v,j) = B^\pm(v)\log\left(\frac{\delta j}{j_c(v)}\right)$$

where

$$B^\pm = -\frac{R(v)}{\sqrt{6m_1/R(v)-1}}F^\pm_\circ(R(v))$$

---

In the weak-field use PM or a PM-PN composite:

$$E^\pm_{5PM/3'PN} = E^\pm_{5PM} + E^\pm_{3'PN} - (E_{3'PN})_{5PM}$$

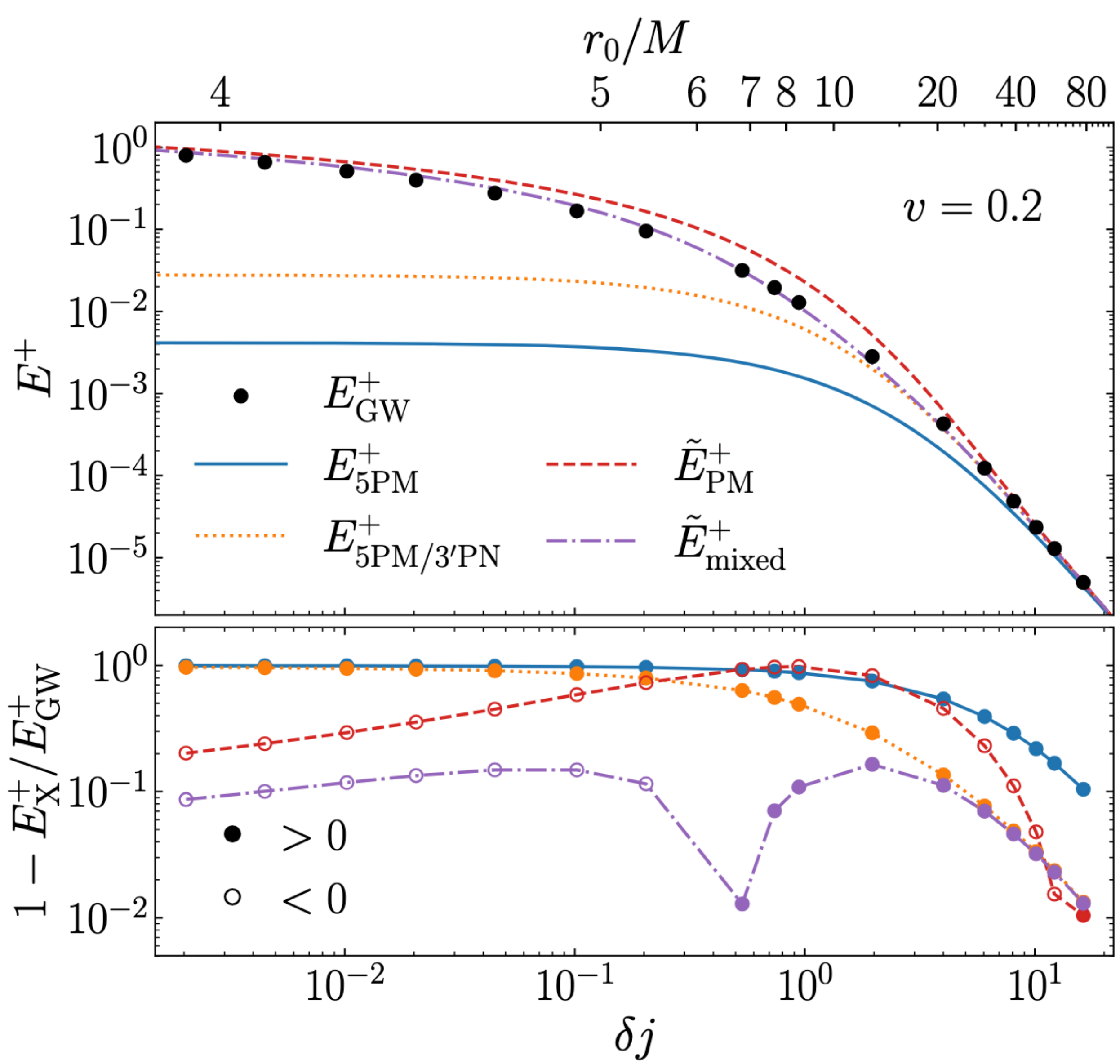
---

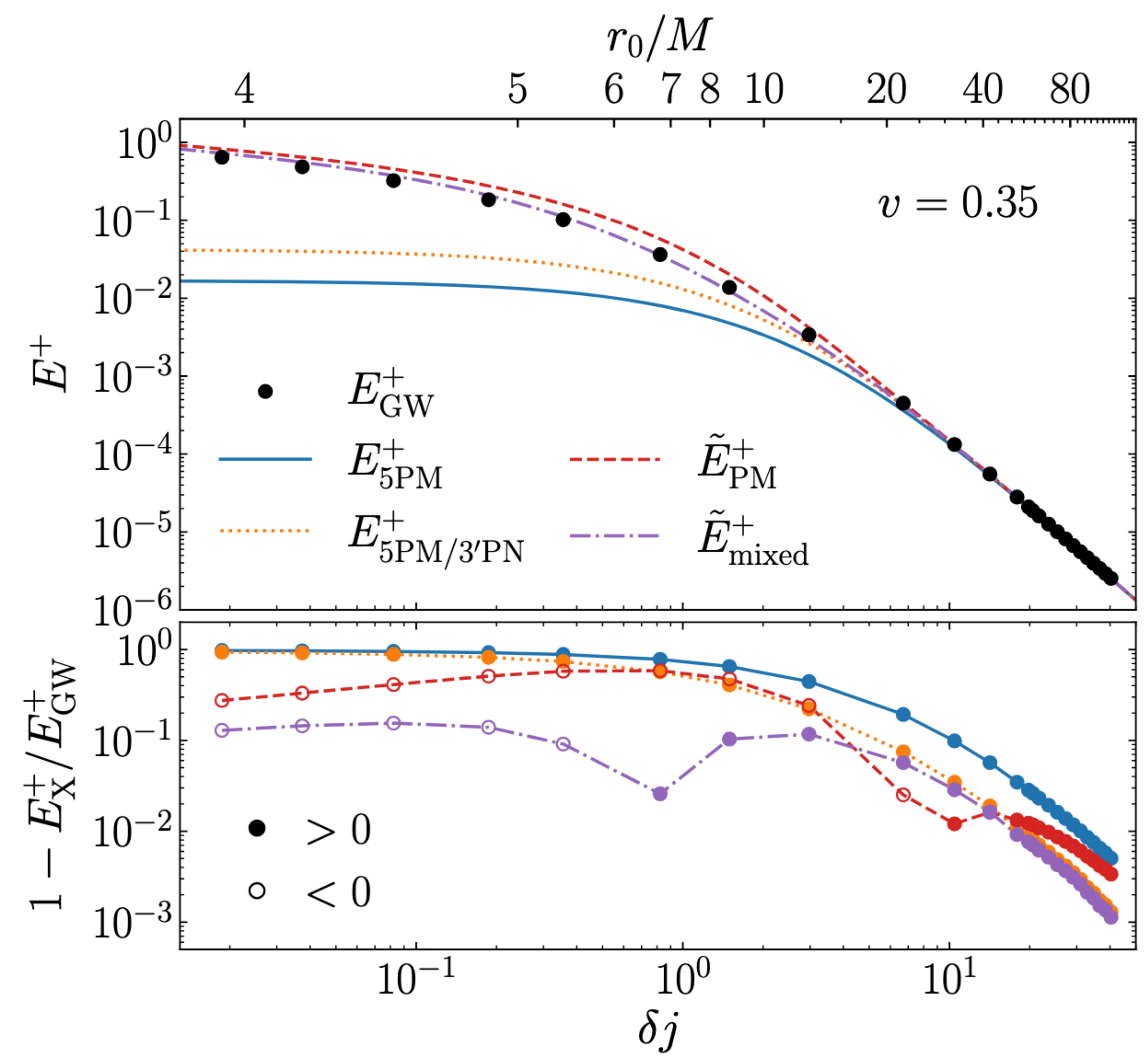
Resummation:

$$\tilde{E}^\pm_X = E^\pm_X + E^\pm_{\rm sing} - CT^\pm_X \qquad \text{where } X \in \{PM, 5PM/3'PN\}$$

Barack, Gonzo, Leather, Long, NW, arXiv:2602.10089

# Resummation: results

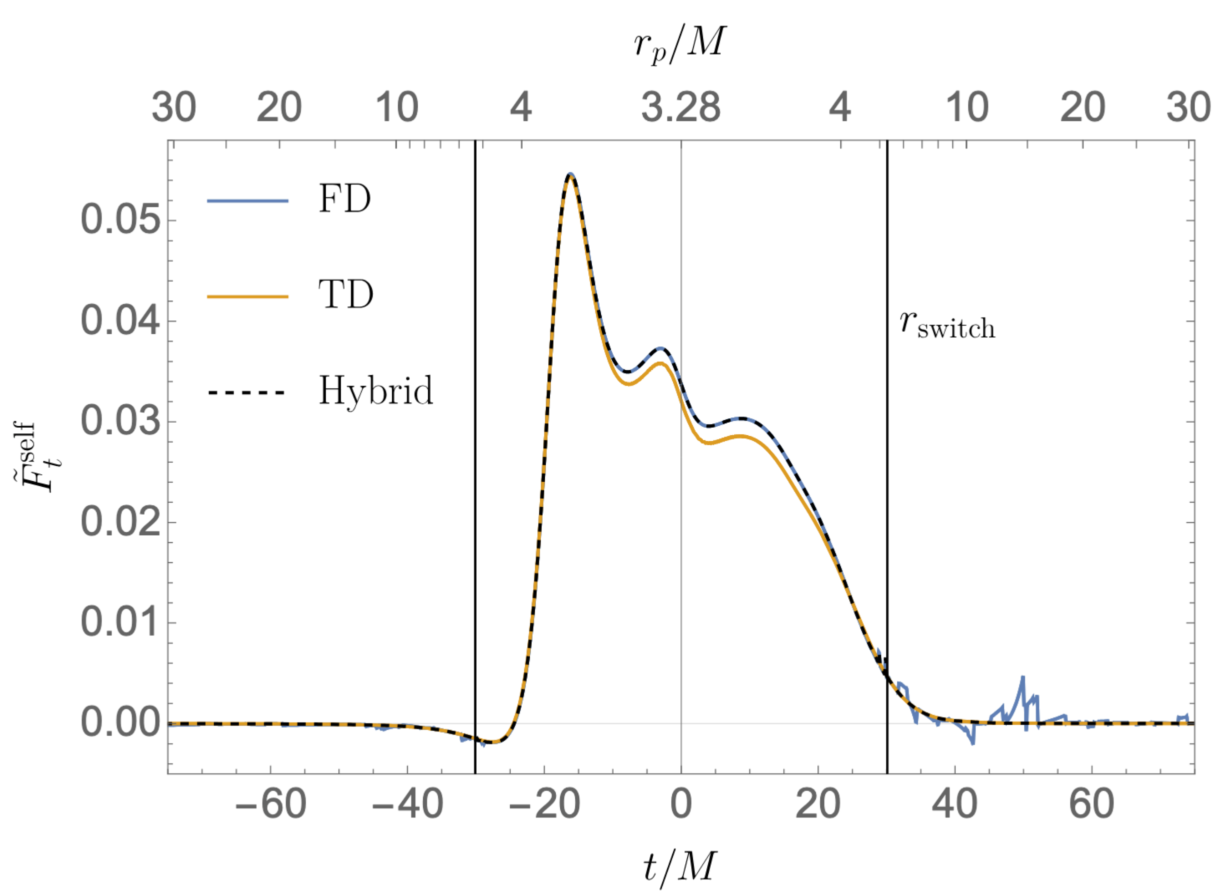


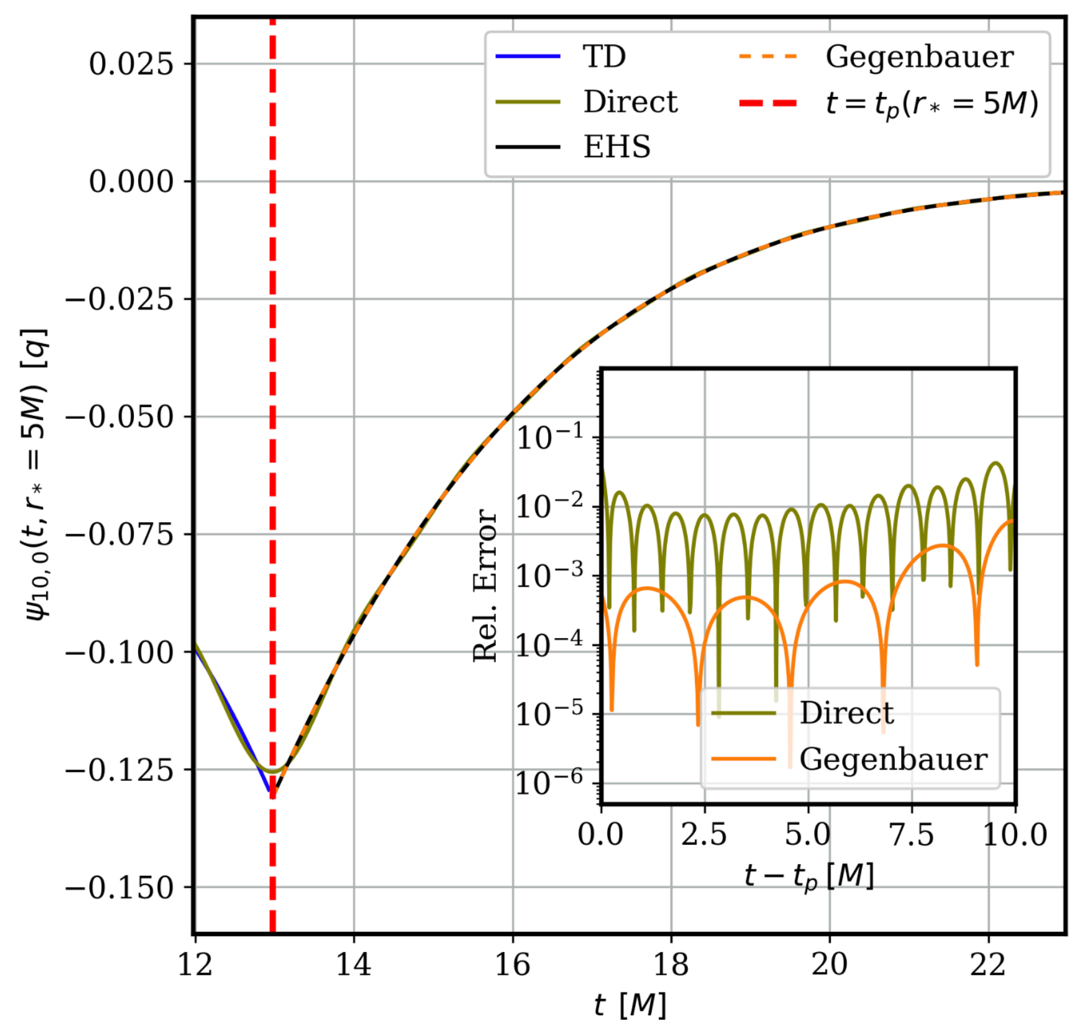

Barack, Gonzo, Leather, Long, NW, arXiv:2602.10089

Conservative effects? Requires calculating the metric perturbation at the particle. Challenging in the frequency domain due to the poor convergence in the source region (Gibbs Phenomenon)

Progress in scalar-field models:

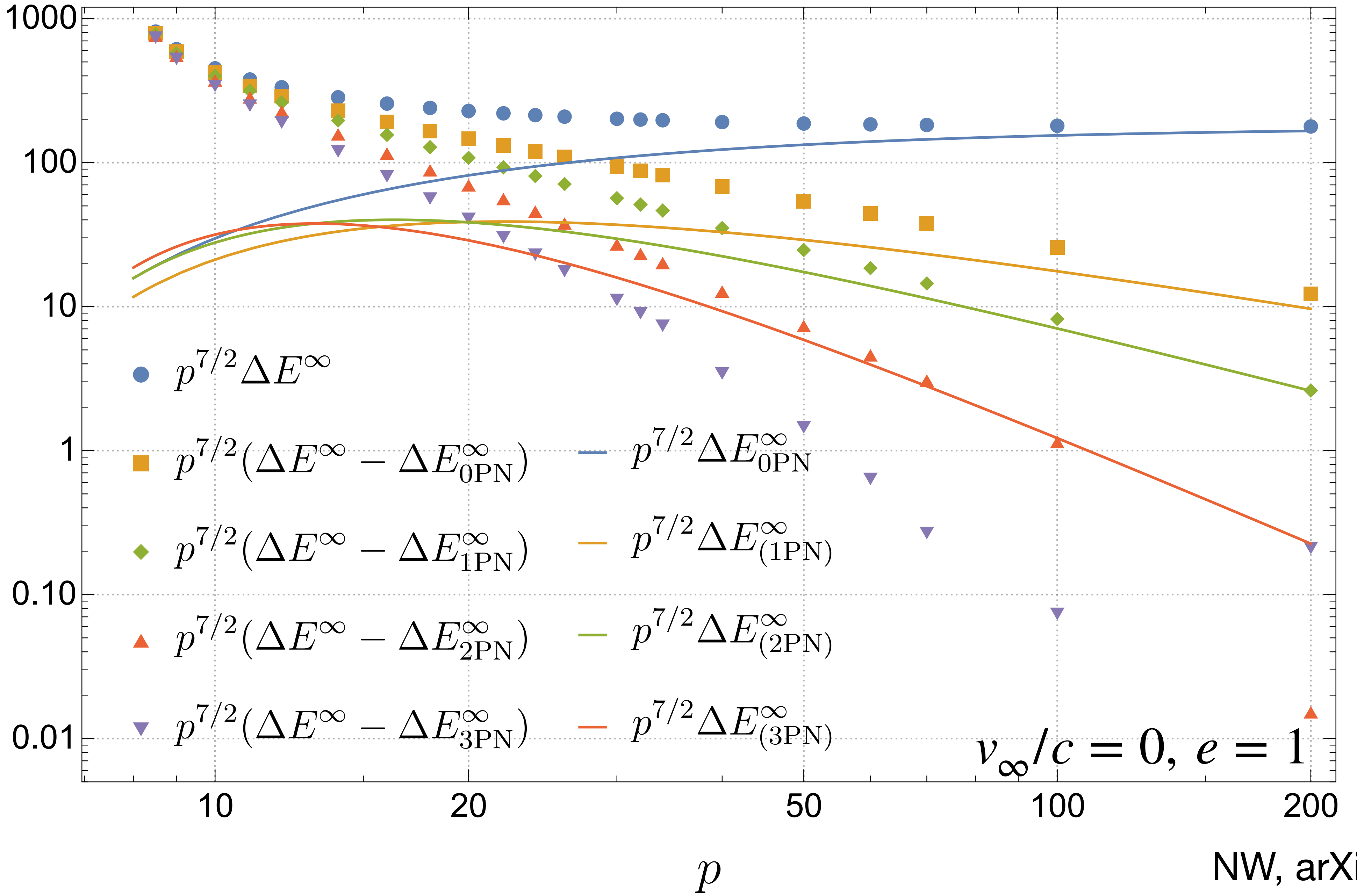



Long, Whittall, Barack, arXiv:2406.08363




Whittall, Barack, Long, arXiv:2509.19439

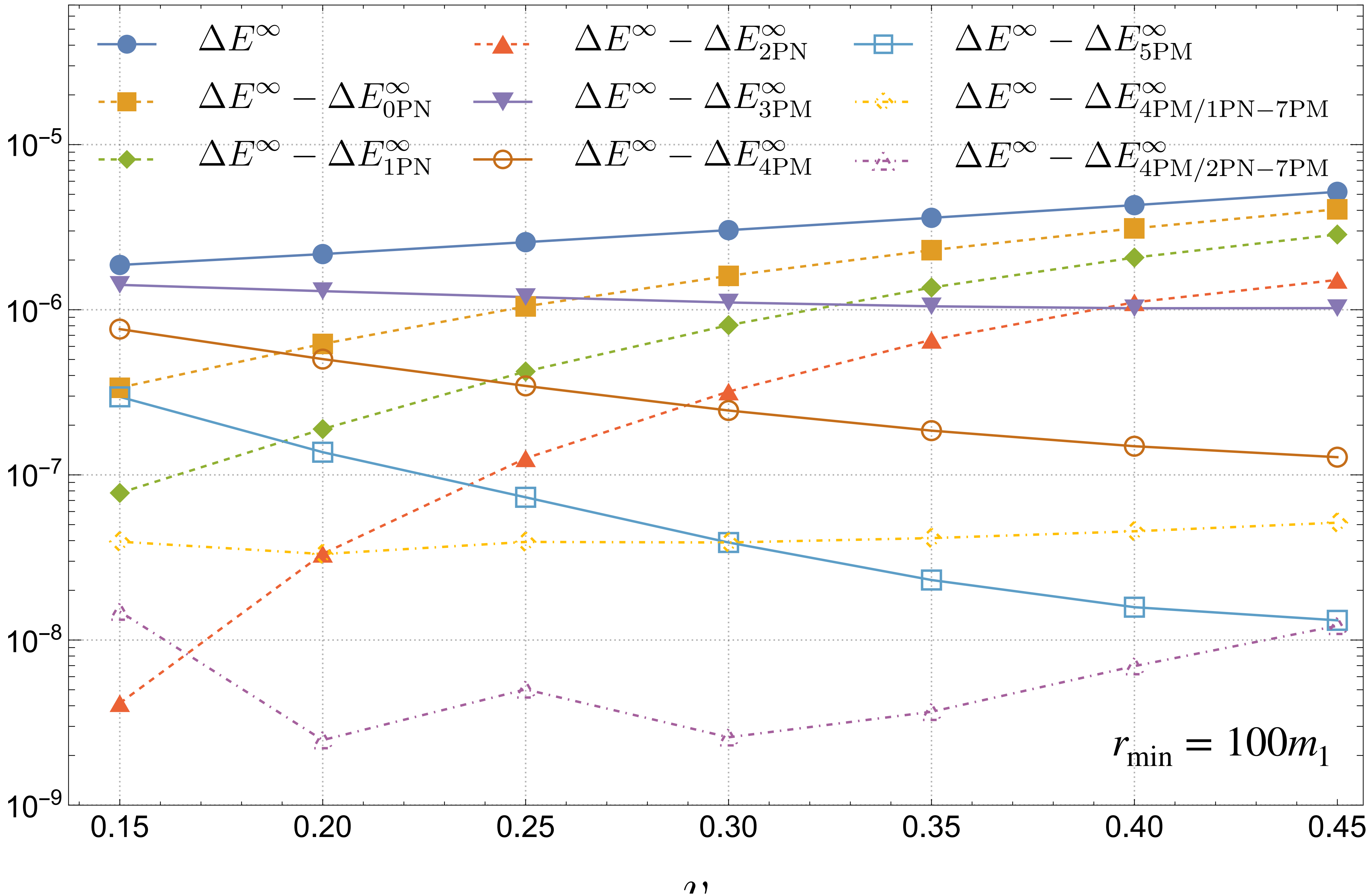
# Parabolic orbits



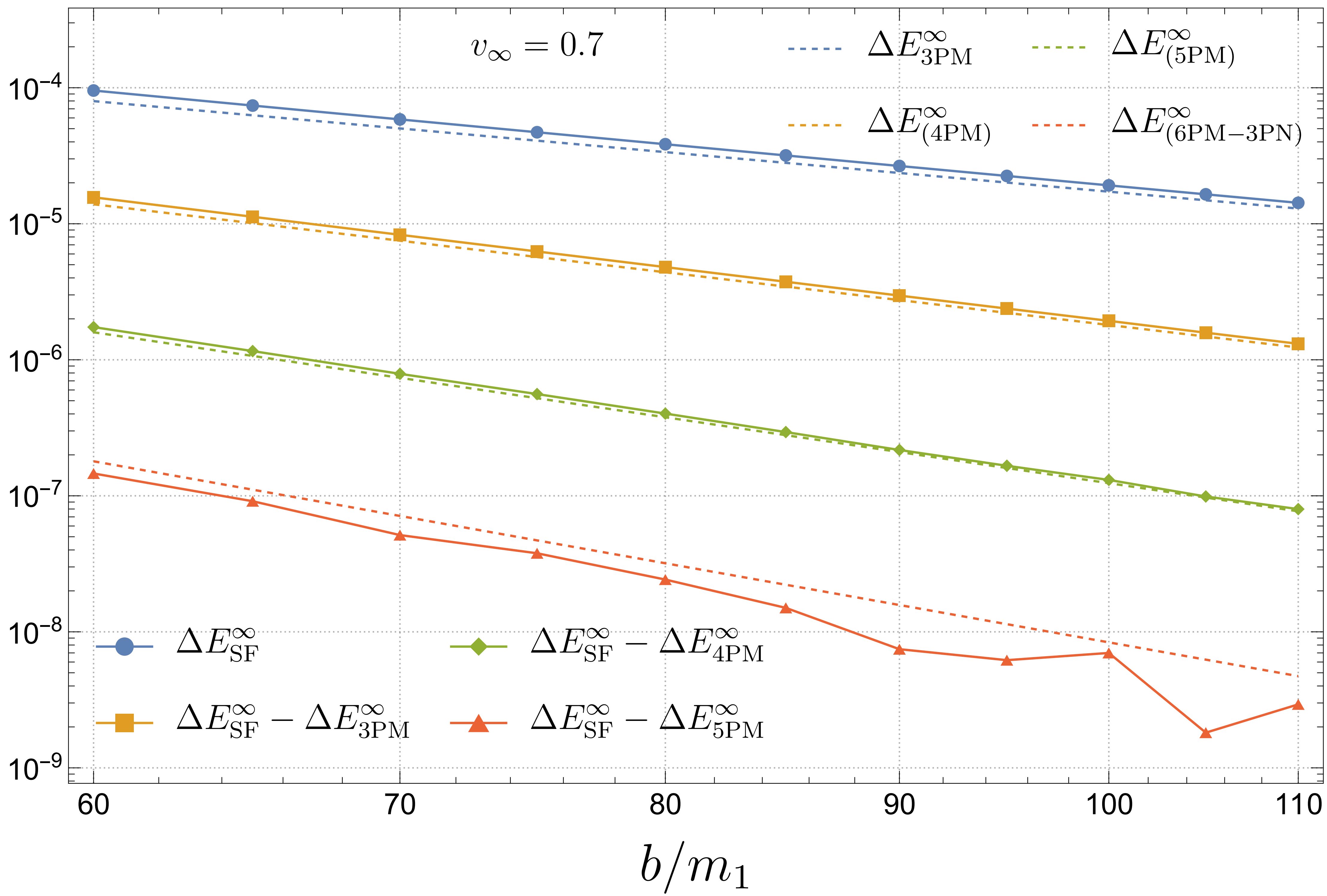

NW, arXiv:2512.02274

- Exact (in G) parabolic results through 3PN given by Cho, arXiv:2203.10872
- ~4% relative error with Martel, arXiv:gr-qc/0311017
- Sub-percent level agreement with H. Khalvati (unpublished)

# Comparison with PN, PM, and PN/PM composite



# Residual with 5PM



# Results: mode data