Hybrid Self-Force and Post-Newtonian Waveform models: An opportunity for Post-Minkowskian Theory?

Josh Mathews

With Loic Honet, Geoffrey Compere, Adam Pound, Barry Wardell, Gabriel Piovano, Maarten Van De Meent, Niels Warburton +…

NUS National University of Singapore

Overview

- **Part 1: Anatomy of self-force waveform models.**
- **Part 2: SF-PN hybrid models.**
- **Part 3: Eccentric generalisations. (SF-PN-PM hybrids)?**

Part 1: Anatomy of self-force waveforms.

Key Definitions

Small mass-ratio $\epsilon \equiv m_2/m_1$ with $m_1 \geq m_2,$

Large mass-ratio $q \equiv 1/\epsilon,$

Symmetric mass-ratio $\nu \equiv m_1 m_2/(m_1 + m_2)^2,$

Dimensionless spin $\chi_i \equiv S_i/m_i^2$

$$G = c = 1$$

Waveform model ingredients

$\mathcal{J}_B$ **Binary parameters e.g** (E, L)

$\dot{\mathcal{J}}_B$ **"Fluxes", e.g** $\left(\frac{dE}{dt}, \frac{dL}{dt}\right)$

ψ_A **A set of phases / action angles**

$\Omega_A \equiv \dot{\psi}_A$ **Frequencies**

$h_{\mathbf{k}}$ **Waveform amplitudes**

Multiscale expansion

Fast timescale

$$\frac{d}{dt} \to \Omega_A \frac{\partial}{\partial \psi_A} + \epsilon \frac{d\mathcal{J}_B}{dt} \frac{\partial}{\partial \mathcal{J}_B}$$

Slow timescale

Into Einstein equations (coupled to SF EOMs)

$$\frac{d\psi_A}{dt} = \Omega_A^{(0)}(\mathcal{J}_B) + \epsilon \Omega_A^{(1)}(\mathcal{J}_B) + \mathcal{O}(\epsilon^2),$$
$$\frac{d\mathcal{J}_B}{dt} = \epsilon F_B^{(0)}(\mathcal{J}_C) + \epsilon^2 F_B^{(1)}(\mathcal{J}_C) + \mathcal{O}(\epsilon^3),$$

See e.g Pound + Wardell 2101.04592

Multiscale expansion II

$$\frac{d\psi_A}{dt} = \Omega_A^{(0)}(\mathcal{J}_B) + \epsilon\Omega_A^{(1)}(\mathcal{J}_B) + \mathcal{O}(\epsilon^2),$$

$$\frac{d\mathcal{J}_B}{dt} = \epsilon F_B^{(0)}(\mathcal{J}_C) + \epsilon^2 F_B^{(1)}(\mathcal{J}_C) + \mathcal{O}(\epsilon^3),$$

Integrate first equation.

$$\psi_A = \frac{1}{\epsilon}\psi_A^{(0)}(\epsilon t) + \psi_A^{(1)}(\epsilon t) + \mathcal{O}(\epsilon)$$

Adiabatic order (0PA)	First post-adiabatic order (1PA)	etc.
1SF dissipative	+1SF conservative +2SF dissipative	

Anatomy of self-force waveforms

Complex amplitudes — **Orbital phases**

$$h = \sum_{\mathbf{k}} \left[\epsilon \mathring{h}^{(1)}_{\mathbf{k}}(\mathcal{J}_B, \theta, \phi) + \epsilon^2 \mathring{h}^{(2)}_{\mathbf{k}}(\mathcal{J}_B, \theta, \phi) \right] e^{-i\psi_{\mathbf{k}}},$$

Computed numerically on a grid of the parameters and stored as interpolating functions.

$$\frac{d\psi_A}{dt} = \Omega^{(0)}_A(\mathcal{J}_B) + \epsilon \Omega^{(1)}_A(\mathcal{J}_B) + \mathcal{O}(\epsilon^2),$$

$$\frac{d\mathcal{J}_B}{dt} = \epsilon F^{(0)}_B(\mathcal{J}_C) + \boxed{\epsilon^2 F^{(1)}_B(\mathcal{J}_C)} + \mathcal{O}(\epsilon^3),$$

2SF limiting factor

Online waveform generation: Integrate ODEs to determine trajectory and sum waveform modes.

Quasi-circular 1PAT1 model

$$\mathcal{J}_A \to (\Omega, \delta m_1, \delta\chi_1)$$
$$\psi_A \to \phi = \int \Omega dt$$

2SF limiting factor

$$\frac{d\Omega}{dt} = \epsilon F_\Omega^{(0)}(\Omega) + \boxed{\epsilon^2 F_\Omega^{(1)}(\mathcal{J}_A)} + \mathcal{O}(\epsilon^3),$$
$$\frac{d\delta m_1}{dt} = \epsilon F_{\delta m_1}^{(1)}(\Omega) + \mathcal{O}(\epsilon^2),$$
$$\frac{d\delta\chi_1}{dt} = \epsilon F_{\delta\chi_1}^{(1)}(\Omega) + \mathcal{O}(\epsilon^2).$$

Currently limited to a slowly spinning primary, generic spinning secondary.

See 2510.16113

1PAT1 model key ingredients

Binding energy

0SF 1SF

$$E = E_0(\Omega) + \epsilon E_1(\Omega) + \ldots$$

Asymptotic energy flux

$$\frac{dE}{dt} = \epsilon \dot{E}_1 + \epsilon^2 \dot{E}_2 + \ldots$$

1SF 2SF

Balance Law (Modulo Schott terms.)

1SFd 2SFd

$$\frac{d\Omega}{dt} = \epsilon \frac{d\Omega}{dE_0}\frac{dE_1}{dt} + \epsilon^2 \left(\frac{d\Omega}{dE_0}\right)^2 \left(\dot{E}_2 \frac{dE_0}{d\Omega} - \dot{E}_1 \frac{dE_1}{d\Omega}\right) + \mathcal{O}(\epsilon^3)$$

1SFc

0PA evolution

1PA evolution

Quasi-circular 1PAT1R model

Effectively re-summed model

1PAT1 $$\frac{d\Omega}{dt} = \epsilon F_\Omega^{(0)}(\Omega) + \epsilon^2 F_\Omega^{(1)}(\mathcal{J}_B) + \mathcal{O}(\epsilon^3)$$

1PAT1R $$\frac{d\Omega}{dt} = -\frac{\mathcal{F} + \mathcal{F}_{\mathcal{H}_i}\partial E/\partial m_i + \dot{\chi}_i\,\partial E/\partial \chi_i}{\partial E/\partial \Omega},$$

Pole at ISCO: DO NOT EXPAND

$$E = \epsilon E_1(\Omega) + \epsilon^2 E_2(\Omega) + \ldots$$

C.f Long et al 2406.08363

From asymmetry to symmetry

To restore some of the symmetry in the interchange of body 1 and body 2, we perform a re-expansion:

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\epsilon \to \nu + 2\nu^2 + \mathcal{O}(\nu^3)$$

$$\tilde{a}_i = \frac{m_i}{m_1 + m_2}\chi_i$$

Rationale: Improve accuracy of self-force expansions in the comparable mass limit.

1PAT1, 1PAT1R, NR

Blue: 1PAT1, Orange: 1PAT1R, Black: SXS:BBH:1132

Red: 1PAT1R, Black: SXS:BBH:1431

Notes on self-force waveforms

- We **can** describe the waveform through **plunge-merger-ringdown.** Work in progress.
- We **can handle precessing spins and eccentricity,** with progress on **2SF as the limiting factor**.
- We **need to improve** our parameter space coverage at **high eccentricity** (which is hard).

Part 2: SF-PN hybrid models.

Waveform model ingredients

$\mathcal{J}_B$	Binary parameters e.g	←	1. Map between SF and PN variables
$\dot{\mathcal{J}}_B$	“Fluxes”, e.g	←	2. Hybridise
ψ_A	A set of phases / action angles		
$\Omega_A \equiv \dot{\psi}_A$	Frequencies	←	2. Hybridise
$h_{\mathbf{k}}$	Waveform amplitudes	←	2. Hybridise

Composite expansions

$$\mathcal{F}_\infty = \overbrace{\nu^2 \mathcal{F}_\infty^{(0)}(x,\chi_1)}^{\text{0PA (1SF)}} + \overbrace{\nu^3 \mathcal{F}_\infty^{(1)}(x,\chi_1,\chi_2)}^{\text{1PA (2SF)}} + \ldots$$

$$= \underbrace{\frac{32}{5}\nu^2 x^5}_{\text{0PN}} \underbrace{- \left(\frac{2494}{105} + \frac{56\nu}{3}\right)\nu^2 x^6}_{\text{1PN}} + \ldots$$

All available SF terms + all available PN terms - common terms to both expansion.

Composite expansions II

SF Quantity

$$Q^{\mathrm{SF}}_{\substack{\bar{k}\,\bar{l}_1\,\bar{l}_2\\ \underline{k}\,\underline{l}_1\,\underline{l}_2}} = \nu^K \sum_{l_i=\underline{l}_i}^{\bar{l}_i} \sum_{k=\underline{k}}^{\bar{k}} Q^{(k)}_{l_1 l_2}(x)\nu^k \chi_1^{l_1}\chi_2^{l_2},$$

PN Quantity

$$Q^{\mathrm{PN}}_{\substack{\bar{n}\,\bar{l}_1\,\bar{l}_2\\ \underline{n}\,\underline{l}_1\,\underline{l}_2}} = x^{N/2} \sum_{l_i=\underline{l}_i}^{\bar{l}_i} \sum_{n=\underline{n}}^{\bar{n}} Q^{\frac{n}{2}\mathrm{PN}}_{l_1 l_2}(\log x) x^{n/2} \chi_1^{l_1}\chi_2^{l_2},$$

SF-PN expansion

$$Q^{\mathrm{SF|PN}}_{\substack{\bar{k}\,\bar{n}\,\bar{l}_1\,\bar{l}_2\\ \underline{k}\,\underline{n}\,\underline{l}_1\,\underline{l}_2}} = \nu^K x^{N/2} \sum_{l_i=\underline{l}_i}^{\bar{l}_i} \sum_{k=\underline{k}}^{\bar{k}} \sum_{n=\underline{n}}^{\bar{n}} Q^{k|n}_{l_1 l_2}(\log x)\nu^k x^{n/2} \chi_1^{l_1}\chi_2^{l_2}$$

All available SF terms + all available PN terms - common terms to both expansion.

$$Q^{\mathrm{SF+PN}} = \sum_{\underline{k},\underline{n},\underline{l}_i,\bar{k},\bar{n},\bar{l}_i} \left(Q^{\mathrm{SF}}_{\substack{\bar{k}\,\bar{l}_1\,\bar{l}_2\\ \underline{k}\,\underline{l}_1\,\underline{l}_2}} + Q^{\mathrm{PN}}_{\substack{\bar{n}\,\bar{l}_1\,\bar{l}_2\\ \underline{n}\,\underline{l}_1\,\underline{l}_2}} - Q^{\mathrm{SF|PN}}_{\substack{\bar{k}\,\bar{n}\,\bar{l}_1\,\bar{l}_2\\ \underline{k}\,\underline{n}\,\underline{l}_1\,\underline{l}_2}} \right).$$

Can exploit PN hierarchy in spins to add available 2SF flux data.

WaSABI-C model

Both spin magnitudes evolve, along with the total mass, mass-ratio and waveform frequency

$$\mathcal{J}_B \to (\omega, \chi_1, \chi_2, \nu, M)$$

$$\psi_A \to \psi$$

$$\Omega_A \to \omega$$

$$\frac{d\psi}{dt} = \omega/M, \quad \frac{d\omega}{dt} = F_\omega(\mathcal{J}_B),$$

$$\frac{d\chi_i}{dt} = F_{\chi_i}(\mathcal{J}_B), \; \frac{d\nu}{dt} = F_\nu(\mathcal{J}_B), \; \frac{dM}{dt} = F_M(\mathcal{J}_B),$$

- Currently limited to BH-BH (BH-NS in development).
- Includes all 1SF, secondary spin 2SF, Schwarzschild 2SF, most available PN data.

See Honet et al, 2510.16112

WaSABI-C model II

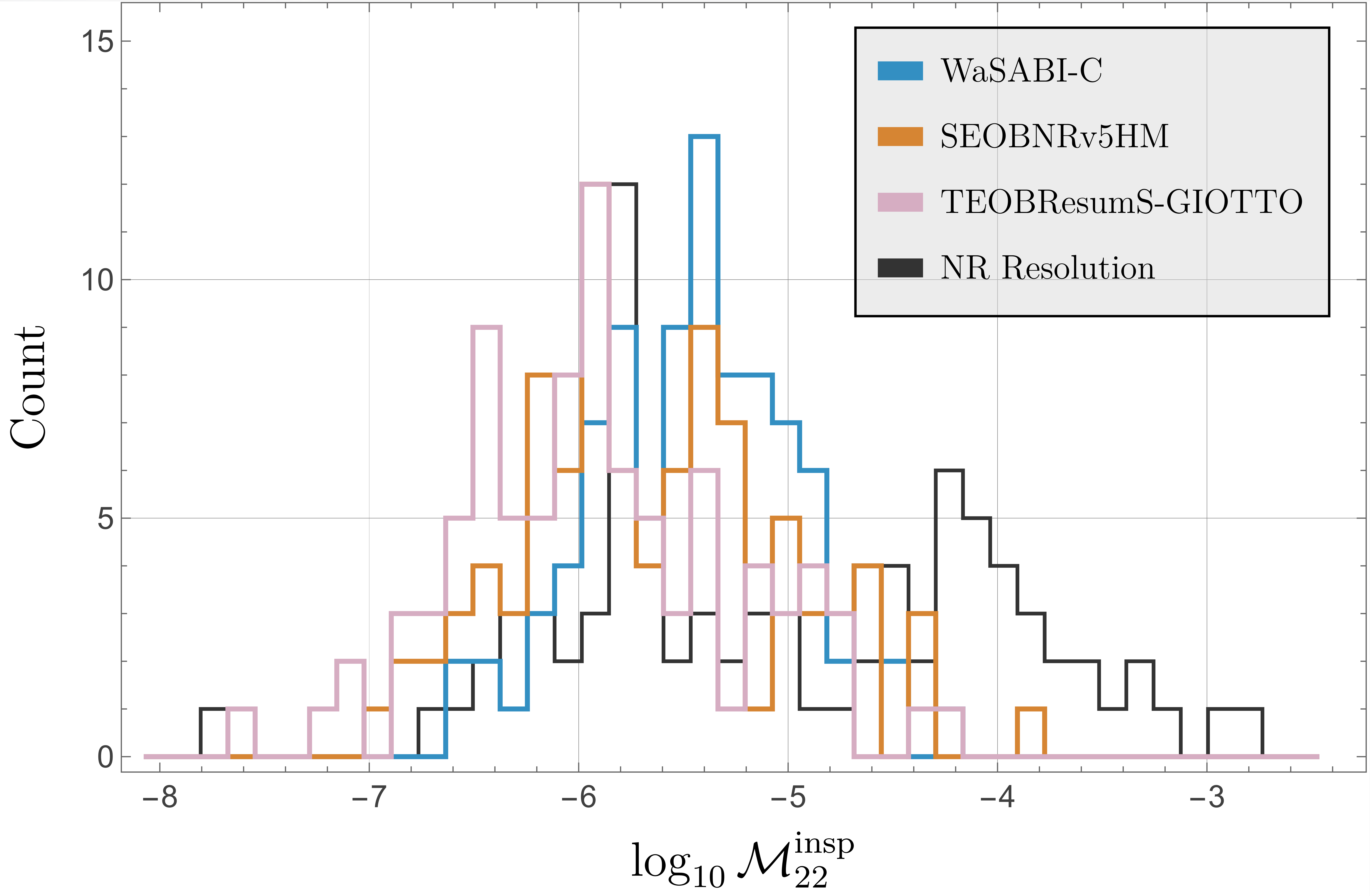
Key point: We hybridise the components of the LHS (binding energy, fluxes) to avoid expanding the pole at the ISCO.

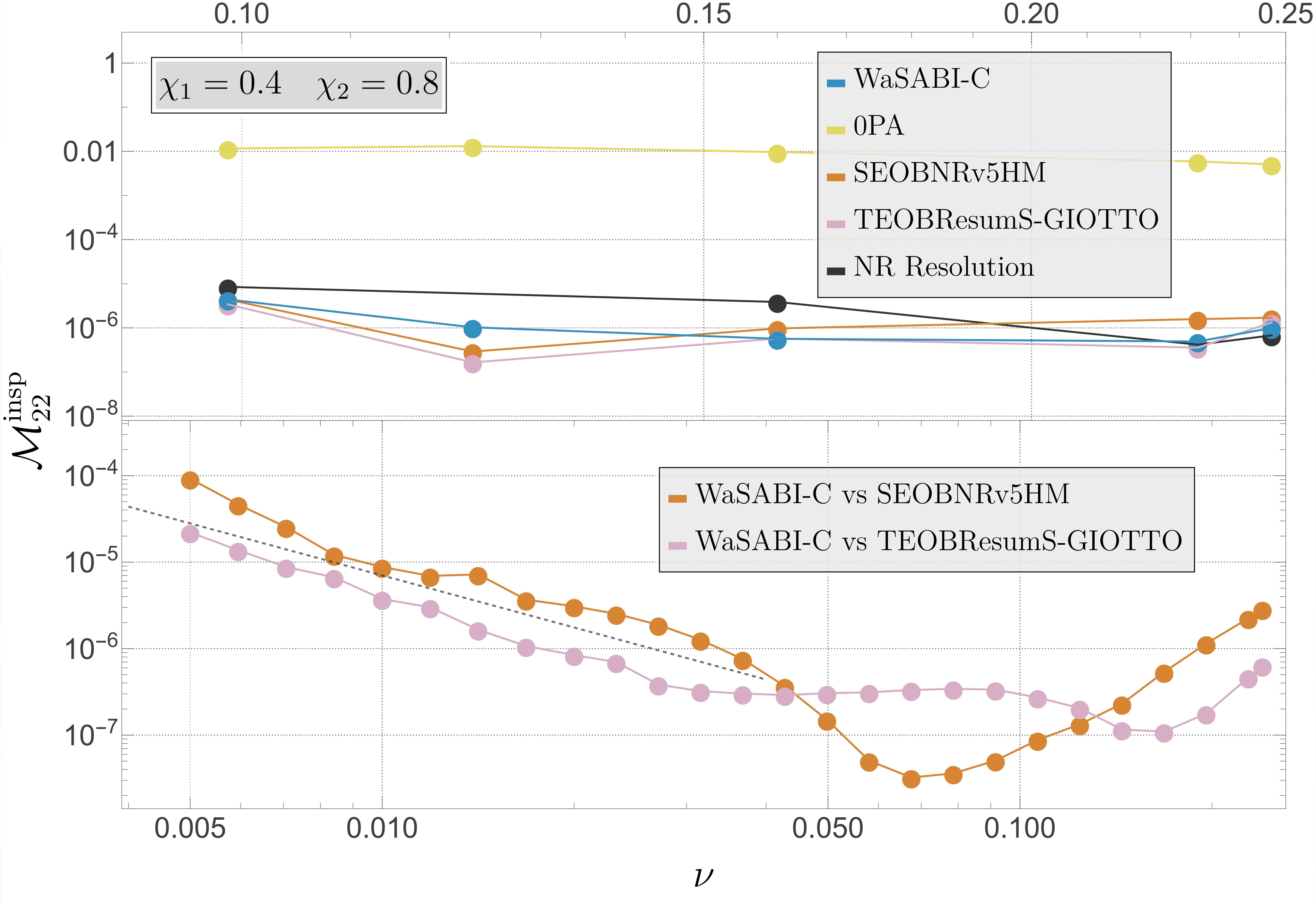
$$\frac{d\Omega}{dt} = -\frac{\mathcal{F} + \mathcal{F}_{\mathcal{H}_i} \partial E / \partial m_i + \dot{\chi}_i \, \partial E / \partial \chi_i}{\partial E / \partial \Omega},$$

$$\omega = \frac{1}{2}\omega_{22} = \Omega - \nu \delta\omega + \dots$$

WaSABI-C, NR

$q = 6.0 \quad \chi_1 = -0.4 \quad \chi_2 = 0.8$
SXS:BBH:2209
WaSABI-C
0.10
0.05
0.00
−0.05
−0.10
$\mathrm{Re}\,(h) \quad [M_0]$
0
500
1000
1500
2000
2500
3000
3500
$t \quad [M_0]$


WaSABI-C
SEOBNRv5HM
TEOBResumS-GIOTTO
NR Resolution
Count
15
10
5
0
−8
−7
−6
−5
−4
−3
$\log_{10} \mathcal{M}_{22}^{\rm insp}$



$\chi_1 = 0.4 \quad \chi_2 = 0.8$
WaSABI-C
0PA
SEOBNRv5HM
TEOBResumS-GIOTTO
NR Resolution
WaSABI-C vs SEOBNRv5HM
WaSABI-C vs TEOBResumS-GIOTTO
$\mathcal{M}_{22}^{\rm insp}$
ν
0.10
0.15
0.20
0.25
1
0.01
10^{-4}
10^{-6}
10^{-8}
10^{-5}
10^{-7}
0.005
0.010
0.050
0.100

Flatten the curve

EOB should recover leading order self-force in the small mass-ratio limits e.g.

$$\mathcal{F}_{EOB} \to \mathcal{F}_{EOB} + D(x)\left(\nu\mathcal{F}_{1SF} - \nu\mathcal{F}^{1SF}_{EOB}\right)$$

Add interpolated self-force data to the inspiral part of the EOB flux without spoiling the re-summations.

Part 3: Eccentric Generalisations

The pitch: SF-PN-PM hybrid for highly eccentric binaries

- Forgetting spin from here but in practice **we do need spins.**
- **1SF conservative quantities** (binding energy and angular momentum) are **difficult to compute at high eccentricity.**
- Hybrid SF-PN fluxes, **hybrid SF-PM binding energy and angular momentum.**

Key Ingredients

$$E = E_0(\mathcal{J}_B) + \epsilon E_1(\mathcal{J}_B) + \ldots$$

Binding energy (SF-PN/SF-PM)

$$L = L_0(\mathcal{J}_B) + \epsilon L_1(\mathcal{J}_B) + \ldots$$

Binding angular momentum (SF-PN/SF-PM)

Hard for SF high eccentricity

$$\dot{E} = \dot{E}_1(\mathcal{J}_B) + \epsilon \dot{E}_2(\mathcal{J}_B) + \ldots$$

Energy fluxes (SF-PN)

$$\dot{L} = \dot{L}_1(\mathcal{J}_B) + \epsilon \dot{L}_2(\mathcal{J}_B) + \ldots$$

Angular momentum fluxes (SF-PN)

2SF eccentric flux unknown

$$\frac{d\mathcal{J}_B}{dt} = \frac{\partial \mathcal{J}_B}{\partial E}\dot{E} + \frac{\partial \mathcal{J}_B}{\partial L}\dot{L}$$

Binary inspiral evolution

Incoming work by Trestini, Nasipak, Pound, +

Why do we need PM?

$$E = E_0(\mathcal{J}_B) + \boxed{\epsilon E_1(\mathcal{J}_B)} + \ldots$$ **Binding energy (SF-PN/SF-PM)**

$$L = L_0(\mathcal{J}_B) + \boxed{\epsilon L_1(\mathcal{J}_B)} + \ldots$$ **Binding angular momentum (SF-PN/SF-PM)**

- Re-summed (for high eccentricity) PN results may be accurate enough.
- Lack of self-force data at very high eccentricity to assess PN efficacy.
- **PM may be more accurate and would provide and independent check.**

What’s needed from PM?

Step 1: Redshift via b2b map from “elapsed proper time”.

$$\delta N^{>,\epsilon}(\mathbb{E},\mathbb{L},\{m_a\}) = -\chi\delta\mathbb{L} + \Delta t^{\epsilon}\delta\mathbb{E} - \sum_{a=1,2}\Delta\tau_a^{\epsilon}\delta m_a$$

$$\delta N^{<}(\mathbb{E},\mathbb{L},\{m_a\}) = \delta N^{>,\epsilon}(\mathbb{E},\mathbb{L},\{m_a\}) - \delta N^{>,\epsilon}(\mathbb{E},-\mathbb{L},\{m_a\})$$

$$\delta N^{<}(\mathbb{E},\mathbb{L},\{m_a\}) = -\Delta\Phi\delta\mathbb{L} + \frac{2\pi}{\Omega_r}\delta\mathbb{E} - \sum_{a=1,2}\frac{2\pi}{\Omega_r}\langle z_a\rangle\delta m_a$$

Gonzo et al. 2409.03437

Step 2: Binding energy and angular momentum via First Law of Binary Black Hole Mechanics.

Lewis et al. 2507.08081
Trestini et al. 2601.05223

Ingredients we need

- 0PA self-force waveforms and fluxes to moderate eccentricity. ✓
- 0PA self-force waveforms and fluxes to high eccentricity.
- Numerical 1SF redshift data to moderate eccentricity. ✓
- High eccentricity 1SF redshift from PN. ✓
- High eccentricity 1SF redshift from PM.
- PN fluxes as a surrogate for 2SF eccentric fluxes. ✓

Concluding remarks

1. **Hybrid models** may be **sufficiently accurate for GW analysis** of asymmetric binaries for LISA and other detectors.

2. **Hybrid models** can cover the parameter space prior to 2SF advancements.

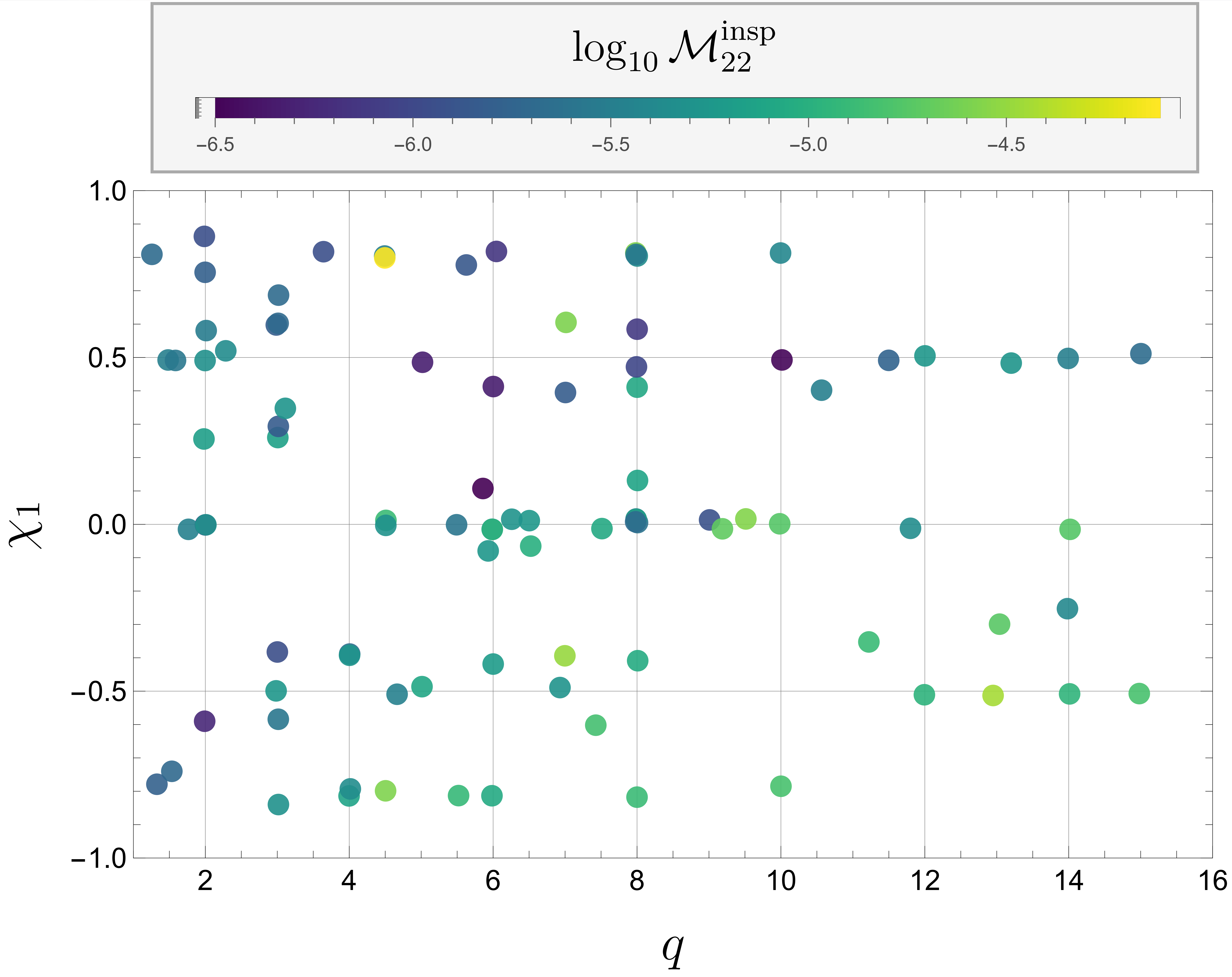
3. There is nearly a WaSABI-E **eccentric model.**

4. **PM** should be incorporated / used to benchmark at high eccentricities.

Public self-force waveform models

Model Type	Quasi-circular	Eccentric	Precessing	Eccentric and precessing
0PA	BHPWave 0PA WaSABI ... **Nasipak 2310.19706**	FEW KerrEccentricEquatorial **Chapman-Bird 2506.09470**	FEW KerrSpherical **(In development)**	Not yet implemented.
1PA	1PAT1 1PAT1R	Not for a good while.	Not for a while.	**Haha**
0PA+hybrid	WaSABI-C	WaSABI-E **(In developement)**	WaSABI-P **(In development)**	WaSABI-G **(Not for a while)**

Bonus slides



$\log_{10} \mathcal{M}_{22}^{\rm insp}$
−6.5
−6.0
−5.5
−5.0
−4.5
1.0
0.5
0.0
−0.5
−1.0
χ_1
2
4
6
8
10
12
14
16
q

SXS simulations

SXS:BBH:2670	SXS:BBH:2677	SXS:BBH:3136	SXS:BBH:2480	SXS:BBH:2474	SXS:BBH:1438
SXS:BBH:0189	SXS:BBH:1436	SXS:BBH:2475	SXS:BBH:1452	SXS:BBH:2486	SXS:BBH:2472
SXS:BBH:1464	SXS:BBH:2018	SXS:BBH:2488	SXS:BBH:1437	SXS:BBH:0371	SXS:BBH:0203
SXS:BBH:2757	SXS:BBH:2155	SXS:BBH:3622	SXS:BBH:4260	SXS:BBH:0202	SXS:BBH:4235
SXS:BBH:4430	SXS:BBH:2508	SXS:BBH:2515	SXS:BBH:3630	SXS:BBH:3891	SXS:BBH:3865
SXS:BBH:2141	SXS:BBH:2179	SXS:BBH:1962	SXS:BBH:2786	SXS:BBH:2495	SXS:BBH:1441
SXS:BBH:0331	SXS:BBH:2466	SXS:BBH:1448	SXS:BBH:2696	SXS:BBH:1470	SXS:BBH:2469
SXS:BBH:1485	SXS:BBH:2467	SXS:BBH:1440	SXS:BBH:2209	SXS:BBH:3122	SXS:BBH:2494
SXS:BBH:0525	SXS:BBH:0513	SXS:BBH:2479	SXS:BBH:2642	SXS:BBH:2470	SXS:BBH:2484
SXS:BBH:1468	SXS:BBH:0185	SXS:BBH:2476	SXS:BBH:2482	SXS:BBH:1460	SXS:BBH:2478
SXS:BBH:2160	SXS:BBH:4432	SXS:BBH:2134	SXS:BBH:2502	SXS:BBH:2161	SXS:BBH:2168
SXS:BBH:2701	SXS:BBH:2755	SXS:BBH:0206	SXS:BBH:4236	SXS:BBH:2132	SXS:BBH:2569
SXS:BBH:2110	SXS:BBH:2119	SXS:BBH:2186	SXS:BBH:1152	SXS:BBH:2127	SXS:BBH:1932
SXS:BBH:0612	SXS:BBH:1961	SXS:BBH:0615	SXS:BBH:4284	SXS:BBH:3924	SXS:BBH:2668
SXS:BBH:3127	SXS:BBH:1445	SXS:BBH:1427	SXS:BBH:3128	SXS:BBH:1444	SXS:BBH:2464
SXS:BBH:2490	SXS:BBH:1428	SXS:BBH:1426			

For SXS:BBH:3136, $\mathcal{M}_{22}^{\rm insp} = 3.15 \times 10^{-6}$, 2.77×10^{-5} and 4.01×10^{-6}

For SXS:BBH:3128, $\mathcal{M}_{22}^{\rm insp} = 7.69 \times 10^{-5}$, 1.54×10^{-4} and 6.83×10^{-5}

(WaSABI-C, SEOBNRv5HM and TEOBResumS)

$(\bar{k},\bar{n})$ orders of $\mathscr{F}_{\infty}^{SF+PN}$

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,8)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,8)	(∅,8)	(∅,∅)	(∅,∅)	(∅,∅)
1	(1,8)	(1,8)	(1,8)	(1,∅)	(1,∅)
0	(1,9)	(0,8)	(0,8)	(0,8)	(0,∅)
χ_2 \ χ_1	0	1	2	3	∞

$(\bar{k},\bar{n})$ orders of $\mathscr{F}_{\mathscr{H}_1}^{SF+PN}$ and $\mathscr{G}_{\mathscr{H}_1}^{SF+PN}$

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
1	(1,3)	(1,3)	(1,3)	(1,∅)	(1,∅)
0	(0,3)	(0,3)	(0,3)	(0,3)	(0,∅)
χ_2 \ χ_1	0	1	2	3	∞

$(\bar{k},\bar{n})$ orders of $\mathscr{F}_{\mathscr{H}_2}^{SF+PN}$ and $\mathscr{G}_{\mathscr{H}_2}^{SF+PN}$

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
1	(∅,3)	(∅,3)	(∅,3)	(∅,∅)	(∅,∅)
0	(∅,3)	(∅,3)	(∅,3)	(∅,3)	(∅,∅)
χ_2 \ χ_1	0	1	2	3	∞

$(\bar{k},\bar{n})$ orders of E^{SF+PN}

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,8)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,8)	(∅,8)	(∅,∅)	(∅,∅)	(∅,∅)
1	(1,8)	(1,8)	(1,8)	(1,∅)	(1,∅)
0	(1,8)	(1,8)	(1,8)	(1,8)	(1,∅)
χ_2 \ χ_1	0	1	2	3	∞

$(\bar{k},\bar{n})$ orders of A_{22}^{SF+PN}

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,7)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,7)	(∅,7)	(∅,∅)	(∅,∅)	(∅,∅)
1	(1,7)	(1,7)	(1,7)	(1,∅)	(1,∅)
0	(1,8)	(0,7)	(0,7)	(0,7)	(0,∅)
χ_2 \ χ_1	0	1	2	3	∞

$(\bar{k},\bar{n})$ orders of $A_{\ell m \neq 22}^{SF+PN}$

∞	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
3	(∅,7)	(∅,∅)	(∅,∅)	(∅,∅)	(∅,∅)
2	(∅,7)	(∅,7)	(∅,∅)	(∅,∅)	(∅,∅)
1	(1,7)	(1,7)	(1,7)	(1,∅)	(1,∅)
0	(1,7)	(0,7)	(0,7)	(0,7)	(0,∅)
χ_2 \ χ_1	0	1	2	3	∞

PN DATA: ENERGY FLUXES AT INFINITY

	PN order	Reference
NS sector	4.5PN	[Blanchet, Faye, Henry, Larrouturou, Trestini, 2304.11185]
SO sector	4PN	[Marsat, Bohr, Blanchet, Buonanno, 1307.6793] [Blanchet, Buonanno, Faye, gr-qc/0605140] [Bohé, Marsat, Blanchet, 1303.7412]
SS sector	4PN	[Boh, Faye, Marsat, Porter, 1501.01529] [Cho, Pardo, Porto, 2103.14612] 4PN: [Cho, Porto, Yang, 2201.05138] [not in Blanchet, Liv.Rev.]
SSS sector	3.5PN	[Marsat, 1411.4118]

PN DATA: ENERGY FLUXES AT THE HORIZON

	PN order	Reference
All sectors	4PN (Starts at 2.5PN in spin sector)	[Tagoshi, Mano, Takasugi, gr-qc/9711072] [Alvi, gr-qc/0107080] [Porto, 0710.5150] [Chatziioannou, Poisson, Yunes, 1211.1686] [Saketh, Steinhoff, Vines, Buonanno, 2212.13095]

PN DATA: BINDING ENERGY

	PN order	Reference
NS sector	4PN	[Damour, Jaranowski, Schäfer, 1401.4548]
SO sector	3.5PN	[Bohé, Marsat, Blanchet, 2013] [Bohé, Marsat, Faye, Blanchet, 2013]
SS sector	4PN	3.5PN: [Damour, Jaranowski, Schäfer, 1401.4548] 4PN: [Cho, Porto, Yang, 2201.05138] (not in Blanchet's Liv.Rev.)
SSS sector	3.5PN	[Marsat, 1411.4118]

PN DATA: AMPLITUDES

- The real-valued $\ell = 2, m = 2$ mode $\hat{h}_{22}^{PN}$ at 4.5PN is provided in [Warburton, Wardell, Trestini, Henry, Pound, Blanchet, Durkan, Faye, Miller, 2407.00366] after the results of **[Blanchet,Faye, Henry, Larrouturou, Trestini, 2304.11186]**; [Blanchet, Faye, Henry, Larrouturou, Trestini, 2304.11185]

- At 3.5PN order, all amplitudes are known including spin effects for non-precessing quasi-circular binaries. They vanish for $\ell \geq 10$. We use the amplitudes $2 \leq \ell \leq 9$ $0 < m \leq \ell$ given in **[Henry, 2210.15602]** which summarize the results of [Faye, Marsat, Blanchet, Iyer, 1204.1043]; [Faye, Blanchet, Iyer, 1409.3546]; [Henry, Faye, Blanchet, 2105.10876]; [Favata, 0812.0069]; [Blanchet, Faye, Iyer, Sinha, 0802.1249], [Henry, Marsat, Khalil, 2209.00374]

- The $m < 0$ amplitudes are deduced from $h_{\ell,-m} = (-1)^{\ell} h^{*}_{\ell m}$.