When eccentricities tick up

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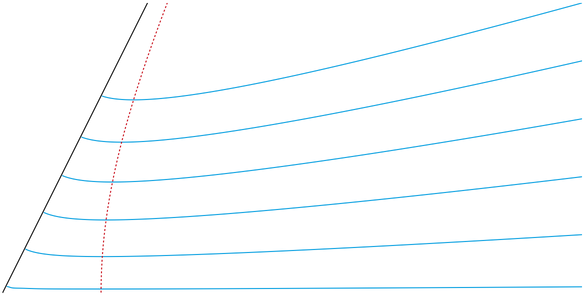

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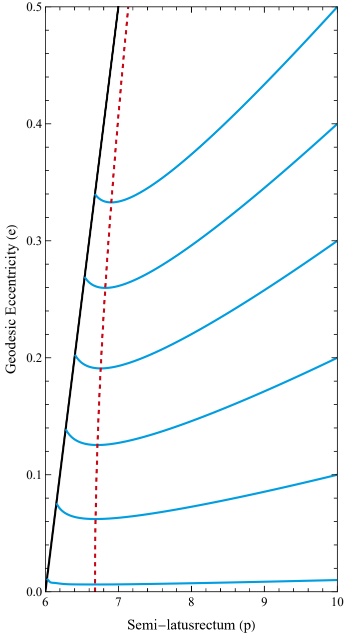
"Emission of gravitational waves causes binaries to become more circular over time."

True in weak field approximations:

Quadrupole approximation [Peters,1964]

$$\dot{e} = -\frac{304}{15} e \frac{G^3 M^3 \nu}{c^5 p^4} (1 - e^2)^{3/2} \left(1 + \frac{121}{304} e^2\right)$$



In the final stage of extreme mass-ratio inspirals eccenticity can increase.

- First described: [Apostolatos+,1993][Tanaka+,1993]
- Well-documented: [Cutler+,1994][Glampedakic&Kennefick,2003] [Warburton+,2011][MvdM+,2018][Lynch+,2022]
- Zoom-Whirl regime ($n_{\text{wind}} \approx 3$)
- Leading order (0PA) effect
- $\dot{e} = 0$ to be used as reference point by LISA SGS

Questions:

- Is this physical?
- Is this expected?
- Is this a unique feature of GR?

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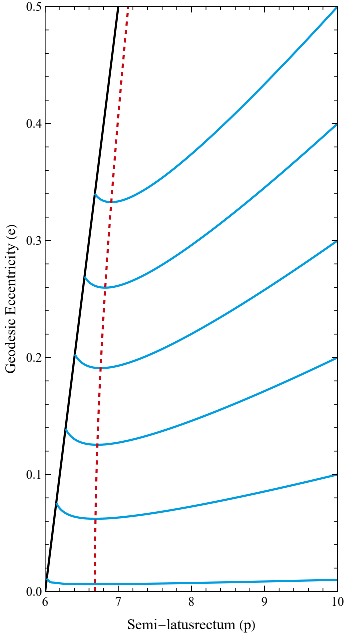

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Rough definition:

Eccentricity measures how non-circular a trajectory is.

Desirable qualitities of a good definition of eccentricity:

1. (Quasi-)circular trajectories have $e = 0$
2. Bound trajectories have $e < 1$
3. Unbound trajectories have $e > 1$
4. Different representations of the same trajectory should agree on e
5. Agrees with Newtonian limit e

THE CENTER OF GRAVITY

What is eccentricity anyway?

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$$u = M/r$$

$$\frac{1}{2}\dot{u}^2 + V[\mathcal{E}, \mathcal{L}](u) = 0$$

$$e_{\text{geo}} = \frac{u_2 - u_1}{u_2 + u_1}$$

Desirable qualitities of eccentricity:

1. (Quasi-)circular trajectories have $e = 0$ ✓
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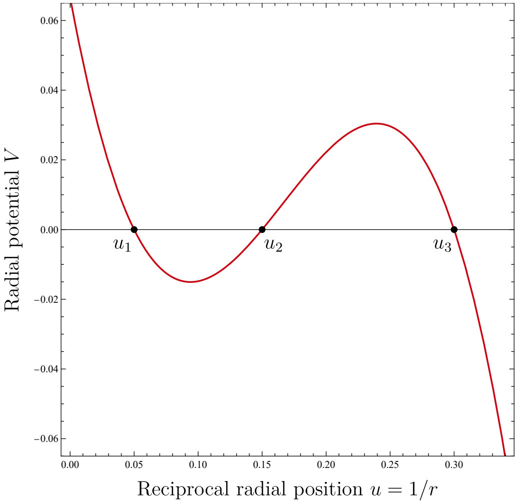
4. Different representations agree on e ✗
5. Agrees with Newtonian limit e ✓



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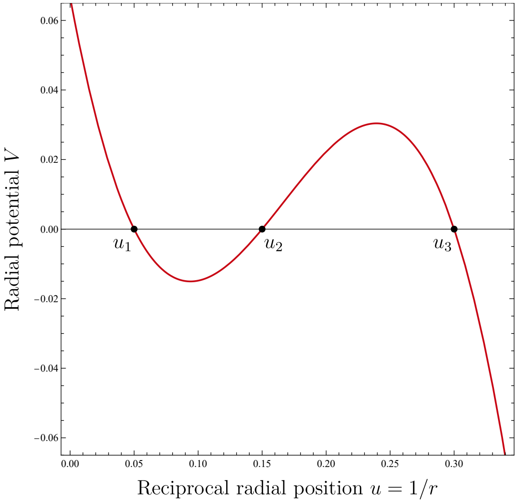
5. Agrees with Newtonian limit e ✓



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GW eccentricity

$$\omega_{22} = \frac{\mathrm{d}}{\mathrm{d}t} \arg h_{22}$$

$$e_{\omega_{22}} = \frac{\sqrt{\omega_{22}^p} - \sqrt{\omega_{22}^a}}{\sqrt{\omega_{22}^p} + \sqrt{\omega_{22}^a}}$$

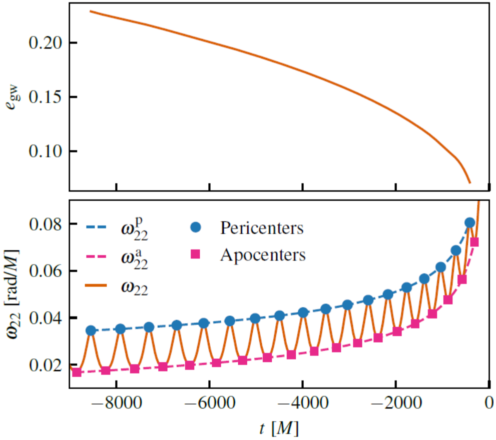
$$e_{\mathrm{GW}} = \cos(\Psi/3) - \sqrt{3}\sin(\Psi/3)$$

$$\Psi = \arctan\left(\frac{1 - e_{\omega_{22}}^2}{2e_{\omega_{22}}}\right)$$

[Ramos-Buades+MvdM,2022]

Desirable qualities of eccentricity:

1. (Quasi-)circular trajectories have $e = 0$ ✓
2. Bound trajectories have $e < 1$ ✓
3. Unbound trajectories have $e > 1$ ✗?
4. Different representations agree on e ✓*
5. Agrees with Newtonian limit e ✓



[Shaikh+MvdM,2023]

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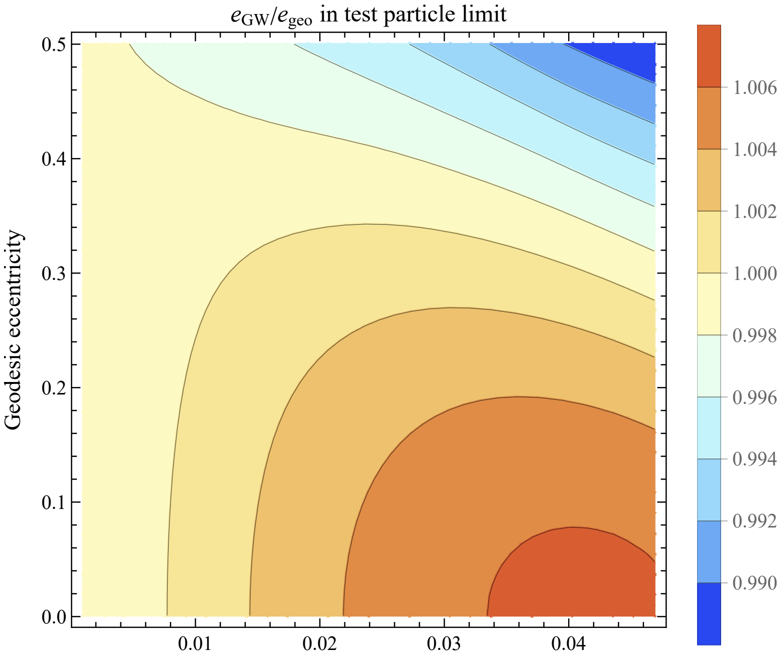
GW and geodesic eccentricity are similar in small mass-ratio limit



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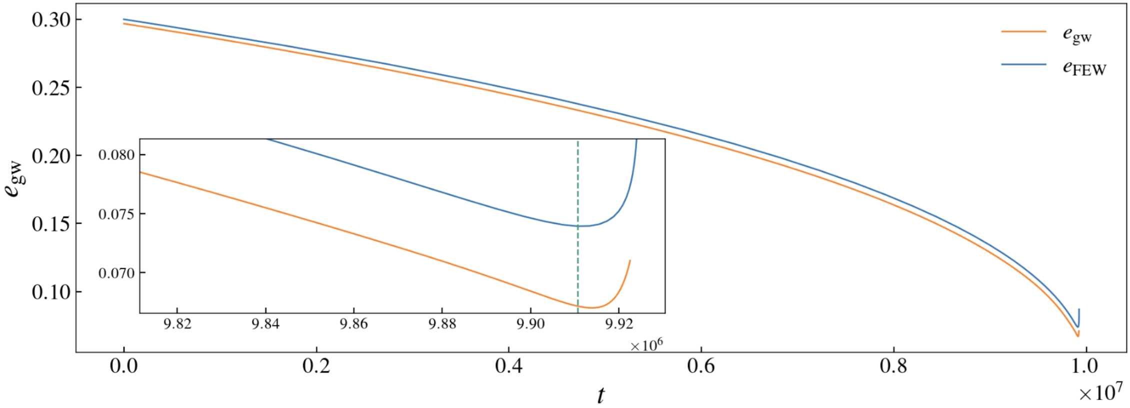
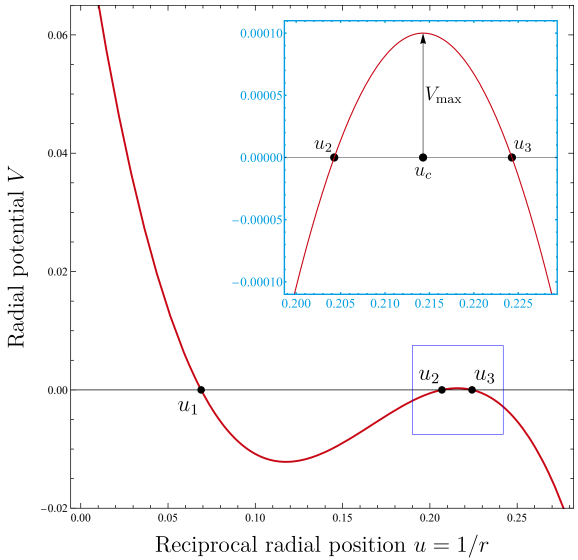
GW and geodesic eccentricity show similar "uptick"

$m_1 : m_2 = 10^4 : 1$



[Courtesy Arif Shaikh]



Transition from stable bound orbit to plunge happens when $u_2 = u_3$.

Near the critical point:

$$V = AV_{\max} - A(u - u_c)^2 + \mathcal{O}(u - u_c)^3$$

(If $V[\mathcal{E}, \mathcal{L}]$is smooth then so are $A[\mathcal{E}, \mathcal{L}]$, $V_{\max}[\mathcal{E}, \mathcal{L}]$, and $r_c[\mathcal{E}, \mathcal{L}]$)

Pericenter can be approximated:

$$u_2 \approx u_c - \sqrt{V_{\max}}$$

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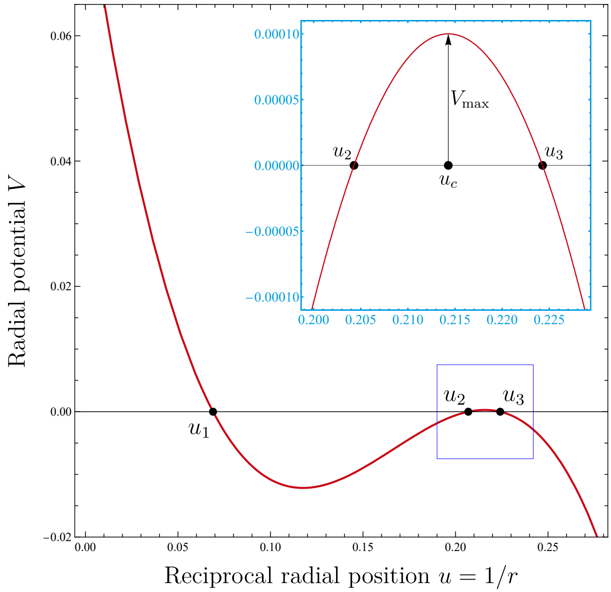
Evolution of eccentricity near LSO separatrix



$$\dot{e} = \frac{2u_1}{(u_1+u_2)^2}\dot{u}_2 - \frac{2u_2}{(u_1+u_2)^2}\dot{u}_1$$

$$\dot{u}_2 \approx \dot{u}_c - \frac{\dot{V}_{\max}}{2\sqrt{V_{\max}}} \approx \dot{u}_c - \frac{\dot{V}_{\max}}{2(u_c-u_2)}$$

Conclusion

$\dot{e}$ diverges to $+\infty$ if $\dot{V}_{\max} < 0$ and does not approach 0.

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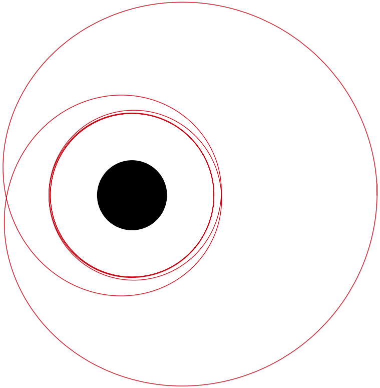
Orbits near the LSO (Zoom-Whirl)

Time spent near pericenter

$$2\int_{u_2-\epsilon}^{u_2} \frac{\mathrm{d}t}{\mathrm{d}u}\,\mathrm{d}u \propto \int_{u_2-\epsilon}^{u_2} \frac{1}{\sqrt{(u-u_c)^2 - V_{\max}}}\,\mathrm{d}u$$
$$\approx \int_{u_2-\epsilon}^{u_2} \frac{1}{u_c - u}\,\mathrm{d}u \approx \log(u_c - u_2)$$

$n_{\mathrm{wind}} = \frac{\Omega_\phi}{\Omega_{\mathrm{rad}}}$: Winding number

$$n_{\mathrm{wind}} \stackrel{?}{=} n_{\mathrm{whirl}} + n_{\mathrm{zoom}}$$



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Time spent near pericenter

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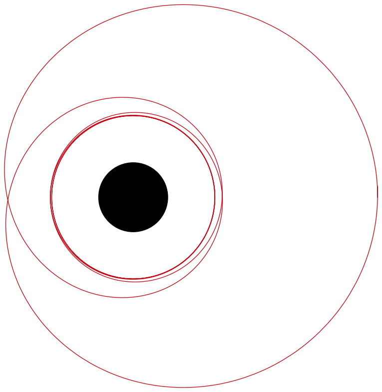
$$n_{\text{wind}} \stackrel{?}{=} n_{\text{whirl}} + n_{\text{zoom}}$$

$n_{\text{zoom}} \approx 1$ and $n_{\text{whirl}} \sim \log(u_c - u_2)$



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Dissipation in one radial cycle

For $\mathcal{C} = \mathcal{E}, \mathcal{L}$

$(\Delta\mathcal{C}$ in radial period$) = n_{\text{whirl}} \times$ $(\Delta\mathcal{C}$ in circular orbit at $u_c) +$ (finite $\Delta\mathcal{C})$

$$\dot{\mathcal{C}} \approx \dot{\mathcal{C}}_{\text{UCO}} + \frac{f(p,e)}{\log(u_c - u_2)}$$

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Evolution of the peak of the potential

$$\dot{V}_{\max} = \frac{\partial V_{\max}}{\partial \mathcal{E}}\dot{\mathcal{E}} + \frac{\partial V_{\max}}{\partial \mathcal{L}}\dot{\mathcal{L}}$$

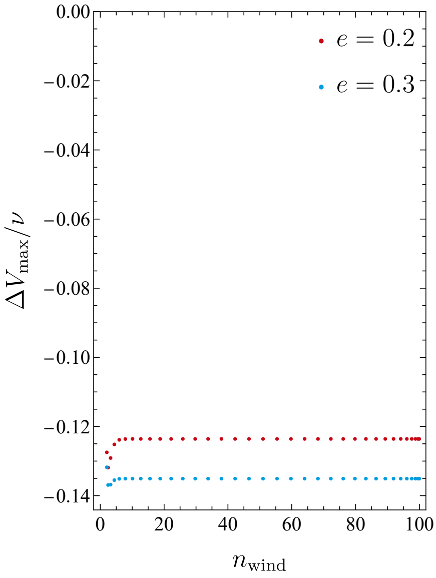
On unstable circular orbits (in GR) $\dot{V}_{\max} = 0$

$$\dot{V}_{\max} \approx 0 + \frac{f(u_c - u_2)}{\log(u_c - u_2)}$$

Corollary

The maximum n_{whirl} encountered in an EMRI scales as $|\log(\nu)|$

In Schwarzschild:



[Data: Lucía Vélez]

Back to $\dot{e}$

$$\dot{e} \sim \frac{1}{(u_c - u_2)\log(u_c - u_2)}$$

Eccentricity will always increase before LSO transition.

Ingredients

- Transition through LSO is a double root.
- (Radiation reaction maintaines circularity)
- Evolution is sufficiently adiabatic.

~~Questions~~ Answers:

- Are eccentricity upticks physical? YES
- Are eccentricity upticks expected? YES
- Are eccentricity upticks unique to GR? NO

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