Modelling black hole scattering using comoving hyperboloidal coordinates

Amplitudes, Strong-Field Gravity, and Resummation (Nordita 2026)

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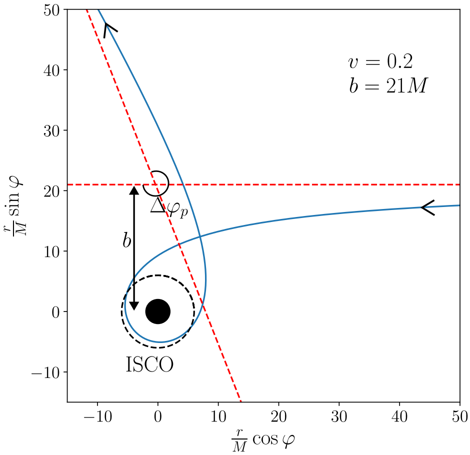
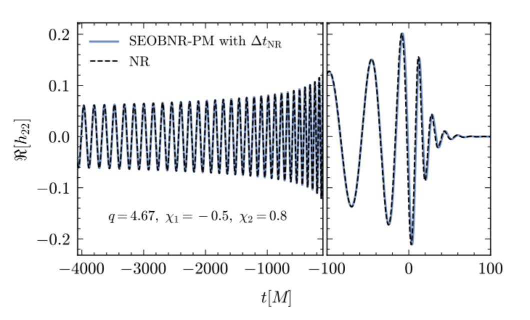
Motivation

High-precision probe of strong-field interaction

- GW astronomy: scattering angle calibrates EOB Hamiltonian
- Amplitudes and EFT ideas: scatter-to-bound relations, double copies, etc...

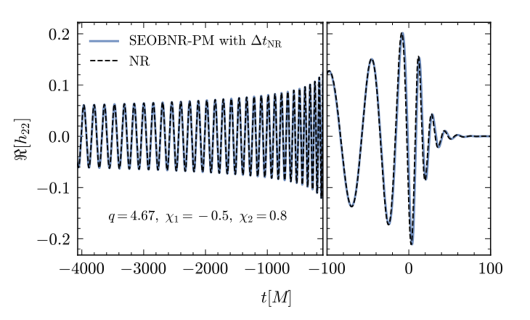


SF, EOB, PM



[Buonanno et al., 2024]

EOB informed by PM. But we can use SF as well.



[Buonanno et al., 2024]

EOB informed by PM. But we can use SF as well.

Complementary expansions of the same system

	1PM	2PM	3PM	4PM	5PM	...
0SF	G	G^2	G^3	G^4	G^5	...
1SF			νG^3	νG^4	νG^5	...
2SF					$\nu^2 G^5$	...
					$\vdots$	$\ddots$

Credit: Oliver Long

$$n\text{SF} \implies \text{all orders in } G,$$
$$n\text{PM} \implies \text{all orders in } \nu$$

Lightning review of GSF

Systematic expansion of the field equations in the small mass ratio $q = \mu/M$

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Sch./Kerr}} + q h_{\mu\nu}^{(1)} + \mathcal{O}(q^2) \quad "+" \quad T_{\mu\nu} = q T_{\mu\nu}^{(1)} + q^2 T_{\mu\nu}^{(2)} + \mathcal{O}(q^3),$$

$$\Big\downarrow \text{EFE}$$

$$z^a(\tau) = z^a_{\text{geod}} + q z^a_{(1)} + \mathcal{O}(q^2)$$

0SF:

- geodesic motion on background BH spacetime
- bound/unbound orbits known analytically

1SF:

- secondary treated as a point-particle $T_{\mu\nu} \propto \delta^{(4)}(x - x_p)$ moving on a fixed geodesic of the background

Toy model: Scalar charge in Schwarzschild

Scalar charge Q, with inertial mass μ but no gravitational mass.
Small parameter $q_s = Q^2/(\mu M)$
Scalar field Φ satisfies Klein-Gordon equation

$$\Box\Phi = -4\pi Q \int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(y^\mu - y_p^\mu(\tau))}{\sqrt{-g}}$$

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Spherical harmonic decomposition: $\Phi(t, r, \theta, \varphi) = (Q/r)\sum_{\ell,m}\psi_{\ell m}(t, r)\, Y_{\ell m}(\theta, \varphi)$

KG equation reduces to 1+1D system

$$\psi_{,uv} + V_\ell(r)\psi = S(\tau)\delta(r - r_p(\tau))$$

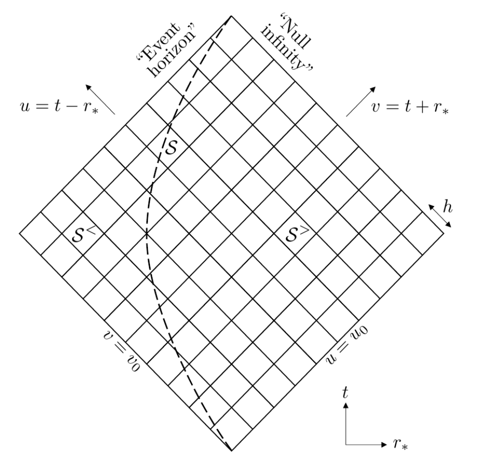
u, v: Eddington-Finkelstein coordinates

- Initial data (usually 0) on the characteristic rays $u = u_0$ and $v = v_0$.

Scalar field ansatz

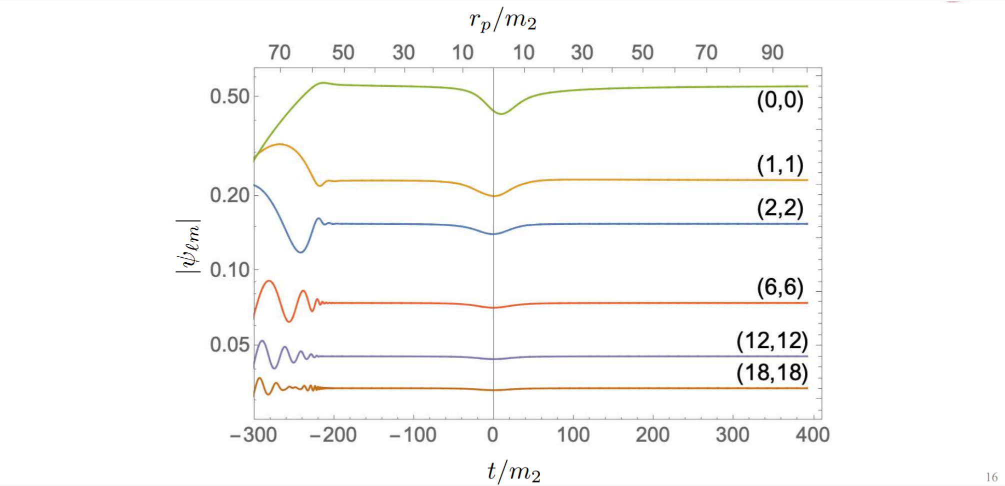
$$\psi(t, r) = \psi^{>}(t, r)\Theta(r - r_p(\tau)) + \psi^{<}(t, r)\Theta(r + r_p(\tau))$$

- Ansatz leads to jump conditions $J_\alpha = (\psi^{+}_{,\alpha} - \psi^{-}_{,\alpha})|_p$
- Discretise KG equation, evolve using jumps



Credit: Oliver Long

Scalar field results



Credit: Oliver Long

Scalar self-force

Self-force needs the"regular" part of the field

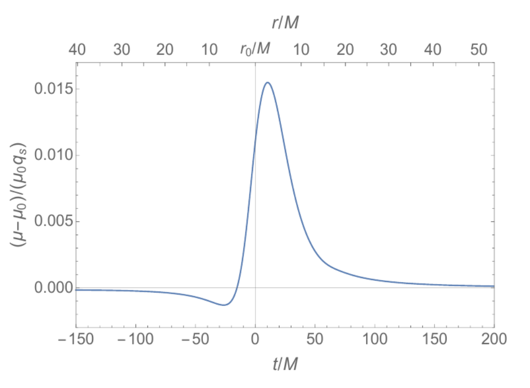
$$u^\beta \nabla_\beta (\mu u^\alpha) = Q \nabla^\alpha \Phi^R = Q^2 \nabla^\alpha \sum_\ell (\psi_\ell - B(t))/r_p$$

Can be projected along u^α and orthogonal to it

Tangent: $\mu(t) = \mu_0 + Q\Phi^R$

Orthogonal: $\mu u^\beta \nabla_\beta u^\alpha = Q(\delta^\alpha_\beta + u^\alpha u_\beta)\nabla^\beta \Phi^R =: \mu q_s F^\alpha$

Mass evolution and SF [Barack and Long, 2022]



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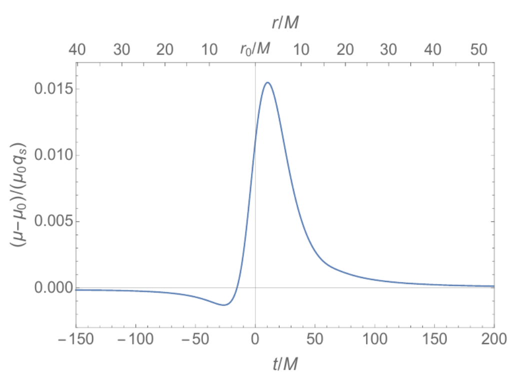
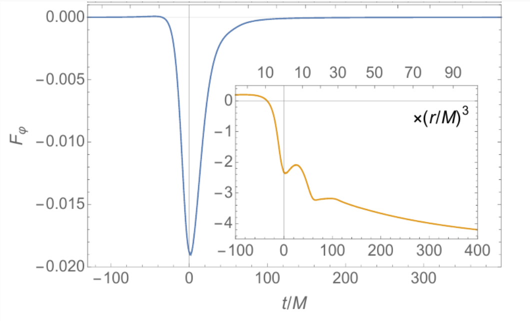


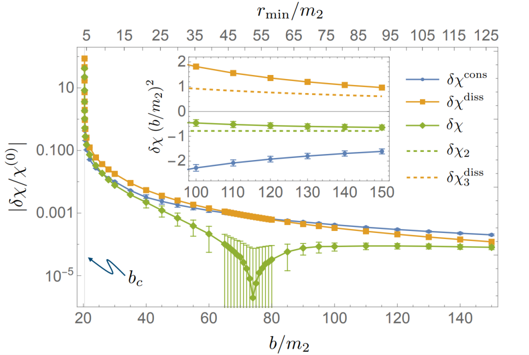


Self-force and scattering angle [Barack and Long, 2022]

Self-force leads to a change in the scattering angle $\chi = \chi^{(0)} + q_s\delta\chi$,

$$\delta\chi = \sum_{\pm}\int_{r_{min}}^{\infty} \mathcal{G}_E^{\pm}(r)F_t^{\pm} + \mathcal{G}_L^{\pm}(r)F_{\varphi}^{\pm}$$

$\mathcal{G}$'s known analytically along the orbit



Credit: Oliver Long

More has been done comparing SF and PM, esp. in the context of resummation (See Olly's talk!)

For gravity, solve the $s = \pm 2$ component of adjoint Teukolsky equation.

In Schwarzschild

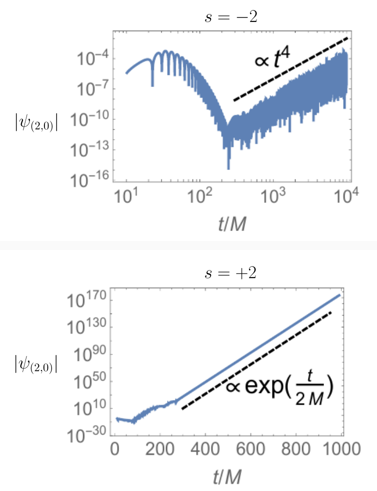
$$\psi^{\pm}_{,uv} + U_s(r)\psi^{\pm}_{,u} + V_s(r)\psi^{\pm}_{,v} + W_s(r)\psi^{\pm} = 0, \qquad (s = \mp 2 \text{ for } \psi^{\pm})$$

Jump conditions for the Hertz potential constructed from the source for the Weyl scalars [Long and Barack, 2021]

Things blow up [Long and Barack, 2021]

Divergent modes are solutions of the Teukolsky equation, but don't satisfy retarded BC's.

Lack of initial data!



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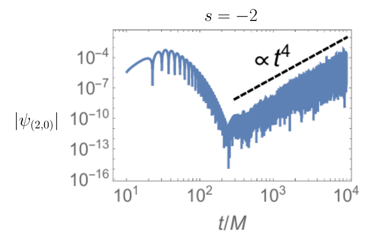
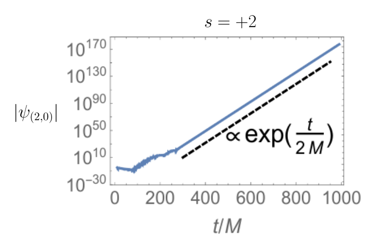
Work-around in Schwarzschild: transform to Regge-Wheeler variable. But this can't be generalised to Kerr!





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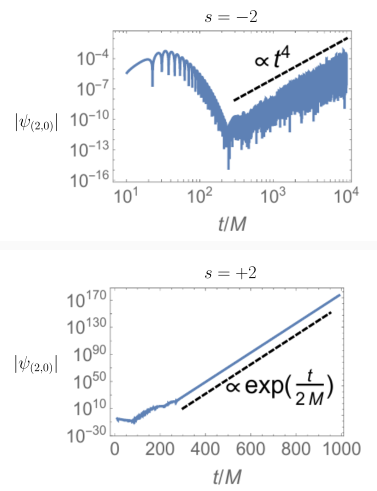
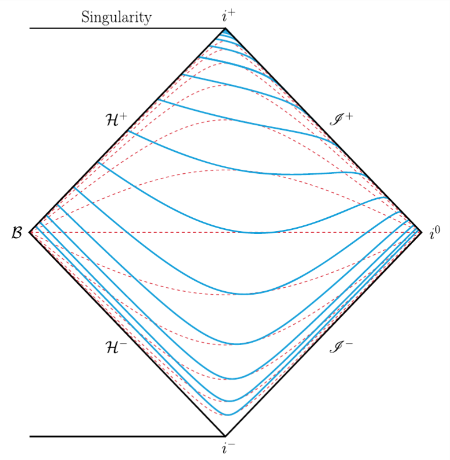
Solution: Hyperboloidal compactification!
This should under-resolve incoming waves from $\mathcal{I}$ and outgoing waves from $\mathcal{H}$ and suppress the divergent modes



Hyperboloidal compactification

Usual t-coordinate singular at spatial infinity and bifurcation sphere.



[Macedo et al., 2022]

Hyperboloidal compactification

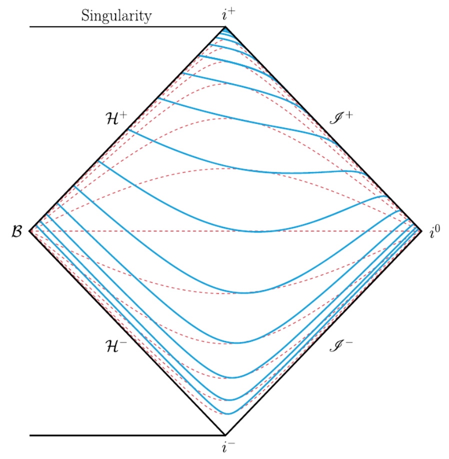
Usual t-coordinate singular at spatial infinity and bifurcation sphere.

Modify t-coordinate by removing divergence near spatial infinity and bifurcation sphere

$$\frac{t}{M} = \tau + \frac{1}{\sigma} - 2\log\left[\sigma(1-\sigma)\right]$$

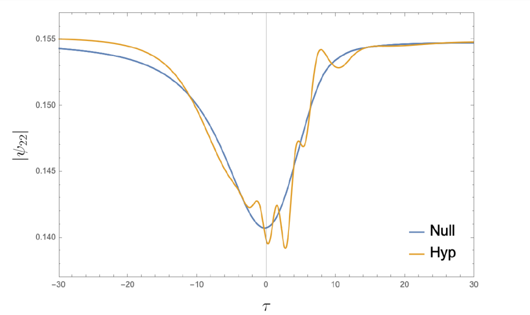
Compactify spatial coordinate $r = 2M/\sigma$

Exterior region $r \in (2M, \infty) \to \sigma \in (0, 1)$



[Macedo et al., 2022]

Issues with minimal gauge



Credit: Oliver Long

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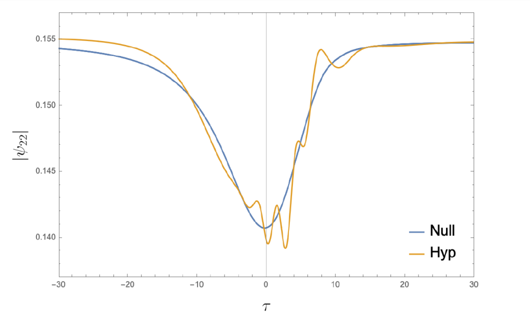

Credit: Oliver Long

Poor resolution of the field, even at periapsis.

Comoving transformation

Keep $t \to \tau$.
Make the radial transformation comoving $\sigma = 1/2 \implies r = r_p$.

$$\frac{t}{M} = \tau + \frac{1}{\sigma} - 2\log\left[\sigma(1-\sigma)\right],$$
$$\frac{r^*}{M} = \frac{r_p^*(\tau)}{M} + \frac{1}{\sigma} - 2 + 2\log\left[\frac{(1-\sigma)}{\sigma}\right]$$

Motivation:

$$\sigma \to 0 \equiv \mathcal{I}^+ \implies u = const., \quad v \to \infty,$$
$$\sigma \to 1 \equiv \mathcal{H}^+ \implies v = const., \quad u \to \infty$$

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At large r_p the grid samples only the region near the particle.
No resolution of the wave zone or the strong field!

Final transformation

$$\frac{t}{M} = \tau + \frac{1}{\sigma^{N_\sigma}} - 2\log\left[\sigma(1-\sigma)\right],$$
$$\frac{r^*}{M} = \frac{r_p^*(\tau)}{M} + \frac{1}{\sigma^{N_\sigma}} - 2^{N_\sigma} + 2\log\left[\frac{(1-\sigma)}{\sigma}\right],$$

N_σ: controls samping of wave zone, $N_\sigma = 1, 2, \cdots$

Final transformation

$$\frac{t}{M} = \tau + \frac{1}{\sigma^{N_\sigma}} - 2\log\left[\sigma(1-\sigma)\right],$$
$$\frac{r^*}{M} = \alpha(\sigma)\frac{r_p^*(\tau)}{M} + \frac{1}{\sigma^{N_\sigma}} - 2^{N_\sigma} + 2\log\left[\frac{(1-\sigma)}{\sigma}\right]$$

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$\alpha(\sigma)$: controls sampling of strong field, $\alpha(1) = 0$

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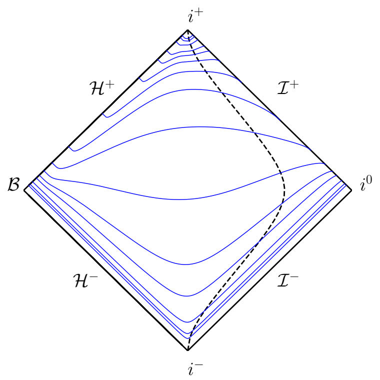
$\alpha(\sigma)$: controls sampling of strong field, $\alpha(1) = 0$

$$\alpha(\sigma) = \begin{cases} 1 - (2\sigma - 1)^{N_\alpha + 1}, & \sigma > 1/2 \\ 1, & \sigma \le 1/2 \end{cases}$$



Scalar field toy model

- Scalar field sourced by point charge in Schwarzschild

$$\Box\Phi = -4\pi q \int d\tau \frac{\delta^4(x^a - x_p^a(\tau))}{\sqrt{-g}}.$$

- Multipole decomposition: $\Phi = \frac{Q}{r}\sum_{\ell,m}\psi_{\ell m}(\tau,\sigma)\,Y_{\ell m}(\theta,\phi)$

Modal KG equation:

$$\ddot{\psi} + C_{\tau\sigma}\dot{\psi}' + C_{\sigma\sigma}\psi'' + C_{\tau}\dot{\psi} + C_{\sigma}\psi' + C_0\psi = S(\tau)\delta(\sigma - 1/2)$$

Numerical implementation

Order reduction:

$$\dot{\Psi} = \Pi,$$
$$\dot{\Pi} = S(\tau)\delta(\sigma - 1/2) - (C_{\tau\sigma}\partial_\sigma + C_\tau)\Pi - (C_{\sigma\sigma}\partial_\sigma^2 + C_\sigma\partial_\sigma + C_0)\Psi$$

- Approximate spatial derivatives by finite differences
- Evolve in time with an RK4 integrator

But how do we incorporate the source into the scheme?

Jump conditions

- Scalar field ansatz satisfying KG equation

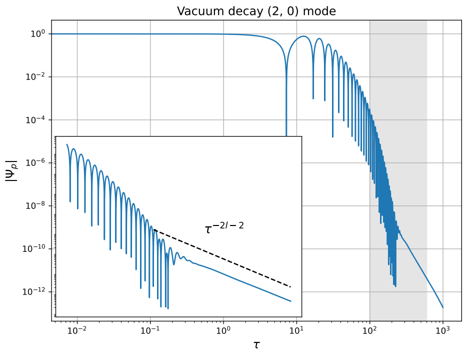
$$\Psi = \Psi^{+}\Theta(\sigma - 1/2) + \Psi^{-}\Theta(\sigma + 1/2)$$

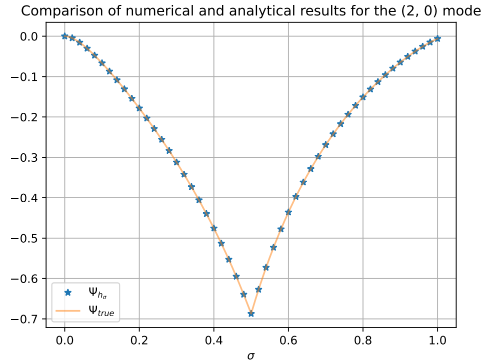
- Field kinked at $\sigma = 1/2$
- Jump conditions relate the field and derivatives on one side to corresponding quantities on the other side in terms of the source

At the particle's location

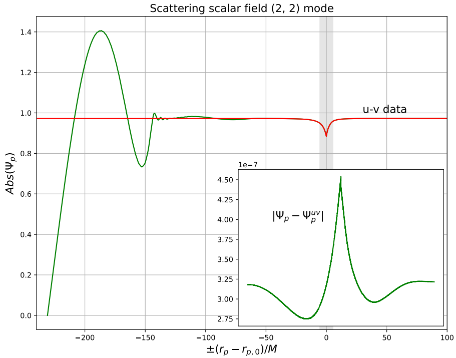
$$\begin{aligned} \Psi^{+} &= \Psi^{-} \\ \partial_\sigma \Psi^{+} &= \partial_\sigma \Psi^{-} + J_\sigma, \qquad J_\sigma = S/C_{\sigma\sigma}|_p \\ \partial_\sigma^2 \Psi^{+} &= \partial_\sigma^2 \Psi^{-} + J_{\sigma\sigma}, \qquad J_{\sigma\sigma} = -(C_{\tau\sigma}\dot{J}_\sigma + C_\sigma J_\sigma)/C_{\sigma\sigma}|_p \end{aligned}$$

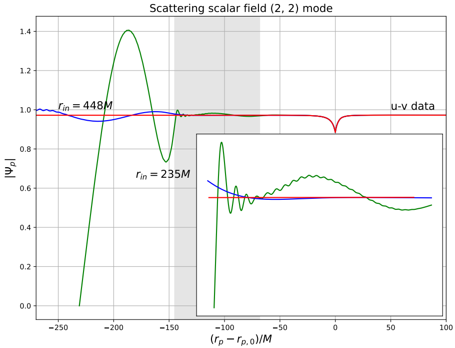
Results: Vacuum + Circular





Scattering: comparison to null code

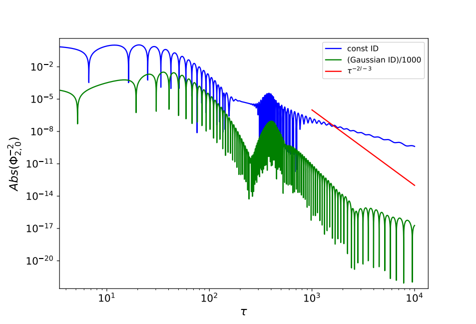




Hertz potential $\psi^{\pm}$ solves the adjoint Teukolsky equation with $s = \mp 2$

In vacuum

$$\ddot{\psi}_{\pm} + D^s_{\tau\sigma}\dot{\psi}'_{\pm} + D^s_{\sigma\sigma}\psi''_{\pm} + D^s_{\tau}\dot{\psi}_{\pm} + D^s_{\sigma}\psi'_{\pm} + D^s_{0}\psi_{\pm} = 0,$$



Teukolsky with hyperboloidal slices

Hertz potential $\psi^{\pm}$ solves the adjoint Teukolsky equation with $s = \mp 2$

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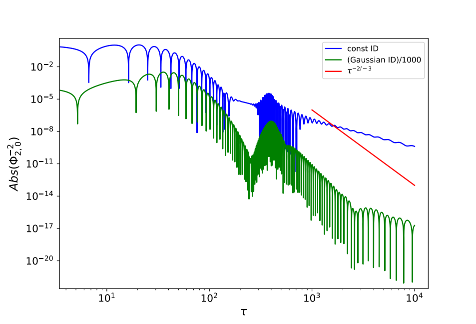
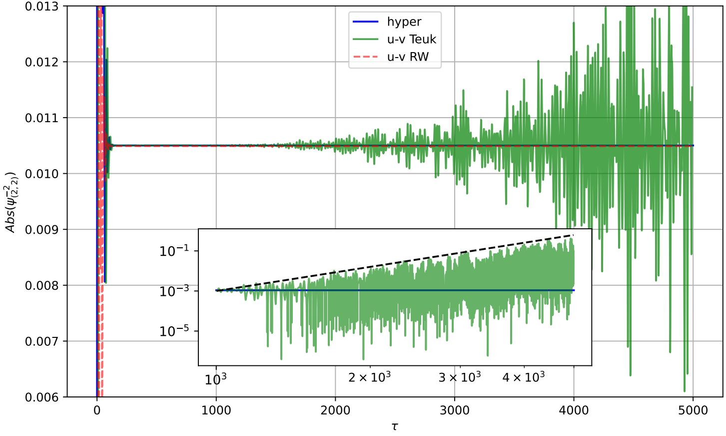
$$\ddot{\psi}_{\pm} + D^s_{\tau\sigma}\dot{\psi}'_{\pm} + D^s_{\sigma\sigma}\psi''_{\pm} + D^s_{\tau}\dot{\psi}_{\pm} + D^s_{\sigma}\psi'_{\pm} + D^s_{0}\psi_{\pm} = 0,$$



The field doesn't blow up!
Poor resolution of $\mathcal{I}^+ \implies$ no clear tail. But we don't need this for GSF calculations.

Gravity: circular geodesics ($r_p^* = 9M$)



Gravity: scattering geodesics

We don't explicitly know the jumps analytically. But it satisfies satisfies

$$\frac{dJ}{dt_p} = F(\Psi_0, \Psi_4, \partial\Psi_0, \partial\Psi_4, \cdots)$$

Algorithm:

- numerically solve $dJ/dt_p = \cdots \to J$
- Compute $J_\sigma, J_{\sigma\sigma}, \cdots$ [Long and Barack, 2021]
- Evolve adjoint Teukolsky for Φ
- Reconstruct the metric perturbation and extract GSF

$$\Phi \xrightarrow{\nabla^2} h^{rec} \to \text{GSF}$$

Conclusion and future work

- Compute the jumps and evolve the full gravity problem.
- Metric reconstruction of sourced case to compute scattering angle, energy- and angular momentum losses
- Strong-field PM resummation

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Thank You!

References

- Barack, L., & Long, O. (2022).Self-force correction to the deflection angle in black-hole scattering: A scalar charge toy model. *Phys. Rev. D*, *106*(10), 104031. https://doi.org/10.1103/PhysRevD.106.104031
- Buonanno, A., Mogull, G., Patil, R., & Pompili, L. (2024).Post-Minkowskian Theory Meets the Spinning Effective-One-Body Approach for Bound-Orbit Waveforms. *Phys. Rev. Lett.*, *133*(21), 211402. https://doi.org/10.1103/PhysRevLett.133.211402
- Long, O., & Barack, L. (2021).Time-domain metric reconstruction for hyperbolic scattering. *Phys. Rev. D*, *104*(2), 024014. https://doi.org/10.1103/PhysRevD.104.024014
- Macedo, R. P., Leather, B., Warburton, N., Wardell, B., & Zengino ğlu, A. ı. (2022).Hyperboloidal method for frequency-domain self-force calculations. *Phys. Rev. D*, *105*, 104033. https://doi.org/10.1103/PhysRevD.105.104033

Extra material

Finite difference stencils: vac + particle

Away from boundaries and next-to-boundary points

$$(\Psi_i')^{(\mathsf{VFD})} = \frac{1}{12h_\sigma}\Big[(\Psi_{i-2} - \Psi_{i+2}) - 8(\Psi_{i-1} - \Psi_{i+1})\Big] + \mathcal{O}(h_\sigma^4),$$

$$(\Psi_i'')^{(\mathsf{VFD})} = \frac{1}{12h_\sigma^2}\Big[-(\Psi_{i-2} + \Psi_{i+2}) + 16(\Psi_{i-1} + \Psi_{i+1}) - 30\Psi_i\Big] + \mathcal{O}(h_\sigma^4).$$

At the particle

$$(\Psi_p^+)' = (\Psi_p')^{(\mathsf{VFD})} + \frac{1}{36}\big(18J_\sigma - 6h_\sigma J_{\sigma\sigma} + h_\sigma^3 J_{\sigma\sigma\sigma\sigma}\big) + \mathcal{O}(h_\sigma^4),$$

$$(\Psi_p^+)'' = (\Psi_p'')^{(\mathsf{VFD})} + \frac{1}{18h_\sigma}\big(-21J_\sigma + 9h_\sigma J_{\sigma\sigma} - 2h_\sigma^2 J_{\sigma\sigma\sigma}\big) + \mathcal{O}(h_\sigma^3).$$

FD stencil at $p \pm 1$

Next to the particle

$$(\Psi^+_{p\pm 1})' = (\Psi'_{p\pm 1})^{(\mathsf{VFD})} + \frac{1}{288}\big(\mp 24 J_\sigma + 12 h_\sigma J_{\sigma\sigma} \mp 4 h_\sigma^2 J_{\sigma\sigma\sigma} + h_\sigma^3 J_{\sigma\sigma\sigma\sigma}\big) + \mathcal{O}(h_\sigma^4),$$

$$(\Psi^+_{p\pm 1})'' = (\Psi''_{p\pm 1})^{(\mathsf{VFD})} + \frac{1}{288 h_\sigma}\big(24 J_\sigma \mp 12 h_\sigma J_{\sigma\sigma} + 4 h_\sigma^2 J_{\sigma\sigma\sigma} \mp h_\sigma^3 J_{\sigma\sigma\sigma\sigma}\big) + \mathcal{O}(h_\sigma^3).$$

At $\sigma = 0$,

$$\Psi'_0 = \frac{1}{12 h_\sigma}\big(-25\Psi_0 + 48\Psi_1 - 36\Psi_2 + 16\Psi_3 - 3\Psi_4\big) + \mathcal{O}(h_\sigma^4),$$

$$\Psi''_0 = \frac{1}{12 h_\sigma^2}\big(45\Psi_0 - 154\Psi_1 + 214\Psi_2 - 156\Psi_3 + 61\Psi_4 - 10\Psi_5\big) + \mathcal{O}(h_\sigma^4).$$

For the stencil at the other boundary, replace index $0 \to N, 1 \to N-1, \cdots$

FD stencil at $1, N-1$

For the next-to-boundary points, use an asymmetric stencil

$$\Psi_1' = \frac{1}{12h_\sigma}\big(-3\Psi_0 - 10\Psi_1 + 18\Psi_2 - 6\Psi_3 + \Psi_4\big) + \mathcal{O}(h_\sigma^4)$$
$$\Psi_1'' = \frac{1}{12h_\sigma^2}\big(11\Psi_0 - 20\Psi_1 + 6\Psi_2 + 4\Psi_3 - \Psi_4\big) + \mathcal{O}(h_\sigma^3)$$

For the corresponding stencil at the $N-1$ point, replace index $0 \to N, 1 \to N-1, \cdots$

Jump conditions

Field ansatz:

$$\psi(\tau,\sigma) = \psi^+(\tau,\sigma)\delta(\sigma - 1/2) + \psi^-(\tau,\sigma)\delta(\sigma + 1/2)$$

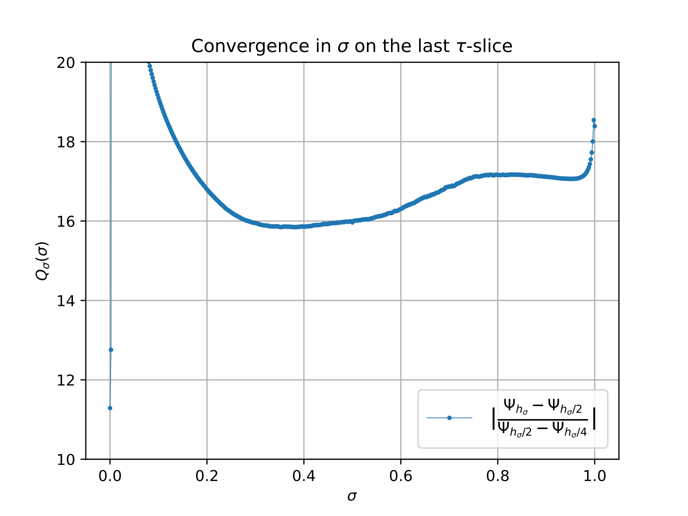
Leads to

$$S_{\ell m}(\tau) = \frac{\pi M f_p (3 - \dot{x}_p)}{8 E r_p} Y^*_{\ell m}(\pi/2, \varphi_p)$$

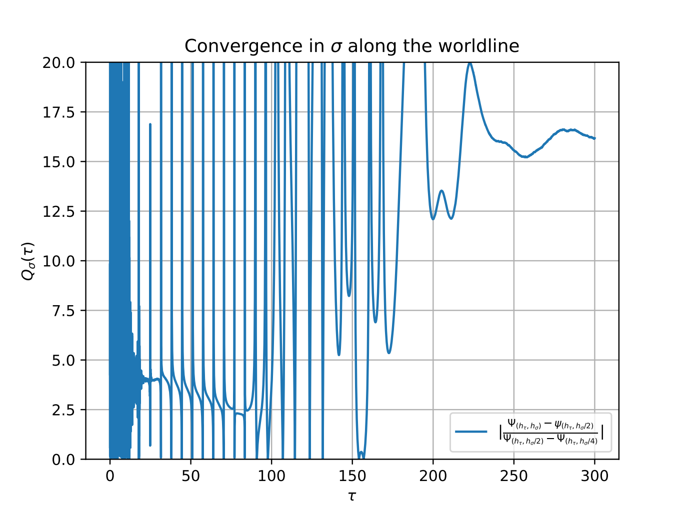
$$J_\sigma(\tau) = S(\tau)/C_{\sigma\sigma}|_p = -\frac{16\pi M f_p}{E r_p} \frac{(3 - \dot{x}_p)}{1 - \dot{x}_p^2} Y^*_{\ell m}(\pi/2, \varphi_p)$$

$$J_{\sigma\sigma}(\tau) = -(C_{\tau\sigma}\dot{J}_\sigma + C_\sigma J_\sigma)/C_{\sigma\sigma}|_p$$

Circular orbit: convergence test in σ



Circular orbit: convergence test in σ (contd.)



Junk comparison