Dissipative observables for black hole scattering

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16 April 2026

Queen Mary University of London

Amplitudes, Strong-Field Gravity and Resummation

Based on:

A. Cipriani, F. Fucito, C. Heissenberg, J.F. Morales, RR:
"Waveforms" at the Horizon, 2602.05766

+ work in progress

See also:

Reconstructing the Gravitational Waveform from Its Probe Limit, 2511.13835

C. Heissenberg, RR

Gravitational waveforms for extreme mass ratio collisions from supersymmetric gauge theories, 2408.07329

F. Fucito, J.F. Morales, RR

Introduction

Black hole binaries are a perfect system to study fundamental issues in gravity and several new analytic results have been derived over the past few years in various different regimes

I'll focus on the case of unbound orbits and study some dissipative observables in the self-force approach (exploiting a QFT insight)

A key dissipative observable is the radiative gravitational field at (future null) infinity, i.e. the waveform from which we derive the energy $E_{\rm rad}$ and angular momentum $J_{\rm rad}$ emitted by the binary

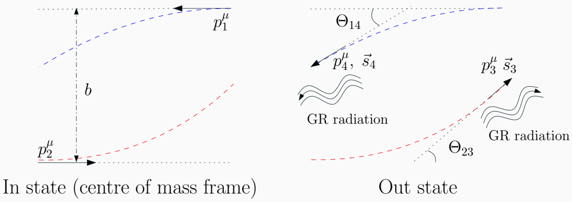
I will mainly focus on the gravitational field close to the bodies (dubbed "waveforms" at the horizon) from which we derive the energy $E_{\rm abs}$ and angular momentum $J_{\rm abs}$ absorbed by each black hole

The leading Post-Minkowskian (PM) expressions for the latter turn out to be simpler than the former (they follow from 1-loop, rather than a 2-loop diagrams in the amplitude language)

The setup

Consider two Schwarzschild BHs interacting gravitationally. What is the final state if they scatter with an initial relative Lorentz factor $\sigma = -\frac{p_1 p_2}{m_1^{\mathrm{BH}} m_2^{\mathrm{BH}}} = \sqrt{1 + p_\infty^2}$ and a large impact parameter b?

We expect the following qualitative picture



Among other effects, we expect the BH masses to increase and to have Kerr BHs in the final state due to the absorption of radiation

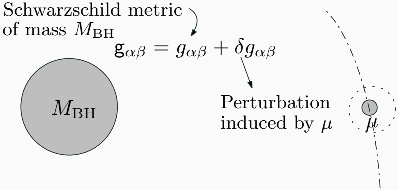
We'll use Black Hole Perturbation Theory (BHPT) to study this phenomenon at LO in the PM expansion, but at arbitrary velocities

Plan of the talk

- Introduce BHPT and how it's (mathematically) related to the analysis of $\mathcal{N}=2$ supersymmetric Gauge Theories
  Aminov, Grassi, Y. Hatsuda; Bianchi, Consoli, Grillo, Morales; Bonelli, Iossa, Lichtig, Tanzini; Fioravanti, Rossi
- Derive an expression for the waveforms at infinity and at the horizon in the regime $m_1^{\rm BH} = M \gg \mu = m_2^{\rm BH}$
- Discuss some general features of this result: highlight the tail contributions and the structure of the PM expansion
- Evaluate explicitly the leading PM expression of the waveforms at the horizon and derive the energy and angular momentum absorbed by the large black hole at leading order
- Derive the leading order energy and angular momentum absorbed by the small black hole
- Discuss the mass dependence of these observables

Linearised perturbations around Schwarzschild

Consider an Extreme Mass Ratio binary



In formulae

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2(\sin\theta)^2 d\phi^2$$

$$f(r) = 1 - \frac{2M}{r} \quad \text{with} \quad M = GM_{\rm BH}$$

$$\mathsf{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \delta C_{\alpha\beta\gamma\delta} + \mathcal{O}(\mu^2)$$

Weyl tensor

We can decouple the EoM of $h_{\alpha\beta} \equiv \delta g_{\alpha\beta}$ by using the Penrose scalars

$$\Psi_0 = \delta C_{\mu\nu\rho\sigma}\ell^\mu m^\nu \ell^\rho m^\sigma + \mathcal{O}(\mu^2)\,, \qquad \Psi_4 = \delta C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma + \mathcal{O}(\mu^2)$$

$$\text{with } \ell = \frac{1}{f(r)}\,\partial_t + \partial_r\,, \quad n = \frac{1}{2}\left(\partial_t - f(r)\partial_r\right), \quad m = \frac{1}{\sqrt{2}\,r}\left(\partial_\theta + \frac{i}{\sin\theta}\,\partial_\phi\right)$$

The rhs of Einstein's equations encodes the presence of the light BH

$$T^{\alpha\beta}(x) = \frac{\mu}{\sqrt{-g(x)}}\int \delta^{(4)}(x - x(\tau))\,\frac{dx^\alpha(\tau)}{d\tau}\frac{dx^\beta(\tau)}{d\tau}\,d\tau$$

$x^\alpha(t)$ parametrizes an open time-like geodesic in Schwarzschild

Teukolsky equations

As usual we separate variables by going to frequency space See Niels and Aditya's talks

spin-weighted spherical harmonics

$$\psi_s = \int \frac{d\omega}{2\pi} \sum_{\ell,m} e^{-i\omega t} R_{\ell m}(\omega, r)\, Y_s^{\ell m}(\theta,\phi)\,, \qquad \mathcal{T} = \int \frac{d\omega}{2\pi} \sum_{\ell,m} e^{-i\omega t} T_{\ell m}(\omega, r)\, Y_s^{\ell m}(\theta,\phi)$$

$s = 2, \ldots, s = -2$

$\psi_s = \{\Psi_0, \ldots, r^4 \Psi_4\}$

obtained from $T_{\alpha\beta}$ and its derivatives (Teukolsky)

We get an ODE for $R_{\ell m}$ ($\Delta(r) = r(r-2M)$)

$$\frac{1}{\Delta(r)^s}\frac{d}{dr}\left[\Delta(r)^{s+1}\frac{d}{dr}R_{\ell m}(r)\right] + \left(\frac{(r^2\omega)^2 - 2is(r-M)r^2\omega}{\Delta(r)} + 4is\omega r - (\ell-s)(\ell+s+1)\right) R_{\ell m}(r) = T_{\ell m}(r)$$

From ψ_s we can reconstruct the "radiative" gravitational field

$$u = t - r_*\,, \quad h_{rr}(u,r,\theta,\phi) = h_{rA}(u,r,\theta,\phi) = 0\,, \quad \gamma^{AB} h_{AB}(u,r,\theta,\phi) = 0$$

with $r_* = r + 2M\log(\frac{r}{2M} - 1)$ and $(h_{uu}, h_{ur}, h_{uA}) \sim \mathcal{O}(r^0)\,,\ h_{AB} \sim \mathcal{O}(r^1)$ as $r \to \infty$

$$\Psi_4 \underset{r\to\infty}{\overset{u\text{ fixed}}{\sim}} -\frac{1}{2}\,\partial_u^2 h\,,\ h = \bar{m}^\mu h_{\mu\nu}\bar{m}^\nu \qquad \Psi_0 \underset{r\to 2M}{\overset{v\text{ fixed}}{\sim}} \frac{(2M)^3}{\Delta(r)^2}(1 - 4M\partial_v)\partial_v \bar{h}\,,\ \bar{h} = m^\mu h_{\mu\nu} m^\nu$$

$$v = t + r_*\,, \quad h_{rr}(v,r,\theta,\phi) = h_{rA}(v,r,\theta,\phi) = 0\,, \quad \gamma^{AB} h_{AB}(v,r,\theta,\phi) = 0$$

with $h_{AB} \sim \mathcal{O}(1)\,,\ (h_{vv}, h_{vA}, h_{vr}) \sim \mathcal{O}(r - 2M)$ as $r \to 2M$

$A, B = (\theta, \phi)$
γ_{AB} round S^2 metric

Heun equation from supersymmetric gauge theories

The homogenous equation can be recast in the confluent Heun form

See Alessandro's talks

$$\text{Useful variables: } x = 4iM\omega\,, \quad y = 2i\omega r\,, \quad z = \tfrac{2M}{r}$$

$$R_{\ell m} = z^s (1-z)^{-\frac{s+1}{2}} \chi(z) \quad \Rightarrow \quad \frac{d^2\chi}{dz^2} + Q\,\chi = 0$$

$$Q = -\frac{x^2}{4z^4} + \frac{x m_3}{z^3} + \frac{1-(m_1-m_2)^2}{4(z-1)z} + \frac{1-(m_1+m_2)^2}{4(z-1)^2 z} + \frac{\mathfrak{u} - \frac{1}{4} + \frac{x}{2}(m_1+m_2-1)}{(z-1)z^2}$$

$$m_1 = \frac{x}{2}\,, \quad m_2 = s + \frac{x}{2}\,, \quad m_3 = s - \frac{x}{2}\,, \quad \mathfrak{u} = \left(\ell + \tfrac{1}{2}\right)^2 + \frac{x}{2} - sx + \frac{x^2}{4}$$

$z = 0$ ($r \to \infty$) is an irregular singularity, $z=1$ and $z \to \infty$ are the horizon and the BH singularity and are regular singularities of the Heun eq.

The parameters above are the masses m_i and the coupling x of a $\mathcal{N} = 2$ gauge theory. In this context the quantum period a appears naturally

$$\mathfrak{u}(a) = a^2 - x\,\partial_x \mathcal{F}_{\text{inst}}(a) \;\Leftrightarrow\; \frac{xM(a+1)}{P(a+1) - \frac{xM(a+2)}{P(a+2)-\cdots}} + \frac{xM(a)}{P(a-1) - \frac{xM(a-1)}{P(a-2)-\cdots}} - P(a) = 0$$

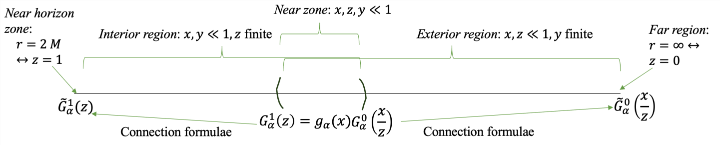
H Poghosyan

$$P(a) = a^2 - \mathfrak{u} + x\left(a + \frac{1}{2} - m_1 - m_2 - m_3\right)\,, \quad M(a) = \prod_{i=1}^{3}\left(a - m_i - \tfrac{1}{2}\right)$$

$$a = \sum_n a_n x^{2n} = \ell + \frac{1}{2} + x^2 \frac{\left(15\ell^4 + 30\ell^3 + 28\ell^2 + 13\ell + 24\right)}{8\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)} + O(x^4)$$

Interior and exterior regions

By using a, we can write perturbative expansions in x of the Heun solutions which cover different regions of the space $2M \leq r < \infty$



From Andrea's poster (see Cipriani, Di Russo, Fucito, Morales, Poghosyan, Poghossian)

$\alpha = \pm$ "Mathematica" parametrization of the Heun eq.

$$R_\alpha(r) \approx e^{-\frac{x}{2z}} (1-z)^{-\frac{s}{2}-\frac{m_1+m_2}{2}} \left(\frac{x}{z}\right)^{-1-s+m_3} \left[G^1_\alpha(z)\right],$$

$$H^1_\alpha(z) = z^{-\frac{1}{2}+\alpha a+m_3} \, {}_2F_1\left(\frac{1}{2}+\alpha a - m_1, \frac{1}{2}+\alpha a - m_2; 1+2\alpha a; z\right)$$

$$G^1_\alpha(z) = P_1(z)\, H^1_\alpha(z) + \widehat{P}_1(z)\, z H^{1\prime}_\alpha(z)$$

At $\mathcal{O}(x^k)$, P_1 and $\widehat{P}_1$ are polynomials in x and z/x of order k

$G^0_\alpha(y)$ is written in a similar way in terms of ${}_1F_1(\frac{1}{2} - m_3 - \alpha a, 1 - 2\alpha a; y)$

Notice that the monodromies around the irregular point depend on a

Also the ratio $g_\alpha(x) = \frac{G^1_\alpha}{G^0_\alpha}$ has a closed expression in terms of $\mathcal{F}_{\text{inst}}$

We can write the ingoing $\mathfrak{R}_{\text{in}}$/upgoing $\mathfrak{R}_{\text{up}}$ solutions in a PM expansion

Waveforms: $W^{\infty}_{\ell m,s}$ and $W^{H}_{\ell m,s}$

Writing the Green function in terms of the homogenous solutions, we get

$$R_{\ell m}(r) = \frac{\mathfrak{R}_{\rm up}(r)}{W}\int_{2M}^{r}\mathfrak{R}_{\rm in}(r')\Delta(r')^s\,T_{\ell m}(r')\,dr' + \frac{\mathfrak{R}_{\rm in}(r)}{W}\int_{r}^{\infty}\mathfrak{R}_{\rm up}(r')\Delta(r')^s\,T_{\ell m}(r')\,dr'$$

Wronskian

$$Z^{\infty}_{\ell m,s} \equiv \int_{2M}^{\infty}\mathfrak{R}_{\rm in}(r)\Delta(r)^s\,T_{\ell m}(r)\,dr \qquad Z^{H}_{\ell m,s} \equiv \int_{2M}^{\infty}\mathfrak{R}_{\rm up}(r)\Delta(r)^s T_{\ell m}(r)\,dr$$

The waveforms are directly related to the $Z_{\ell m,-2}$

$$h \underset{r\to\infty}{\sim} \frac{4G}{r}\int\frac{d\omega}{2\pi}\sum_{\ell m} e^{-i\omega u}W^{\infty}_{\ell m,-2}(\omega)Y^{\ell m}_{-2}(\theta,\phi)\,, \quad \bar{h} \underset{r\to 2M}{\sim} \frac{4G}{2M}\int\frac{d\omega}{2\pi}\sum_{\ell m} e^{-i\omega v}W^{H}_{\ell m,+2}(\omega)Y^{\ell m}_{+2}(\theta,\phi)$$

$$W^{\infty}_{\ell m,-2} = \frac{1}{2G\omega^2}\,Z^{\infty}_{\ell m,-2} \qquad W^{H}_{\ell m,+2} = -\frac{4\,(2M)^2(1-2iM\omega)(1-4iM\omega)}{G\ell(\ell^2-1)(\ell+2)+(-1)^{\ell+m}\,12iM\omega}Z^{H}_{\ell m,-2}$$

There are features that are independent of the source

$W^{\infty}_{\ell m,s}$ involves $\mathfrak{R}_{\rm in} = x^{s-m_3}\sum_{\alpha} C_\alpha R_\alpha$

$x^{-m_3} \sim (\omega M)^{2i\omega M}$: 1.5PN tail contribution

The dominant term is $\sim R_-(y) \sim (\omega r)^a$:

it yields the 2.5PN the tail of tail terms

$W^{H}_{\ell m,s}$ involves $\mathfrak{R}_{\rm up} = x^{a+s+\frac{1}{2}}\sum_{\alpha} D_\alpha R_\alpha$

$x^a \sim (\omega M)^{l+\frac{1}{2}+\cdots}$: higher harmonics are PM suppressed!

a determines both the electric and the magnetic tails of tails

Fucito, Morales, RR 2408.07329; it holds beyond BHPT and has a nice EFT interpretation, see Ivanov, Li, Parra-Martinez, Zhou

“Boostrapping” the waveforms from the probe limit

At LO as $M_{\rm BH} \gg \mu$ we can use $T^{\alpha\beta}$ as a source

This is sufficient to reconstruct the full $W^{\infty}_{\ell m,s}$ at leading PM order

Kovacs Thorne 1978

This is true also at the NLO (as for the deflection angle)

2511.13835 Heissenberg, RR

Waveform stripped of $\frac{4G}{r m_1 m_2}$

$$\tilde{\mathcal{W}}^{\mu\nu} = \tilde{\mathcal{W}}^{\mu\nu}_0 + \tilde{\mathcal{W}}^{\mu\nu}_1 + \tilde{\mathcal{W}}^{\mu\nu}_2 + \cdots$$

0PM 1PM 2PM

$$\tilde{\mathcal{W}}^{\mu\nu}_0 = \tilde{\mathcal{W}}^{\mu\nu}_0(v_1, v_2, b, k)\,,$$
$$\tilde{\mathcal{W}}^{\mu\nu}_1 = m_1\,\tilde{\mathcal{W}}^{\mu\nu}_{m_1}(v_1, v_2, b, k) + m_2\,\tilde{\mathcal{W}}^{\mu\nu}_{m_2}(v_1, v_2, b, k)\,,$$
$$\tilde{\mathcal{W}}^{\mu\nu}_2 = m_1^2\,\tilde{\mathcal{W}}^{\mu\nu}_{m_1^2}(v_1, v_2, b, k) + m_1 m_2\,\tilde{\mathcal{W}}^{\mu\nu}_{m_1 m_2}(v_1, v_2, b, k) + m_2^2\,\tilde{\mathcal{W}}^{\mu\nu}_{m_2^2}(v_1, v_2, b, k)\,,$$

From $1 \leftrightarrow 2$ exchange

From the probe limit $M \to m_1^{\rm BH}$ and $\mu \to m_2^{\rm BH}$

Genuine “1SF” term

When written in the appropriate (Lorentz + BMS) frame the waveform satisfies $\tilde{\mathcal{W}}^{\mu\nu}_{m_1^\alpha m_2^\beta}(v_1, v_2, b, k) = \tilde{\mathcal{W}}^{\mu\nu}_{m_1^\beta m_2^\alpha}(v_2, v_1, -b, k)$

At the end we can boost/translate the result back to the CoM frame

"Waveforms" at the horizon

Finding an explicit result for $Z_{\ell m,-2}$ involves an integral over the trajectory of the light BH. At leading PM order we need just $\ell = 2$

$$Z^H_{2(\pm 2),-2}(\mathsf{u}) = GM\mu\sqrt{\frac{\pi}{5}}\left(-\frac{2\mathsf{u}^2(\sigma^2+1)}{b^2\sqrt{\sigma^2-1}}K_0(\mathsf{u}) - \frac{2\mathsf{u}(-1\pm 2\mathsf{u}\sigma+2\sigma^2)}{b^2\sqrt{\sigma^2-1}}K_1(\mathsf{u})\right)$$

$$Z^H_{2(\pm 1),-2}(\mathsf{u}) = GM\mu\sqrt{\frac{\pi}{5}}\left(\mp\frac{4i\mathsf{u}^2\sigma}{b^2}K_0(\mathsf{u}) - \frac{4i\mathsf{u}(\mathsf{u}\pm 2\sigma)}{b^2}K_1(\mathsf{u})\right)$$

$$Z^H_{20,-2}(\mathsf{u}) = GM\mu\sqrt{\frac{24\pi}{5}}\left(\frac{\mathsf{u}^2\sqrt{\sigma^2-1}}{b^2}K_0(\mathsf{u}) + \frac{\mathsf{u}(2\sigma^2-1)}{b^2\sqrt{\sigma^2-1}}K_1(\mathsf{u})\right)$$

$$\mathsf{u} = \frac{\omega b}{p_\infty}$$

By recalling that $W^H_{\ell m,+2} \sim Z^H_{\ell m,-2}$, we can derive the energy and angular momentum absorbed by the large black hole

$$E_{\rm abs} = \int_0^\infty d\omega\,\frac{dE_{\rm abs}}{d\omega} = \frac{G}{\pi^2}\int_0^\infty d\omega\,\omega^2\sum_{\ell m}\left|W^H_{\ell m,+2}\right|^2$$

$$J_{\rm abs} = \int_0^\infty d\omega\,\frac{dJ_{\rm abs}}{d\omega} = \frac{G}{\pi^2}\int_0^\infty d\omega\,\omega\sum_{\ell m} m\left|W^H_{\ell m,+2}\right|^2$$

Absorbed energy and angular momentum

We have

$$\frac{dE_{\rm abs}}{d\mathsf{u}} = \frac{128G^7\mu^2M_{\rm BH}^6\mathsf{u}^4\sqrt{\sigma^2-1}}{45\pi b^7}\Big[\mathsf{u}^2\left(1-2\sigma^2+2\sigma^4\right)K_0(\mathsf{u})^2$$
$$+\mathsf{u}\left(1-8\sigma^2+8\sigma^4\right)K_0(\mathsf{u})K_1(\mathsf{u})+(1-8\sigma^2+8\sigma^4+\mathsf{u}^2(2\sigma^2-1))K_1(\mathsf{u})^2\Big]$$

$$\frac{dJ_{\rm abs}}{d\mathsf{u}} = \frac{256G^7\mu^2M_{\rm BH}^6\mathsf{u}^4\sigma K_1(\mathsf{u})\left[2\mathsf{u}\sigma^2K_0(\mathsf{u})+\left(4\sigma^2-3\right)K_1(\mathsf{u})\right]}{45\pi b^6}$$

$$E_{\rm abs} = \frac{5\pi G^7\mu^2M_{\rm BH}^6\sqrt{\sigma^2-1}\left(21\sigma^4-14\sigma^2+1\right)}{16b^7}\,,\quad J_{\rm abs} = \frac{\pi G^7\mu^2M_{\rm BH}^6\sigma\left(7\sigma^2-3\right)}{2b^6}$$

A balance law requires that $E_{\rm abs}$ is ΔM, the LO variation of the mass of the big black hole, which is the case! see the ΔM derived by Goldberger, Rothstein and by Jones, Ruf

Similarly $J_{\rm abs}$ should match the LO variation of spin of the large BH

to be checked

The viewpoint of the light BH

At distances $\sim G\mu$ from the small BH the gravitational field should be Schwazschild of mass μ plus tidal deformations induced by the large BH

The tidal corrections are written in terms of the background Weyl tensor (Schwarzschild in our case) and the geodesic of the light BH

Manassi, Alvi, Detweiler, Poisson

One can get the energy ($E_{\rm abs}^{(\mu)}$) and angular momentum ($J_{\rm abs}^{(\mu)}$) absorbed by the small black hole by using the formulas by Poisson gr-qc/0407050

$$E_{\rm abs}^{(\mu)} = \frac{5\pi G^7 M_{\rm BH}^2 \mu^6 \sqrt{\sigma^2-1}\left(21\sigma^4 - 14\sigma^2 + 1\right)}{16 b^7}\,, \quad J_{\rm abs}^{(\mu)} = \frac{\pi G^7 M_{\rm BH}^2 \mu^6 \sigma\left(7\sigma^2 - 3\right)}{2b^6}$$

At leading PM order they agree with $E_{\rm abs}$, $J_{\rm abs}$ after swapping $\mu \leftrightarrow M_{\rm BH}$

This is consistent with the EFT results at LO for $\Delta m_{1,2}$

Goldberger, Rothstein and by Jones, Ruf

At NLO, there are two independent structures: corrections in $GM_{\rm BH}/b$ and in $G\mu/b$. They will be swapped $\mu \leftrightarrow M_{\rm BH}$ to be checked

Conclusions

Dissipative observables include contributions related to absorption

Even if absorption effects are small in PM (G^7), it is interesting to calculate the LO contributions to this new phenomenon

We derived the LO absorbed fluxes for BH scattering

Another set of observables where to compare amplitudes and BHPT (with an unexpected insight from non-perturbative QFT!)

We considered also the case of scalar ($s = 0$) and electromagnetic ($s = \pm 1$) perturbations

PM results provide useful tests for numerical implementations of BHPT

Niels Warburton 2512.02274

It is interesting to derive the first subleading corrections for instance to test the expected mass dependence

work in progress